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THE MULTIMATCHING PROPERTY OF NESTED SETS

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RÉSUMÉ – La propriété de multicouplage d'ensembles de paires non croisées.

Présentation d'une généralisation de la propriété de couplage entre deux ensembles de paires non croisées et des résultats correspondants. Certains mots de Motzkin sont associés à cette notion, ainsi qu'aux permutations non planaires.

MOTS-CLÉS□– Mot de Dyck, Mot de Motzkin, Couplage, Non croisée, Permutation planaire.

SUMMARY – *A generalization of the matching property of nested sets and of the relevant results is presented. Motzkin words are associated to this notion and are related to non planar permutations.*

KEYWORDS – Dyck word, Motzkin word, Matching, Nested, Planar permutation.

1. INTRODUCTION

The matching property of nested sets was introduced in [3] and was applied on planar permutations in [2].

In this paper the matching property is generalized with the introduction of k-matching nested sets. Four recursive constructions are presented for the generation of k-matching nested sets; every pair of k-matching nested sets can be constructed this way. The k-matching nested sets are related to Motzkin words. A Motzkin word is assigned to each non planar permutation, thus giving a correspondence between non planar permutations and k-matching nested sets.

k-matching nested sets are represented by non intersecting, closed, plane curves, creating figures in the shape of level curves (isotherms, indifference curves, etc.). Furthermore, they could be related to components of electrical circuits (see [5]).

We recall that a set U of pairwise disjoint sets of pairs of elements of $\mathbb{N}^*$ is called nested set of pairs if for every $\{a,b\},\{c,d\} \in U$ we never have $a < c < b < d$. Let domU be the set of all the elements of $\mathbb{N}^*$ that belong to some pair of U and N_{2n} the set of all nested sets of pairs U with $\text{dom}U = [2n]$. We say that two nested sets U, L are *matching* iff $\text{dom}U = \text{dom}L$ and $\text{dom}A = \text{dom}B$, $A \subseteq U$, $B \subseteq L$ imply that either $A = B = \varnothing$ or $\text{dom}A = \text{dom}U$.

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2. k-MATCHING PROPERTY

Let U be a nested set and B ⊆ domU; we call B *complete* if for every a ∈ B with {a,b} ∈ U, we have b ∈ B. We write U/B = {{a,b} ∈ U : a ∈ B}.

PROPOSITION 2.1. *Let* U, L *be two nested sets of pairs with* domU = domL. *Then, there exists a partition* $B_1, B_2, …, B_k$ *of* domU *with* B_i *complete, such that the sets* $U/B_i, L/B_i, i ∈ [k]$ *are matching.*

PROOF. Let $U_1 ⊆ U$, $L_1 ⊆ L$ with $domU_1 = domL_1$ and $\left| domU_1 \right|$ is minimum but not zero (such a pair exists, since domU = domL). U_1, L_1 are matching nested sets. Since, obviously, the sets $U\backslash U_1$ and $L\backslash L_1$ are also nested, we repeat the procedure for these sets, thus obtaining the matching nested sets U_2, L_2. We continue recursively until we get that $U_k = U \backslash (U_1 ∪ U_2 ∪ … ∪ U_{k-1})$, $L_k = L \backslash (L_1 ∪ L_2 ∪ … ∪ L_{k-1})$ are nested, thus determining uniquely the required partition $\{domU_1, domU_2,…, domU_k\}$ of domU. ∎

We call two nested sets of pairs *k-matching*, k ∈ ℕ * in order to indicate the number of blocks contained in the k-partition of the above proposition. In this paper we deal with k ≥ 2, since for k = 1 we get the notion of matching nested sets (see [3]).

We find the blocks of the partition formed by the k-matching nested sets U, L by the following recursive procedure.

Suppose we have already found the blocks $B_1, B_2, …, B_m$. We choose the pair {a,b} ∈ U containing the smallest element a of [2n] not already included in $B_1 ∪ B_2 ∪ … ∪ B_m$. Then we choose the pair {b,c} ∈ L, then the pair {c,d} ∈ U and so on, until we choose the pair of L containing a. The union of these pairs forms the block B_{m+1}.

So, the sets   U={{1,10}, {2,3}, {4,5}, {6,9}, {7,8}, {11,12}}

L={{1,4}, {2,3}, {5,6}, {7,12}, {8,11}, {9,10}}

are 3-matching.

Indeed we have:

U⊡{1,10}   {9,6}   {5,4}   ‖ {2,3}   ‖ {7,8}   {11,12}   ‖

L⊡   {10,9}   {6,5}   {4,1} ‖   {3,2} ‖   {8,11}   {12,7} ‖

thus getting the blocks B_1={1,4,5,6,9,10}, B_2={2,3}, B_3={7,8,11,12} and the matching nested sets:

U/B_1={{1,10}, {4,5}, {6,9}} U/B_2={{2,3} U/B_3={{7,8}, {11,12}}

L/B_1={{1,4}, {5,6}, {9,10}} L/B_2={{2,3}} L/B_3={{7,12}, {8,11}}
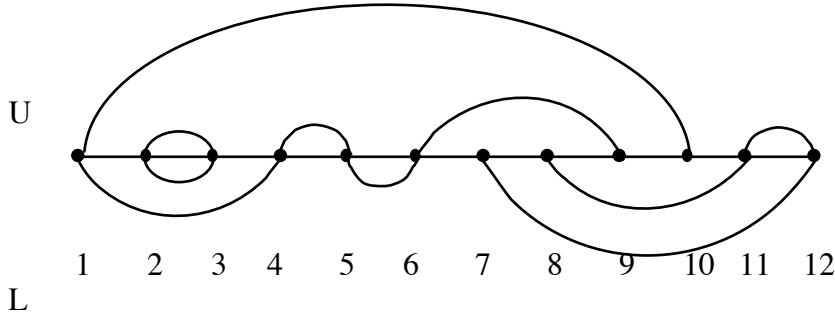
(see Figure 1).

Figure 1. 3-matching nested sets

As we have already pointed out, the k-matching property is a generalization of the matching property of nested sets, which has been studied in [3]. So, we present here the generalization of the relevant propositions, after giving the following definition.

Given $U \in N_{2n}$ and $c \in Z$, we define the translation $U + c = \{\{a + c, b + c\} : \{a,b\} \in U\}$, where all numbers are taken mod2n, so that $U + c \in N_{2n}$.

PROPOSITION 2.2. *If* U, L *are k-matching, then* U + c, L + c *are also k-matching.*

A pair $\{a,b\} \in U$ is called *outer pair* if there is no pair $\{c,d\} \in U$ such that $c \le a < b < d$, whereas it is called *short pair* if there is no $c \in domU$ with $a < c < d$.

In [6] it has been shown that every element of N_{2n+2} may be written in either one or the other of the forms

$U^* = U \cup \{\{2n + 1, 2n + 2\}\}$ and $U_{ab} = (U \setminus \{\{a,b\}\}) \cup \{\{a, 2n + 2\},\{b, 2n+ 1\}\}$,

where $U \in N_{2n}$ and $\{a,b\}$ is an outer pair of U.

PROPOSITION 2.3. *If the sets* L, U *of* N_{2n} *are k-matching, then the sets* L*, U_{ab} *of* N_{2n+2} *are also k-matching.*

PROPOSITION 2.4. *Let* L, U *be two k-matching nested sets of* N_{2n}. *If* {a,b} *and* {c,d} *are outer pairs of* L *and* U *respectively, that do not belong to the same block of the k-partition, then the sets* L_{ab}, U_{cd} *are (k-1)-matching nested sets of* N_{2n+2}.

The proofs of propositions 2.2, 2.3 and 2.4 are straightforward extensions of the respective proofs of [3].

For each nested set U of N_{2n} and each outer pair {a,b} of U, we now define the nested set of N_{2n+2}

$$U_a = U_1 \cup (U_2 + 1) \cup \{\{a, 2n +2\}\}$$

where $U_1 = U/[a - 1]$, $U_2 = U/[a, 2n]$ and $[0] = \varnothing$. Notice that this partition of U into U_1, U_2 always exists, since {a,b} is an outer pair.

PROPOSITION 2.5. *If* L, U *are k-matching nested sets of* N_{2n} *and* $\{a,b\}, \{a,c\}$ *are outer pairs of* L, U *respectively then* L_a, U_a *are (k+1)-matching nested sets of* N_{2n+2}.

PROPOSITION 2.6. *If* L, U *are k-matching nested sets of* N_{2n} *then* $L \cup \{2n + 1, 2n + 2\}, U \cup \{2n + 1, 2n + 2\}$ *are (k+1)-matching nested sets of* N_{2n+2}.

Propositions 2.3, 2.4, 2.5 and 2.6 suggest four constructions for the generation of k-matching nested sets of N_{2n+2}, using h-matching nested sets of N_{2n}, with $h \le k$.

The generalization of the result of [3] for the converse procedure is given in the following proposition.

PROPOSITION 2.7. *Every pair* U, L *of k-matching nested sets of* N_{2n+2}, *may be generated by h-matching nested sets of* N_{2n}, *with* $h \le k$.

Let U be a nested set of N_{2n}. We define $\tilde{U}$ to be the nested set such that $\{a,b\} \in U$ if and only if $\{2n + 1 - b, 2n + 1 - a\} \in \tilde{U}$.

PROPOSITION 2.8. U, L *are k-matching nested sets if and only if* $\tilde{U}, \tilde{L}$ *are k-matching nested sets.*

3. MOTZKIN WORDS

We recall that a word $w \in \{x, \overline{x}\}^*$ is called *Dyck* if $|w|_x = |w|_{\overline{x}}$ and for every factorization $w = uv$ of w, we have $|u|_x \ge |u|_{\overline{x}}$ where $|w|_x, |u|_x$ (resp. $|w|_{\overline{x}}, |u|_{\overline{x}}$) is the number of occurrences of the letter x (resp. $\overline{x}$) in w, u.

A word $w \in \{x, \overline{x}, y, \overline{y}\}^*$ is called *Motzkin* if the word obtained by deleting all occurrences of $y, \overline{y}$ from w, is a Dyck word.

Now, to each pair of k-matching nested sets $U, L \in N_{2n}$ we assign a word $w = z_1 z_2 \ldots z_{2n}$ of $\{x, \overline{x}, y, \overline{y}\}^*$ with

$$z_i = \begin{cases} x, & \text{if } i < j, h \\ \overline{x}, & \text{if } h, j < i \\ \\ y, & \text{if } h < i < j \\ \overline{y}, & \text{if } j < i < h \end{cases}$$

where $\{i,j\} \in U$, $\{i,h\} \in L$.

So, from our example of section 2, we get the following word

$$w = x \quad x \quad \overline{x} \quad y \quad \overline{y} \quad y \quad x \quad \overline{y} \quad \overline{y} \quad \overline{x} \quad y \quad \overline{x}$$

which is a Motzkin word.

In general, we have the following proposition.

PROPOSITION 3.1. *The word* w *is a Motzkin word, having the same number of occurrences of* y *and* $\overline{y}$.

PROOF. There is a Dyck word, and hence a set of left and right parentheses, corresponding to each nested set U and L (see [1]). Let X be the subset of $[2n]$ corresponding to the positions of the word w, where both parentheses are left; similarly X' for both right parentheses, Y for left U parenthesis and right L parenthesis and finally Y' for right U parenthesis and left L parenthesis. Notice that if in the i-th position both parentheses are left then $i < j, k$ and so $z_i = x$; hence, $|X|$ equals the number of occurrences of x in w. Similarly $|X'|, |Y|$ and $|Y'|$ equal the number of occurrences of $\overline{x}, y, \overline{y}$ respectively.

Let X_p (resp. X'_p, Y_p, Y'_p) be the subset of X (resp. X', Y, Y') that we get if we deal with the first p pairs of parentheses only. Since U corresponds to a Dyck word we have that $|X_p \cup Y_p| \geq |X'_p \cup Y'_p|$, since $X_p \cup Y_p$ (resp. $X'_p \cup Y'_p$) refers to the positions where the U parentheses are left (resp. right). Since X_p, Y_p and X'_p, Y'_p are obviously disjoint, we get that $|X_p| + |Y_p| \geq |X'_p| + |Y'_p|$, i.e.

$$|X_p| - |X'_p| \geq |Y'_p| - |Y_p| \tag{1}$$

From L we similarly get $|X_p \cup Y'_p| \geq |X'_p \cup Y_p|$ and hence, finally

$$|X_p| - |X'_p| \geq |Y_p| - |Y'_p| \tag{2}$$

From (1), (2) we get that $|X_p| - |X'_p| \geq ||Y_p| - |Y'_p|| \geq 0$

i.e.
$$|X_p| \geq |X'_p| \tag{3}$$

From the definition of Dyck words we also have that

$$|X| + |Y| = |X'| + |Y'| \quad \text{and} \quad |X| + |Y'| = |X'| + |Y|$$

getting
$$|X| = |X'| \tag{4}$$
and
$$|Y| = |Y'| \tag{5}$$

(3) and (4) prove that the word which we get if we delete all occurrences of y and $\overline{y}$ from w is a Dyck word (and so w is a Motzkin word), whereas (5) proves that the number of occurrences of y and $\overline{y}$ in w is the same. ∎

We also have the following result.

PROPOSITION 3.2. *Each digram* $x\bar{x}, x\bar{y}, y\bar{x}, y\bar{y}$ *(resp.* $x\bar{x}, xy, \bar{y}\bar{x}, \bar{y}y$*) in* w, *corresponds to a short pair of* U *(resp.* L*).*

So, applying recursively the above proposition, we get from the Motzkin word w of our example the nested sets U and L as follows⊡

$$
\begin{array}{ccccccccccccc}
w = & x & x & \bar{x} & y & \bar{y} & y & x & \bar{y} & \bar{y} & \bar{x} & y & \bar{x} \\
& 1 & \boxed{2 \quad 3} & \boxed{4 \quad 5} & 6 & \boxed{7 \quad 8} & 9 & 10 & \boxed{11 \quad 12}
\end{array}
$$

$$
\begin{array}{cccc}
x & y & \bar{y} & \bar{x} \\
1 & \boxed{6 \quad 9} & 10
\end{array}
$$

$$
\begin{array}{cc}
x & \bar{x} \\
\boxed{1 \quad 10}
\end{array}
$$

So, U={{1,10},{2,3},{4,5},{6,9},{7,8},{11,12}.

Similarly, for L we have⊡

$$
\begin{array}{ccccccccccccc}
w = & x & x & \bar{x} & y & \bar{y} & y & x & \bar{y} & \bar{y} & \bar{x} & y & \bar{x} \\
& 1 & \boxed{2 \quad 3} & 4 & \boxed{5 \quad 6} & 7 & 8 & \boxed{9 \quad 10} & 11 & 12
\end{array}
$$

$$
\begin{array}{cccccc}
x & y & x & \bar{y} & y & \bar{x} \\
\boxed{1 \quad 4} & 7 & \boxed{8 \quad 11} & 12
\end{array}
$$

$$
\begin{array}{cc}
x & \bar{x} \\
\boxed{7 \quad 12}
\end{array}
$$

So, L={{1,4},{2,3},{5,6},{7,12},{8,11},{9,10}}.

We define the level of a pair {a,b} of a nested set U recursively, as follows:

 {a,b} is of level 1 if it is a short pair.

 {a,b} is of level k + 1 if every pair {c,d} with a < c < b is of level less or equal to k and there is a least one such pair of level k.

Obviously, each pair of maximum level in U is an outer pair.

Notice that the above procedure determines not only the pairs of each nested set, but their level too: the pairs using the elements of the block B_i, i ∈ [k] of the corresponding partition of proposition 2.1, have level i.

So, in our previous example, the pairs {2,3},{4,5},{7,8},{11,12} of U are short pairs, whereas {6,9} is of level 2 and {1,10} is of level 3.

4. NON PLANAR PERMUTATIONS

Planar permutations were defined by Rosenstiehl in [4]. A permutation σ on [2n] with σ(1) = 1 is called planar (p.p.) if the sets U_σ={{σ (2i - 1), σ (2i)} : i ∈ [n]},

$L_\sigma = \{\{\sigma(2i), \sigma(2i + 1)\} : i \in [n - 1]\} \cup \{\{\sigma(2n), 1\}\}$ are both nested;

e.g. $\sigma = 1\;\; 6\;\; 7\;\; 8\;\; 5\;\; 4\;\; 9\;\; 10\;\; 11\;\; 12\;\; 3\;\; 2$ is a p.p. with

$U_\sigma = \{\{1,6\},\{2,3\},\{4,5\},\{7,8\},\{9,10\},\{11,12\}\}$,

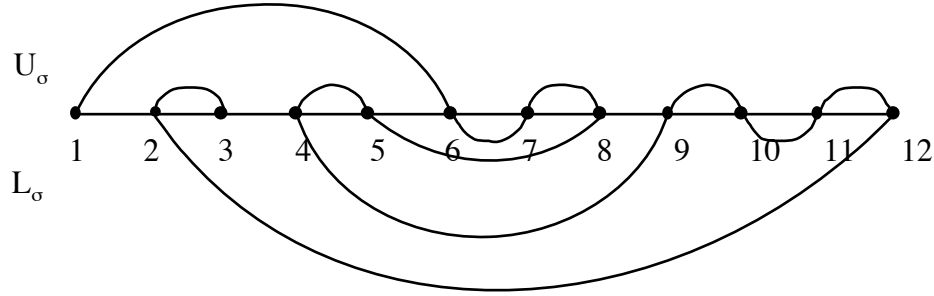$L_\sigma = \{\{1,2\},\{3,12\},\{4,9\},\{5,8\},\{6,7\},\{10,11\}\}$; (see Figure 2).

Figure 2. The nested sets U_σ, L_σ of a p.p. σ

Let Π_{2n} be the set of all p.p.'s of lenght 2n. It is easy to check that for every $\sigma \in \Pi_{2n}$ we have that $\sigma(i)$ is odd iff i is odd.

Propositions 3.4 and 3.5 of [3] give a correspondence between pairs of matching nested sets and p.p.'s on [2n]. This result is generalized by the following proposition:

PROPOSITION 4.1. *To each non planar permutation σ of length 2n corresponds a pair of k-matching nested sets*.

Indeed, as we have already developed a procedure giving the two matching nested sets of pairs from the word $w = w_1\, w_2 \ldots w_{2n}$ of $\{x, \bar{x}, y, \bar{y}\}^*$, it is enough to assign a particular word w to each non planar permutation $\sigma = \sigma(1)\, \sigma(2) \ldots \sigma(2n)$.

We do this as follows:

$$w_{\sigma(i)} = \begin{cases} x, & \text{if} \quad \sigma(i - 1) > \sigma(i) < \sigma(i + 1) \\[4pt] \bar{x}, & \text{if} \quad \sigma(i - 1) < \sigma(i) > \sigma(i + 1) \\[4pt] y, & \text{if} \quad \sigma(i - 1) < \sigma(i) < \sigma(i + 1) \quad \text{and}\;\; i\;\; \text{is odd, or} \\[4pt] & \qquad \sigma(i - 1) > \sigma(i) > \sigma(i + 1) \quad \text{and}\;\; i\;\; \text{is even} \\[4pt] \bar{y}, & \text{if} \quad \sigma(i - 1) < \sigma(i) < \sigma(i + 1) \quad \text{and}\;\; i\;\; \text{is even, or} \\[4pt] & \qquad \sigma(i - 1) > \sigma(i) > \sigma(i + 1) \quad \text{and}\;\; i\;\; \text{is odd} \end{cases}$$

for every $i \in [2n]$ (where the indices are taken mod2n).

e.g. For the non planar permutation $\sigma = 1\;\; 3\;\; 2\;\; 5\;\; 6\;\; 9\;\; 12\;\; 11\;\; 7\;\; 10\;\; 8\;\; 4$ we get $w = x\;\; x\;\; \bar{x}\;\; y\;\; \bar{y}\;\; y\;\; x\;\; \bar{y}\;\; \bar{y}\;\; \bar{x}\;\; y\;\; \bar{x}$ giving the 3-matching nested sets U, L of the example of section 2.

It is easy to establish a correspondence between the 1-1 mappings of length $2n$ to a subset of $\mathbb{N}^*$, and the set Π_{2n} of p.p.'s on $[2n]$; indeed, if $\tau = \tau(1) \tau(2)\ldots\tau(2n)$ is such a mapping, we can consider the corresponding $\sigma = \sigma(1) \sigma(2) \ldots \sigma(2n)$ of Π_{2n} to be the unique p.p. for which $\sigma(i) < \sigma(j)$ if and only if $\tau(i) < \tau(j)$, $i, j \in [2n]$.

e.g. If $\tau = 1\ \ 6\ \ 5\ \ 4\ \ 9\ \ 10$, then the corresponding $\sigma \in \Pi_6$ is $\sigma = 1\ \ 4\ \ 3\ \ 2\ \ 5\ \ 6$.

The matching nested sets U/B_i, L/B_i, $i = 1, 2, \ldots, k$ of proposition 2.1 give rise to 1-1 mappings of length $2n_i$ to subsets of $\mathbb{N}^*$.

So, from proposition 4.1 we get the following corollary.

COROLLARY 4.2. *To each non-planar permutation corresponds a family of p.p.'s.*

e.g. We have already seen that from $\sigma = 1\ \ 3\ \ 2\ \ 5\ \ 6\ \ 9\ \ 12\ \ 11\ \ 7\ \ 10\ \ 8\ \ 4$ we get the 3-matching nested sets U,L which, in turn, give $\tau_1 = 1\ \ 10\ \ 9\ \ 6\ \ 5\ \ 4$, $\tau_2 = 2\ \ 3$ and $\tau_3 = 7\ \ 8\ \ 11\ \ 12$, giving the corresponding p.p.'s $\sigma_1 = 1\ \ 6\ \ 5\ \ 4\ \ 3\ \ 2$, $\sigma_2 = 1\ \ 2$ and $\sigma_3 = 1\ \ 2\ \ 3\ \ 4$.

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