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The axiomatic characterizations of majority voting and scoring rules

Les caractérisations axiomatiques du vote majoritaire et des classements par points

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THE AXIOMATIC CHARACTERIZATIONS OF MAJORITY VOTING AND SCORING RULES[1]

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RÉSUMÉ – Les caractérisations axiomatiques du vote majoritaire et des classements par points. *Le cadre arrowien de la théorie des choix collectifs est suffisament flexible pour entreprendre une étude axiomatique précise des règles de vote qui sont communément utilisées dans des élections politiques, lors de compétitions sportives ou par des comités d'experts etc. comme le vote à la majorité ou les classements par points. L'objectif de cet article est de rendre compte des résultats qui ont été obtenus dans cette direction depuis 1951. Nous présentons d'abord les conditions qui garantissent qu'une règle de choix collectif est démocratique. Ensuite, nous exposons en détails deux résultats fondamentaux : la caractérisation de la règle de décision à la majorité par May, et l'axiomatisation de la famille des classements par points par Young. Par la suite, en utilisant ces résultats, des classements par points particuliers, comme le vote uninominal à un tour ou la méthode de Borda, ont aussi pu être axiomatisés. Quelques remarques sur d'autres voies de recherche et des questions ouvertes concluent l'article.*

MOTS CLÉS – Choix collectif, Vote majoritaire, Classement par points, Vote uninominal, Méthode de Borda.

SUMMARY – *The Arrovian framework of social choice theory is flexible enough to allow for a precise axiomatic study of the voting rules that are used in political elections, sport competitions or expert committees, etc. such as the majority rule or the scoring rules. The objective of this paper is to give an account of the results that have been obtained in this direction since 1951. We first present some basic conditions for a collective decision rule to be democratic. Next, we expound in detail two fundamental results: the characterization of the majority rule by May, and the axiomatization of the family of scoring rules by Young. Afterwards, using these results, some specific scoring rules, such as the plurality vote or the Borda count, have also been characterized. Some remarks on other directions of research and open issues conclude the paper.*

KEYWORDS – Social Choice, Majority Voting, Scoring Rules, plurality voting, Borda count.

JEL classification number: D71.

1.   INTRODUCTION

Arrow's theorem [Arrow, 1963] is often considered as an impossibility result: there is no democratic voting rule that satisfies *a priori* benign requirements. On the other hand, it is the first characterization result in social choice theory: dictatorship is the unique social welfare function that satisfies Independence and Pareto. In fact, Arrow's model provides an adequate framework for the analysis and the comparison of the various voting processes that are used in every day life by committees, voting bodies, parliaments, juries, international institutions, etc. As Young [1975] pointed out:

> "What is needed is an axiomatic framework for comparing the merits of these various methods. This type of study was begun by Arrow, who identified a set of conditions that permit only dictatorship when three or more alternatives are involved. This result (and later refinement of it) tell us much about what cannot be done, but leaves open the problem of defining what can be done."

In particular, Arrow's theorem tells nothing about the most popular voting rules, that is the majority rule and the scoring rules. In a two-candidate election, the majority rule is the voting method that picks out as a winner the one who obtains the greatest number of votes. The class of scoring rules contains many rules, but they all rely on the same principle: when the choice set contains m candidates, each voter ranks the m alternatives from her first choice to her least preferred candidate, and each alternative obtains a fixed number of points, s_k, each time it is ranked k^{th} by one voter. The winner is the candidate who gets the greatest amount of points over the whole population. For example, the plurality rule which attributes one point for a first place and zero point for the other ranks is used in many country for parliamentary elections (e.g. United Kingdom, Australia, United States). Two centuries ago, Borda [1781] suggested another natural way to give points: a candidate should receive $m-1$ points each time she is ranked first by some voter, $m-2$ points each time she is ranked second, and so on down to one point for the next to the last and zero point for the last candidate in her preference ordering.

If we except the characterization of the majority rule by May in 1952 [May, 1952], the extensive use of Arrow's framework for the analysis of voting rules dates back to the seventies. At that time, Fishburn [1973], Gärdenfors [1973], Fine and Fine [1974(a)(b)], Smith [1973], Young [1974(a)(b), 1975], Richelson [1978], Nitzan and Rubinstein [1981] proposed several characterizations of the scoring rules. Since these pioneering works, several other voting rules have been characterized, and a floodgate of papers has compared the various merits and flaws of all kinds of voting rules (for recent books on this subjects, see for example Nurmi [1989] or Saari [1994]). Among all these works, we present two of them in details. The first one is May's characterization of the majority rule: his result provides theoretical arguments for using majority voting in many models of public economy. Moreover, his paper completes Arrow's 1951 book, by the study of the case $m = 2$. The second result we present is a characterization of scoring rules. Chebotarev and Shamis [1998] listed more than forty different characterizations of scoring rules in different contexts

(social choice, statistics, tournament theory). The one we provide here is based upon Young's articles [1974(b), 1975]. Several reasons justify this choice. The result in Young [1974(b)] is given for social welfare functions, which permits a direct comparison with the characterization of dictatorship. Moreover, the framework is almost Arrow's model, the only improvement being that the size of the electorate may vary (but is still finite). Smith [1973] presented a similar result one year before, but the proof given by Young in [1974(b)] is shorter and clearer, although it uses several lemmas from a paper he wrote before [Young, 1975]. It also focuses on the relationships between the notion of convexity and the scoring rules, which was not the case in previous works. And, to some extent, Young's results foreshadow the framework of the geometry of voting, that has been developed by Saari [1994] in the mid eighties and early nineties. At last, from a historical point of view, one may notice that this body of research on the axiomatization of majority rule and scoring rules show that the Arrovian framework is flexible enough to continue in modern terms the two hundred-year old debate between Borda, the proponent of the specific scoring rule which now bears his name, and Condorcet, a partisan of the majority vote.

The remaining of this paper is organized as follows. Section 2 recalls basic definitions and introduces the concept of social welfare function (SWF): it is a mapping that associates a social ranking of these candidates to each list of preferences on candidates. Similarly, a social choice correspondence (SCC) associates a non empty subset of candidates to each list of preferences on candidates. Section 3 is a preparatory section. It presents basic democratic requirements that a SWF should satisfy. The interesting point is that, whenever one of these axioms is satisfied, the domain of a SWF f can be extended from the set of all possible profiles of preferences to $\mathbb{N}^{m!}$, $\mathbb{Z}^{m!}$ (the set of all the integers), or even $\mathbb{Q}^{m!}$ (the set of all the rationals). From a technical point of view, this change allows to switch from combinatorial techniques (like in classical social choice) to linear algebra and geometry. Section 4 is devoted to the characterization of the majority vote. Section 5 presents the axiomatization of scoring rules. Several technical lemmas on $\mathbb{Q}$-convexity that are needed for the proof of the main theorem are also presented in this section. In Section 6, we consider several consequences of these results for SCCs. In particular, we present the characterizations of the Borda count, the plurality rule and the antiplurality rule that are based upon Young's results. To conclude, we complete the picture by describing other ways to analyze voting rules in social choice theory and raise some unsolved problems.

2. THE MODEL

Let $A = \{x, y, z, \ldots\}$ (or $\{a_1, a_2, \ldots, a_m\}$ if necessary) be a finite set of $m \geq 2$ elements called here alternatives (and elsewhere issues, decisions, proposals, candidates, allocations, policies,etc.). Each voter (or agent, individual, committee member) is identified with one element of the set of natural integers $\mathbb{N}$. A population V is a finite and non empty subset of $\mathbb{N}$. In this paper, we focus on the case where the preference of each voter $i \in \mathbb{N}$ is represented by a linear order L_i on A, but some-

times, we consider that the preference is a complete preorder R_i. If R is a complete preorder, we denote by I its symmetric part, and by P its asymmetric part.

A profile (of individual preferences) is a function π from $\mathbb{N}$ into $\mathcal{L}$, the set of linear orders (or $\mathcal{R}$, the set of complete preorders). $\mathcal{L}^\infty$ (respectively $\mathcal{R}^\infty$) is the set of all profiles of linear orders (respectively complete preorders). When we restrict our attention to a population $V \subset \mathbb{N}$ of size v, the profile of individual preferences $\pi(V) = (L_i)_{i \in V}$ in $\mathcal{L}^v$ is a v-tuple of linear orders (with the preferences presented in the order of increasing indices). A similar definition holds when the preferences are complete preorders ($\pi(V) \in \mathcal{R}^v$). For two disjoint populations V and V', with $V \cap V' = \emptyset$, and two profiles $\pi(V) = (L_i)_{i \in V}$ and $\pi(V') = (L_j)_{j \in V'}$, we define the union of the two profiles by the $2v$-tuple $\pi(V \cup V') = (L_k)_{k \in V \cup V'}$. We denote by $2^{\mathbb{N}}$ the set of all finite parts of $\mathbb{N}$. Thus, the set of all the possible profiles for any finite population is $\bar{\mathcal{R}} = \mathcal{R}^\infty \times 2^{\mathbb{N}}$ or $\bar{\mathcal{L}} = \mathcal{L}^\infty \times 2^{\mathbb{N}}$.

A **social welfare function** (SWF) is a function $f : \bar{\mathcal{R}} \to \mathcal{R}$ that assigns to each profile of preferences $\pi(V)$ a complete preorder $f(\pi(V)) = R_V$. A **social choice correspondence** (SCC) is a function $g : \bar{\mathcal{R}} \to 2^A \setminus \emptyset$ that assigns to each profile of preferences $\pi(V)$ a non empty set of alternatives. Of course, we can restrict the domain of definition of SWFs and SCCs to $\bar{\mathcal{L}}$.

3. BASIC DEMOCRATIC REQUIREMENTS AND EXTENSION OF THE DO-MAIN

Several democratic properties may be required when we want to describe a real life decision process by the mean of a social welfare function[3]: each voter should have the same weight in the decision process, all the candidates should be treated equally, the size of the population should not influence the results, etc. In social choice theory, such conditions are called the **anonymity**, the **neutrality**, the **homogeneity** and the **independence of symmetric profiles**. Moreover, these properties induce extensions of the domain of f from $\bar{\mathcal{L}}$ to $\mathbb{N}^{m!}$, $\mathbb{Z}^{m!}$, $\mathbb{Q}^{m!}$ or to the unit simplex $S(m!) = \{x \in \mathbb{Q}^{m!} \mid \sum_{t=1}^{m!} x_t = 1, \, x_t \geq 0 \, \forall t \in \{1, \ldots, m\}\}$. In several papers, the voting rules are directly defined on these sets, which means that one or several of the above conditions are implicitly assumed.

DEFINITION 1. *Let V and V' be two sets with $cardV = cardV'$ and let $\Gamma(V, V')$ be the set of all the one to one mappings $\gamma : V \to V'$. For any profile $\pi(V) = (R_i)_{i \in V}$, we define the profile $\gamma(\pi(V)) = (R_j)_{j \in V'}$ on V' such that $R_i = R_j$ if and only if $j = \gamma(i)$. A social welfare function f is anonymous if and only if for any $\gamma \in \Gamma(V, V')$ and any $\pi(V) \in \bar{\mathcal{R}}$, $f(\gamma(\pi(V))) = f(\pi(V))$.*

This definition is not the traditional one, presented by May [1952] or Arrow [1963]. Like many authors in social choice theory, they assume that the set of voters cannot vary, and they directly assume that the set of voters is the fixed set $V = \{1, \ldots, v\}$. The anonymity condition described by Definition 1 has two

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[3]All these axioms have a natural counterpart when the aim of the society is to pick out a unique winner or select a subset of socially best alternatives with the help of a SCC rather than ranking all the alternatives.

consequences: first, it establishes a relationship on the social ordering of two different populations (this fact is generally omitted in social choice models) and secondly, it implies that $f(\pi(V))$ only depends upon the numbers of voters having a definite preference. There are $m!$ preference types in $\mathcal{L}$, indexed from 1 to $m!$ by p according to the lexicographic order.

EXAMPLE 1. *Let $A = \{a, b, c\}$. The set $\mathcal{L}$ contains six linear orders, labelled from 1 to 6 in the lexicographic order (each column describes a linear order, the top alternative being the most preferred, etc.):*

$$
\begin{array}{cccccc}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \\
a & a & b & b & c & c \\
b & c & a & c & a & b \\
c & b & c & a & b & a
\end{array}
$$

DEFINITION 2. *For any $\pi(V) \in \bar{\mathcal{L}}$, the vector $\tilde{n}(\pi(V)) = (n(\pi(V))_1, \ldots, n(\pi(V)_{m!}) \in \mathbb{N}^{m!}$ (also denoted by $\tilde{n}(\pi)$ or simply $\tilde{n}$ for short), where $n(\pi(V))_p$ is the number of individuals having the linear ordering L_p as a preference in the profile $\pi(V)$ is called a voting situation.*

THEOREM 1. *Let $\pi(V)$ and $\pi'(V')$, two profiles on two populations of the same size (possibly with common voters) such that $\tilde{n}(\pi(V)) = \tilde{n}(\pi'(V')) = \tilde{n}$. If f is anonymous, $f(\pi(V)) = f(\pi'(V')) = f(\tilde{n})$.*

Proof. Take V, V', π, π' such that $\tilde{n}(\pi(V)) = \tilde{n}(\pi'(V')) = \tilde{n}$. First, consider a new population $V'' = \{1, \ldots, v\}$, with $v = \text{card}V = \text{card}V'$. Let $\gamma \in \Gamma(V, V'')$ such that $\gamma(\pi(V)) = \pi(V'')$ and $\gamma' \in \Gamma(V', V'')$ such that $\gamma'(\pi'(V')) = \pi'(V'')$. By anonymity, $f(\pi(V)) = f(\gamma(\pi(V))) = f(\pi(V''))$ and $f(\pi'(V')) = f(\gamma(\pi'(V'))) = f(\pi'(V''))$. This shows that we can directly label the voters from 1 to v in the natural order without loss of generality. Thus, $\pi(V'') = (L_1, \ldots, L_v)$ and $\pi'(V'') = (L'_1, \ldots, L'_v)$. Secondly, let $\gamma'' \in \Gamma(V'', V'')$ such that $\gamma(i) = j$ if and only if $L_i = L'_j$. Such a mapping exists, as $\tilde{n}(\pi(V'')) = \tilde{n}(\pi(V)) = \tilde{n}(\pi'(V')) = \tilde{n}(\pi'(V''))$. By anonymity, $f(\pi(V'')) = f(\gamma''(\pi(V''))) = f(\pi'(V''))$ and $f(\pi(V)) = f(\pi'(V')) = f(\tilde{n})$. QED.

Thus, anonymity implies that each individual has the same power and that the names of the voters have no importance. We can directly define f on the domain of voting situations $\tilde{n}$, that is $\mathbb{N}^{m!}$. In the three candidate case, for any $\pi(V) \in \bar{\mathcal{L}}$, we have $f(\pi(V)) = f(n_1, n_2, n_3, n_4, n_5, n_6)$ whenever f is anonymous. The second democratic requirement concerns the alternatives.

DEFINITION 3. *Let $\Sigma(A)$ be the set of permutations on $A = \{a_1, \ldots, a_m\}$. For any binary relation R on A, we define $\sigma(R)$ by $a_{\sigma(i)}\sigma(R)a_{\sigma(j)}$ iff $a_i R a_j$. For any profile of linear orders $\pi(V) \in \bar{\mathcal{L}}$, we define $\sigma(\pi(V)) = (\sigma(L_{i_1}), \ldots, \sigma(L_{i_v}))$. Then, a social welfare function f is neutral iff, for any $\sigma \in \Sigma(A)$ and any profile $\pi(V) \in \bar{\mathcal{L}}$, $f(\sigma(\pi(V))) = \sigma(f(\pi(V)))$.*

Neutrality means that no alternative (like the statu quo) plays a particular role in the decision process. A neutral and anonymous SWF is called **symmetric**. If

f is a symmetric SWF with domain $\mathbb{N}^{m!}$, then the permutation of coordinates of $\tilde{n}$ induced by σ, a permutation on A, can be conveniently represented by a permutation matrix M_σ, and we have:

$$\forall \tilde{n} \in \mathbb{N}^{m!}, \ f(M_\sigma(\tilde{n})) = \sigma(f(\tilde{n})).$$

Notice that if f is symmetric and $\tilde{n}$ is a fixed point of M_σ, then $f(\sigma(\tilde{n})) = \sigma f(\tilde{n})$ must contain some indifference relations. Moreover, for the voting situation $e = (1, \ldots, 1) \in \mathbb{N}^{m!}$, $f(e) = I(A)$, the complete indifference relation on A.

DEFINITION 4. *Let f be a symmetric SWF, defined on $\mathbb{N}^{m!}$. Then f is independent of symmetric profiles if and only if $\forall \tilde{n} \in \mathbb{N}^{m!}$, $\forall k \in \mathbb{N}$, $f(\tilde{n} + ke) = f(\tilde{n})$.*

THEOREM 2. *Let f be a symmetric and independent of symmetric profiles SWF, defined on $\mathbb{N}^{m!}$. Then, there exists a unique extension of f from $\mathbb{N}^{m!}$ to $\mathbb{Z}^{m!}$, defined by $\forall k \in \mathbb{N}$, $\forall \tilde{n} \in \mathbb{N}^{m!}$, $f(\tilde{n} - ke) = f(\tilde{n})$*

Proof. Assume that for $(\tilde{n}, \tilde{m}) \in (\mathbb{N}^{m!})^2, (k, k') \in \mathbb{N}^2$, $\tilde{n} - ke = \tilde{m} - k'e$. Without loss of generality, $k \geq k'$. Then, $f(\tilde{n} - ke) = f(\tilde{n}) = f(\tilde{m} + (k - k')e) = f(\tilde{m}) = f(\tilde{m} - k'e)$. This extends f to $\mathbb{Z}^{m!}$. QED.

The next condition is homogeneity: if each voter is replicated k times with the same preference to create a new population of size kv, the social outcome should not be changed.

DEFINITION 5. *Let $V^1, V^2, \ldots, V^k$, be k disjoint populations of the same size v, and a profile π such that $\forall t \in \{1, \ldots, k \}$, $\pi(V^t) = \pi(V^1)$. Let $\pi(kV^1) = \pi(V^1 \cup V^2 \cup \ldots \cup V^k)$. Then a social welfare function f is homogeneous if and only if $f(\pi(kV^1)) = f(\pi(V^1))$.*

THEOREM 3. *Let f be a homogeneous, symmetric and independent of symmetric profiles social welfare function defined on $\mathbb{Z}^{m!}$. Then, there exists a unique extension of f from $\mathbb{Z}^{m!}$ to $\mathbb{Q}^{m!}$, defined by: $\forall k \in \mathbb{N}$, $\forall \tilde{n} \in \mathbb{Z}^{m!}$ $f(\frac{\tilde{n}}{k}) = f(\tilde{n})$.*

Proof. For $\tilde{n}, \tilde{m} \in (\mathbb{Z}^{m!})^2$, $k, k' \in \mathbb{N}^2$, let $\frac{\tilde{n}}{k} = \frac{\tilde{m}}{k'}$. Then $f(\frac{\tilde{n}}{k}) = f(\tilde{n}) = f(k'\tilde{n}) = f(k\tilde{m}) = f(\tilde{m}) = f(\frac{\tilde{m}}{k'})$. The domain of f is now $\mathbb{Q}^{m!}$. QED

Another possibility is to consider symmetric and homogeneous SWF. Then their natural domain can be viewed as the set of rational points in the unit simplex,

$$S(m!) = \left\{ x = (x_1, \ldots, x_{m!}) \in \mathbb{Q}^{m!} \mid \sum_{t=1}^{m!} x_t = 1, \ and \ for \ 1 \leq t \leq m, \ x_t \geq 0 \right\}$$

where x_t is the fraction of voters having L_t as a preference. In particular, this domain is extensively used by Saari [1988, 1990, 1994].

4. MAJORITY VOTING

Majority voting plays a central role in voting theory: when two alternatives are in contention, it is the most natural way to order the candidates. Its characterization is due to May [1952]. His result is a natural complement of Arrow's theorem, as it is a possibility theorem for the case $m = 2$. Unfortunately, for more than two candidates, the majority relation may lead to cycles; Arrow's theorem [Arrow, 1963] can be also considered as generalizations of the cyclicity result. Nevertheless, in some contexts, natural restrictions on the preferences make the majority relation transitive and the main objection to its use disappears[4]. This is why it is so often used in public economy as the most natural way to model the voting process.

Without loss of generality, consider that $V = \{1, \ldots, v\}$ is a finite set of voters. Let $\pi = (R_1, \ldots, R_v) \in \bar{\mathcal{R}}$ be a profile of complete preorders. For all $(x, y) \in A^2$, we write:

$$N_\pi(x, y) = \{i \in V \mid xP_iy\}.$$

DEFINITION 6. *Let $A = \{x, y\}$. A SWF is the simple majority rule if and only if,*

$$\forall \pi(V) \in \bar{\mathcal{R}}, \ xRy \Leftrightarrow CardN_\pi(x, y) \geq CardN_\pi(y, x)$$

where $R = f(\pi)$.

The majority voting is obviously symmetric. It also satisfies the **monotonicity** property. We here define it for any number of candidates.

DEFINITION 7. *Let $\pi(V) \in \bar{\mathcal{R}}$ and f a SWF be such that xP_Vy or xI_Vy where $R_V = f(\pi(V))$. Let $\pi'(V) \in \bar{\mathcal{R}}$ be a second profile such that there exists $j \in V$ with, , $(xI_jy$ and $xP'_jy)$ or $(yP_jx$ and $xI'_jy)$ and with $\forall i \neq j$, $R'_i = R_i$. Then f is monotone if and only if xP'_Vy, with $R'_V = f(\pi'(V))$.*

THEOREM 4 (May). *Let $A = \{x, y\}$. A SWF is anonymous, neutral and monotone if and only if it is the simple majority rule.*

Proof. Using the same argument as in Theorem 1, we can state that f is defined on $\mathbb{N}^3$. Thus, $f(\pi) = f(n_1, n_2, n_3) = f(\tilde{n})$, where $n_1 = CardN_\pi(x, y)$, $n_3 = CardN_\pi(y, x)$ and $n_2 = v - n_1 - n_3$. By symmetry, we will prove that

$$n_1 = n_3 \ \Rightarrow \ xI_{\tilde{n}}y, \tag{1}$$

with the convention $R_{\tilde{n}} = f(\tilde{n})$. Assume the contrary, $xP_{\tilde{n}}y$. Let σ be such that $\sigma(x) = y$, and $\sigma(y) = x$, and γ be a permutation on V such that $\gamma(i) = j$ if and only if xP_iy and yP_jx (such a permutation exists as $n_1 = n_3$). Then, by neutrality, $yP_{\sigma(\tilde{n})}x$. By anonymity, $yP_{\gamma(\sigma(\tilde{n}))}x$, but $\gamma(\sigma(\tilde{n})) = \tilde{n}$, which implies $yP_{\tilde{n}}x$, a contradiction.

Suppose now that π is such that $n_1 = n_3 + 1$. By (1) and monotonicity, $xP_{\tilde{n}}y$. By induction,

$$n_1 > n_3 \ \Rightarrow \ xPy. \tag{2}$$

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[4]This fact has been noticed first by Duncan Black [1998] in 1948.

Similarly,

$$n_1 < n_3 \Rightarrow yPx. \tag{3}$$

Thus, equations (1), (2) and (3) define the majority rule. Independence of the three conditions is left to the reader as a simple exercise. QED.

From the axiomatic point of view, there is almost nothing more to say about the simple majority rule for the case of two alternatives. Fishburn [1973] discussed extensively possible modifications of the three axioms, and characterized the family of qualified majorities, where n_1 must attain some threshold greater than n_3 to declare x socially better than y.

5.   CHARACTERIZATION OF SCORING RULES

5.1.   SCORING RULES AND THEIR PROPERTIES

Scoring rules are popular decision processes. Given m alternatives and a profile, assign a s_k points (s_k is a real number) to each voter's k^{th} most preferred alternative. Then x will be ranked before y in the social ordering if its total score over the whole population is higher. Any SWF defined in this way is a simple scoring function, denoted by f^s, where s is the m-dimensional vector $(s_1, s_2, \ldots, s_m) \in \mathbb{R}^m$. The vector s is called a scoring vector.

Formally, we may define f^s on the domain $\mathbb{N}^{m!}$ as follows. Given $L_p \in \mathcal{L}$, let E_p be the $m \times m$ permutation matrix with "1" in the $(i,j)^{th}$ position if and only if a_j is the ith most preferred alternative in the preference order L_p. For every $\tilde{n}$ in $\mathbb{N}^{m!}$, define $T(\tilde{n}) = \sum_{L_p \in \mathcal{L}} n_p E_p$. The column vector $T_j(\tilde{n})$ gives the positions of a_j on the m ranks for a profile $\tilde{n}$.

Let $\pi(V)$ be a profile, and $\tilde{n}$ the associated voting situation. Let s be a scoring vector. Then, the simple scoring function f^s is a SWF, where $R_{\tilde{n}}^s = f^s(\tilde{n})$, defined by:

$$a_i R_{\tilde{n}}^s a_j \Leftrightarrow s.T_i(\tilde{n}) \geq s.T_j(\tilde{n}).$$

Thus, the product $s.T_i(\tilde{n})$ is the total score of the alternative a_i with the scoring vector s and the voting situation $\tilde{n}$.

Among the class of simple scoring functions, we may distinguish the Borda count ($s = (m-1, m-2, \ldots, 1, 0)$), the plurality rule ($s = (1, 0, \ldots, 0)$), the antiplurality rule ($s = (1, \ldots, 1, 0)$) and the trivial social welfare function ($s = (0, \ldots, 0)$). One may easily check the equality $f^s = f^t$ whenever s is a positive linear transformation of t. At this point, we do not require the coordinates in s to be decreasing.

It is possible to refine further the concept of simple scoring functions, when ties relative to the simple scoring rule f^{s^1} are resolved by the mean of another simple scoring function f^{s^2}. The composition of f^{s^2} with f^{s^1}, denoted by $f^{s^2} \circ f^{s^1}$, is defined, for all $a, b \in A$ and all $\pi(V) \in \bar{\mathcal{L}}$, by:

$$aI^{s^2 \circ s^1}b \quad \Leftrightarrow \quad (aI^{s^1}b) \; and \; (aI^{s^2}b) \tag{4}$$

$$aP^{s^2 \circ s^1}b \quad \Leftrightarrow \quad (aP^{s^1}b) \; or \; (aI^{s^1}b \; and \; aP^{s^2}b). \tag{5}$$

DEFINITION 8. *Let $s^1, s^2, \ldots s^\alpha$ be a sequence of α m-dimensional scoring vectors. Then, the composition $f^{s^\alpha} \circ f^{s^{\alpha-1}} \circ \ldots \circ f^{s^2} \circ f^{s^1}$, called a composite scoring function, is defined by applying recursively formulas (4) and (5).*

By definition, scoring functions (simple or composite) are symmetric. Another important property of scoring rules is Young's **consistency** [1974(a)(b), 1975], also called separability by Smith [1973], elimination by Fine and Fine [1974(a)(b)] or reinforcement by Moulin [1988] and Myerson [1995].

DEFINITION 9. *Let f be a social welfare function defined on $\bar{\mathcal{L}}$. The SWF f is consistent if for any two profiles $\pi(V), \pi(V')$ defined on disjoint populations:*

$$aP_V b \text{ and } aR_{V'}b \quad imply \quad aP_{V \cup V'}b \tag{6}$$

$$aI_V b \text{ and } aI_{V'}b \quad imply \quad aI_{V \cup V'}b \tag{7}$$

where $f(\pi(V)) = R_V$, $f(\pi(V')) = R_{V'}$ and $f(\pi(V \cup V')) = R_{V \cup V'}$.

In other words, consistency means that if candidate a is considered as better than candidate b with the social welfare function f in two or more different sub-populations of voters, this conclusion should remain true when we directly use f on the whole population. Notice that we do not care about the ranking of the other candidates to reach this conclusion.

The fact that all the scoring functions (simple or composite) satisfy the consistency condition follows from the additivity of the scores. This later property also implies homogeneity and independence of symmetric profiles. Thus, there is a unique way to extend f^s and compositions of scoring rules from $\bar{\mathcal{L}}$ to $\mathbb{Q}^{m!}$, as shown by Theorems 1, 2, and 3. This enables us to introduce a last condition, directly defined on $\mathbb{Q}^{m!}$.

DEFINITION 10. *Let f be a SWF defined on $\mathbb{Q}^{m!}$. It is continuous (or Archimedean in Smith's terminology [Smith, 1973]) if, for every $(a, b) \in A^2$ and a sequence of profiles $\{x_n\}$ from $\mathbb{Q}^{m!}$ such that $\forall n \geq \bar{n} \; aR_{x_n}b$, then $lim_{n \to \infty} x_n = x$ imply $aR_x b$.*

Continuity distinguishes between simple and composite scoring functions. For example, consider $A = \{a, b, c\}$. Take a sequence of profiles x_n in $\mathbb{Q}^6$ where the coordinate corresponding to the linear order $aLbLc$ is 1, the coordinate corresponding to the linear order $bLcLa$ is $1 - \frac{1}{n}$ and all the other components 0. With the adequate labelling (see Example 1), $x_n = (1, 0, 0, 1 - \frac{1}{n}, 0, 0)$ and $Lim_{n \to \infty} x_n = (1, 0, 0, 1, 0, 0)$. Take now the composite scoring function $f^{s_2} \circ f^{s_1}$ with the scoring vectors $s_1 = (1, 0, 0)$ and $s_2 = (0, 1, 0)$: we first rank the alternatives on the basis of he number of first place and use the extra information given by the number of second place to break the possible ties. For all x_n, $aR_{x_n}b$, but $bP_x a$. The composition of simple scoring rules does not satisfy the continuity, while simple scoring rules do so.

THEOREM 5. *Young [1974(b)]. A social welfare function defined on $\bar{\mathcal{L}}$ is symmetric and consistent if and only if it is a (simple or composite) scoring function. It is also continuous if and only if it is a simple scoring function.*

In his proof, Young uses a key fact: the set of profiles that lead to a social outcome is a convex set in $\mathbb{Q}^{m!}$ for the scoring rules. We present in detail this feature of the scoring rules in the next paragraphes.

5.2. CONVEXITY

To characterize all symmetric and consistent social welfare functions, we will extensively use the notion of $\mathbb{Q}$-convex sets. In general, we say that a set $S \subseteq \mathbb{R}^n$ is $\mathbb{Q}$-convex if $S \subseteq \mathbb{Q}^n$, and for all $(x,y) \in S^2$, and all rational $\lambda \in \mathbb{Q}$, $0 \le \lambda \le 1$, we have $\lambda x + (1 - \lambda)y \in S$.

For a given pair of alternatives, $(a,b) \in A^2$, and a simple scoring function f^s, define $D_a^s = \{x \in \mathbb{Q}^{m!} \mid aR_xb\}$. It is the set of profiles where a does better than b on the ground of the scoring vector s. Then, it is easy to see that D_a^s is a $\mathbb{Q}$-convex set. For any two profiles x and y in D_a^s, $s.T_a(x) \ge s.T_b(x)$ and $s.T_a(y) \ge s.T_b(y)$. Thus, for any rational $\lambda \in [0,1]$, $s.(\lambda T_a(x)+(1-\lambda)T_a(y)) \ge s.(\lambda T_b(x)+(1-\lambda)T_b(y))$. Thus, $aR_{\lambda x+(1-\lambda)y}b$ and $\lambda x + (1-\lambda)y \in D_a^s$. Similarly, if we define $B_a^s = \{x \in \mathbb{Q}^{m!} \mid aP_xb\}$, we can prove that B_a^s is $\mathbb{Q}$-convex.

Before going to the proof of the main theorem, it is useful to recall the following facts. For $S \subseteq \mathbb{R}^n$, let $cvxS$ be the convex hull of S, afS the affine hull of S, and $\overline{S}$ the closure of S. If $S \subseteq W \subseteq \mathbb{R}^n$, where W is an affine set, let int_WS denote the interior of S relative to W, and $riS = int_{afS}S$ the relative interior of S. The dimension of S, $dimS$, is the dimension of afS. We shall also use the following well known facts: if S is convex, then $\overline{S}$ and $ri\overline{S}$ are convex, $riS = ri\overline{S}$, and $\overline{riS} = \overline{S}$.

In order to characterize the scoring functions, Young [1975] also needs to establish two technical lemmas. The proofs are given in appendix.

LEMMA 1. *Young [1975]. S is $\mathbb{Q} - convex$ if and only if $S = \mathbb{Q}^n \cap cvxS$.*

LEMMA 2. *Young [1975]. If S is $\mathbb{Q} - convex$, then $\overline{S} = \overline{cvxS}$ and $\overline{S}$ is convex.*

5.3. PROOF OF THEOREM 5

Proof. The part "if" is left to the reader as an exercise. Conversely, let f defined on $\bar{\mathcal{L}}$ be symmetric and consistent. Without loss of generality, we can uniquely extend its domain of definition to $\mathbb{Q}^{m!}$, as shown by Theorems 1, 2 and 3. For $m = 1$, the theorem is trivial, so let $m \ge 2$, and fix $(a,b) \in A^2$. For every profile $x \in \mathbb{Q}^{m!}$ define the linear mapping $\alpha : \mathbb{Q}^{m!} \to \mathbb{Q}^m$, $\alpha(x) = T_a(x) - T_b(x)$, such that $\alpha_i(x)$ is the number of times a occurs in rank i minus the number of times b occurs in rank i in x. Evidently, $\sum_{i=1}^m \alpha_i(x) = 0$ for all x, and the dimension of the image by α is $m - 1$. Hence, $\ker(\alpha) = \{x \in \mathbb{Q}^{m!} \mid \alpha(x) = 0\}$ has dimension $m! - (m-1)$.

Let us now find a base for $\ker(\alpha)$. We will use two kinds of vectors in $\mathbb{Q}^{m!}$. For every $L_p \in \mathcal{L}$, let e_p denote the profile with "1" in coordinate p, and "0" elsewhere.

First group. The set $\mathcal{L}'$ of preferences in which neither a nor b has rank m has cardinality $(m-1)!$. With each L_p in $\mathcal{L}'$, associate $\eta_p = e_p + e_{\sigma(p)} + e_{\sigma^2(p)}$, where σ permutes a,b and the mth ranked alternative in a 3-cycle. The η_p's are linearly independent, as they are not based upon the same vectors e_p's. Moreover, $\alpha(\eta_p) = 0$.

Second group. Consider the remaining preferences in $\mathcal{L} \setminus \mathcal{L}'$ as the vertices of a graph in which L_p and L_q are adjacent if L_p is obtained from L_q by interchanging a and b, or by interchanging a with b and c with d, c,d different from a,b. This graph has $(m-1)$ connected components T (one for each $i < m$), with $2(m-2)!$ edges and vertices in each component. With each edge $\{L_p, L_q\}$ in T associate the profile

$\epsilon_{pq} = e_p + e_q$. There are $(m-1)(2(m-2)! - 1)$ such vectors which are independent as a linear order L_p only belongs to one connected graph, and is only connected with $m - 2\,2 + 1$ other linear orders. Moreover, $\alpha(\epsilon_{pq}) = 0$.

EXAMPLE 2. *Let $A = \{a, b, c\}$. The preference types are the ones presented in Example 1. $\mathcal{L}' = \{L_1, L_3\}$, $\eta_1 = (1, 0, 0, 1, 1, 0)$ and $\eta_2 = (0, 1, 1, 0, 0, 1)$. Next, we obtain that $\mathcal{L} \setminus \mathcal{L}' = \{L_2, L_4, L_5, L_6\}$. The linear orders L_5 and L_6 are adjacent, so we get $\epsilon_{56} = (0, 0, 0, 0, 1, 1)$. Similarly, L_2 and L_4 are adjacent, and we obtain $\epsilon_{24} = (0, 1, 0, 1, 0, 0)$. These four independent vectors form a base for $\ker(\alpha)$ in the case $m = 3$.*

Thus, we have $(m-1)(m-2)(m-2)! + 2(m-1)! - (m-1) = m! - (m-1)$ linearly independent vectors η's and ϵ's which form a set W. The vectors in W span $\ker(\alpha)$. Each $x \in W$ is fixed by some permutation $\sigma \in \Sigma(A)$ taking a to b, so symmetry and the transitivity of R_x imply $aI_x b$. As any profile in $\ker(\alpha)$ is a linear combination of the vectors in W (with rational weights), by consistency (equation (3)), $aI_x b$ for all x in $\ker(\alpha)$.

If $\alpha(x) = \alpha(y)$, then the profile $x - y$ is in $\ker(\alpha)$ so $aI_{x-y} b$, and $aP_x b$ if and only if $aP_y b$ (by equation (4)). Thus, f, relative to a and b, depends only on $\alpha(x)$, and we may consider f to have domain $D = \{x \in \mathbb{Q}^m \mid \sum_{i=1}^m x_i = 0\}$.

The sets $D_1 = \{x \in D \mid aP_x b\}$ and $D_2 = \{x \in D \mid bP_x a\}$ are $\mathbb{Q}$-convex (by consistency), and disjoint (by the antisymmetry of f). Moreover, unless f is the trivial function, they are non empty. By lemma 2, $\overline{D_1}$ and $\overline{D_2}$ are non empty convex sets. If $\overline{D_1} \cup \overline{D_2}$ is not the whole set $\overline{D} = \{x \in R^m \mid \sum_{i=1}^m x_i = 0\}$, then the set $\overline{D} - (\overline{D_1} \cup \overline{D_2})$ is open in $\overline{D}$ and contains a rational point x for which $aI_x b$. For any $y \in \overline{D_1}$, and sufficiently small rational $\lambda > 0$, $(1 - \lambda)x + \lambda y$ is rational and in $\overline{D} - (\overline{D_1} \cup \overline{D_2})$, $aP_{((1-\lambda)x + \lambda y)} b$, a contradiction. Thus, $\overline{D} = \overline{D_1} \cup \overline{D_2}$. By symmetry, $\overline{D_1} = -\overline{D_2}$, so both sets must have non empty interior relative to $\overline{D}$. If the interior of $\overline{D_1}$ meets the interior of $\overline{D_2}$, then their intersection contains a rational point x, and, since $int(\overline{D_i}) = int(\overline{cvxD_i}) = int(cvxD_i)$, we have by $\mathbb{Q}$-convexity and lemma 1 that $x \in (cvxD_1 \cap cvxD_2) \cap D = D_1 \cap D_2$, contradicting the disjointness of D_1 and D_2.

Thus $\overline{D_1}$ and $\overline{D_2}$ are nonempty convex sets with disjoint interiors (relative to $\overline{D}$), so the separation theorem for convex sets implies that there exists a non zero vector $s^1 \in \overline{D}$ such that $s^1.x \geq 0$ for all $x \in \overline{D_1}$, and $s^1.x \leq 0$ for all $x \in \overline{D_2}$.

If $x \in D$ and $s^1.x > 0$, then x is in $\overline{D_1} - \overline{D_2}$; hence, x is in the interior of $\overline{D_1}$, and since it is rational, it must be in D_1. Thus, $s^1.x > 0$ implies $aP_x b$. Similarly, $s^1.x < 0$ implies $bP_x a$.

If the set $D' = \{x \in D \mid s^1.x = 0\}$ contains points x such that $aP_x b$, we can define $D'_1 = \{x \in D' \mid aP_x b\}$ and $D'_2 = \{x \in D' \mid bP_x a\}$. By applying the preceding argument to D', we obtain a non zero vector s^2 such that $s^2.x > 0$ implies $aP_x b$ for all $x \in D'_1$, and so forth. This construction terminates in at most $\dim D = m - 1$ steps with a sequence of vectors $s^1, s^2, \ldots, s^k$ belonging to D that represents f as a scoring function for the pair a, b. By symmetry, the same numbers must apply to all pairs from A, so f is a scoring function.

If f is continuous, then aI_xb for all x in D', meaning that f is determined by a single scoring vector s^1. QED.

To conclude this section, notice that the scoring vectors s need not be decreasing. One needs a further condition, like a kind of monotonicity, Pareto optimality (see below), or other weaker conditions, to get $s_j \geq s_{j+1}$.

6. FURTHER AXIOMATIC RESULTS

The axiomatic approach led to many positive results in the analysis of voting rules. After the pioneering research of Young [1974(b), 1975] and Smith [1973], several authors characterized specific scoring functions. We can quote, among others, Young [1974(a)], Fishburn and Gehrlein [1976] Nitzan and Rubinstein [1981] and Saari [1994] for the Borda count, Richelson [1978], Lepelley [1992], Ching [1996] and Merlin and Naeve [1999] for the plurality rule, Barberà and Dutta [1982] for the antiplurality rule. All these results are given for social choice correspondences. In this framework, for two disjoint populations V and V' and a SCC g defined on $\bar{\mathcal{L}}$, consistency reads :

$$g(\pi(V)) \cap g(\pi(V')) \neq \emptyset \Rightarrow g(\pi(V \cup V')) = g(\pi(V)) \cap g(\pi(V')) \tag{8}$$

and Young's result applies (see [Young, 1975] for details): a symmetric and consistent SCC is described by a scoring rule (simple if we add a continuity axiom). Most of the additional characterizations proceed in the same way: keeping (up to some minor changes) Young's framework and axioms, and adding one or two extra properties. Most of the proofs also follow the same pattern[5]: it is easy to prove that the considered scoring rule satisfies the extra axioms, and the key point of the proof is to check it is the only one in this family by providing a well chosen counterexample. We omit the proofs of the results that we will now present, but the reader can try to demonstrate them in this way.

These extra axioms can be broadly classified in two groups. The first group gathers conditions which all share a majoritarian flavor. In the second group, we encounter weakened version of Maskin Monotonicity [Maskin, 1985].

6.1. MAJORITY-LIKE CONDITIONS

The Borda count has been proposed for the first time in 1786 by Jean Charles de Borda [1781], a member of the French Académie Royale des Sciences: he suggested that each voter should award $m-1$ points to her first best candidate, $m-2$ points to her second, and so on down to 1 point for the next to the last and zero point for the last one. A few years later, Condorcet [1785] pointed out that all the scoring rules, including the Borda count, suffer from the same flaw: for some profiles, they fail to pick out as a winner the candidate who is able to beat all the other candidates in majority pairwise comparisons. Such a candidate now bears Condorcet's name in

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[5]The three exceptions are the characterizations of the Borda Count by Young [1974(a)] and Nitzan and Rubinstein [1981] and the characterization of the plurality rule by Ching [1996].

the literature. Thus, it seems a priori useless to distinguish among the scoring rules on their ability to select a Condorcet winner. But in fact, many positive results have been obtained by using weakened version of Condorcet's majority criterion.

The first result is due to Young [1974(a)], who had in fact characterized the Borda count as a SCC one year before his general result. He uses two extra axioms, but was able to get rid off anonymity.

DEFINITION 11. *Consider a single voter population :* $\pi(V) = (L_i) \in \mathcal{L}$. *Let* a_i *be the top element in the preference* L_i : $\forall b \in A \setminus \{a_i\}$, $a_i L_i b$. *Then, a SCC g is faithful if and only if :*

$$g(L_i) = a_i.$$

When the population is reduced to a single voter, she is able to impose her most preferred alternative.

DEFINITION 12. *Consider a profile* $\pi(V) \in \bar{\mathcal{L}}$ *such that for any pair of alternative* $CardN_\pi(x, y) = CardN_\pi(y, x)$*s. Thus, g satisfies the cancellation property if and only if*

$$g(\pi(V)) = A.$$

Thus, the cancellation property asserts that when all the majority comparisons end up in a tie, the SCC should select all the alternatives.

THEOREM 6. *Young [1974(a)]. For any fixed number of alternatives, there is one and only one social choice correspondence that is neutral, consistent, faithful, and has the cancellation property - namely the Borda count.*

Notice that Nitzan and Rubinstein [1981] proposed a much related result for SWFs: the Borda count is the only SWF which satisfies neutrality, consistency, and the adequate adapted versions of monotonicity and cancellation. The cancellation property emphasizes a particularity of the Borda count: the Borda scores can be computed from the values of the pairwise comparisons obtained through majority voting. This fact had been pointed out by Fishburn and Gehrlein [1976] and Smith [1973] who proved that formula (9) is true for any profiles and any alternative $x \in A$ if and only if $s = (m - 1, m - 2, \dots, 1, 0)$; it asserts that the total score of an alternative under a scoring vector s is equal to the sum of its margins of victory in majority comparisons.

$$s.T_x(\tilde{n}) \quad = \quad \sum_{y \in A, \, y \neq x} CardN_\pi(x, y) \tag{9}$$

Formula (9) can also be used to prove Theorem 7.

DEFINITION 13. *An alternative* $x \in A$ *is a Condorcet loser for a profile* $\pi(V)$ *if*

$$\forall y \in A \setminus \{x\}, \; N_\pi(x, y) < N_\pi(y, x)$$

A SCC satisfies the Condorcet Loser (CL) property if and only if it never selects a Condorcet loser whenever such a candidate exists.

THEOREM 7. *Smith [1973], Fishburn and Gehrlein [1976]. The Borda count is the only simple scoring rule which satisfies the Condorcet loser property.*

Symmetrically, one can easily show that a Condorcet winner is never ranked last with the Borda count. However, in the Borda ordering, a Condorcet loser can be ranked up to the second place, and a Condorcet winer down to the next to last position. A complete description of all the relationships between majority pairwise voting and the Borda count can be found in Saari [1988, 1990]. In fact, Saari did more: he completely characterized all the possible relationships among the rankings of scoring rules on all the different subsets $B \subseteq A$ of candidates. Again, the Borda count emerges as the unique scoring rule which minimizes the number of inconsistencies among the rankings of different subsets. Theorem 6 and 7 can now be seen as corollaries of his more general theory.

However, the Borda count is not the only scoring rule which can be characterized by the mean of some extra majority-like axiom. The plurality rule, which ranks the alternatives according to the number of first positions in the individual preferences, also meet some weak Condorcet type properties. Richelson [1978] was the first author who provided a characterization of the plurality rule. He slightly changed Young's framework, allowing the set of alternatives to change too. Thus a SCC is now a mapping g from the set $(2^A \setminus \emptyset) \times \mathcal{L}^\infty \times 2^\mathbb{N}$ to $2^A \setminus \emptyset$ with, for any $B \subseteq A$, $g(B, \pi(V)) \subset B$; a preference on B is defined as the restriction of L_i on B.

DEFINITION 14. *Let $\pi(V) \in \bar{\mathcal{L}}$ be a profile such that there exist x and y with $N_\pi(x, y) = V$. Thus, a SCC satisfies the reduction property if and only if:*

$$g(A, \pi(V)) = g(A \setminus \{y\}, \pi(V)).$$

In words, the social outcome should be unaffected when we remove a Pareto dominated alternative from the choice set.

THEOREM 8. *Richelson [1978], Ching [1996]. There is one and only one social choice correspondence that satisfies anonymity, neutrality, consistency and the reduction property - namely, the plurality rule.*

Richelson also included the continuity as a necessary condition, but Ching prove later that this condition could be omitted. Another characterization of the plurality rule is based upon the notion of a strong Condorcet winner.

DEFINITION 15. *An alternative $x \in A$ is a strong Condorcet winner for a profile $\pi(V) \in \bar{\mathcal{L}}$ if*

$$Card\{\ i \in V \mid x P_i y\ \forall y \in A \setminus \{x\}\ \} > v/2$$

A SCC satisfies the strong Condorcet winner (SCW) property if and only if it selects the strong Condorcet winner whenever such a candidate exists.

THEOREM 9. *Lepelley [1992]. There is one and only one social choice correspondence that satisfies anonymity, neutrality, consistency, continuity and the strong Condorcet winner property - namely, the plurality rule.*

6.2. A SYMMETRIC RESULT FOR PLURALITY AND ANTIPLURALITY RULES

Up to our knowledge, there is only one characterization of the antiplurality rule in the literature: it has been provided by Barberà and Dutta [1982] in the appendix of a more general paper on implementation theory. In this literature, a famous axiom is Maskin Monotonicity:

DEFINITION 16. *A SCC g satisfies* Maskin Monotonicity *if for all voter $i \in V$, for all profile $\pi(V) \in \bar{\mathcal{L}}^v$, and for all preference $L_i' \in \mathcal{L}$, the following implication holds.*

$$[a \in g(\pi(V)) \ and \ (\forall \, b \in A, \ aL_ib \Rightarrow aL_i'b)] \Rightarrow a \in g(\pi'(V)),$$

π' being a new profile where the preference L_i has been removed and replaced by the preference L_i'.

As noticed by Barberà and Dutta, this condition can be split in three parts which together are equivalent to Maskin Monotonicity. The first one is the monotonicity condition we presented in Definition 1 for the SWFs. For the case of SCCs, it reads:

DEFINITION 17. *A SCC g satisfies monotonicity if for all voter $i \in V$, for all profile $\pi(V) \in \mathcal{L}^v$, and for all preference $L_i' \in \mathcal{L}$ the following implication holds.*

$$\left. \begin{array}{c} a \in g(\pi(V)) \\ L_i \ and \ L_i' \ agree \ on \ A\backslash\{a\} \\ \forall \, b \in A \ (aL_ib \Rightarrow aL_i'b) \end{array} \right\} \Rightarrow a \in g(\pi'(V)),$$

π' being a new profile where the preference L_i has been removed and replaced by the preference L_i'.

For $L \in \mathcal{L}$ and $r \in \{1, \dots, m\}$, we denote the rth ranking worst alternative in L by $b_r(L)$ and the kth ranking best alternative in L by $t_k(L)$. Also we define the ℓ-bottom $B(\ell, L) = \{b_r(L) \in A \mid r \leq \ell\}$, and the ℓ-top $T(\ell, L) = \{t_k(L) \in A \mid k \leq \ell\}$.

DEFINITION 18. *A SCC g satisfies top-invariance if for all voter $i \in V$, for all profile $\pi(V) \in \bar{\mathcal{L}}$, and for all preference $L_i' \in \mathcal{L}$ the following implication holds.*

$$\left. \begin{array}{c} b_r(L_i) \in g(\pi) \\ B(r, L_i) = B(r, L_i') \\ L_i \ and \ L_i' \ agree \ on \ B(r, L_i) \end{array} \right\} \Rightarrow b_r(L_i) \in g(\pi'(V)),$$

π' being a new profile where the preference L_i has been removed and replaced by the preference L_i'.

The top invariance condition asserts that a voter cannot exclude a winner from the choice set by reshuffling her preferences above this winner. Similarly, the bottom invariance condition states that a change of preference below a winner cannot be used by a voter to remove it from the choice set: in some sense, the choice set is not affected by a change in the bottom preferences of the voters.

DEFINITION 19. *A SCC g satisfies bottom-invariance if for all voter $i \in I$, for all profile $\pi(V) \in \bar{\mathcal{L}}$, and for all preference $L_i' \in \mathcal{L}$ the following implication holds.*

$$\left.\begin{array}{c} t_k(L_i) \in g(\pi) \\ T(k, L_i) = T(k, L_i') \\ L_i \text{ and } L_i' \text{ agree on } T(k, L_i) \end{array}\right\} \Rightarrow t_k(L_i) \in g(\pi'(V)),$$

π' being a new profile where the preference L_i has been removed and replaced by preference L_i'.

THEOREM 10. *Barberà and Dutta [1982]. There is one and only one social choice correspondence that satisfies anonymity, neutrality, consistency, monotonicity and top invariance - namely, the simple antiplurality rule.*

Though Barberà and Dutta did not notice it, it is easy to get a symmetrical result for the plurality rule:

THEOREM 11. *Merlin and Naeve [1999]. There is one and only one social choice correspondence that satisfies anonymity, neutrality, consistency, monotonicity and bottom invariance - namely, the simple plurality rule.*

7. CONCLUSION

As one may guess, the theorems that we presented in this paper do not exhaust all the results on voting rules that have been undertaken in the Arrovian framework. Several other directions of research have emerged and have been developed since the seventies.

Though majority voting is known to posses nice axiomatic properties, the supporters of the Condorcet criterion have to come up with extra arguments to choose among alternatives when there is a cycle. This drawback led to the development of new solution concepts that respect the Condorcet criteria. For example, the Copeland method selects as a winner the candidate who obtains the greater number of victories in majority pairwise comparisons. Fishburn [1978] listed and classified in 1977 nine so-called Condorcet social choice rules.

More generally, since thirty years, tournament theory has been a very active field. A binary relation T on the set of alternative A is a tournament if, for any x and y in A, one and only one of the following is true:

$$x = y, \ xTy \text{ or } yTx.$$

Denote by $\mathcal{T}$ the set of all the tournaments defined on A. Then, a tournament solution is a mapping from $\mathcal{T}$ to $2^A \setminus \emptyset$. McGarvey [1953] prove that any tournament could be obtained by using the majority rule on a preference profile. Thus, the search for new Condorcet voting rules largely coincide with the study of tournaments solutions. The recent works by Laslier [1996, 1997] give an account of the research in this area, presenting many solutions and characterization results.

The research on the axiomatic of scoring rules has been much less active. However, three papers are worth to notice. They all consider modifications of the consistency axiom but these new properties still convey the notion of convexity. Fishburn [1978] and Sertel [1988] propose characterizations of the approval voting rule: each voters indicates whether she approves or disapproves each alternative. The candidate who obtains the highest degree of support is elected. In this model, the preferences of the voter are restricted: they can be interpreted as a weak ordering with two classes of equivalence (approve or disapprove). Nevertheless, the consistency axioms can be adapted to this framework, and again helps to characterize this peculiar scoring rule.

Saari [1991] pointed out that Young's consistency was too strong in some sense. If the objective is to design a SCC g such that the set of alternatives leading to a definite outcome is convex in the set of profiles $S(m!) = \{x \in \mathbb{Q}^{m!} : \sum_{t=1}^{m} x_1 = 1, \text{ and } \forall t \in \{1, \ldots, m\}, \ x_t \geq 0\}$, it suffices to use a weak consistency axiom:

$$\forall B \subseteq A, \ \forall (x, y) \in (S(m!))^2, \ \forall \lambda \in [0, 1], \ g(x) = g(y) = B \Rightarrow g(\lambda x + (1 - \lambda)y) = B.$$

Together with neutrality and anonymity, this weaker axiom enables Saari to characterize the class of scoring rules with thresholds. Such a rule is also based upon the total scores computed with a scoring vector, but the victory conditions are weakened. To be selected, a candidate does not need the maximal number of points. For example, the rule which selects all the alternatives that get more than the average number of points awarded by all the voters with the Borda count belongs to this class. A rule which removes an alternative from the choice set only if another candidate gets 10% more vote with the plurality rule also belongs to this class.

The notion of consistency has also been used by Young and Levenglick, [1978] for the analysis of the Kemeny rule [1959]. A preference aggregation function g is a mapping from $\bar{\mathcal{L}}$ into $2^{\mathcal{L}}$, the set of subsets of linear orderings. The objective is to associate to each profile a subset of socially good strict rankings of the candidate (ideally, only one). In this framework, consistency condition is also defined by equation (7) except that $g(x)$ is now a subset of linear orders rather than a subset of alternatives. Define the distance between two linear orders as the number of pairs for which they disagree. Thus, the Kemeny rule [1959] selects the linear ordering(s) whose sum of the symmetric difference distance to the individual preferences is minimum. Using the same technique as in the proof of Theorem 5, Young and Levenglick could characterize the Kemeny preference function with anonymity, neutrality, consistency, and an extra Condorcet type axiom.

I believe that these papers [Fishburn, 1978], [Sertel, 1988], [Saari, 1991], [Young, Levenglick, 1978], although they seem isolated in the whole literature on voting rules, clearly indicate that consistency-like conditions are a powerful tool. A detailed exploration of the consequences of convexity properties in social choice theory still awaits.

To some extent, we only reported in this paper the successes of the axiomatic approach, telling nothing about its limitations and drawbacks. First, some very simple rules have not been characterized yet. For example, consider the way to elect the French president: the top two candidates with the plurality rule are selected for

a final head to head comparison. This rule belongs to the larger class of scoring run-off rules, where the alternatives are eliminated progressively by the mean of scoring rules. Social choice theory gives us no more clue about these rules than some partial results by Smith [1973] and Richelson [1980]. Similarly, the axiomatic results for tournament solutions cannot be adapted for social choice correspondences: their domain of definition is the set of preference profiles, not the set of tournament (see for example the axiomatization of the Copeland method by Henriet [1985]). The same argument can be used against the work of Roberts [1991]: this author proposesy several interesting characterization results for the plurality rule, but assumes from the very beginning that each voter only reports her top alternative. It is not possible to define many other voting rules in this framework (like the Borda count, the antiplurality rule, the tournament solutions), and we cannot compare his results with the ones we presented in this paper.

So, if we consider that the implicit objective of this research programme is to characterize all the voting rules that are used in practice, in order to be able to make some recommendation on their respective merits and flaws, we must conclude that there is still lot of work to accomplish! However, in order to compare the voting rules, we do not need characterization results. We only need a list of "good" properties, and check which rule satisfies which axiom. Such exercises are proposed by Fishburn [1977], Richelson [1975, 1978(a)(b)(c), 1980] or Nurmi [1989] in order to identify the best voting rules. Nevertheless, the conclusions of these authors strongly depend upon their opinion on which properties are the most important ones; having different opinions on the axioms that a voting rule should satisfy lead inevitably to different conclusions.

This overall mosaic of results would not be complete without some criticisms on this axiomatic programme. The results presented in Section 6 can be all proved in the same way: show that rule X is the only scoring rule that satisfies the axiom Y with the help of a well chosen example. But we do not know whether the fact that rule Z does not meet axiom Y is a rare event or not. If condition Y is violated for only 1% of the profiles, can we really say that rule Z does not satisfies it ? This type of consideration justifies the growing literature on the computation of the likelihood of voting paradoxes (a paradox occurs each time a desired axiom is violated at some preference profile). We cannot give an account of all this literature here, but the reader can find an extensive survey in Gehrlein [1997].

Another direction of research which departs from the axiomatic approach has been initiated by Saari in a series of papers on scoring rules that have been published at the end of the eighties and beginning of the nineties (see for example [Saari, 1988, 1990, 1991] and his book [Saari, 1994]). His objective is not to characterize which rule satisfies which axioms, but to describe, for a given rule, all its possible behavior when the set of alternatives shrinks or when the preferences varies. His techniques are also quite innovative: voting rules (and especially scoring rules) are depicted as linear mappings from the set $S(m!)$ into an adequate image space for the scores. For example, assume that the objective is to describe the relationships between the scores obtained with a scoring rules on a set A of m alternatives and the results of the pairwise comparisons among these alternatives. Saari reduces this problem

to the analysis of a linear mapping from $S(m!)$ into a $\frac{(m-1)^2 m}{2}$-dimensional space where all these scores ly. The study of the properties of voting rules is then reduced to the analysis of a linear mapping. Since this pioneering works, this approach has been successfully used by Merlin and Saari for the analysis of the Copeland method [Saari and Merlin, 1996], [Merlin and Saari, 1997] and of the Kemeny rule [Saari and Merlin, 2000(a)(b)].

There is no more place to talk about other important issues such as, for example, voting in economic environments or the strategic aspects of voting. But we hope this review of the literature has convinced the reader that the study of voting rules has been an active field over the last thirty years. There is no doubt that the axiomatic approach, the probability models and the linear algebra analysis will continue to supply this field with new and important results and facts in the next decades.

APPENDIX

Proof of Lemma 1. If $S = \mathbb{Q}^n \cap cvxS$, then clearly S is $\mathbb{Q}$-convex. Conversely, if S is $\mathbb{Q}$-convex, then certainly $S \subseteq \mathbb{Q}^n \cap cvxS$. Assume for the moment that S is a $\mathbb{Q}$-convex cone containing the origin. For any $q \in (\mathbb{Q}^n \cap cvxS)$ such that $q \neq 0$,

$$q = \sum_{i=1}^{k} \lambda_i q^i, \ (q^1, q^2, \ldots, q^k) \in S^k, \ and \ for \ 1 \leq i \leq k, \lambda_i > 0, \ \lambda_i \in \mathbb{R}. \qquad (10)$$

Assume that k is the smallest among all the expressions (10). We shall show that for all $1 \leq i \leq k$, $\lambda_i \in \mathbb{Q}$. Letting $\lambda_0 = -1$ and $q^0 = q$, we can rewrite (10) as

$$\sum_{i=0}^{k} \lambda_i q^i = 0. \qquad (11)$$

If, say, $\lambda_1 \notin \mathbb{Q}$, then, considering $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$, let $\{\lambda_0, \ldots, \lambda_l\}$, $l \geq 1$, be a basis for $\{\lambda_0, \lambda_1, \ldots, \lambda_k\}$ (renumbering the λ's if necessary), where for $0 \leq i \leq k$, $\lambda_i = \sum_{j=0}^{l} b_{ij} \lambda_j$, $b_{ij} \in \mathbb{Q}$ and $b_{ij} = 0$, $0 \leq i \leq i \neq j \leq l$. Then (11) implies

$$\sum_{j=0}^{l} \left(\sum_{i=0}^{k} b_{ij} q^i \right) \lambda_j = 0.$$

So by independence,

$$\sum_{i=0}^{k} b_{i1} q^i = 0,$$

and since $b_{01} = 0$

$$\sum_{i=1}^{k} b_{i1} q^i = 0. \qquad (12)$$

Let λ be the greatest real such that for $1 \leq i \leq k$, $\lambda_i' = \lambda_i - \lambda b_{i1} \geq 0$. Then $q = \sum_{i=1}^{k} (\lambda_i - \lambda b_{i1}) q^i$ yields a shorter expression for q, a contradiction. Hence

$\forall 1 \leq i \leq k$, $\lambda_i \in \mathbb{Q}$, and so $q \in S$ by $\mathbb{Q}$-convexity. Thus, $S = \mathbb{Q}^n \cap cvxS$ is S if a $\mathbb{Q}$-convex cone containing the origin. If $S \in \mathbb{Q}^n$ is $\mathbb{Q}$-convex, consider the $\mathbb{Q}$-convex cone $K = \{\lambda(1, x) \mid \lambda \geq 0, \lambda \in \mathbb{Q}, x \in S\} \subseteq \mathbb{Q}^{n+1}$. Then,

$$cvxS = \{x \in \mathbb{R}^n \mid (1, x) \in cvxK\}.$$

Hence if $x \in \mathbb{Q}^n \cap cvxC$, then $(1, x) \in \mathbb{Q}^{n+1} \cap cvxK = K$, so $x \in S$. Thus, we get $S = \mathbb{Q}^n \cap cvxS$ in any case. QED

Proof of Lemma 2. If $x \in cvxS$, then $x = \sum_{i=1}^{k} \lambda_i q^i$ for some finite collection $(q^1, q^2, \ldots, q^k) \in S^k$, and $\lambda_i \in \mathbb{R}$, $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$. For each i, $1 \leq i \leq k-1$, let $\{\lambda_i^n\}$ be a sequence of rationals converging to λ_i, such that $0 \leq \lambda_i^n \leq \lambda_i$, and let $\lambda_k^n = 1 - \sum_{i=1}^{k-1} \lambda_i^n$. Then, $\lambda_k^n \in \mathbb{Q}$, $\lambda_k^n \geq 0$, and $\sum_{i=1}^{k} \lambda_i^n = 1$ for every n, so $x^n = \sum_{i=1}^{k} \lambda_i^n q^i \in S$ by $\mathbb{Q}$-convexity. Since x^n converges to x, $x \in \overline{S}$. Thus, $S \subseteq cvxS \subseteq \overline{S}$, which implies $\overline{S} \subseteq \overline{cvxS} \subseteq \overline{S}$. So $\overline{S} = \overline{cvxS}$, the latter of which is convex. QED.

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