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L^2-BURAU MAPS AND L^2-ALEXANDER TORSIONS

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Abstract

It is well known that the Burau representation of the braid group can be used to recover the Alexander polynomial of the closure of a braid. We define L^2-Burau maps and use them to compute some L^2-Alexander torsions of links. As an application, we prove that the L^2-Burau maps distinguish more braids than the Burau representation.

1. Introduction

The L^2-Alexander torsions of 3-manifolds were introduced by Dubois, Friedl and Lück in [12] as generalizations of both the classical Reidemeister torsions and the L^2-Alexander invariant of knots of Li-Zhang [16]. These L^2-Alexander torsions are topological invariants that are functions on the positive real numbers. On the one hand, the L^2-Alexander torsions share many features with the Alexander polynomial: for instance they are symmetric [11] and provide information on the Thurston norm of the considered manifold [12, 14]. In the classical case, the Alexander polynomial is related to the (reduced) Burau representation of the braid group [7]. It is thus natural to ask whether a similar relation exists in the L^2 case. On the other hand, the L^2-Alexander torsions are, in a sense, stronger invariants than their classical counterparts: not only do they contain the simplicial volume of a 3-manifold [18], they also detect an infinite number of knots [3, 4] whereas the Alexander polynomial does not. Therefore, if an L^2-analogue of the Burau representation were to exist, one may expect it to distinguish more braids than the classical Burau representation.

In the present article, we introduce L^2-Burau maps and reduced L^2-Burau maps (see Section 4 for the precise definitions) and study their properties. Although these maps do not provide (anti-)representations of the braid group, they remain computable by recursive formulas and Fox calculus (see Lemma 4.1 and Proposition 4.2). Moreover, we show that one can extract the classical Burau representation from any L^2-Burau map (Proposition 4.5). Furthermore, we relate particular L^2-Burau maps of braids to L^2-Alexander torsions of the closures of these braids (Theorem 4.9). As an application, we provide an example of two braids indistinguishable under the Burau representation but which can be told apart by the L^2 version (Corollary 4.11). Our main tools rely on well-known results from the theory of L^2-invariants together with the homological interpretation of the Burau representation.

The paper is organized as follows. First, in Section 2, we recall some theory of L^2-invariants. Then, in Section 3, we fix notations regarding the braid group and recall the definition of the L^2-Alexander torsion together with its relation to Fox calculus. Finally, in

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Section 4, we introduce the L^2-Burau maps (Subsections 4.1 and 4.2) and prove the main results (Subsection 4.3).

2. Hilbert $\mathcal{N}(G)$-modules and the L^2-torsion

In this section we briefly review some theory of L^2-invariants. We begin with the von Neumann dimension of a finitely generated Hilbert $\mathcal{N}(G)$-module (Subsection 2.1) before moving on to the Fuglede-Kadison determinant (Subsection 2.2) and discussing L^2-homology (Subsection 2.3). We mostly follow [18] and [12].

2.1. The von Neumann dimension. Given a countable discrete group G, the completion of the algebra $\mathbb{C}[G]$ endowed with the scalar product $\left\langle \sum_{g\in G}\lambda_g g, \sum_{g\in G}\mu_g g\right\rangle := \sum_{g\in G}\lambda_g\overline{\mu_g}$ is the Hilbert space

$$\ell^2(G) := \left\{ \sum_{g\in G}\lambda_g g \mid \lambda_g \in \mathbb{C}, \sum_{g\in G}|\lambda_g|^2 < \infty \right\}$$

of square-summable complex functions on G. We denoted by $B(\ell^2(G))$ the algebra of operators on $\ell^2(G)$ that are bounded with respect to the operator norm.

Given $h \in G$, we define the corresponding *left-* and *right-multiplication operators* L_h and R_h in $B(\ell^2(G))$ as extensions of the automorphisms $(g \mapsto hg)$ and $(g \mapsto gh)$ of G. One can extend the operators R_h $\mathbb{C}$-linearly to operators $R_w\colon \ell^2(G) \to \ell^2(G)$ for any $w \in \mathbb{C}[G]$. Moreover, if $\ell^2(G)^n$ is endowed with its usual Hilbert space structure and $A = \left(a_{i,j}\right) \in M_{p,q}(\mathbb{C}[G])$ is a $\mathbb{C}[G]$-valued $p \times q$ matrix, then the right multiplication

$$R_A := \left(R_{a_{i,j}}\right)_{1\leqslant i\leqslant p, 1\leqslant j\leqslant q}$$

provides a bounded operator $\ell^2(G)^q \to \ell^2(G)^p$. Note that we shall consider elements of $\ell^2(G)^n$ as *column vectors* and suppose that matrices with coefficients in $B(\ell^2(G))$ act on the *left* (even though the coefficients may be *right*-multiplication operators).

The *von Neumann algebra* $\mathcal{N}(G)$ of the group G is the sub-algebra of $B(\ell^2(G))$ made up of G-equivariant operators (i.e. operators that commute with all left multiplications L_h). A *finitely generated Hilbert* $\mathcal{N}(G)$*-module* consists in a Hilbert space V together with a left G-action by isometries such that there exists a positive integer m and an embedding φ of V into $\ell^2(G)^m$. A *morphism of finitely generated Hilbert* $\mathcal{N}(G)$*-modules* $f\colon U \to V$ is a linear bounded map which is G-equivariant.

Denoting by e the identity element of G, the von Neuman algebra of G is endowed with the *trace* $\mathrm{tr}_{\mathcal{N}(G)}\colon \mathcal{N}(G) \to \mathbb{C}, \phi \mapsto \langle\phi(e), e\rangle$ which extends to $\mathrm{tr}_{\mathcal{N}(G)}\colon M_{n,n}(\mathcal{N}(G)) \to \mathbb{C}$ by summing up the traces of the diagonal elements.

Definition. The *von Neumann dimension* of a finitely generated Hilbert $\mathcal{N}(G)$-module V is defined as

$$\dim_{\mathcal{N}(G)}(V) := \mathrm{tr}_{\mathcal{N}(G)}(\mathrm{pr}_{\varphi(V)}) \in \mathbb{R}_{\geqslant 0},$$

where $\mathrm{pr}_{\varphi(V)}\colon \ell^2(G)^m \to \ell^2(G)^m$ is the orthogonal projection onto $\varphi(V)$.

The von Neumann dimension does not depend on the embedding of V into the finite direct sum of copies of $\ell^2(G)$.

2.2. The Fuglede-Kadison determinant. The *spectral density* $F(f)\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ of a morphism $f\colon U \to V$ of finitely generated Hilbert $\mathcal{N}(G)$-modules maps $\lambda \in \mathbb{R}_{\geqslant 0}$ to

$$F(f)(\lambda) := \sup\{\dim_{\mathcal{N}(G)}(L) | L \in \mathcal{L}(f,\lambda)\},$$

where $\mathcal{L}(f,\lambda)$ is the set of finitely generated Hilbert $\mathcal{N}(G)$-submodules of U on which the restriction of f has a norm smaller than or equal to λ. Since $F(f)(\lambda)$ is monotonous and right-continuous, it defines a measure $dF(f)$ on the Borel set of $\mathbb{R}_{\geqslant 0}$ solely determined by the equation $dF(f)(]a,b]) = F(f)(b) - F(f)(a)$ for all $a < b$.

Definition. The *Fuglede-Kadison determinant of f* is defined by

$$\det_{\mathcal{N}(G)}(f) = \begin{cases} \exp\left(\int_{0^+}^{\infty} \ln(\lambda)\, dF(f)(\lambda)\right) & \text{if } \int_{0^+}^{\infty} \ln(\lambda)\, dF(f)(\lambda) > -\infty, \\ 0 & \text{otherwise .} \end{cases}$$

Moreover, when $\int_{0^+}^{\infty} \ln(\lambda)\, dF(f)(\lambda) > -\infty$, one says that f *is of determinant class*.

If U and V have the same von Neumann dimension, we define the *regular Fuglede-Kadison determinant of f* denote by $\det^r_{\mathcal{N}(G)}(f)$ as $\det_{\mathcal{N}(G)}(f)$ when f is injective, and zero otherwise. For later use, let us mention the following property of the determinant (see [18] for the proof).

Proposition 2.1. *Let G be a countable discrete group. If $g \in G$ is of infinite order, then for all $t \in \mathbb{C}$ the operator $Id - tR_g$ is injective and* $\det^r_{\mathcal{N}(G)}(Id - tR_g) = \max(1, |t|)$.

2.3. L^2-torsion of a finite Hilbert $\mathcal{N}(G)$-chain complex. A *finite Hilbert $\mathcal{N}(G)$-chain complex C_** is a sequence of morphisms of finitely generated Hilbert $\mathcal{N}(G)$-modules

$$C_* = \left(0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0\right)$$

such that $\partial_p \circ \partial_{p+1} = 0$ for all p. The *p-th L^2-homology* of C_* is the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_p^{(2)}(C_*) := \operatorname{Ker}(\partial_p)/\overline{\operatorname{Im}(\partial_{p+1})}.$$

The *p-th L^2-Betti number of C_** is defined as $b_p^{(2)}(C_*) := \dim_{\mathcal{N}(G)}(H_p^{(2)}(C_*))$. A finite Hilbert $\mathcal{N}(G)$-chain complex C_* is *weakly acyclic* if its L^2-homology is trivial (i.e. if all its L^2-Betti numbers vanish) and of *determinant class* if all the operators ∂_p are of determinant class.

The following result is a reformulation of [18, Theorem 1.21 and Theorem 3.35 (1)]:

Proposition 2.2. *Let $0 \to C_* \xrightarrow{\iota_*} D_* \xrightarrow{\rho_*} E_* \to 0$ be an exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes. If two of the finite Hilbert $\mathcal{N}(G)$-chain complexes C_*, D_*, E_* are weakly acyclic (respectively weakly acyclic and of determinant class), then the third is as well.*

DEFINITION. The L^2-*torsion* of a finite Hilbert $\mathcal{N}(G)$-chain complex C_* is defined as

$$T^{(2)}(C_*) := \prod_{i=1}^{n} \det{}_{\mathcal{N}(G)}(\partial_i)^{(-1)^i} \in \mathbb{R}_{>0}$$

when C_* is weakly acyclic and of determinant class, and as $T^{(2)}(C_*) := 0$ otherwise.

Let $C_* = (0 \to \ell^2(G)^k \xrightarrow{\partial_2} \ell^2(G)^{k+l} \xrightarrow{\partial_1} \ell^2(G)^l \to 0)$ be a finite Hilbert $\mathcal{N}(G)$-chain complex and let $J \subset \{1, \ldots, k+l\}$ be a subset of size l. Viewing ∂_1, ∂_2 as matrices with coefficients in $B(\ell^2(G))$, denote by $\partial_1(J)$ the operator composed of the columns of ∂_1 indexed by J, and by $\partial_2(J)$ the operator obtained from ∂_2 by deleting the rows indexed by J. We refer to [12, Lemma 3.1] for the proof of the following proposition.

Proposition 2.3. *Assume that $\partial_1(J)$ is injective and of determinant class. Then*

$$T^{(2)}(C_*) = \frac{\det^r_{\mathcal{N}(G)}(\partial_2(J))}{\det^r_{\mathcal{N}(G)}(\partial_1(J))}.$$

In particular, $\partial_2(J)$ is injective and of determinant class if and only if C_* is weakly acyclic and of determinant class, and in this case one can write

$$T^{(2)}(C_*) = \frac{\det_{\mathcal{N}(G)}(\partial_2)}{\det_{\mathcal{N}(G)}(\partial_1)} = \frac{\det^r_{\mathcal{N}(G)}(\partial_2(J))}{\det^r_{\mathcal{N}(G)}(\partial_1(J))}.$$

3. Topological preliminaries

In this section, we start by reviewing the braid group (Subsection 3.1), before discussing the L^2-homology of CW-complexes (Subsection 3.2) and the L^2-Alexander torsions together with their relation to Fox calculus (Subsection 3.3).

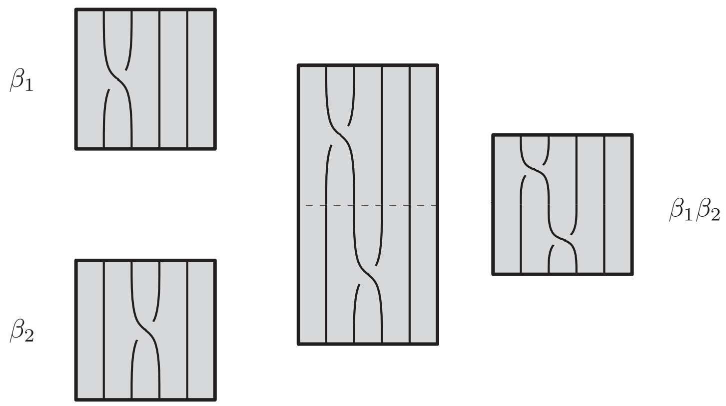

Fig. 1. Two braids β_1, β_2 and their composition, the braid $\beta_1\beta_2$.

3.1. The braid group. Following Birman [6], we start by recalling some well-known properties of the braid group B_n including its *right* action on the free group F_n. In contrast with Birman however, the composition of maps will be written in the usual way (from right to left) which leads to an *anti*-representation $B_n \to \mathrm{Aut}(F_n)$.

Let D^2 be the closed unit disk in $\mathbb{R}^2$. Fix a set of $n \geq 1$ punctures $p_1, p_2, \ldots, p_n$ in the interior of D^2. We shall assume that each p_i lies in $(-1, 1) = \mathrm{Int}(D^2) \cap \mathbb{R}$ and $p_1 < p_2 < \cdots < p_n$. A *braid with n strands* is an n-component piecewise linear one-dimensional submanifold β of the cylinder $D^2 \times [0, 1]$ whose boundary is $\bigsqcup_{i=1}^n p_i \times \{0, 1\}$, and where the projection to $[0, 1]$ maps each component of β homeomorphically onto $[0, 1]$. Two braids β_1 and β_2 are *isotopic* if there is a self-homeomorphism H of $D^2 \times [0, 1]$ which keeps $D^2 \times \{0, 1\} \cup \partial D^2 \times [0, 1]$ fixed and such that $H(\beta_1) = \beta_2$. The *braid group* B_n consists of the set of isotopy classes of braids. The identity element is given by the *trivial braid* $\xi_n := \{p_1, p_2, \ldots, p_n\} \times [0, 1]$ while the composition $\beta_1\beta_2$ consists in gluing β_1 on top of β_2 and shrinking the result by a factor 2 (see Figure 1).

The braid group B_n can also be seen as the set of isotopy classes of orientation-preserving homeomorphisms of $D_n := D^2 \setminus \{p_1, \ldots, p_n\}$ fixing the boundary pointwise. Either way, B_n admits a presentation with $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ subject to the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for each i, and $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i - j| > 2$. Topologically, the generator σ_i is the braid whose i-th component passes over the $i+1$-th component.

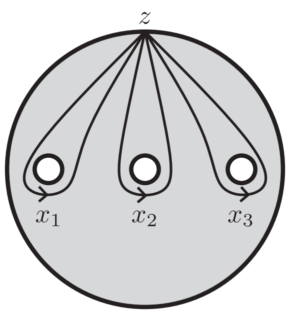

Fig. 2. The punctured disk D_3.

Fix a base point z of D_n and denote by x_i the simple loop based at z turning once around p_i counterclockwise for $i = 1, 2, \ldots, n$ (see Figure 2). The group $\pi_1(D_n)$ can then be identified with the free group F_n on the x_i. If H_β is a homeomorphism of D_n representing a braid β, then the induced automorphism h_β of the free group F_n depends only on β. It follows from the way we compose braids that $h_{\alpha\beta} = h_\beta \circ h_\alpha$, and the resulting *right* action of B_n on F_n can be explicitly described by

$$h_{\sigma_i}(x_j) = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{if } j = i, \\ x_i & \text{if } j = i+1, \\ x_j & \text{otherwise.} \end{cases}$$

The *closure* of a braid $\beta \in B_n$ is the oriented link $\hat{\beta}$ in the three-sphere obtained from β by adding n parallel strands in $S^3 \setminus (D^2 \times [0, 1])$ (see Figure 3).

3.2. L^2-homology of CW-complexes. Following [18, 12], we recall the definition of the L^2-homology of a CW-complex associated with an admissible triple. We then make an explicit computation in the case of the punctured disk.

Let X be a compact connected CW-complex endowed with a basepoint z, and let Y be a connected CW-subcomplex of X. We denote by $p\colon \widetilde{X} \to X$ the universal cover of X and

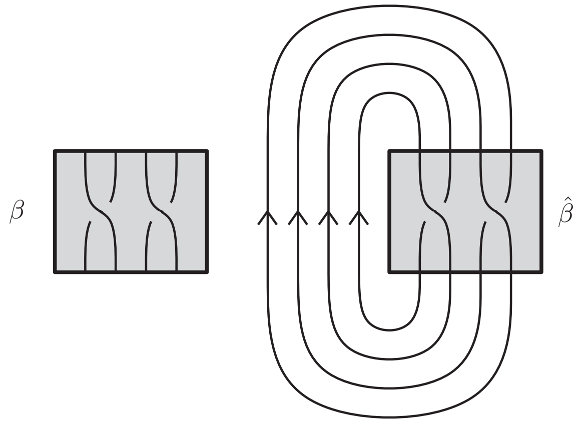

Fig. 3. The closure of a braid.

write $\widetilde{Y} = p^{-1}(Y)$. Setting $\pi = \pi_1(X, z)$, an *admissible triple* (π, ϕ, γ) consists in homomorphisms $\phi\colon \pi \to \mathbb{Z}$ and $\gamma\colon \pi \to G$ such that ϕ factors through γ. Given such a triple and $t > 0$, if we denote by

$$\kappa(\pi, \phi, \gamma, t)\colon \mathbb{Z}[\pi] \to \mathbb{R}[G]$$

the ring homomorphism determined by $g \mapsto t^{\phi(g)}\gamma(g)$ for $g \in \pi$, then there is a right action of π on $\ell^2(G)$ given by $a \cdot g = R_{\kappa(\pi,\phi,\gamma,t)(g)}(a)$, where $a \in \ell^2(G)$ and $g \in \pi$; this turns $\ell^2(G)$ into a right $\mathbb{Z}[\pi]$-module.

On the other hand, the natural left action of $\pi = \pi_1(X, z)$ on $\widetilde{X}$ gives rise to a left $\mathbb{Z}[\pi]$-module structure on the cellular chain complex $C_*(\widetilde{X}, \widetilde{Y})$. The *$\mathcal{N}(G)$-cellular chain complex* of the pair (X, Y) associated to (ϕ, γ, t) is the finite Hilbert $\mathcal{N}(G)$-chain complex

$$C_*^{(2)}(X, Y; \phi, \gamma, t) = \ell^2(G) \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}, \widetilde{Y}),$$

and the *L^2-homology* of (X, Y) associated to (ϕ, γ, t), denoted $H_*^{(2)}(X, Y; \phi, \gamma, t)$, is obtained by taking the L^2-homology of $C_*^{(2)}(X, Y; \phi, \gamma, t)$.

Lemma 3.1. *Given $z \in D_n$, for all admissible (π, ϕ, γ) and all $t > 0$, the finitely generated Hilbert $\mathcal{N}(G)$-module $H_1^{(2)}(D_n, z; \phi, \gamma, t)$ has von Neumann dimension n.*

Proof. The punctured disk D_n is simple homotopy equivalent to X, the wedge of the n loops representing the generators of $\pi_1(D_n)$ described in Subsection 3.1. As a consequence, it follows from [18, Theorem 1.21] and the proof of [2, Theorem 2.12] that $H_1^{(2)}(X, z; \phi, \gamma, t)$ and $H_1^{(2)}(D_n, z; \phi, \gamma, t)$ have the same von Neumann dimension. Thus it suffices to prove the claim for X.

Choose a cellular decomposition of this latter space X consisting of the 0-cell z (the basepoint of the wedge) and one 1-cell x_i for each loop. For $i = 1, 2, \ldots, n$, let $\widetilde{x_i}$ be the lift of x_i starting at an (arbitrary) fixed lift of z. With this cell structure, the $\mathcal{N}(G)$-cellular chain complex of the pair (X, z) associated to (ϕ, γ, t) is $0 \to C_1^{(2)}(X, z; \phi, \gamma, t) \to C_0^{(2)}(X, z; \phi, \gamma, t) \to 0$, where

$$C_1^{(2)}(X, z; \phi, \gamma, t) = \ell^2(G) \otimes_{\mathbb{Z}[\pi]} C_1(\widetilde{X}, \widetilde{z}) \cong \bigoplus_{i=1}^{n} \ell^2(G)\widetilde{x_i}.$$

Since $C_0^{(2)}(\widetilde{X},\widetilde{z})$ vanishes, $H_1^{(2)}(X,z;\phi,\gamma,t) = C_1^{(2)}(X,z;\phi,\gamma,t)$ and the claim follows. □

Given an admissible triple $(\pi_1(X',z'),\phi,\gamma\colon \pi_1(X',z') \to G)$, note that a basepoint-preserving homeomorphism of pairs $F\colon (X,Y) \to (X',Y')$ induces an isomorphism $f\colon \pi_1(X) \to \pi_1(X')$ and isomorphisms of finitely generated Hilbert $\mathcal{N}(G)$-modules

$$H_i^{(2)}(F)\colon H_i^{(2)}(X,Y;\phi\circ f,\gamma\circ f,t) \to H_i^{(2)}(X',Y';\phi,\gamma,t).$$

The precomposition by f is required for the homeomorphism F to induce a well-defined chain map.

EXAMPLE 3.2. Fix a basepoint $z \in \partial D_n$ as in Figure 2. Let $H_\beta : D_n \to D_n$ be a homeomorphism representing a braid $\beta \in B_n$. As H_β fixes the boundary of the disk, it lifts uniquely to a homeomorphism $\widetilde{H}_\beta\colon \widetilde{D}_n \to \widetilde{D}_n$ which preserves a fixed lift of z. Up to homotopy, this lift depends uniquely on the isotopy class of H_β and consequently the map induced on the chain group $C_1(\widetilde{D}_n,\widetilde{z})$ depends uniquely on the braid β.

Denote by $\phi\colon \pi_1(D_n) \to \mathbb{Z}$ the epimorphism defined by $x_i \mapsto 1$. Fixing $t > 0$ and a homomorphism $\gamma\colon \pi_1(D_n) \to G$ through which ϕ factors, each braid β induces a well-defined isomorphism of finitely generated Hilbert $\mathcal{N}(G)$-modules:

$$H_1^{(2)}(H_\beta)\colon H_1^{(2)}(D_n,z;\phi\circ h_\beta,\gamma\circ h_\beta,t) \to H_1^{(2)}(D_n,z;\phi,\gamma,t).$$

Since $\phi\circ h_\beta = \phi$ for all braids $\beta \in B_n$, from now on we shall write ϕ instead of $\phi \circ h_\beta$.

3.3. L^2-Alexander torsion and Fox calculus. Following [12], we define the L^2-Alexander torsion and outline its relation to Fox calculus. We also recall the definition of the L^2-Alexander torsion associated to a link and discuss its behavior on split links.

Given a compact connected CW-complex X, fix an admissible triple $(\pi_1(X),\phi,\gamma)$.

DEFINITION. The *L^2-Alexander torsion* of (X,ϕ,γ) at $t > 0$ is defined as

$$T^{(2)}(X,\phi,\gamma)(t) := T^{(2)}\left(C_*^{(2)}(X;\phi,\gamma,t)\right).$$

Observe that $T^{(2)}(X,\phi,\gamma)(t) \neq 0$ if and only if $C_*^{(2)}(X;\phi,\gamma,t)$ is weakly acyclic and of determinant class.

Note that L^2-Alexander torsions are only defined up to multiplication by $(t \mapsto t^k)$ with $k \in \mathbb{Z}$. For this reason, we shall write $f(t) \doteq g(t)$ if f is equal to g up to multiplication by $(t \mapsto t^k)$ for $k \in \mathbb{Z}$. Moreover the L^2-Alexander torsions are invariant by simple homotopy equivalence [9, 18, 12, 2]. Using this fact, we briefly review Fox calculus and outline how it can be used to compute the L^2-Alexander torsion.

Denoting by F_n the free group on $x_1, x_2, \ldots, x_n$, the *Fox derivative* (first introduced by Fox [13]) $\dfrac{\partial}{\partial x_i} : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$ is the linear extension of the map defined on elements of F_n by

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij}, \qquad \frac{\partial x_j^{-1}}{\partial x_i} = -\delta_{ij}x_j^{-1}, \qquad \frac{\partial(uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} + u\frac{\partial v}{\partial x_i}.$$

If $P = \langle x_1,\ldots,x_n | r_1,\ldots,r_m\rangle$ is a presentation of a group π, construct the 2-complex W_P with one 0-cell v, n oriented 1-cells labeled $x_1, x_2, \ldots, x_n$ and m oriented 2-cells $c_1, c_2, \ldots, c_m$ with each ∂c_j glued to the 1-cells according to the word r_i. Note that $\pi_1(W_P) = \pi$ and let $\widetilde{v}, \widetilde{x_i}$ and $\widetilde{c}_j$ be corresponding lifts to the universal cover $p : \widetilde{W}_P \to W_P$ (i.e. each $\widetilde{x_i}$ starts at $\widetilde{v}$ and the first word in the boundary of $\widetilde{c}_j$ is of the form $\widetilde{x_i}$).

Denote by $\mathrm{pr} : \mathbb{Z}[F_n] \to \mathbb{Z}[\pi]$ the ring homomorphism induced by the quotient map. The $\mathbb{Z}[\pi]$-module $C_1(\widetilde{W}_P, p^{-1}(v))$ is generated by the $\widetilde{x_i}$, and if w is a word in the x_i, then its lift $\widetilde{w}$ (viewed as a 1-chain in the universal cover) can be written as

$$\widetilde{w} = \sum_{i=1}^{n} \mathrm{pr}\left(\frac{\partial w}{\partial x_i}\right)\widetilde{x_i}.$$

Since the boundary map ∂_2 of the chain complex $C_*(\widetilde{W}_P)$ sends $\widetilde{c}_j$ to the lift of r_j beginning at $\widetilde{v}$, the previous equation specializes to

$$\partial_2(\widetilde{c}_j) = \sum_{i=1}^{n} \mathrm{pr}\left(\frac{\partial r_j}{\partial x_i}\right)\widetilde{x_i}.$$

We shall assume that the elements in the chain complex $C_*(\widetilde{W}_P)$ of free left $\mathbb{Z}[\pi]$-modules are column vectors and that the matrices of the differentials act by left multiplication. Consequently, ∂_2 is represented by the $(n \times m)$ matrix whose (i, j)-coefficient is $\mathrm{pr}\left(\frac{\partial r_j}{\partial x_i}\right)$.

Combining these remarks with Propositions 2.1, 2.3 and the fact that for any integer k and $t > 0$, $\max\left(1, t^k\right) = t^{\frac{k-|k|}{2}} \max(1, t)^{|k|} \doteq \max(1, t)^{|k|}$, the following result is immediate.

Proposition 3.3. *Let $P = \langle x_1, \ldots, x_n | r_1, \ldots, r_{n-1}\rangle$ be a deficiency one presentation of a group π, fix $t > 0$ and let $(\pi, \phi\colon \pi \to \mathbb{Z}, \gamma\colon \pi \to G)$ be an admissible triple. If one denotes by A the matrix in $M_{n-1,n-1}(\mathbb{C}[G])$ whose (i, j) component is*

$$\kappa(\pi, \phi, \gamma, t)\left(\mathrm{pr}\left(\frac{\partial r_j}{\partial x_i}\right)\right)$$

and one assumes that $\gamma(x_n)$ has infinite order in G, then

$$T^{(2)}(W_P, \phi, \gamma)(t) \doteq \frac{\det^r_{\mathcal{N}(G)}(R_A)}{\det^r_{\mathcal{N}(G)}\left(t^{\phi(x_n)} R_{\gamma(x_n)} - Id\right)} \doteq \frac{\det^r_{\mathcal{N}(G)}(R_A)}{\max(1, t)^{|\phi(x_n)|}}.$$

Moreover, if M is an irreducible 3-manifold with non-empty toroidal boundary and infinite $\pi = \pi_1(M)$, then

$$T^{(2)}(M, \phi, \gamma)(t) \doteq \frac{\det^r_{\mathcal{N}(G)}(R_A)}{\det^r_{\mathcal{N}(G)}(t^{\phi(x_n)} R_{\gamma(x_n)} - Id)} \doteq \frac{\det^r_{\mathcal{N}(G)}(R_A)}{\max(1, t)^{|\phi(x_n)|}}.$$

The second part of the above proposition uses the fact that the L^2-Alexander torsions are invariant under simple homotopy equivalence and the following Lemma 3.4. Although the content of this lemma is known [1, Section 3.2], we present a short proof that (somewhat appropriately) uses L^2-Betti numbers.

Lemma 3.4. *Let M be an irreducible 3-manifold with non-empty toroidal boundary and infinite fundamental group. If P is a deficiency one presentation of $\pi_1(M)$, then M is simple homotopy equivalent to W_P.*

Proof. As M is an irreducible 3-manifold with infinite fundamental group, it is aspherical [1, Paragraph C.1]. Since the Whitehead group of the fundamental group of a compact, orientable, non-spherical irreducible 3-manifold is trivial [1, Paragraph C.36], one only needs to show that M and W_P are homotopy equivalent. Consequently it remains to prove that W_P is aspherical: indeed both spaces would then be $K(\pi_1(M), 1)$'s. The first L^2-Betti number of a finite CW-complex depends only on its fundamental group [15, Section 2.2], therefore $b_1^{(2)}(W_P) = b_1^{(2)}(\pi_1(M)) = b_1^{(2)}(M)$. Since M is prime and has infinite fundamental group, $b_1^{(2)}(M) = 0$ by [18, Theorem 4.1] and therefore $b_1^{(2)}(W_P) = 0$. As P has deficiency one, $\chi(W_P) = 0$. From [15, Theorem 2.4], a connected finite 2-dimensional CW-complex X satisfying $-b_1^{(2)}(X) = \chi(X)$ is aspherical. Since $-b_1^{(2)}(W_P) = 0 = \chi(W_P)$, it follows that W_P is aspherical. □

Given an oriented link $L = L_1 \cup \ldots \cup L_\mu$ in S^3, denote by M_L its exterior and by $G_L = \pi_1(M_L)$ its group. Since any homomorphism $\phi\colon G_L \to \mathbb{Z}$ factors through the abelianization $\alpha_L\colon G_L \to H_1(M_L) \cong \mathbb{Z}^\mu$, it is determined by integers $n_1, \ldots, n_\mu$. Following the notation of [2], we denote by $(n_1, \ldots, n_\mu)\colon H_1(M_L) \to \mathbb{Z}$ the map sending the i-th meridian of L to n_i (thus $\phi = (n_1, \ldots, n_\mu) \circ \alpha_L$) and we call

$$T^{(2)}_{L,(n_1,\ldots,n_\mu)}(\gamma)(t) := T^{(2)}(M_L, \phi, \gamma)(t)$$

the *L^2-Alexander torsion associated to the link L* and the morphism $\gamma\colon G_L \to G$ at the value $t > 0$.

Although the next lemma is certainly well-known to the experts, to the best of our knowledge, it does not appear in this form in the literature. Therefore, we include it in our discussion.

Lemma 3.5. *Let L be a link, $t > 0$ and $n_1, \ldots, n_\mu \in \mathbb{Z}$. The following assertions are equivalent:*

(1) *L is split.*
(2) *$C_*^{(2)}(M_L, (n_1, \ldots, n_\mu) \circ \alpha_L, id, t)$ is not weakly acyclic.*
(3) *The L^2-Alexander torsion $T^{(2)}_{L,(n_1,\ldots,n_\mu)}(id)(t)$ vanishes.*

Proof. If the μ-component link L is not split, then its exterior M_L is irreducible, and it follows from [19] that for all integers $n_1, \ldots, n_\mu$ and all $t > 0$, $C_*^{(2)}(M_L, (n_1, \ldots, n_\mu) \circ \alpha_L, id, t)$ is weakly acyclic and of determinant class. Thus, in this case $T^{(2)}_{L,(n_1,\ldots,n_\mu)}(id)(t)$ is non-zero, proving that (3) ⇒ (1). Moreover, (2) ⇒ (3) is immediate.

Let us prove (1) ⇒ (2). If L is split, then M_L is not irreducible, and one can write $M_L = M_1 \sharp \ldots \sharp M_r$, where the M_i are irreducible link exteriors in S^3. Let us order the M_i so that

$$M_L = (M_1 \setminus B^3) \cup \left(\bigcup_{i=2}^{r-1} (M_i \setminus (B^3 \sqcup B^3)) \right) \cup (M_r \setminus B^3),$$

where the intersection is a disjoint union of $r-1$ spheres S^2. Fix $t > 0$, integers $n_1, \ldots, n_\mu \in \mathbb{Z}$ (we denote $\phi_L = (n_1, \ldots, n_\mu) \circ \alpha_L$) and let j_i be the group monomorphism induced by the inclusion of M_i minus one or two balls into M_L. An immediate generalization of the proof of [2, Theorem 3.1] (see also [18]) implies that

$$0 \to \bigoplus_{i=1}^{r-1} C_*^{(2)}(S^2, 1, 1, t) \to \begin{matrix} C_*^{(2)}(M_1 \setminus B^3, \phi_L \circ j_1, j_1, t) \oplus \\ \bigoplus_{i=2}^{r-1} C_*^{(2)}(M_i \setminus (B^3 \sqcup B^3), \phi_L \circ j_i, j_i, t) \\ \oplus C_*^{(2)}(M_r \setminus B^3, \phi_L \circ j_r, j_r, t) \end{matrix} \to C_*^{(2)}(M_L, \phi_L, id, t) \to 0$$

is an exact sequence of finite Hilbert $\mathcal{N}(G_L)$-chain complexes.

Now, for all $i = 1, \ldots, r-1$, we add a term $\ell^2(G_L)\widetilde{B^3} \oplus \ell^2(G_L)\widetilde{B^3}$ to the i-th summand of the left part of the sequence and to the i-th and $(i+1)$-th summands of the middle part (one ball for each), where the boundary operators send one $\widetilde{B^3}$ to the corresponding $\widetilde{S^2}$ and the other to a corresponding $-\widetilde{S^2}$. Since this process does not change exactness of the sequence, it follows that

$$0 \to \bigoplus_{i=1}^{r-1} C_*^{(2)}(S^3, 1, 1, t) \to \bigoplus_{i=1}^{r} C_*^{(2)}(M_i, \phi_L \circ j_i, j_i, t) \to C_*^{(2)}(M_L, \phi_L, id, t) \to 0$$

remains an exact sequence of finite Hilbert $\mathcal{N}(G_L)$-chain complexes.

Recall that if $i\colon H \hookrightarrow G$ is an injective group homomorphism, we can construct an induction functor i_* from the category (finitely generated Hilbert $\mathcal{N}(H)$-modules, morphisms of finitely generated Hilbert $\mathcal{N}(H)$-modules) to (finitely generated Hilbert $\mathcal{N}(G)$-modules, morphisms of finitely generated Hilbert $\mathcal{N}(G)$-modules) such that $i_*(\ell^2(H)) = \ell^2(G)$, as explained in [18, Section 1.1.5]. Each j_i is an injective group homomorphism and thus induces an induction functor $(j_i)_*$. As weak acyclicity is unaffected by these induction functors $(j_i)_*$ (see [18, Lemma 1.24 (4)]), the first part of the proof applied to the irreducible pieces M_i shows that $\bigoplus_{i=1}^{r} C_*^{(2)}(M_i, \phi_L \circ j_i, j_i, t)$ is weakly acyclic. Since the left part of the above short exact sequence is not weakly acyclic (see [18, Theorem 1.35 (8)]), neither is $C_*^{(2)}(M_L, (n_1, \ldots, n_\mu) \circ \alpha_L, id, t)$ (by Proposition 2.2). □

4. The L^2-Burau maps and the L^2-Alexander torsions

In this section, we define the L^2-Burau maps (Subsection 4.1), the reduced L^2-Burau maps (Subsection 4.2) and relate the latter to some L^2-Alexander torsions of links (Subsection 4.3).

4.1. The L^2-Burau map. In this subsection, we define L^2-Burau maps and show how to compute them using Fox calculus. We wish to emphasize that since our conventions differ from [10] (see Subsection 3.1), the resulting maps nearly behave as *anti*-representations (instead of representations).

Denote by $\phi\colon \pi_1(D_n) \to \mathbb{Z}$ the epimorphism defined by $x_i \mapsto 1$. Fix $t > 0$ and a homomorphism $\gamma\colon \pi_1(D_n) \to G$ through which ϕ factors. Given a basepoint $z \in \partial D_n$, we saw in Example 3.2 that each braid $\beta \in B_n$ induces a well-defined isomorphism of finitely generated $\mathcal{N}(G)$-modules

$$H_1^{(2)}(H_\beta)\colon H_1^{(2)}(D_n, z; \phi, \gamma \circ h_\beta, t) \longrightarrow H_1^{(2)}(D_n, z; \phi, \gamma, t).$$

Using the same notations as in the proof of Lemma 3.1, we shall call the basis resulting from the isomorphism

$$H_1^{(2)}(D_n, z; \phi, \gamma, t) \cong \bigoplus_{i=1}^{n} \ell^2(G)\widetilde{x_i}$$

the *good basis* of $H_1^{(2)}(D_n, z; \phi, \gamma, t)$. With respect to the good bases of $H_1^{(2)}(D_n, z; \phi, \gamma \circ h_\beta, t)$ and $H_1^{(2)}(D_n, z; \phi, \gamma, t)$, the isomorphism of finitely generated $\mathcal{N}(G)$-modules $H_1^{(2)}(H_\beta)$ gives rise to a $n \times n$ matrix $\mathcal{B}_{t,\gamma}^{(2)}(\beta)$ with coefficients in $B(\ell^2(G))$.

Definition. The *L^2-Burau map* $\mathcal{B}_{t,\gamma}^{(2)}$ associated to the value $t > 0$ and the homomorphism γ sends a braid $\beta \in B_n$ to the matrix $\mathcal{B}_{t,\gamma}^{(2)}(\beta) \in M_{n,n}\left(B(\ell^2(G))\right)$ representing the isomorphism of finitely generated Hilbert $\mathcal{N}(G)$-modules defined above.

The next lemma shows that while the L^2-Burau map is generally not an (anti-) representation, it is nevertheless determined by the generators of B_n.

Lemma 4.1. *Given two braids $\alpha, \beta \in B_n$, the equation*

$$\mathcal{B}_{t,\gamma}^{(2)}(\alpha\beta) = \mathcal{B}_{t,\gamma}^{(2)}(\beta) \circ \mathcal{B}_{t,\gamma\circ h_\beta}^{(2)}(\alpha)$$

holds for all $t > 0$ and for all $\gamma\colon \pi_1(D_n) \to G$ through which ϕ factors.

Proof. Since the lift of $H_{\alpha\beta}$ to the universal cover coincides with the lift of $H_\beta \circ H_\alpha$, the composition

$$H_1^{(2)}(D_n, z; \phi, \gamma \circ h_{\alpha\beta}, t) \xrightarrow{\mathcal{B}_{t,\gamma\circ h_\beta}^{(2)}(\alpha)} H_1^{(2)}(D_n, z; \phi, \gamma \circ h_\beta, t) \xrightarrow{\mathcal{B}_{t,\gamma}^{(2)}(\beta)} H_1^{(2)}(D_n, z; \phi, \gamma, t)$$

coincides with the map $\mathcal{B}_{t,\gamma}^{(2)}(\alpha\beta)$. □

In particular, Lemma 4.1 shows that if one picks a homomorphism γ satisfying $\gamma \circ h_\beta = \gamma$ for each $\beta \in B_n$, then the L^2-Burau maps $\mathcal{B}_{t,\gamma}^{(2)}$ yield anti-representations of the braid group. More generally, fixing $\gamma\colon \pi_1(D_n) \to G$, the L^2-Burau maps $\mathcal{B}_{t,\gamma}^{(2)}$ provide anti-representations of $B_n^\gamma := \{\beta \in B_n \mid \gamma \circ h_\beta = \gamma\}$.

The next proposition shows that the L^2-Burau map can be computed via Fox calculus.

Proposition 4.2. *Let $\beta \in B_n$ be a braid. If one denotes by A the $(n\times n)$-matrix whose (i, j) component is*

$$\kappa(\pi_1(D_n), \phi, \gamma, t)\left(\frac{\partial(h_\beta(x_j))}{\partial x_i}\right) \in \mathbb{C}[G],$$

then $\mathcal{B}_{t,\gamma}^{(2)}(\beta)$ is equal to R_A.

Proof. Fix a lift of z to the universal cover $p\colon \widetilde{D}_n \to D_n$. Given a homeomorphism H_β representing a braid β, let $\widetilde{H}_\beta$ be the map induced by the lift of H_β on the chain group $C_1(\widetilde{D}_n, \widetilde{z})$ (where $\widetilde{z} = p^{-1}(z)$). As $H_1^{(2)}(D_n, z; \phi, \gamma, t) \cong \ell^2(G) \otimes_{\mathbb{Z}[\pi_1(D_n)]} C_1(\widetilde{D}_n, \widetilde{z})$, it remains to compute the operator $id \otimes \widetilde{H}_\beta$. Clearly $\widetilde{H}_\beta(\widetilde{x_j})$ is the lift of a loop representing $h_\beta(x_j)$ to the universal cover. Fox calculus then shows that on the chain level

$$\widetilde{H}_\beta(\widetilde{x_j}) = \sum_{i=1}^{n} \frac{\partial(h_\beta(x_j))}{\partial x_i}\widetilde{x_i}.$$

As we view elements of the left $\mathbb{Z}[\pi_1(D_n)]$-module $C_1(\widetilde{D_n}, \widetilde{z})$ as column vectors, $\widetilde{H_\beta}$ is represented by the $(n \times n)$ matrix whose (i, j) component is $\frac{\partial(h_\beta(x_j))}{\partial x_i}$. The claim now follows from the right $\mathbb{Z}[\pi_1(D_n)]$-module structures of $\ell^2(G)$. □

EXAMPLE 4.3. A short computation involving Fox calculus shows that

$$\frac{\partial(h_{\sigma_i}(x_i))}{\partial x_i} = \frac{\partial(x_i x_{i+1} x_i^{-1})}{\partial x_i} = 1 - x_i x_{i+1} x_i^{-1}, \quad \text{and} \quad \frac{\partial(h_{\sigma_i}(x_i))}{\partial x_{i+1}} = \frac{\partial(x_i x_{i+1} x_i^{-1})}{\partial x_{i+1}} = x_i.$$

Consequently, with respect to the good bases, the L^2-Burau maps of σ_i are given by

$$\mathcal{B}^{(2)}_{t,\gamma}(\sigma_i) = Id^{\oplus(i-1)} \oplus \begin{pmatrix} Id - tR_{\gamma(x_i x_{i+1} x_i^{-1})} & Id \\ tR_{\gamma(x_i)} & 0 \end{pmatrix} \oplus Id^{\oplus(n-i-1)}.$$

EXAMPLE 4.4. Using Proposition 4.2, let us illustrate Lemma 4.1 with an example. For $\sigma_1, \sigma_2 \in B_3$, one has

$$\mathcal{B}^{(2)}_{t,\gamma}(\sigma_2) = \begin{pmatrix} Id & 0 & 0 \\ 0 & Id - tR_{\gamma(x_2 x_3 x_2^{-1})} & Id \\ 0 & tR_{\gamma(x_2)} & 0 \end{pmatrix}, \mathcal{B}^{(2)}_{t,\gamma\circ h_{\sigma_2}}(\sigma_1) = \begin{pmatrix} Id - tR_{\gamma(x_1 x_2 x_3 x_2^{-1} x_1^{-1})} & Id & 0 \\ tR_{\gamma(x_1)} & 0 & 0 \\ 0 & 0 & Id \end{pmatrix},$$

and their composition is equal to

$$\mathcal{B}^{(2)}_{t,\gamma}(\sigma_2) \circ \mathcal{B}^{(2)}_{t,\gamma\circ h_{\sigma_2}}(\sigma_1) = \begin{pmatrix} Id - tR_{\gamma(x_1 x_2 x_3 x_2^{-1} x_1^{-1})} & Id & 0 \\ tR_{\gamma(x_1)} - t^2 R_{\gamma(x_1 x_2 x_3 x_2^{-1})} & 0 & Id \\ t^2 R_{\gamma(x_1 x_2)} & 0 & 0 \end{pmatrix},$$

which coincides with $\mathcal{B}^{(2)}_{t,\gamma}(\sigma_1\sigma_2)$.

Let us now relate the L^2-Burau maps to the classical Burau representation $\mathcal{B}$. Given $\beta \in B_n$, recall that a matrix for $\mathcal{B}(\beta) \in M_{n,n}(\mathbb{Z}[T, T^{-1}])$ can be obtained by computing $T^\phi\left(\frac{\partial h_\beta(x_i)}{\partial x_j}\right)$, where the ring homomorphism $T^\phi : \mathbb{Z}[F_n] \to \mathbb{Z}[T, T^{-1}]$ sends x_i to the indeterminate T. For any given γ and t, the L^2-Burau map $\mathcal{B}^{(2)}_{t,\gamma}$ holds at least as much information as the classical Burau representation, in the following sense:

Proposition 4.5. *Let $R_{M_{n,n}(\mathbb{C}[G])}$ denote $\{R_A \in B(\ell^2(G)^n) | A \in M_{n,n}(\mathbb{C}[G])\}$. Given $\beta \in B_n$, for any $t > 0$ and $\gamma : F_n \to G$, there exists a map $\Theta : R_{M_{n,n}(\mathbb{C}[G])} \to M_{n,n}(\mathbb{Z}[T, T^{-1}])$ such that $\Theta\left(\mathcal{B}^{(2)}_{t,\gamma}(\beta)\right) = \mathcal{B}(\beta)$. In particular, if $\alpha, \beta \in B_n$ and $\mathcal{B}^{(2)}_{t,\gamma}(\alpha) = \mathcal{B}^{(2)}_{t,\gamma}(\beta)$, then $\mathcal{B}(\alpha) = \mathcal{B}(\beta)$.*

Proof. Using Proposition 4.2, $\mathcal{B}^{(2)}_{t,\gamma}(\beta)$ is the right-multiplication operator R_A where the matrix A has $\kappa(F_n, \phi, \gamma, t)\left(\frac{\partial(h_\beta(x_j))}{\partial x_i}\right) \in \mathbb{C}[G]$ as its (i, j)-coefficient. By considering the map $\theta : B(\ell^2(G)^n) \to M_{n,n}(\ell^2(G))$ which evaluates an operator S on the n canonical (column) vectors of $\ell^2(G)^n$, one can extract $A = \theta(R_A)$ from R_A. Thus it only remains to recover $\mathcal{B}(\beta)$ from A.

Since $(\pi_1(D_n), \phi, \gamma)$ is an admissible triple, there exists a homomorphism $\psi : G \to \mathbb{Z}$ such that $\phi = \psi \circ \gamma$. Defining the homomorphism $T^\psi : G \to \{T^m; m \in \mathbb{Z}\} \subset \mathbb{Z}[T, T^{-1}]$ by $g \mapsto T^{\psi(g)}$, the (i, j)-coefficient of $\kappa(G, \psi, T^\psi, t^{-1})(A)$ is

$$\left(\kappa(G,\psi,T^{\psi},t^{-1})\circ\kappa(F_n,\phi,\gamma,t)\right)\left(\frac{\partial(h_\beta(x_j))}{\partial x_i}\right)=T^{\phi}\left(\frac{\partial h_\beta(x_j)}{\partial x_i}\right),$$

which is precisely the (j,i)-coefficient of $\mathcal{B}(\beta)$. The map $\Theta=\mathsf{tra}\circ\kappa(G,\psi,T^{\psi},t^{-1})\circ\theta|_{R_{M_{n,n}(\mathbb{C}[G])}}$ thus satisfies the assumptions of the proposition (where tra is the transpose operator). □

Remark 4.6. Although all L^2-Burau maps recover the Burau representation, different choices of $\gamma\colon F_n\to G$ produce various effects on the injectivity of the resulting maps and their defect to being anti-representations. On one end of the spectrum, if γ is the identity, the L^2-Burau maps $\mathcal{B}^{(2)}_{t,id}\colon B_n\to B(\ell^2(F_n)^n)$ are injective for all $t>0$ (since $B_n\to\mathrm{Aut}(F_n),\beta\mapsto h_\beta$ is injective and automorphisms of the free group are determined by their Fox jacobian [8, Proposition 9.8]). As G becomes smaller, the L^2-Burau maps $\mathcal{B}^{(2)}_{t,\gamma}$ lose in injectivity but edge closer to being actual anti-representations. As the proof of Proposition 4.5 demonstrates, a critical step appears when γ reaches T^{ϕ}: in this case, $\mathcal{B}^{(2)}_{t,T^{\phi}}(\beta)$ is an anti-representation which is equal to $R_{\mathsf{tra}(\mathcal{B}(\beta))}$ up to a change of variable; in particular it is known not to be faithful for $n\geq 5$ [17, 5].

Summarizing, the various L^2-Burau maps interpolate between the injective ones induced by the Artin representation and the classical Burau representation. Indeed, they all distinguish at least as many braids as the Burau representation (as shown in Proposition 4.5) but sometimes do better, as Corollary 4.11 will show (for L^2-Burau maps associated to a link group).

4.2. The reduced L^2-Burau map. In this subsection, we shall generalize the definition of the reduced Burau representation to the L^2-setting.

Instead of working with the free generators $x_1,x_2\dots,x_n$ of $\pi_1(D_n)$, consider the elements $g_1,g_2,\dots,g_n$, where $g_i=x_1x_2\cdots x_i$. The action of the braid group B_n on this new set of free generators for $\pi_1(D_n)$ is given by

$$h_{\sigma_i}(g_j)=\begin{cases}g_j & \text{if } j\neq i,\\ g_{i+1}g_i^{-1}g_{i-1} & \text{if } j=i\neq 1,\\ g_2g_1^{-1} & \text{if } j=i=1.\end{cases}$$

Let $\widetilde{g_i}$ be the lift of g_i starting at a fixed lift of z (note that $\widetilde{g_i}=\widetilde{x_1}+\ldots+(x_1\ldots x_{i-1})\widetilde{x_i}$). Using the same argument as in Lemma 3.1, one obtains the splitting

$$H_1^{(2)}(D_n,z;\phi,\gamma,t)=\bigoplus_{i=1}^{n-1}\ell^2(G)\widetilde{g_i}\oplus\ell^2(G)\widetilde{g_n}$$

for any $\gamma\colon F_n\to G$ through which ϕ factors. As g_n is always fixed by the action of the braid group, its lift $\widetilde{g_n}$ is fixed by the lift $\widetilde{H_\beta}$ of a homeomorphism H_β representing a braid β.

Definition. The *reduced L^2-Burau map* sends a braid β to the restriction $\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\beta)$ of the L^2-Burau map to the subspace of $H_1(D_n,z;\phi,\gamma\circ h_\beta,t)$ generated by $\widetilde{g}_1,\dots,\widetilde{g}_{n-1}$.

The next proposition now follows immediately.

Proposition 4.7. *If $\widetilde{\mathcal{B}}^{(2)}_{t,\gamma}(\beta)$ denotes the L^2-Burau matrix of a braid $\beta \in B_n$ with respect to the basis of the $\widetilde{g_i}$, then*

$$\widetilde{\mathcal{B}}^{(2)}_{t,\gamma}(\beta) = \begin{pmatrix} \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\beta) & 0 \\ V & Id \end{pmatrix},$$

where $V \in M_{1,n-1}(B(\ell^2(G)))$.

One can see the matrix of operators $\widetilde{\mathcal{B}}^{(2)}_{t,\gamma}(\beta)$ as a conjugate of $\mathcal{B}^{(2)}_{t,\gamma}(\beta)$ via a trigonal change of basis matrix between the good basis $(\widetilde{x_i})_{1\leqslant i\leqslant n}$ and the new basis $(\widetilde{g_i})_{1\leqslant i\leqslant n}$. In particular the reduced L^2-Burau map also satisfies the property of Lemma 4.1 :

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha\beta) = \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\beta) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_\beta}(\alpha).$$

EXAMPLE 4.8. Combining Proposition 4.2 and Proposition 4.7, the reduced L^2-Burau map of $\sigma_i \in B_n$ is given by

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_i) = Id^{\oplus(i-2)} \oplus \begin{pmatrix} Id & tR_{\gamma(g_{i+1}g_i^{-1})} & 0 \\ 0 & -tR_{\gamma(g_{i+1}g_i^{-1})} & 0 \\ 0 & Id & Id \end{pmatrix} \oplus Id^{\oplus(n-i-2)}$$

for $1 < i < n-1$, and for σ_1 and σ_{n-1} it is represented by

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_1) = \begin{pmatrix} -tR_{\gamma(g_2g_1^{-1})} & 0 \\ Id & Id \end{pmatrix} \oplus Id^{\oplus(n-3)},$$

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_{n-1}) = Id^{\oplus(n-3)} \oplus \begin{pmatrix} Id & tR_{\gamma(g_ng_{n-1}^{-1})} \\ 0 & -tR_{\gamma(g_ng_{n-1}^{-1})} \end{pmatrix}.$$

4.3. Relation to the L^2-Alexander torsions of links. In this subsection, we show how a particular L^2-Alexander torsion associated to a link can be computed from some reduced L^2-Burau maps. As an application, we exhibit two braids which are distinguished by the L^2-Burau maps but can not be told apart by the classical Burau representation.

Let X_β be the exterior of a braid $\beta \in B_n$ in the cylinder $D^2\times[0,1]$, and recall that ξ_n denotes the trivial braid with n strands. The manifold obtained by gluing X_β and X_{ξ_n} along $D_n \sqcup D_n$ is nothing but the exterior of the link $L' := \hat{\beta} \cup \partial D_n$ in S^3. Identify the free group F_n with $\pi_1(D_n)$ so that the free generators x_i correspond to the loops described in Subsection 3.1. As in Subsection 4.2, the elements $g_1, g_2, \ldots, g_n$ then also form a free generating set of $\pi_1(D_n)$. If x is a meridian of ∂D_n, then the fiberedness of $M_{L'}$ implies that $G_{L'}$ admits the presentation

$$P' = \langle g_1, \ldots, g_n, x | h_\beta(g_1) = xg_1x^{-1}, \ldots, h_\beta(g_n) = xg_nx^{-1}\rangle.$$

The exterior M_L of $L = \hat{\beta}$ can now be recovered by canonically pasting a solid torus on the boundary component of $M_{L'}$ corresponding to ∂D_n. Since $h_\beta(g_n) = g_n$ in the free group F_n, G_L thus admits the following deficiency one presentation:

$$P = \langle g_1, \ldots, g_n | h_\beta(g_1) = g_1, \ldots, h_\beta(g_{n-1}) = g_{n-1}\rangle.$$

Finally, denote by $\gamma_L \colon F_n \to G_L$ the resulting quotient map. This way, if one sets $\phi_L :=$

$(1,\dots,1)\circ\alpha_L$, then the map $\phi\colon \pi_1(D_n)\to\mathbb{Z}$ described in Subsection 4.1 factors as $\phi_L\circ\gamma_L$.

Theorem 4.9. *Given an oriented link L obtained as the closure of a braid $\beta\in B_n$, one has*

$$T^{(2)}_{L,(1,\dots,1)}(id)(t)\cdot\max(1,t)^n \doteq \det{}^r_{\mathcal{N}(G_L)}\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma_L}(\beta)-Id^{\oplus(n-1)}\right)$$

for all $t>0$.

Proof. Fix $t>0$ and assume that L is non-split. Performing Fox calculus on the presentation P yields

$$\frac{\partial(h_\beta(g_j)g_j^{-1})}{\partial g_i}=\frac{\partial(h_\beta(g_j))}{\partial g_i}-\delta_{ij}.$$

Since M_L is irreducible and the previously described presentation P of G_L has deficiency one, combining Proposition 3.3 with the definition of the reduced Burau representation then gives

$$T^{(2)}(M_L,\phi_L,id)(t)\doteq\frac{\det^r_{\mathcal{N}(G_L)}\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma_L}(\beta)-Id^{\oplus(n-1)}\right)}{\det^r_{\mathcal{N}(G_L)}(t^nR_{g_n}-Id)}=\frac{\det^r_{\mathcal{N}(G_L)}\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma_L}(\beta)-Id^{\oplus(n-1)}\right)}{\max(1,t)^n},$$

which proves the theorem in the non-split case.

Next, assume that L is split. Since Lemma 3.5 implies that $T^{(2)}_{L,(1,\dots,1)}(id)(t)=0$, it only remains to prove that $\det{}^r_{\mathcal{N}(G_L)}\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma_L}(\beta)-Id^{\oplus(n-1)}\right)$ also vanishes. By Proposition 3.3, the latter claim reduces to proving that $C^{(2)}_*(W_P,\phi_L,id,t)$ is not weakly acyclic. As $C^{(2)}_*(M_L,\phi_L,id,t)$ is not weakly acyclic (by Lemma 3.5), the L^2-version of the Torres formula [2, Theorem 3.8] implies that $C^{(2)}_*(M_{L'},\phi_L\circ Q,Q,t)$ is not weakly acyclic either, where $Q\colon G_{L'}\to G_L$ is the epimorphism induced by the inclusion $M_{L'}\subset M_L$. Since L' is non split, $M_{L'}$ is simply homotopy equivalent to $W_{P'}$ (by Lemma 3.4) and it follows that $C^{(2)}_*(W_{P'},\phi_L\circ Q,Q,t)$ is not weakly acyclic, by [2, Theorem 2.12]. Let v be the 0-cell of W_P, $g_1,\dots,g_n$ be its 1-cells and $r_1,\dots,r_{n-1}$ be its 2-cells. Similarly let v' be the 0-cell of $W_{P'}$, $g_1,\dots,g_n,x$ be its 1-cells and $r'_1,\dots,r'_n$ be its 2-cells. Denote the lifts to the universal covers as in Subsection 3.3 and set $D_1=\ell^2(G)\widetilde{x}$, $D_2=\ell^2(G)\widetilde{r}'_n$. A straightforward matrix computation involving Fox calculus now shows that

$$0\to C^{(2)}_*(W_P,\phi_L,id,t)\xrightarrow{\iota}C^{(2)}_*(W_{P'},\phi_L\circ Q,Q,t)\xrightarrow{\rho}D_*\to 0$$

is an exact sequence of finite Hilbert $\mathcal{N}(G_L)$-chain complexes, where $\iota_1(\widetilde{g_i})=\widetilde{g}'_i$, $\iota_2(\widetilde{r_i})=\widetilde{r}'_i$ for $i=1,\dots,n-1$ and ρ_1,ρ_2 are the obvious projections. As the boundary operator $D_2\to D_1$ is given by the injective operator $Id-t^nR_{g_n}$, the chain complex D_* is weakly acyclic. Since $C^{(2)}_*(W_{P'},\phi_L\circ Q,Q,t)$ is not weakly acyclic, neither is $C^{(2)}_*(W_P,\phi_L,id,t)$ by Proposition 2.2. This concludes the proof. □

Remark 4.10. If L is a knot K, then Theorem 4.9 can be expressed as

$$\Delta^{(2)}_K(t)\cdot\max(1,t)^{n-1}\doteq\det{}^r_{\mathcal{N}(G_K)}\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma_K}(\beta)-Id^{\oplus(n-1)}\right),$$

where $\Delta^{(2)}_K(t)$ is the L^2-Alexander invariant of K defined by Li-Zhang [16].

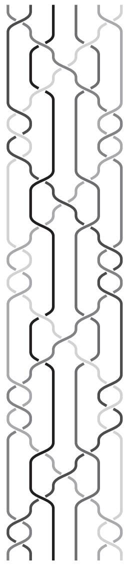

Fig.4. The braid $\beta \in B_6$.

Corollary 4.11. *There exist two braids which have the same image under the classical Burau representation but have different images under an L^2-Burau map with a non-injective γ.*

Proof. Long and Paton [17] proved that the braid $\beta \in B_6$ depicted in Figure 4 has the same image under the classical Burau representation as the trivial braid $\xi_6 \in B_6$. Taking any $t > 0$, we will prove that $\mathcal{B}^{(2)}_{t,\gamma_\beta}(\beta) \neq \mathcal{B}^{(2)}_{t,\gamma_\beta}(\xi_6)$, and to do this we will show that $\overline{\mathcal{B}}^{(2)}_{t,\gamma_\beta}(\beta) \neq \overline{\mathcal{B}}^{(2)}_{t,\gamma_\beta}(\xi_6)$: this is enough since the reduced L^2-Burau map is the upper left matricial part of the L^2-Burau map expressed in the basis of the $\widetilde{g_i}$. We claim that the closure L of β is a 6-component non-split link. To see this, define $\Gamma(L)$ to be the graph whose vertices are the components L_i of L and such that there is an edge between L_i and L_j when there exists a third component L_k such that $L_i \cup L_j \cup L_k$ is a non-split link. Since L being split implies $\Gamma(L)$ being disconnected, it suffices to show that $\Gamma(L)$ is connected. One can observe that all sublinks of L with three components are either trivial or the non-split link L_{10a140}, and there are enough of the second type so that $\Gamma(L)$ is connected.

Consequently, as L is non-split, $T^{(2)}_{L,(1,\dots,1)}(t)$ is non-zero for all $t > 0$ (by Lemma 3.5) and thus Theorem 4.9 implies that the operator $\overline{\mathcal{B}}^{(2)}_{t,\gamma_L}(\beta) - Id^{\oplus(n-1)}$ has non-zero regular Fuglede-Kadison determinant and is thus injective. The result follows immediately. □

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