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GLOBAL EXISTENCE OF SOLUTIONS TO AN n-DIMENSIONAL PARABOLIC-PARABOLIC SYSTEM FOR CHEMOTAXIS WITH LOGISTIC-TYPE GROWTH AND SUPERLINEAR PRODUCTION

Dedicated to Professor Masayasu Mimura on the occasion of his 75th birthday.

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Abstract

We study the global existence of solutions to an n-dimensional parabolic-parabolic system for chemotaxis with logistic-type growth. We introduce superlinear production of a chemoattractant. We then show the global existence of solutions in L_p space ($p > n$) under certain relations between the degradation and production orders.

1. Introduction

In the present paper we study a chemotaxis system with logistic growth:

$$\text{(E)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u) & \text{in } \Omega \times (0, \infty), \\ \tau \dfrac{\partial v}{\partial t} = \Delta v - v + g(u) & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) & \text{in } \Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and the space dimension $n \in \mathbb{N}$ is an arbitrary positive integer. The unknown functions $u(x, t)$ and $v(x, t)$ are the population density of bacteria and the concentration of a chemical substance at the position x and time t, respectively. The term $-\chi \nabla \cdot (u \nabla v)$ expresses the advection of bacteria due to chemotaxis. The coefficient χ is a positive constant, which shows chemotactic intensity. The function $f(u)$ is the proliferation and the reduction in numbers due to death of bacteria (we refer to the combined effects of proliferation and reduction in numbers simply as growth). Typical $f(u)$'s are quadratic $u(1 - u)$ and cubic $u(1 - u)(u - \gamma)$, $0 < \gamma < 1/2$, logistic growth functions [12]. The coefficient τ is a positive constant, which shows the time scale of reaction and diffusion of v. The function $g(u)$ is the secretion of chemical substance v by bacteria. A typical $g(u)$ is a linear function; and some nonlinear forms of $g(u)$ have been proposed, such as the saturating function $u/(1 + \gamma u)$, as used in the nonlinear signal kinetics

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model. For these topics, see the book by Murray [15], and the review articles by Hillen and Painter [4] and by Tindall, Maini, Porter and Armitage [23].

We consider the global existence of solutions to (E). In the context of global existence, the degradation of the growth $f(u)$ can be considered as an inhibitory effect on the increase of u. Indeed, if there is no growth ($f(u) \equiv 0$) and the production $g(u)$ is linear, then the system (E) reduces to the classical parabolic-parabolic Keller-Segel system [10]. In the Keller-Segel system, it is known that when $n = 2$, a finite-time blow-up with a δ-function singularity of u occurs if $\chi \|u_0\|_{L_1}$ is sufficiently large [3, 7]. In contrast, when $n \geq 3$, no restriction on χ and $\|u_0\|_{L_1}$ is necessary for the occurrence of blow-up [26]. For other topics on the Keller-Segel system, see Horstmann's review papers [5, 6] and the references therein. On the contrary, if $f(u)$ is quadratic and $g(u)$ is linear, then blow-up does not occur and global existence of solutions is assured even if $\|u_0\|_{L^1}$ and χ are large. This has been shown for $n = 2$ by one of the authors et al. [19] and for $n \geq 1$ with convex Ω and large μ by Winkler [27]. See also the recent related works [1, 11, 14].

We henceforth assume that the function $f(u)$ is a real, smooth function of $u \in [0, \infty)$ such that $f(0) \geq 0$ and

$$f(u) = u - \mu u^{\alpha} \quad \text{for sufficiently large } u \geq 0;$$

and the function $g(u)$ is given by

$$g(u) = u(1 + u)^{\beta - 1} \quad \text{for } u \geq 0,$$

where the exponents α and β satisfy the relations

$$\alpha > 1 \quad \text{and} \quad 0 < \beta \leq 2, \tag{1}$$

and μ is a positive constant. From the results quoted above, we find that in the n-dimensional domain ($n \geq 2$), a blow-up can occur when $\alpha = 1$ and $\beta = 1$ with a special choice of $\mu = 1$, and the blow-up of solutions is prevented and the global existence of solutions is assured when $\alpha = 2$ and $\beta = 1$. We can then conjecture that the critical degradation order α_{cr} is in the interval $1 \leq \alpha_{\mathrm{cr}} \leq 2$ under linear production $\beta = 1$; however, it has not been determined for the parabolic-parabolic chemotaxis-growth system (E). Recently, Xiang [29] showed global existence of solutions under $\beta = 1$ when $\alpha > 19/9$ if $n = 3$ and when $\alpha > n - 1$ if $n > 3$.

In the two- and three-dimensional cases, the authors [16, 17] introduced sublinear production order $\beta < 1$, and showed a sufficient condition $2(n + 4)/(n + 6) < \alpha \leq 2$ and $0 < \beta < (n + 6)(\alpha - 1)/[2(n + 2)]$ for the existence of global and bounded solutions to (E) in a Hilbert space $H_2^{(n/2)-1}(\Omega) \times H_2^{(n/2)+\varepsilon}(\Omega) \subset L_n(\Omega) \times C(\overline{\Omega})$ (their results would include Xiang's results [29] when $n = 3$ if the existence of local solutions were assured for $\alpha > 2$). The authors have also shown in the previous paper [18] the global existence of solutions in L_p-space of arbitrary space dimension n with $p > n$, where (α, β) is merely allowed for $0 < \beta < (\alpha - 1)/2$.

In this paper, we revise the results obtained in [18] considerably by combining the semigroup method and the energy estimates and by applying the technique of trace operator [9, 13] (see Step 2 of Proof of Lemma 9). The main theorem of this paper is as follows:

Theorem 1. *Assume that the exponents α and β satisfy the relations* (1) *and*

$$\beta \le \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha - 1). \tag{2}$$

Let p be an arbitrarily fixed exponent with

$$\max\{2, n, (\alpha - 2)n\} < p < \infty. \tag{3}$$

Then, for each pair of nonnegative initial functions $(u_0, v_0) \in L_p(\Omega)\times H_p^1(\Omega) \subset L_n(\Omega)\times C(\overline{\Omega})$, the system (E) *admits a unique global solution (u, v) in the function space*

$$\begin{cases} 0 \le u \in C([0,\infty); L_p(\Omega)) \cap C((0,\infty); H^2_{p,N}(\Omega)) \cap C^1((0,\infty); L_p(\Omega)), \\ 0 \le v \in C([0,\infty); H^1_p(\Omega)) \cap C((0,\infty); H^3_{p,N}(\Omega)) \cap C^1((0,\infty); H^1_p(\Omega)). \end{cases} \tag{4}$$

Moreover the solution satisfies the estimate

$$\|u(t)\|_{L_p} + \|v(t)\|_{H^1_p} \le \psi\left(\|u_0\|_{L_p} + \|v_0\|_{H^1_p}\right), \quad t \ge 0 \tag{5}$$

with some increasing function $\psi(\cdot)$.

The definition and notation of function spaces will be given below and in Section 2. Theorem 1 above does not yet cover the case $(\alpha, \beta) = (2, 1)$ for $n \ge 2$ shown by Winkler [27], but the theorem requires no assumption on the largeness of μ nor the convexity of Ω considered in [27]. Our new results also contain the uniform boundedness of solutions with respect to the size of initial data.

We conclude this introduction by referring the results on the parabolic-elliptic chemotaxis systems. The parabolic-elliptic simplifications correspond to the situation where the chemical substance diffuses very quickly, which implies that the time scale τ tends to 0 in (E). For the n-dimensional parabolic-elliptic system with α-th order growth and linear secretion, that is, in the case of $\tau = 0$ and $\beta = 1$ in (E), the problem on the global existence and blow-up of solutions has largely been solved by Winkler [25, 28]: global existence and boundedness are assured when $\alpha > \max\{n/2, 2 - (1/n)\}$ [25]; also, there exists a blow-up solution when $1 < \alpha < 3/2 + 1/(2n - 2)$ with $n \ge 5$ [28].

This paper is organized as follows. We provide preliminary results that we utilize in subsequent sections. In Section 3 we show the local existence of solutions by using a semigroup method (Theorem 5). In the final section we construct several a priori energy estimates by combining semigroup and energy methods. After obtaining the a priori estimates, we give the proof of the main theorem.

Notations. Let Ω be a smooth bounded domain in $\mathbb{R}^n$. For $1 \le p \le \infty$, the space of complex-valued L_p functions in Ω is denoted by $L_p(\Omega)$ with the usual norm $\|\cdot\|_{L_p}$. The complex Sobolev space in Ω of order k, $k = 0, 1, 2, \ldots$, and exponent p, $1 \le p \le \infty$, is denoted by $H^k_p(\Omega)$ with norm $\|\cdot\|_{H^k_p}$. More generally, the Sobolev space of fractional order $s > 0$ and exponent $1 \le p \le \infty$ is denoted by $H^s_p(\Omega)$ with norm $\|\cdot\|_{H^s_p}$. The space of complex-valued continuous functions on $\overline{\Omega}$ is denoted by $C(\overline{\Omega})$ with norm $\|\cdot\|_C$. Let X be a Banach space and I an interval of $\mathbb{R}$. $C(I; X)$ and $C^1(I; X)$ denote the space of X-valued continuous functions and of X-valued continuously differentiable functions, respectively. $\mathcal{B}(I; X)$ denotes the space of X-valued bounded functions. For simplicity, we will use a universal notation C to denote various constants that are determined for each occurrence by Ω in a specific way. In a situation where C also depends on some parameter, say η, it will be denoted by C_η. In addition, by a universal notation $\psi(\cdot)$ we will denote continuous increasing functions, which may change depending on the context.

2. Preliminaries

In this section we shall list some well-known results in the theories of function spaces and linear operators [19, 22, 24, 30].

Interpolation of Sobolev spaces. For $0 \le s_0 < s < s_1 < \infty$ and $1 < p < \infty$, $H^s_p(\Omega)$ is the interpolation space $[H^{s_0}_p(\Omega), H^{s_1}_p(\Omega)]_\theta$ between $H^{s_0}_p(\Omega)$ and $H^{s_1}_p(\Omega)$, where $s = (1-\theta)s_0 + \theta s_1$, with the estimate

$$\|w\|_{H^s_p} \le C\|w\|^{1-\theta}_{H^{s_0}_p}\|w\|^{\theta}_{H^{s_1}_p} \quad \text{for } w \in H^{s_1}_p(\Omega). \tag{6}$$

See [30, Theorem 1.35].

Embedding theorem of Sobolev spaces. Let $1 < p < \infty$.

If $0 \le s < n/p$, then $H^s_p(\Omega) \subset L_r(\Omega)$ for any $p \le r \le pn/(n-ps) = [(1/p)-(s/n)]^{-1}$ with continuous embedding

$$\|w\|_{L_r} \le C_{s,p}\|w\|_{H^s_p} \quad \text{for } w \in H^s_p(\Omega). \tag{7}$$

If $s = n/p$, then $H^s_p(\Omega) \subset L_r(\Omega)$ for any finite $p \le r < \infty$ with continuous embedding

$$\|w\|_{L_r} \le C_{s,p}\|w\|_{H^s_p} \quad \text{for } w \in H^s_p(\Omega). \tag{8}$$

If $n/p < s < \infty$, then $H^s_p(\Omega) \subset C(\overline{\Omega})$ with continuous embedding

$$\|w\|_{C} \le C_{s,p}\|w\|_{H^s_p} \quad \text{for } w \in H^s_p(\Omega). \tag{9}$$

See [30, Theorem 1.36].

If $1 \le r \le p < \infty$, then $L_r(\Omega)$ is embedded in $(H^s_{p'}(\Omega))'$, the dual space of $H^s_{p'}(\Omega)$ with respect to L_2-inner product, for $(n/r) - (n/p) \le s < \infty$ and $p' = p/(p-1)$ with continuous embedding

$$\|w\|_{(H^s_{p'})'} \le C_r\|w\|_{L_r} \quad \text{for } w \in L_r(\Omega). \tag{10}$$

Gagliardo-Nirenberg's inequality. Let $1 \le q \le p \le \infty$. Then the embedding $H^1_p(\Omega) \cap L_q(\Omega) \subset L_r(\Omega)$ holds for

$$\begin{cases} q \le r \le pn/(n-p) & \text{if } 1 \le p < n; \\ q \le r < \infty & \text{if } p = n; \\ q \le r \le \infty & \text{if } n < p \le \infty, \end{cases} \tag{11}$$

with the estimate

$$\|w\|_{L_r} \le C_{p,q,r}\|w\|^{a}_{H^1_p}\|w\|^{1-a}_{L_q} \quad \text{for } w \in H^1_p(\Omega), \tag{12}$$

where a is given by

$$\frac{1}{r} = a\left(\frac{1}{p} - \frac{1}{n}\right) + \frac{1-a}{q}. \tag{13}$$

See [30, Theorem 1.37].

Norms of a product of two functions. For $1 < p < \infty$ and $s > n/p$, from (9),

$$\|uv\|_{L_p} \le C_p\|u\|_{L_p}\|v\|_{L_\infty} \le C_{p,s}\|u\|_{L_p}\|v\|_{H_p^s} \quad \text{for } u \in L_p(\Omega),\ v \in H_p^s(\Omega). \tag{14}$$

As a corollary,

$$\begin{aligned} \|\nabla\cdot(u\nabla v)\|_{L_p} &\le \|\nabla u\cdot\nabla v\|_{L_p} + \|u\Delta v\|_{L_p} \le \|\nabla u\|_{L_p}\|\nabla v\|_{L_\infty} + \|u\|_{L_\infty}\|\Delta v\|_{L_p} \\ &\le C_{p,s}(\|u\|_{H_p^1}\|v\|_{H_p^{1+s}} + \|u\|_{H_p^s}\|v\|_{H_p^2}) \\ &\qquad \text{for } u \in H_p^1(\Omega)\cap H_p^s(\Omega),\ v \in H_p^2(\Omega)\cap H_p^{1+s}(\Omega). \end{aligned} \tag{15}$$

When $n < p < \infty$, since $H_p^1(\Omega) \subset L_\infty(\Omega)$ by (9), it holds that

$$\|\nabla\cdot(u\nabla v)\|_{L_p} \le C_p\|u\|_{H_p^1}\|v\|_{H_p^2} \quad \text{for } u \in H_p^1(\Omega),\ v \in H_p^2(\Omega). \tag{16}$$

Domains of fractional powers of Laplace operators in L_p-spaces. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and $A_0 = -\Delta + 1$, Δ being the Laplace operator with Neumann boundary condition. Then, for each $1 < p < \infty$, A_0 is considered as a closed operator in $L_p(\Omega)$, the domain of which is $H^2_{p,N}(\Omega)$ (see [2, Theorem 2.4.1.3], [24, Theorem 5.3.4] or [30, Theorem 2.15]). Let us denote $A_p = A_0|_{L_p}$; then $\mathcal{D}(A_p) = H^2_{p,N}(\Omega)$. Moreover, by the shift property (see [2, Theorem 2.5.1.1] or [24, Theorems 5.3.4 and 5.4.1]) it holds that $\mathcal{D}(A_p|_{H_p^1}) = H^3_{p,N}(\Omega)$ with norm equivalence.

The domains of fractional powers of A_p are characterized by

$$\mathcal{D}(A_p^\theta) = \begin{cases} H_p^{2\theta}(\Omega) & \text{for } 0 \le \theta < \dfrac{1}{2} + \dfrac{1}{2p} \\ H_{p,N}^{2\theta}(\Omega) & \text{for } \dfrac{1}{2} + \dfrac{1}{2p} < \theta \le \dfrac{3}{2} \end{cases} \tag{17}$$

with norm equivalence. Here, $H^s_{p,N}(\Omega)$ for $s > 1 + (1/p)$ denotes a closed subspace of $H^s_p(\Omega)$ such that

$$H^s_{p,N}(\Omega) = \left\{ w \in H^s_p(\Omega);\ \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega \right\} \quad \text{for } s > 1 + \frac{1}{p}.$$

Indeed, we can see that A_p has a bounded H_∞ functional calculus (see Yagi [30, Sec.16.1.2]) in $L_p(\Omega)$ and $H_p^1(\Omega)$, and by Yagi [30, Theorem 16.5], that the interpolation $\mathcal{D}(A_p^\theta) = [L_p(\Omega), H^2_{p,N}(\Omega)]_\theta$ and $\mathcal{D}((A_p|_{H_p^1})^\theta) = [H_p^1(\Omega), H^3_{p,N}(\Omega)]_\theta$ hold for $0 < \theta < 1$ with norm equivalence. Then, carefully following the proof of [30, Theorem 16.11], we can verify the rest part of (17). For the detail see Appendix.

Analytic semigroups generated by Laplace operators in L_p-spaces. For each $1 < p < \infty$, A_0 defined above generates in L_p-space an analytic semigroup e^{-tA_0} (it is independent of p in the sense that $e^{-tA_p}w = e^{-tA_2}w$ for $w \in L_p(\Omega)\cap L_2(\Omega)$). For $\gamma \ge 0$ it satisfies the estimate

$$\|A_0^\gamma e^{-tA_0}w\|_{L_p} \le Ct^{-\gamma}e^{-\delta_0 t}\|w\|_{L_p}, \qquad t > 0,\ w \in L_p(\Omega), \tag{18}$$

with some fixed constant $\delta_0 > 0$. See [8, Sec. 2] (see also [26, Lemma 1.3], [30, Theorems 2.19 and 2.27] and [22, Sec. 13.7]).

A differential geometric property of functions with Neumann boundary condition. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. If the function $w \in C^2(\overline{\Omega})$ satisfies $\partial w/\partial \nu = 0$ on $\partial\Omega$, then it holds that

$$\frac{\partial |\nabla w|^2}{\partial \nu} \le 2\kappa_\Omega |\nabla w|^2 \quad \text{on } \partial\Omega, \tag{19}$$

where κ_Ω is an upper bound for the curvatures of $\partial\Omega$; $\kappa_\Omega = 0$ when Ω is convex. See [13, Lemma 4.2]. See also [9].

Boundedness of trace operators. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Let $1 < p < \infty$ and $s > 1/p$. Then, the trace $T : f \mapsto f|_{\partial\Omega}$ is a bounded linear operator from $H^s_p(\Omega)$ to $L_p(\partial\Omega)$. Hence, we have

$$\|w\|_{L_p(\partial\Omega)} \le C_{s,p}\|w\|_{H^s_p(\Omega)}, \qquad w \in H^s_p(\Omega). \tag{20}$$

See [30, Theorem 1.39] or [24, Theorem 4.7.1].

3. Local solutions

By similar argument to that in [17, 18, 19] or [30, Chap. 12], we can show the existence of local solutions to (E). We first review the existence theorem by Yagi [30, Chap. 4] (see also [20]) for local solutions to an abstract equation in a Banach space. Let X be a Banach space with norm $\|\cdot\|_X$. We consider the following Cauchy problem for a semilinear abstract evolution equation in X:

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), & t > 0, \\ U(0) = U_0. \end{cases} \tag{21}$$

Here A is a sectorial operator of X satisfying that its spectral set is contained in a sectorial domain $\Sigma = \{\lambda \in \mathbb{C};\ |\arg \lambda| \le \phi\}$ with some $0 \le \phi < \pi/2$, and $\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \le M/(|\lambda| + 1)$, $\lambda \notin \Sigma$ with constant M. The nonlinear operator F is a mapping from $\mathcal{D}(A^\eta)$ to X, where $0 < \eta < 1$, and it also satisfies a Lipschitz condition:

$$\begin{aligned} \|F(U) - F(\tilde{U})\|_X &\le \varphi\left(\|A^\gamma U\|_X + \|A^\gamma \tilde{U}\|_X\right) \\ &\quad \times \left[\|A^\eta(U - \tilde{U})\|_X + \left(\|A^\eta U\|_X + \|A^\eta \tilde{U}\|_X\right)\|A^\gamma(U - \tilde{U})\|_X\right], \quad U, \tilde{U} \in \mathcal{D}(A^\eta), \end{aligned} \tag{22}$$

where γ is an exponent such that $0 < \gamma \le \eta < 1$, and $\varphi(\cdot)$ is some increasing continuous function. The initial value U_0 is taken in $\mathcal{D}(A^\gamma)$. Then, from [30, Theorem 4.1] (or [20, Theorem 3.1]) we have the existence theorem of the local solutions to (21):

Theorem 2 ([30, Theorem 4.1]). *Under the above assumptions, for any $U_0 \in \mathcal{D}(A^\gamma)$, (21) possesses a unique local solution U in the function space:*

$$\begin{cases} U \in C((0, T_{U_0}]; \mathcal{D}(A)) \cap C([0, T_{U_0}]; \mathcal{D}(A^\gamma)) \cap C^1((0, T_{U_0}]; X), \\ t^{1-\gamma} U \in \mathcal{B}((0, T_{U_0}]; \mathcal{D}(A)) \end{cases}$$

with the estimate

$$t^{1-\gamma}\|AU(t)\|_X + \|A^\gamma U(t)\|_X \le C_{U_0}, \quad 0 < t \le T_{U_0},$$

where T_{U_0} and C_{U_0} are positive constants depending only on the norm $\|A^\gamma U_0\|_X$.

By applying Theorem 2, we can show the existence of the local solutions to (E). The following proposition has been proved in [18].

Proposition 3 ([18, Proposition 3]). *Let $n \in \mathbb{N}$, assume the relation* (1) *for α and β, and let p be an exponent satisfying*

$$\max\{n, (\alpha - 2)n\} < p < \infty. \tag{23}$$

Then, for each pair of initial functions $(u_0, v_0) \in L_p(\Omega)\times H_p^1(\Omega) \subset L_n(\Omega)\times C(\overline{\Omega})$, the problem (E) *admits a unique local solution (u, v) in the function space*

$$\begin{cases} u \in C((0,T]; H_p^1(\Omega)) \cap C([0,T]; L_p(\Omega)) \cap C^1((0,T]; (H_{p'}^1(\Omega))'), \\ v \in C((0,T]; H_{p,N}^2(\Omega)) \cap C([0,T]; H_p^1(\Omega)) \cap C^1((0,T]; L_p(\Omega)) \end{cases} \tag{24}$$

with the estimate

$$t^{\frac{1}{2}}\left\{\|u(t)\|_{H_p^1} + \|v(t)\|_{H_p^2}\right\} + \left\{\|u(t)\|_{L_p} + \|v(t)\|_{H_p^1}\right\} \le C, \quad 0 < t \le T,$$

where $p' = p/(p-1)$, and T and C are positive constants depending only on the norm $\|u_0\|_{L_p} + \|v_0\|_{H_p^1}$.

By a solution (u, v) to (E) in the function space (24) we mean that the pair of functions (u, v) contained in (24) satisfies

$$\begin{cases} \dfrac{d}{dt}\langle u, w\rangle_{L_2} = -\langle \nabla u, \nabla w\rangle_{L_2} + \chi\langle u\nabla v, \nabla w\rangle_{L_2} + \langle f(u), w\rangle_{L_2} \\ \qquad\qquad\qquad\qquad \text{for any } w \in H_{p'}^1(\Omega) \text{ and } 0 < t < \infty, \\ \tau\dfrac{\partial v}{\partial t} = \Delta v - v + g(u) \quad \text{in } \Omega \times (0,\infty). \end{cases}$$

Next, we will show the local existence of solutions in the second function space:

Proposition 4. *Let $n \in \mathbb{N}$, assume the relation* (1) *for α and β, and let p be an exponent satisfying $n < p < \infty$. Then, for each pair of initial functions $(u_0, v_0) \in H_p^1(\Omega) \times H_{p,N}^2(\Omega)$, the problem* (E) *admits a unique local solution (u, v) in the function space*

$$\begin{cases} u \in C((0,T]; H_{p,N}^2(\Omega)) \cap C([0,T]; H_p^1(\Omega)) \cap C^1((0,T]; L_p(\Omega)), \\ v \in C((0,T]; H_{p,N}^3(\Omega)) \cap C([0,T]; H_{p,N}^2(\Omega)) \cap C^1((0,T]; H_p^1(\Omega)) \end{cases}$$

with the estimate

$$t^{\frac{1}{2}}\left\{\|u(t)\|_{H_p^2} + \|v(t)\|_{H_p^3}\right\} + \left\{\|u(t)\|_{H_p^1} + \|v(t)\|_{H_p^2}\right\} \le C, \quad 0 < t \le T,$$

where T and C are positive constants depending only on the norm $\|u_0\|_{H_p^1} + \|v_0\|_{H_p^2}$.

Proof. The system (E) can be expressed as a semilinear parabolic equation

$$(25)\qquad \begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad t > 0, \\ U(0) = U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \end{cases}$$

in a product Banach space $X = L_p(\Omega) \times H_p^1(\Omega)$. Here, we define the linear operator A by

$$A = \begin{bmatrix} -\Delta + 1 & 0 \\ 0 & \tau^{-1}(-\Delta + 1) \end{bmatrix}, \quad \mathcal{D}(A) = H^2_{p,N}(\Omega) \times H^3_{p,N}(\Omega).$$

The nonlinear operator F is defined by

$$F(U) = \begin{bmatrix} -\chi\nabla \cdot (u\nabla v) + \bar{f}(u) + u \\ \bar{g}(u) \end{bmatrix}, \quad U = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{D}(A^\eta) = H^1_p(\Omega) \times H^2_{p,N}(\Omega)$$

with $\eta = 1/2$. Here, $\bar{f}(u)$ and $\bar{g}(u)$ denote some smooth extensions of $f(u)$ and $g(u)$ for the variable $u \in \mathbb{C}$ satisfying $f(u) \geq 0$ for $u < 0$ and $g(u) = 0$ for $u < -1$, respectively. The initial value U_0 is taken in the function space $\mathcal{D}(A^\gamma) = \mathcal{D}(A^\eta)$, that is $\gamma = \eta$. Under this setting, we need to verify only the Lipschitz condition (22). For $U = \begin{bmatrix} u \\ v \end{bmatrix}, \tilde{U} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \in \mathcal{D}(A^\eta)$,

$$\begin{aligned} \|F(U) - F(\tilde{U})\|_X \leq{}& \chi\|\nabla \cdot (u\nabla v - \tilde{u}\nabla\tilde{v})\|_{L_p} \\ &+ \|u - \tilde{u}\|_{L_p} + \|\bar{f}(u) - \bar{f}(\tilde{u})\|_{L_p} + \|\bar{g}(u) - \bar{g}(\tilde{u})\|_{H^1_p}. \end{aligned}$$

For the first term, applying (16), we see

$$\begin{aligned} \|\nabla \cdot (u\nabla v) - \nabla \cdot (\tilde{u}\nabla\tilde{v})\|_{L_p} &\leq \|\nabla \cdot ((u - \tilde{u})\nabla v)\|_{L_p} + \|\nabla \cdot (\tilde{u}\nabla(v - \tilde{v}))\|_{L_p} \\ &\leq C_{p,s}(\|u - \tilde{u}\|_{H^1_p}\|v\|_{H^2_p} + \|\tilde{u}\|_{H^1_p}\|v - \tilde{v}\|_{H^2_p}) \\ &\leq C(\|A^\gamma U\|_X + \|A^\gamma \tilde{U}\|_X)\|A^\gamma(U - \tilde{U})\|_X. \end{aligned}$$

For the third and forth terms, using (1) and $H^s_p(\Omega) \subset L_\infty(\Omega)$ by (9), we can easily see that

$$\begin{aligned} \|\bar{f}(u) - \bar{f}(\tilde{u})\|_{L_p} &\leq C(1 + \|u\|_{L_\infty} + \|\tilde{u}\|_{L_\infty})^{\alpha-1}\|u - \tilde{u}\|_{L_p} \\ &\leq C(1 + \|u\|_{H^1_p} + \|\tilde{u}\|_{H^1_p})^{\alpha-1}\|u - \tilde{u}\|_{L_p}, \end{aligned}$$

$$\begin{aligned} \|\bar{g}(u) - \bar{g}(\tilde{u})\|_{H^1_p} &\leq \|\bar{g}(u) - \bar{g}(\tilde{u})\|_{L_p} + \|\bar{g}'(u)\nabla(u - \tilde{u})\|_{L_p} + \|\{\bar{g}'(u) - \bar{g}'(\tilde{u})\}\nabla\tilde{u}\|_{L_p} \\ &\leq C(1 + \|u\|_{L_\infty} + \|\tilde{u}\|_{L_\infty})(\|u - \tilde{u}\|_{L_p} + \|\nabla(u - \tilde{u})\|_{L_p}) + C\|u - \tilde{u}\|_{L_\infty}\|\nabla\tilde{u}\|_{L_p} \\ &\leq C(1 + \|u\|_{H^1_p} + \|\tilde{u}\|_{H^1_p})\|u - \tilde{u}\|_{H^1_p}. \end{aligned}$$

Thus $F(U)$ satisfies the Lipschitz condition (22). We complete the proof. □

Now we can state our main theorem of this section:

Theorem 5. *Let $n \in \mathbb{N}$, assume the relation* (1) *for α and β, and let p be an arbitrarily fixed exponent satisfying* (23). *Then, for each pair of nonnegative initial functions $(u_0, v_0) \in L_p(\Omega) \times H^1_p(\Omega) \subset L_n(\Omega) \times C(\overline{\Omega})$, the problem* (E) *admits a unique local solution (u, v) in the function space*

$$(26)\qquad \begin{cases} 0 \le u \in \mathcal{C}([0,T]; L_p(\Omega)) \cap \mathcal{C}((0,T]; H^2_{p,N}(\Omega)) \cap \mathcal{C}^1((0,T]; L_p(\Omega)), \\ 0 \le v \in \mathcal{C}([0,T]; H^1_p(\Omega)) \cap \mathcal{C}((0,T]; H^3_{p,N}(\Omega)) \cap \mathcal{C}^1((0,T]; H^1_p(\Omega)) \end{cases}$$

with the estimate

$$(27)\qquad \|u(t)\|_{L_p} + \|v(t)\|_{H^1_p} \le C, \quad 0 < t \le T,$$

where T and C are positive constants depending only on the norm $\|u_0\|_{L_p} + \|v_0\|_{H^1_p}$.

Proof. It is clear that the local solutions belong to the function space (26) from Propositions 3 and 4. The nonnegativity of solutions has been proved in [18, Theorem 4] with the aid of the truncation method [30, Section 12.1.3]. Hence we conclude the proof. □

4. A priori estimates and global solutions

In this section we will construct several a priori estimates. The a priori estimates hold with each of the inequalities of α and β in the lemmas. Throughout this section, except for in the global existence theorem, we assume that $0 \le u_0 \in H^2_{p,N}(\Omega) \subset H^1_\infty(\Omega)$ and $0 \le v_0 \in H^3_{p,N}(\Omega) \subset H^2_\infty(\Omega)$ with $n < p < \infty$. In this case, applying [30, Theorem 4.2], we can verify that $0 \le u \in \mathcal{C}([0,T]; H^2_{p,N}(\Omega))$ and $0 \le v \in \mathcal{C}([0,T]; H^3_{p,N}(\Omega))$ with the estimate $\|u(t)\|_{H^2_p} + \|v(t)\|_{H^3_p} \le C_{U_0}$ for $0 \le t \le T$, where C_{U_0} is some positive constant. For a local solution (u, v) to (E) and exponents $z > 0$ and $\omega > 0$, we define

$$I^z_\omega(t) = \int_0^t \omega e^{-\omega(t-s)} \int_\Omega u^z dx ds.$$

The following lemma will be used frequenty in this section.

Lemma 6 (Gronwall's inequality)**.** *Assume that a smooth real function $h(t)$ satisfies the differential inequality*

$$h'(t) + ah(t) \le K(t), \quad t_0 \le t \le T,$$

with a positive constant a and an integrable real function $K(t)$. Then, $h(t)$ is estimated by

$$h(t) \le h(t_0)e^{-a(t-t_0)} + \int_{t_0}^t e^{-a(t-s)} K(s) ds, \quad t_0 \le t \le T.$$

Lemma 7. *Let (u, v) be a local solution to* (E)*, and assume that*

$$\alpha > 1.$$

Then, it holds that

$$(28)\qquad \|u\|_{L_1} = \int_\Omega u\, dx \le e^{-t}\|u_0\|_{L_1} + a_1|\Omega|$$

with a constant $a_1 = \max\{f(u) + u;\ u \ge 0\}$*. In addition, for an arbitrary constant $\omega > 0$,*

$$(29)\qquad I^\alpha_\omega(t) \le \frac{2}{\mu}\{(a + a_1\omega)|\Omega| + \omega\|u_0\|_{L_1}\} \equiv \bar{I}^\alpha_\omega$$

holds with a constant $a = \max\{f(u) + \mu u^\alpha/2;\ u \ge 0\}$.

Proof. (Just the same as [17, Lemmas 4.1 and 4.2] or the first half part of [18, Lemma 5].) Integrating the first equation of (E) over Ω, we have

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega f(u)\,dx \le \int_\Omega (a_1 - u)dx.$$

Then, by Lemma 6, we obtain (28). From these inequalities, we see that

$$\frac{\mu}{2} I_\omega^\alpha(t) \le \int_0^t \omega e^{-\omega(t-s)} \int_\Omega \{a - f(u)\}dxds = \int_0^t \omega e^{-\omega(t-s)} \left\{a|\Omega| - \frac{d}{ds}\|u\|_{L_1}\right\} ds$$
$$\le a|\Omega|(1 - e^{-\omega t}) + \omega e^{-\omega t}\|u_0\|_{L_1} + \int_0^t \omega^2 e^{-\omega(t-s)}\|u\|_{L_1} ds$$
$$\le (a + a_1\omega)|\Omega| + \omega\|u_0\|_{L_1},$$

which yields (29). □

Lemma 8. *Let* (u, v) *be a local solution to* (E)*, and assume that*

$$\alpha > 1 \quad \text{and} \quad 0 < \beta \le \frac{\alpha}{2}.$$

Then, for any exponent $2 \le q \le \alpha/\beta$*,*

$$\|v\|_{H_q^1}^q \le C_q e^{-\delta_q t}\|v_0\|_{H_q^1}^q + C_q \left(|\Omega| + \|u_0\|_{L_1}\right) \tag{30}$$

holds with some positive constants C_q *and* δ_q*.*

Proof. When $q = 2$ (see [17, Proposition 4.4]), multiplying the second equation of (E) by $-\Delta v + v$ and integrating it over Ω, we see that

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega \left(|\nabla v|^2 + v^2\right) dx \le -\frac{1}{2}\int_\Omega (\Delta v)^2 dx - 2\int_\Omega |\nabla v|^2 dx - \frac{1}{2}\int_\Omega v^2 dx + \int_\Omega (1+u)^{2\beta} dx,$$

that is,

$$\tau\frac{d}{dt}\int_\Omega \left(|\nabla v|^2 + v^2\right) dx + \int_\Omega \left(|\nabla v|^2 + v^2\right) dx + \int_\Omega (\Delta v)^2 dx + 3\int_\Omega |\nabla v|^2 dx \le 2\int_\Omega (1+u)^{2\beta} dx. \tag{31}$$

Thus, by Lemma 6 again, we verify

$$\|v\|_{H_2^1}^2 \le e^{-t/\tau}\|v_0\|_{H_2^1}^2 + 2\int_0^t e^{-(t-s)/\tau}\int_\Omega (1+u)^{2\beta} dxds.$$

When $q > 2$ (just in the same way as the second half part of [18, Lemma 5]), we utilize the semigroup $e^{-tA_0/\tau}$ of $A_0 = -\Delta + 1$, Δ be the Laplace operator with Neumann boundary condition. Then the second equation of (E) gives

$$v(t) = e^{-tA_0/\tau} v_0 + \frac{1}{\tau}\int_0^t e^{-(t-s)A_0/\tau} g(u(s))ds. \tag{32}$$

Operating $A_0^{1/2}$ to this equality and applying (17) and (18), we have

$$\begin{aligned}
\|v\|_{H^1_q} &\le C_q\|A_0^{1/2}v\|_{L_q} \\
&\le C_q\|A_0^{1/2}e^{-tA_0/\tau}v_0\|_{L_q} + \frac{1}{\tau}\int_0^t C_q\|A_0^{1/2}e^{-(t-s)A_0/\tau}g(u)\|_{L_q}ds \\
&\le C_q e^{-\delta_0 t/\tau}\|A_0^{1/2}v_0\|_{L_q} + \int_0^t C_q(t-s)^{-1/2}e^{-\delta_0(t-s)/\tau}\|g(u)\|_{L_q}ds \\
&\le C_q e^{-\delta_0 t/\tau}\|v_0\|_{H^1_q} + \int_0^t C_q(t-s)^{-1/2}e^{-\delta_0(t-s)/\tau}\|(1+u)^\beta\|_{L_q}ds.
\end{aligned}$$

The last term can be estimated as

$$\begin{aligned}
&\int_0^t (t-s)^{-1/2}e^{-\delta_0(t-s)/\tau}\|(1+u)^\beta\|_{L_q}ds \\
&\qquad = \int_0^t (t-s)^{-1/2}e^{-\delta_0(t-s)/\tau}\|1+u\|^\beta_{L_{q\beta}}ds \\
&\qquad \le \left(\int_0^t (t-s)^{-q'/2}e^{-\delta_0(t-s)/\tau}ds\right)^{1/q'}\left(\int_0^t e^{-\delta_0(t-s)/\tau}\|1+u\|^{q\beta}_{L_{q\beta}}ds\right)^{1/q},
\end{aligned}$$

where $q' = q/(q-1)$. Here we notice that $q'/2 < 1$ and the singular integral converges. Hence we have

$$\|v\|_{H^1_q} \le C_q e^{-\delta_0 t/\tau}\|v_0\|_{H^1_q} + C_q\left(\int_0^t e^{-\delta_0(t-s)/\tau}\int_\Omega (1+u)^{q\beta}dxds\right)^{1/q}.$$

Combining both cases when $q = 2$ and when $q > 2$, we have

$$\|v\|^q_{H^1_q} \le C_q e^{-\delta_q t/\tau}\|v_0\|^q_{H^1_q} + C_q\int_0^t e^{-\delta_q(t-s)/\tau}\int_\Omega (1+u)^{q\beta}dxds \tag{33}$$

for $q \ge 2$ with some positive constants C_q and δ_q. Applying (29), we prove (30) for $2 \le q \le \alpha/\beta$. □

Lemma 9. *Let* (u, v) *be a local solution to* (E)*, and assume that*

$$\alpha > 1, \quad 0 < \beta \le \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha - 1).$$

Then, for any $2 \le q \le \alpha/\beta$ *satisfying* $q > 2n\alpha/[(n+2)(\alpha-1)]$*, and for any exponent* $1 < \theta \le \{q(n+2)/(2n) - 1\}(\alpha - 1)$*,*

$$\|1+u\|^\theta_{L_\theta} \le e^{-qt/(2\tau)}\|1+u_0\|^\theta_{L_\theta} + \psi_{\theta,q}\left(\|1+u_0\|_{L_1} + \|v_0\|_{H^1_q}\right) \tag{34}$$

holds with some increasing function $\psi_{\theta,q}(\cdot)$*. In addition, for an arbitrary constant* $\omega > 0$*,*

$$\begin{aligned}
I^{\alpha+\theta-1}_\omega(t) \le \frac{4}{\mu}\Bigg\{&\left(1+\frac{\omega}{\theta}\right)\psi_{\theta,q}\left(\|u_0\|_{L_1} + \|v_0\|_{H^1_q}\right) \\
&+ \omega\left(\frac{1}{\theta}\|1+u_0\|^\theta_{L_\theta} + \zeta\frac{2\tau}{q}\|v_0\|^q_{H^1_q}\right)\Bigg\} \equiv \bar{I}^{\alpha+\theta-1}_\omega
\end{aligned} \tag{35}$$

holds with some constant $\zeta > 0$*.*

Proof. We describe the proof in several steps.

Step 1. Multiplying the first equation of (E) by $(1+u)^{\theta-1}$ and integrating it over Ω, we see that

$$\begin{aligned}\frac{1}{\theta}\frac{d}{dt}\int_\Omega (1+u)^\theta dx &= -(\theta-1)\int_\Omega (1+u)^{\theta-2}|\nabla u|^2 dx\\ &\quad + \chi(\theta-1)\int_\Omega u(1+u)^{\theta-2}\nabla u\cdot\nabla v dx + \int_\Omega (1+u)^{\theta-1} f(u)dx\\ &\le -\frac{\theta-1}{2}\int_\Omega (1+u)^{\theta-2}|\nabla u|^2 dx + \frac{\chi^2(\theta-1)}{2}\int_\Omega (1+u)^\theta|\nabla v|^2 dx\\ &\quad + \int_\Omega (1+u)^{\theta-1} f(u)dx.\end{aligned}$$

For the second term on the right-hand side, using (12), we note that

$$\begin{aligned}\frac{\chi^2(\theta-1)}{2}\int_\Omega (1+u)^\theta|\nabla v|^2 dx &\le \frac{\chi^2(\theta-1)}{2}\left\|(1+u)^\theta\right\|_{L_{\kappa/(\kappa-1)}}\left\||\nabla v|^2\right\|_{L_\kappa}\\ &= \frac{\chi^2(\theta-1)}{2}\left\|(1+u)^\theta\right\|_{L_{\kappa/(\kappa-1)}}\left\||\nabla v|^{q/2}\right\|_{L_{4\kappa/q}}^{4/q}\\ &\le \frac{\chi^2(\theta-1)}{2}\left\|(1+u)^\theta\right\|_{L_{\kappa/(\kappa-1)}}\cdot C_q\left\||\nabla v|^{q/2}\right\|_{H_2^1}^{2/\kappa}\left\||\nabla v|^{q/2}\right\|_{L_2}^{(4/q)-(2/\kappa)}\\ &\le C_q\eta^{-\kappa+1}\chi^{2\kappa}(\theta-1)^\kappa\left\||\nabla v|^{q/2}\right\|_{H_2^1}^2 + \eta\|\nabla v\|_{L_q}^{(2\kappa-q)/(\kappa-1)}\int_\Omega (1+u)^{\theta\kappa/(\kappa-1)}dx\end{aligned}$$

with $\kappa = q(n+2)/(2n)$ and an arbitrary $\eta > 0$. Hence we have

$$\begin{aligned}(36)\quad \frac{1}{\theta}\frac{d}{dt}\int_\Omega (1+u)^\theta dx &\le -\frac{\theta-1}{2}\int_\Omega (1+u)^{\theta-2}|\nabla u|^2 dx + C_q\eta^{-\kappa+1}\chi^{2\kappa}(\theta-1)^\kappa\left\||\nabla v|^{q/2}\right\|_{H_2^1}^2\\ &\quad + \int_\Omega\left[\eta\|\nabla v\|_{L_q}^{(2\kappa-q)/(\kappa-1)}(1+u)^{\theta\kappa/(\kappa-1)} + (1+u)^{\theta-1}f(u)\right]dx.\end{aligned}$$

Step 2. We present the differential inequality on $\|v\|_{H_q^1}^q$ for $q \ge 2$. For the present assume $q > 2$. Firstly, multiplying the second equation of (E) by v^{q-1} and integrating it over Ω, we see that

$$\begin{aligned}(37)\quad \frac{\tau}{q}\frac{d}{dt}\int_\Omega v^q dx &= -(q-1)\int_\Omega v^{q-2}|\nabla v|^2 dx - \int_\Omega v^q dx + \int_\Omega v^{q-1}g(u)\\ &\le -(q-1)\int_\Omega v^{q-2}|\nabla v|^2 dx - \frac{1}{2}\int_\Omega v^q dx + \frac{C_q'}{2}\int_\Omega (1+u)^{q\beta}dx.\end{aligned}$$

Next, differentiating the second equation of (E), we have

$$\tau\frac{\partial}{\partial t}|\nabla v|^2 = 2\tau\nabla v\cdot\nabla v_t = 2\nabla v\cdot\nabla\Delta v - 2|\nabla v|^2 + 2\nabla v\cdot\nabla g(u).$$

Noting that $\Delta|\nabla v|^2 = 2|D^2v|^2 + 2\nabla v\cdot\nabla\Delta v$ and $(\Delta v)^2 \le n|D^2v|^2$, where $|D^2v|^2 = \sum_{i,j}\left|D_iD_jv\right|^2$, we see

$$\tau\frac{\partial}{\partial t}|\nabla v|^2 \leq \Delta|\nabla v|^2 - \frac{2}{n}(\Delta v)^2 - 2|\nabla v|^2 + 2\nabla v\cdot\nabla g(u).$$

Multiplying this inequality by $|\nabla v|^{q-2}$, integrating it over Ω and applying (19), we obtain

$$\begin{aligned}
\frac{2\tau}{q}\frac{d}{dt}\int_\Omega |\nabla v|^q dx &\leq \int_\Omega |\nabla v|^{q-2}\left\{\Delta|\nabla v|^2 - \frac{2}{n}(\Delta v)^2 - 2|\nabla v|^2 + 2\nabla v\cdot\nabla g(u)\right\}dx\\
&= \int_{\partial\Omega}|\nabla v|^{q-2}\,\frac{\partial|\nabla v|^2}{\partial\nu}dx - \int_\Omega \nabla|\nabla v|^{q-2}\cdot\nabla|\nabla v|^2 dx\\
&\quad - \int_\Omega |\nabla v|^{q-2}\left\{\frac{2}{n}(\Delta v)^2 + 2|\nabla v|^2\right\}dx + \int_\Omega 2|\nabla v|^{q-2}\nabla v\cdot\nabla g(u)dx\\
&\leq 2\kappa_\Omega\int_{\partial\Omega}|\nabla v|^q dx - \int_\Omega \frac{q-2}{2}|\nabla v|^{q-4}\left|\nabla|\nabla v|^2\right|^2 dx - \int_\Omega \frac{2}{n}|\nabla v|^{q-2}(\Delta v)^2 dx\\
&\quad - \int_\Omega 2|\nabla v|^q dx + \int_\Omega 2|\nabla v|^{q-2}\nabla v\cdot\nabla g(u)dx.
\end{aligned}$$

For the first term on the right-hand side, applying (20) and (6) with any $1/2 < s < 1$ and $\varepsilon > 0$, we see that

$$\begin{aligned}
2\kappa_\Omega\int_{\partial\Omega}|\nabla v|^q dx &= 2\kappa_\Omega\left\||\nabla v|^{q/2}\right\|^2_{L_2(\partial\Omega)} \leq C\left\||\nabla v|^{q/2}\right\|^2_{H_2^s(\Omega)}\\
&\leq C\left\||\nabla v|^{q/2}\right\|^{2s}_{H_2^1}\left\||\nabla v|^{q/2}\right\|^{2(1-s)}_{L_2} \leq \varepsilon\left\|\nabla\left(|\nabla v|^{q/2}\right)\right\|^2_{L_2} + C_\varepsilon\left\||\nabla v|^{q/2}\right\|^2_{L_2}.
\end{aligned}$$

For the last term on the right-hand side, we see

$$\begin{aligned}
\int_\Omega 2|\nabla v|^{q-2}\nabla v\cdot\nabla g(u)dx &= -\int_\Omega\left\{(q-2)|\nabla v|^{q-4}\nabla|\nabla v|^2\cdot\nabla v + 2|\nabla v|^{q-2}\Delta v\right\}g(u)dx\\
&\leq \int_\Omega (q-2)|\nabla v|^{q-3}\left|\nabla|\nabla v|^2\right|(1+u)^\beta dx + \int_\Omega 2|\nabla v|^{q-2}|\Delta v|\,(1+u)^\beta dx\\
&\leq \frac{q-2}{4}\int_\Omega |\nabla v|^{q-4}\left|\nabla|\nabla v|^2\right|^2 dx + \frac{1}{n}\int_\Omega |\nabla v|^{q-2}(\Delta v)^2 dx + \int_\Omega |\nabla v|^q dx\\
&\quad + C_q''\,(n+q-2)^{q/2}\int_\Omega (1+u)^{q\beta}dx.
\end{aligned}$$

Hence, noting that $\left|\nabla\left(|\nabla v|^{q/2}\right)\right|^2 = (q^2/16)|\nabla v|^{q-4}\left|\nabla|\nabla v|^2\right|^2$, we have

$$\begin{aligned}
\text{(38)}\quad \frac{2\tau}{q}\frac{d}{dt}\int_\Omega |\nabla v|^q dx &+ \frac{4(q-2)}{q^2}\int_\Omega\left|\nabla\left(|\nabla v|^{q/2}\right)\right|^2 dx\\
&+ \frac{1}{n}\int_\Omega |\nabla v|^{q-2}(\Delta v)^2 dx + \int_\Omega |\nabla v|^q dx\\
\leq \varepsilon\int_\Omega\left|\nabla\left(|\nabla v|^{q/2}\right)\right|^2 dx &+ C_\varepsilon\int_\Omega |\nabla v|^q dx + C_q''\,(n+q-2)^{q/2}\int_\Omega (1+u)^{q\beta}dx.
\end{aligned}$$

Adding (38) to (37) and taking $\varepsilon = 2(q-2)/q^2$, we see

$$(39)\quad \frac{2\tau}{q}\frac{d}{dt}\int_\Omega(|\nabla v|^q+v^q)dx+\int_\Omega(|\nabla v|^q+v^q)dx \\ +\frac{2(q-2)}{q^2}\int_\Omega\left|\nabla\left(|\nabla v|^{q/2}\right)\right|^2dx+\frac{1}{n}\int_\Omega|\nabla v|^{q-2}(\Delta v)^2dx+\frac{8(q-1)}{q^2}\int_\Omega\left|\nabla\left(v^{q/2}\right)\right|^2dx \\ \le C\int_\Omega|\nabla v|^q dx+C_q\int_\Omega(1+u)^{q\beta}dx.$$

This inequality holds also for $q=2$ (see (31)). The right-hand side is bounded in terms of $\|u_0\|_{L_1}$ and $\|v_0\|_{H^1_q}$ in view of Lemmas 7 and 8 since $q\beta\le\alpha$.

Step 3. Adding (39) multiplied by some weight $\zeta>0$ to (36) , we see

$$(40)\quad \frac{1}{\theta}\frac{d}{dt}\int_\Omega(1+u)^\theta dx+\zeta\left(\frac{2\tau}{q}\frac{d}{dt}\|v\|^q_{H^1_q}+\|v\|^q_{H^1_q}+\frac{2(q-2)}{q^2}\left\||\nabla v|^{q/2}\right\|^2_{H^1_2}\right) \\ \le C_q\eta^{-\kappa+1}\chi^{2\kappa}(\theta-1)^\kappa\left\||\nabla v|^{q/2}\right\|^2_{H^1_2}+C\|\nabla v\|^q_{L_q} \\ +\int_\Omega\left[\eta\,\|\nabla v\|^{(2\kappa-q)/(\kappa-1)}_{L_q}(1+u)^{\theta\kappa/(\kappa-1)}+(1+u)^{\theta-1}f(u)+\zeta C_q(1+u)^{q\beta}\right]dx.$$

Since $q\beta\le\alpha<\alpha+\theta-1$ and $\theta\kappa/(\kappa-1)\le\alpha+\theta-1$ from the assumptions, suitable choice of η and ζ yields

$$(41)\quad \frac{d}{dt}\left(\frac{1}{\theta}\|1+u\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v\|^q_{H^1_q}\right)+\frac{q}{2\tau}\left(\frac{1}{\theta}\|1+u\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v\|^q_{H^1_q}\right) \\ \le\psi_{\theta,q}\left(\|v\|_{H^1_q}\right)-\frac{\mu}{4}\int_\Omega u^{\alpha+\theta-1}dx,$$

and hence, by Lemma 6,

$$\frac{1}{\theta}\|1+u\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v\|^q_{H^1_q}\le e^{-qt/(2\tau)}\left(\frac{1}{\theta}\|1+u_0\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v_0\|^q_{H^1_q}\right) \\ +\int_0^t e^{-q(t-s)/(2\tau)}\psi_{\theta,q}\left(\|v\|_{H^1_q}\right)ds.$$

Application of (30) to the right-hand side of this inequality leads to (34).

Step 4. The proof of (35) is very similar to that of (29), as follows: using (41),

$$\frac{\mu}{4}I^{\alpha+\theta-1}_\omega(t)\le\int_0^t\omega e^{-\omega(t-s)}\left\{\psi_{\theta,q}\left(\|v\|_{H^1_q}\right)-\frac{d}{dt}\left(\frac{1}{\theta}\|1+u\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v\|^q_{H^1_q}\right)\right\}ds \\ \le\psi_{\theta,q}\left(\sup_{t\ge0}\|v\|_{H^1_q}\right)+\omega e^{-\omega t}\left(\frac{1}{\theta}\|1+u_0\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v_0\|^q_{H^1_q}\right) \\ +\int_0^t\omega^2e^{-\omega(t-s)}\left(\frac{1}{\theta}\|1+u\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v\|^q_{H^1_q}\right)ds \\ \le\left(1+\frac{\omega}{\theta}\right)\psi_{\theta,q}\left(\sup_{t\ge0}\|v\|_{H^1_q}\right)+\omega\left(\frac{1}{\theta}\|1+u_0\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v_0\|^q_{H^1_q}\right).$$

Thus we complete the proof of the lemma. □

Lemma 10. *Let* (u, v) *be a local solution to* (E)*, and assume that*

$$\alpha > 1, \quad 0 < \beta \le \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha - 1).$$

Suppose that for some exponent $\sigma > 1$ *and* $r > 2n\alpha/[(n+2)(\alpha-1)]$ *the integral* $I_\omega^{\alpha+\sigma-1}(t)$ *is bounded by*

$$I_\omega^{\alpha+\sigma-1}(t) \le (1+\omega)\,\psi_{\sigma,r}\left(\|1+u_0\|_{L_\sigma} + \|v_0\|_{H_r^1}\right) \equiv \bar{I}_\omega^{\alpha+\sigma-1} \tag{42}$$

for an arbitrary constant $\omega > 0$*. Then, for any exponent* $q \ge 2$ *satisfying* $r \le q \le (\alpha + \sigma - 1)/\beta$*,*

$$\|v\|_{H_q^1}^q \le C_q e^{-\delta_q t}\|v_0\|_{H_q^1}^q + C_q\,\psi_{\sigma,r}\left(\|1+u_0\|_{L_\sigma} + \|v_0\|_{H_r^1}\right) \tag{43}$$

holds with some positive constants C_q *and* δ_q*. Moreover, for any exponent* $\sigma \le \theta \le \{q(n+2)/(2n) - 1\}(\alpha - 1)$*,*

$$\|1+u\|_{L_\theta}^\theta \le e^{-qt/(2\tau)}\|1+u_0\|_{L_\theta}^\theta + \psi_{\theta,q}\left(\|1+u_0\|_{L_\sigma} + \|v_0\|_{H_q^1}\right) \tag{44}$$

holds with some increasing function $\psi_{\theta,q}(\cdot)$*. In addition, for an arbitrary constant* $\omega > 0$*, it holds that*

$$I_\omega^{\alpha+\theta-1}(t) \le (1+\omega)\,\psi_{\theta,q}\left(\|1+u_0\|_{L_\theta} + \|v_0\|_{H_q^1}\right) \equiv \bar{I}_\omega^{\alpha+\theta-1}. \tag{45}$$

Proof. We can prove the lemma in the similar argument as in Lemmas 8 and 9.

Firstly, the inequality (33) holds also in this case. Since $q\beta \le \alpha + \sigma - 1$, by Lemma 6 again, we verify

$$\|v\|_{H_q^1}^q \le C_q e^{-\delta_q t}\|v_0\|_{H_q^1}^q + C_q\, b_{q,\alpha+\sigma-1}\left(|\Omega| + I_{q/2\tau}^{\alpha+\sigma-1}(t)\right).$$

By (42), we obtain (43).

The estimate (44) is verified from the inequality (41) together with (43), since $q\beta \le \alpha + \sigma - 1 \le \alpha + \theta - 1$ and $\theta\kappa/(\kappa - 1) \le \alpha + \theta - 1$ with $\kappa = q(n+2)/(2n)$. The proof of (45) is just the same as that of (35).

Thus we complete the proof of the lemma. □

For obtaining the final a priori estimate, we apply Lemma 10 iteratively. We then show the following a priori estimate.

Proposition 11. *Let* (u, v) *be a local solution to* (E)*, and assume that*

$$\alpha > 1, \quad 0 < \beta \le \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha - 1).$$

Then, for any exponent $p > 2$*, it holds that*

$$\|1+u\|_{L_p}^p + \|v\|_{H_p^1}^p \le Ce^{-pt/(2\tau)}\left(\|1+u_0\|_{L_p}^p + \|v_0\|_{H_p^1}^p\right) + \psi_p\left(\|1+u_0\|_{L_\sigma} + \|v_0\|_{H_r^1}\right) \tag{46}$$

with some exponents $1 < \sigma < p$*,* $\alpha/\beta < r < p$ *and some increasing function* $\psi_p(\cdot)$*.*

Proof. The proof is given by induction. Firstly we have estimates (28) on $\|u\|_{L_1}$. Let

$$\theta_0 = 1.$$

Secondly we have (30) on $\|v\|_{H^1_q}$ for $2 \le q \le \alpha/\beta$ by Lemma 8 and (34) on $\|1+u\|_{L_\theta}$ for $1 < \theta \le \{q(n+2)/(2n) - 1\}(\alpha - 1)$ by Lemma 9. Let

$$q_1 = \frac{\theta_0 + \alpha - 1}{\beta} = \frac{\alpha}{\beta}, \quad \theta_1 = \left(\frac{n+2}{2n} q_1 - 1\right)(\alpha - 1).$$

For each integer k and given θ_k, we can obtain by Lemma 10 the estimates (43) on $\|v\|_{H^1_q}$ for $2 < q \le (\theta_k + \alpha - 1)/\beta$ and (44) on $\|1+u\|_{L_\theta}$ for $1 < \theta \le \{q(n+2)/(2n) - 1\}(\alpha - 1)$. Define

$$q_{k+1} = \frac{\theta_k + \alpha - 1}{\beta} = \frac{(n+2)(\alpha-1)}{2n\beta} q_k, \quad \theta_{k+1} = \left(\frac{n+2}{2n} q_{k+1} - 1\right)(\alpha - 1).$$

Since $(n+2)(\alpha-1)/(2n\beta) > 1$ by assumption, we can easily see that

$$q_k \to \infty \quad \text{and} \quad \theta_k \to \infty \quad \text{as} \quad k \to \infty.$$

Hence, for any $p > 1$, there exists a finite integer k_0 such that $q_{k_0} > p$ and $\theta_{k_0} > p$, and the desired estimates are obtained. □

By using the a priori estimates shown above, we prove the main theorem for the global existence of the solutions.

Proof of Theorem 1. From Theorem 5 for each pair of nonnegative initial functions (u_0, v_0) there exists a unique nonnegative local solution (u, v) on the interval $[0, T]$ with the estimate (27), and the existence time T depends only on the norm $\|u_0\|_{L_p} + \|v_0\|_{H^1_p}$. In addition, from Proposition 11, the norm $\|u(t)\|_{L_p} + \|v(t)\|_{H^1_p}$, $0 \le t \le T$, is estimated from above by a uniform constant C_{U_0} also depending only on the norm $\|u_0\|_{L_p} + \|v_0\|_{H^1_p}$. Hence, the interval can be extended to $[0, T + \tilde{T}]$, where the extended time $\tilde{T}$ and the norm $\|u(t)\|_{L_p} + \|v(t)\|_{H^1_p}$, $0 \le t \le T + \tilde{T}$, are estimated by the same constant C_{U_0}. The existence interval can be again extended, to $[0, T + 2\tilde{T}]$. Repeating this procedure proves the global existence theorem with the estimate (5). □

Appendix . On the domains of fractional powers of Laplace operators in L_p-spaces

Here we discuss the characterization of the domains of definition of fractional powers of Laplace operator $A_0 = -\Delta + 1$ with Neumann boundary condition on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, as a closed operator in $L_p(\Omega)$ for each $1 < p < \infty$.

We have already known the following facts.

Theorem A.1 ([2, Theorem 2.4.1.3], [24, Theorem 5.3.4], [30, Theorem 2.15])**.** *For each $1 < p < \infty$, A_0 is considered as a closed operator in $L_p(\Omega)$, the domain of which is $H^2_{p,N}(\Omega)$. If we denote $A_p = A_0|_{L_p}$, then it holds that $\mathcal{D}(A_p) = H^2_{p,N}(\Omega)$ with norm equivalence.*

Theorem A.2 ([2, Theorem 2.5.1.1], [24, Theorems 5.3.4 and 5.4.1])**.** *Let k be a positive integer and $1 < p < \infty$. Then $u \in H^{k+2}_p(\Omega) \cap H^2_{p,N}(\Omega)$ yields $A_0 u \in H^k_p(\Omega)$. Moreover, if $u \in H^2_{p,N}(\Omega)$ satisfies $A_0 u \in H^k_p(\Omega)$, then $u \in H^{k+2}_p(\Omega)$. That means, as the first example, if we denote $\mathfrak{A}_p = A_0|_{H^1_p}$, that the identity $\mathcal{D}(\mathfrak{A}_p) = H^3_{p,N}(\Omega)$ holds with norm equivalence.*

To interpolate these results between $k = 0$ and $k = 2$, we apply the theory of bounded H_∞ functional calculus in $L_p(\Omega)$ given by Yagi [30, Sec.16.1.2].

Theorem A.3. *For the operator $A_p = A_0|_{L_p}$, the identity*

$$\text{(A.1)} \qquad \mathcal{D}(A_p^\theta) = [L_p(\Omega), H^2_{p,N}(\Omega)]_\theta = \begin{cases} H_p^{2\theta}(\Omega) & \text{for } 0 \le \theta < \frac{1}{2} + \frac{1}{2p} \\ H_{p,N}^{2\theta}(\Omega) & \text{for } \frac{1}{2} + \frac{1}{2p} < \theta \le 1 \end{cases}$$

holds with norm equivalence.

Proof. Firstly, it is obvious that $L_p(\Omega)$ is a reflexive Banach space and A_p is a sectorial operator in $L_p(\Omega)$ with angle $\omega_A = 0$. Hence, we can directly verify the following condition given in [30, Theorem 16.5] with $A = A_p$, $X = L_p(\Omega)$, $X^* = L_{p'}(\Omega)$ and $\langle \cdot, \cdot \rangle$ their duality product:

(H) For every angle $\omega_A < \omega < \pi$ and every exponent $0 < \theta < 1$, the integrable condition along the V-shaped contour $\Gamma_\omega : \lambda = \rho e^{\pm i\omega}$ $(0 \le \rho < \infty)$

$$\text{(A.2)} \quad I_{\omega,\theta} = \int_{\Gamma_\omega} |\lambda|^{2\theta-1} \left|\langle A^{2(1-\theta)}(\lambda - A)^{-2}F, G\rangle\right| |d\lambda| \le C_{\omega,\theta}\|F\|\|G\|_*, \ F \in X, G \in X^*,$$

holds with some constant $C_{\omega,\theta} > 0$.

We omit the detail here. Then, by [30, Theorem 16.5] it is verified that A_p has a bounded H_∞ functional calculus in $L_p(\Omega)$. Again by [30, Theorem 16.5], we have the first identity of (A.1). The rest part of the theorem has been already shown in [30, Theorem 16.11]. □

The next theorem shows the interpolation result between $k = 1$ and $k = 3$.

Theorem A.4. *For the operator $\mathfrak{A}_p = A_0|_{H_p^1}$, the identity*

$$\text{(A.3)} \qquad \mathcal{D}(\mathfrak{A}_p^\theta) = [H^1_p(\Omega), H^3_{p,N}(\Omega)]_\theta = \begin{cases} H_p^{1+2\theta}(\Omega) & \text{for } 0 \le \theta < \frac{1}{2p} \\ H_{p,N}^{1+2\theta}(\Omega) & \text{for } \frac{1}{2p} < \theta \le 1 \end{cases}$$

holds with norm equivalence.

Proof. It is also obvious that $H^1_p(\Omega)$ is a reflexive Banach space with duality product $\langle\langle \cdot, \cdot \rangle\rangle$ of $H^1_p(\Omega) \times H^1_{p'}(\Omega)$, and that $\mathfrak{A}_p$ is a sectorial operator in $H^1_p(\Omega)$ with angle $\omega_A = 0$. Hence, again we can prove the condition (H) above by direct calculation to see that the Laplace operator $\mathfrak{A}_p = A_0|_{H^1_p}$ has a bounded H_∞ functional calculus in $H^1_p(\Omega)$. Then, we have the first identity of (A.3) again by [30, Theorem 16.5]. For the proof of the rest part of the theorem, we must follow carefully the proof of [30, Theorem 16.11].

Step 1: $\mathcal{D}(\mathfrak{A}_p^\theta) \subset H^{1+2\theta}_{p,(N)}(\Omega)$. We can easily see that $[H^1_p(\Omega), H^3_{p,N}(\Omega)]_\theta \subset [H^1_p(\Omega), H^3_p(\Omega)]_\theta = H^{1+2\theta}_p(\Omega)$. To see the boundary condition for $1/(2p) < \theta < 1$, take a sequence $\{u_k\} \subset \mathcal{D}(\mathfrak{A}_p) = H^3_{p,N}(\Omega)$ converging to u in $\mathcal{D}(\mathfrak{A}_p^\theta)$. This implies that every $u \in \mathcal{D}(\mathfrak{A}_p^\theta)$ satisfies the Neumann boundary condition on $\partial\Omega$.

Step 2: $\mathcal{D}(\mathfrak{A}_p^\theta) \supset H^{1+2\theta}_{p,(N)}(\Omega)$. We divide this step into three parts.

(i) $\frac{1}{2} \le \theta \le 1$. Let $u \in H^{1+2\theta}_{p,N}(\Omega)$. Then, for any $v \in H^3_{p',N}(\Omega) = \mathcal{D}(\mathfrak{A}_p^*)$, similarly in the proof of [30, Theorem 16.11], we see

$$\left|\langle\langle u, (\mathfrak{A}_p^*)^\theta v\rangle\rangle\right| = \left|\frac{1}{2\pi i}\int_\Gamma \lambda^\theta \langle\langle u, (\lambda - \mathfrak{A}_p^*)^{-1} v\rangle\rangle\, d\lambda\right|$$
$$\le \|A_0 u\|_{H_p^{2\theta-1}} \left\|\frac{1}{2\pi i}\int_\Gamma \lambda^\theta (\lambda - A_0)^{-1} v\, d\lambda\right\|_{(H_p^{2\theta-1})'}$$
$$\le C\|A_0 u\|_{H_p^{2\theta-1}} \left\|A_0^{\theta-\frac{1}{2}} v\right\|_{H_{p'}^{2-2\theta}} \le C\|u\|_{H_p^{1+2\theta}}\|v\|_{H_{p'}^1}.$$

Here we utilize [30, Theorem 1.43] for the boundedness of $A_0 = 1 - \sum_k D_k^2 : H_p^{2\theta+1}(\Omega) \to H_p^{2\theta-1}(\Omega)$ and Theorem 3 for the boundedness of the fractional powers of $A_0|_{L_p} = A_p$. This inequality yields that, for each fixed $u \in H_{p,N}^{1+2\theta}(\Omega)$, the linear form $\langle\langle u, (\mathfrak{A}_p^*)^\theta v\rangle\rangle$ is a bounded linear functional of $v \in H_{p'}^1(\Omega)$, that is, there exists $w \in H_p^1(\Omega)$ such that $\langle\langle u, (\mathfrak{A}_p^*)^\theta v\rangle\rangle = \langle\langle w, v\rangle\rangle$ for any $v \in H_{p'}^1(\Omega)$. Hence, $w = \mathfrak{A}_p^\theta u$ and $u \in \mathcal{D}(\mathfrak{A}_p^\theta)$.

(ii) $\frac{1}{2p} < \theta < \frac{1}{2}$. Let $u \in H_{p,N}^{1+2\theta}(\Omega)$. Then, for any $v \in H_{p',N}^3(\Omega) = \mathcal{D}(\mathfrak{A}_p^*)$, by an argument quite similar to (i), we see

$$\left|\langle\langle u, (\mathfrak{A}_p^*)^\theta v\rangle\rangle\right| \le C\|u\|_{H_p^{2\theta+1}} \left\|\frac{1}{2\pi i}\int_\Gamma \lambda^\theta (\lambda - A_0)^{-1} v\, d\lambda\right\|_{H_{p'}^{1-2\theta}} \le C\|u\|_{H_p^{2\theta+1}}\|v\|_{H_{p'}^1}.$$

Here we utilize again [30, Theorem 1.43] for the boundedness of $D_k : H_{p'}^{1-2\theta}(\Omega) \to (H_p^{2\theta}(\Omega))'$. Thus, for each fixed $u \in H_{p,N}^{1+2\theta}(\Omega)$, there exists $w \in H_p^1(\Omega)$ such that $\langle\langle u, (\mathfrak{A}_p^*)^\theta v\rangle\rangle = \langle\langle w, v\rangle\rangle$ for any $v \in H_{p'}^1(\Omega)$. Hence, $w = \mathfrak{A}_p^\theta u$ and $u \in \mathcal{D}(\mathfrak{A}_p^\theta)$.

(iii) $0 < \theta < \frac{1}{2p}$. We can verify that $u \in H_p^{1+2\theta}(\Omega)$ is contained in $\mathcal{D}(\mathfrak{A}_p^\theta)$ by the same argument as (ii) except for the boundary conditions.

Hence we complete the proof. □

As the consequence, we conclude (17) in Section 2.

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