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Unique Implementation in Auctions and in Public Goods Problems

Claude d'Aspremont *
Jacques Crémer **
Louis-André Gérard-Varet † ***

Summary

We present new conditions that guarantee the existence of mechanisms with a unique or essentially unique equilibrium in auction and public goods problems with quasi-linear utility functions. These conditions bear only on the information structures of the agents.

Résumé

Des conditions nouvelles sont présentées, assurant l'existence de mécanismes d'enchères ou de production de biens publics menant à un équilibre unique, ou essentiellement unique, lorsque les préférences des agents sont quasi-linéaires. Ces conditions portent exclusivement sur les croyances des agents.

*. CORE, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.
**. CNRS, GREMAQ and IDEI, Université des Sciences Sociales de Toulouse, France.
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Keywords: Auction mechanism, unique implementation, Bayesian incentive compatibility.

Mots clés : Mécanismes d'enchères, unicité de la mise en œuvre, incitation bayésienne.

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1. Introduction

The main problem raised by the multiplicity of noncooperative equilibria in strategic form games – that is the difficulty for the players to coordinate their actions properly – has its counterpart in mechanism design. Even if a mechanism has an equilibrium outcome with some desirable property, it may have other equilibrium outcomes and a lack of coordination may lead to undesirable ones. However, in mechanism design, by the very definition of the exercise, the selection arguments used for games may be supplemented by some adequate modification of the constructed mechanism. Starting from a given mechanism with mutiple equilibrium outputs, a new mechanism could be constructed having the "good" outcome as the unique equilibrium.

Since Maskin (1977, 1986) contributions on Nash implementation – introducing this line of research – much effort has been oriented towards the identification of conditions that characterize, for various classes of environments, unique (or full) implementation of desirable social outcomes via mechanisms, under both complete and incomplete information. For Bayesian implementation, most of the work has dealt with the extension of Maskin's monotonicity condition, namely Bayesian monotonicity [1]. Palfrey (1992) presents a good survey of the state of the literature on this topic.

Bayesian monotonicity restricts jointly the utilities and the probabilities. But, in the same paper, Palfrey shows (for direct mechanisms and allowing unlimited transferability of the utilities) that unique implementation may reduce to incentive compatibility under some conditions imposed on the belief structure only. One of these conditions, however, is specially restrictive by requiring that at least one agent to be uninformed (*i.e.* of a single possible type), thus allowing the "modified" mechanism to base the elimination of undesirable equilibria on this agent's behavior. The modification of the direct mechanism relies on an augmentation of

1. This was introduced by Postlewaite and Schmeidler (1986).

the message space to specific non-type messages, of the kind already introduced in Ma, Moore and Turnbull (1988) and used by Mookherjee and Reichelstein (1990).

In the present paper, while keeping transferable utility and using the same kind of augmented mechanism to “selectively eliminate” undesirable equilibria, we get unique implementation under a weaker set of belief restrictions that do not require the presence of an uninformed agent. We first consider the traditional auction problem, with independent valuations. With no additional assumption, a modification of the rules of the auction – implying a larger message space – guarantees to the seller the desired expected revenue. This is a property of essentially unique implementability. More precisely, taking any auction, and a “good” equilibrium of this auction for the seller, we construct an other auction in which every equilibrium provides him the same expected payoff.

Then, for direct auction mechanisms and without restricting to independent beliefs, we show that unique implementation obtains under a simple (and generic) condition on the variability of the beliefs. In fact it will appear that this result is not linked to the auction interpretation of the model. It applies to a wide class of environments. As we will finally show, this includes the provision of a public good by a central planner, imposing transfers and balancing its budget.

2. Auctions with Independent Valuations: Essentially Unique Implementation

2.1. Beliefs and Utilities

We consider a situation in which a seller denoted 0, sells an object to a number of potential buyers $N = \{1,\dots,i,\dots n\}$. The characteristic, or type, of buyer $i \in N$, takes values in a finite set A_i. We denote $A = \times_{i\in N} A_i$ the set of states of nature (and $A_{-i} = \times_{j\neq i} A_j$). An outcome is a vector $x = (x_1,\dots,x_i,\dots,x_n) \in \Re^n_+$, where $x_i \geq 0$ is the probability that buyer $i \in N$ gets the object, assuming $\sum_{i\in N} x_i \leq 1$ (there is a probability that the seller keeps the object). Let X be the set of possible vectors x. The valuation for buyer $i \in N$ is given by the real-valued function $u_i(x;\alpha)$, $x \in X$, $\alpha \in A$. Valuations are measured in money and the payoff for agent $i \in N$ making a payment $y_i \in \Re$ when the outcome is $x \in X$ and $\alpha \in A$ is the state of nature is $u_i(x;\alpha) + y_i$. Observe that the utility of each buyer, as defined, might be affected by the types of all others. A standard case in auction theory, however, where this influence disappears, is given by $u_i(x;\alpha) = x_i W_i(\alpha_i)$, where $W_i(\alpha_i)$ stands for the willingness to pay for the object of player i when of type α_i (this is the case used in the example below in 2.4). To make things simple we consider that the object has no value for the seller who only collects the payments from the buyers (although the reader will easily see that this does not change the results at all). The seller is an uninformed player, whereas buyers have private information about their own types.

Before the auction starts, the seller has a probability distribution p over the set A of states of nature. The buyer $i \in N$ knows his true type $\alpha_i \in A_i$ and we assume that his beliefs over A_{-i} given $\alpha_i \in A_i$ are consistent with p and given by $p(\alpha_{-i} \mid \alpha_i)$. We also assume (without loss of generality) that for every $i \in N$ and $\alpha_i \in A_i$, the marginal $p_i(\alpha_i) = \sum_{\alpha_{-i}} p(\alpha_{-i}, \alpha_i)$ is strictly positive. In this section we concentrate on the independent case where $p(\alpha) = \times_{i \in N} p_i(\alpha_i)$.

An auction problem is given by $(A, p, X, (u_i)_{i \in N})$, where (A, p) is also called the belief structure.

2.2. Auction Mechanisms

The auction is conducted as follows. Each potential buyer $i \in N$ reports a bid to the seller. This bid is a message $m_i \in M_i$ in a finite set M_i. For $m = (m_1, \ldots, m_n) \in M = \times_{i \in N} M_i$, bidder i pays an amount $t_i(m)$ and receives the object with probability $s_i(m)$.

An auction mechanism is a triple (M, s, t) where $s : M \to X$ is called the outcome function and $t : M \to \Re^N$ is the payment scheme. Note that we are making no assumption about the type of auction.

An auction mechanism determines a game with incomplete information. A *Bayesian equilibrium* is a vector of strategies $\tilde{m} = (\tilde{m}_1, \ldots, \tilde{m}_i, \ldots, \tilde{m}_n)$ where, for every $i \in N$, $\tilde{m}_i$ is a function from A_i into M_i and

$$\forall i \in N, \quad \forall \alpha_i \in A_i, \quad \forall m_i \in M_i : \tag{1}$$

$$\sum_{\alpha_{-i}} (u_i(s(\tilde{m}(\alpha)); \alpha) + t_i(\tilde{m}(\alpha)))\, p(\alpha_{-i} \mid \alpha_i) \geq \tag{2}$$
$$\sum_{\alpha_{-i}} (u_i(s(m_i, \tilde{m}_{-i}(\alpha_{-i})); \alpha) + t_i(m_i, \tilde{m}_{-i}(\alpha_{-i})))\, p(\alpha_{-i} \mid \alpha_i).$$

2.3. Essentially Unique Implementation

Given an auction mechanism, the associated game of incomplete information may have several equilibria leading to more or less advantageous expected payoff to the seller. For instance, following Myerson (1981) the optimal auction that leads to the maximum expected revenue to the seller can be computed. However, this maximum expected revenue is only obtained for one equilibrium, and nothing guarantees that this equilibrium will obtain. In this section we show how to build a new (associated) mechanism in which all equilibria give the same revenue to the seller than the "good" equilibrium.

Formally, consider a mechanism (M,s,t) and an equilibrium $\tilde{m}$ where the seller's expected payoff is given by the expected revenue $\sum_\alpha (\sum_i t_i(\tilde{m}(\alpha))\, p(\alpha)$. Another auction mechanism $(\mathcal{M},\sigma,\tau)$ is said to *implement essentially uniquely* the equilibrium $\tilde{m}$ of (M,s,t) if, for all equilibria $\tilde{\mu}$ of $(\mathcal{M},\sigma,\tau)$ the expected revenue of the seller is the same:

$$\sum_\alpha \left(\sum_i \tau_i(\tilde{\mu}(\alpha))\right) p(\alpha) = \sum_\alpha \left(\sum_i t_i(\tilde{m}(\alpha))\right) p(\alpha). \tag{3}$$

Proposition 2.1 – *Take any equilibrium $\tilde{m}$ associated to any auction mechanism (M,s,t). With independent beliefs, there exists an auction mechanism $(\mathcal{M},\sigma,\tau)$ that implements $\tilde{m}$ essentially uniquely.*

Proof. Let Θ_i be the set of all functions $\theta_i : M_{-i} \to \mathfrak{R}$ such that either $\theta_i(m_{-i}) = 0$ for all $m_{-i} \in M_{-i}$ or $\sum_{\alpha_{-i}} \theta_i(\tilde{m}_{-i}(\alpha_{-i}))p(\alpha_{-i}) < 0$. That is θ_i belongs to Θ_i if at the truthtelling equilibrium of the original auction, it would either be identically 0, or would yield strictly negative expected payoff to bidder i.

We consider the auction mechanism $(\mathcal{M},\sigma,\tau)$ where $\mathcal{M}_i = M_i \times \Theta_i$ and for every $i \in N$,

$$\sigma_i((m_1,\theta_1),\dots(m_n,\theta_n)) = s_i(m_1,\dots,m_n),$$

and

$$\tau_i((m_1,\theta_1),\dots,(m_n,\theta_n)) = t_i(m) + \theta_i(m_{-i}).$$

Since, in the extended auction mechanism, a nonzero transfer θ_i gives a negative expected payoff to bidder i, it is an equilibrium for every bidder $i \in N$ of type $\alpha_i \in A_i$ to play $(\tilde{m}_i(\alpha_i), 0) \in \mathcal{M}_i$. Then, the expected revenue of the seller is the same as at the original equilibrium of the original auction mechanism.

We now show that all equilibria of the new mechanism satisfy condition (3). In fact we shall show something stronger: All equilibria of $(\mathcal{M},\sigma,\tau)$ generate the same distribution on $\mathcal{M}$ as that induced by the good equilbrium of the original mechanism. Assume that this were not the case for some equilibrium denoted $\tilde{\mu}' = (\tilde{m}', \tilde{\theta}')$, with $\tilde{m}'_i : A_i \to M_i$ and $\tilde{\theta}'_i : A_i \to \Theta_i$. There would then exist i, m_i^+ and m_i^- such that

$$\sum_{\{\alpha_i : \tilde{m}_i(\alpha_i) = m_i^+\}} p_i(\alpha_i) \quad > \quad H^+ > \sum_{\{\alpha_i : \tilde{m}'_i(\alpha_i) = m_i^+\}} p_i(\alpha_i),$$
$$\sum_{\{\alpha_i : \tilde{m}_i(\alpha_i) = m_i^-\}} p_i(\alpha_i) \quad < \quad H^- < \sum_{\{\alpha_i : \tilde{m}'_i(\alpha_i) = m_i^-\}} p_i(\alpha_i).$$

For any $j \neq i$, let

$$\begin{aligned}\theta_j(m_i^+, m_{-\{i,j\}}) &= -KH^-,\\ \theta_j(m_i^-, m_{-\{i,j\}}) &= KH^+,\\ \theta_j(m) &= 0, \text{ if } m_i \notin \{m_i^+, m_i^-\}.\end{aligned}$$

Then for any $K > 0$, the function θ_j is an acceptable second component of the annoucement by bidder j, as

$$\sum_{\{\alpha_i : \tilde{m}_i(\alpha_i) = m_i^-\}} p_i(\alpha_i)H^+ - \sum_{\{\alpha_i : \tilde{m}_i(\alpha_i) = m_i^+\}} p_i(\alpha_i)H^- < 0.$$

Furthermore given that

$$\sum_{\{\alpha_i : \tilde{m}_i'(\alpha_i) = m_i^-\}} p_i(\alpha_i)H^+ - \sum_{\{\alpha_i : \tilde{m}_i'(\alpha_i) = m_i^+\}} p_i(\alpha_i)H^- > 0,$$

bidder j will find it profitable to deviate, for K large enough, since any loss of utility stemming from the change in the allocation of the good will be more than offset by the increase in his income. This eliminates the "bad" equilibrium. ■

In this proof the original mechanism is transformed by augmenting the message spaces. Each bidder i is allowed to propose an additional transfer scheme, as long as these transfers give him negative expected payoff in the original equilibrium. The seller, being the mechanism designer, should play the role of a guarantee for these transfers. Since this at the same time ensures the realization of the "good" equilibrium, where no additional transfers are made, it is in his own interest, as illustrated by the simple example that we present next.

2.4. An Example

In this example we use the technique described above on a first price auction to eliminate bad equilibria. There are two bidders, each with valuation equal to 0 or 1, with probability 1/2 each. In the optimal first price auction, they are allowed to make a closed bid of 0 or 2/3. We have the standard rules of a first price auction with the good being allocated to each bidder with probability 1/2 if they bid the same amount. With these rules, there are two equilibria. With obvious notation in the "good" equilibrium

$$\tilde{m}_i(0) = 0,\ \tilde{m}_i(1) = 2/3, \text{for all } i,$$

and in the bad equilibrium

$$\tilde{m}_i(0) = \ \tilde{m}_i(1) = 0.$$

It is easy to see that the bad equilibrium is better from the viewpoint of the bidders.

Change now the rules as follows. Each bidder is allowed to a give a special signal to the seller (raise a ”flag”, for example) amounting to propose an additional transfer scheme. When one bidder raises a flag, the auction goes on as before but the bidder who raised the flag receives 10 from the seller, if the other bid is 0, and pays 12 to the seller, if the other bid is 2/3. In the good equilibrium, it does not pay to raise a flag, but it does in the bad equilibrium, which is therefore eliminated. It is possible to show that no new equilibrium is introduced.

3. Auctions: Unique Implementation

The preceding result has two limitations. First it uses a weak notion of uniqueness. Second it imposes the strong condition of independence on the valuation distribution. In this section we consider auctions as “direct mechanisms” and, in this simplified framework, we remove these limitations, first, by imposing on the beliefs another condition which does not require independence and is generic, and, second, by using a stronger notion of uniqueness.

3.1. Direct Mechanisms and Unique Implementation

For a given auction problem $(A, p, \mathcal{X}, (u_i)_{i\in N})$, an associated *direct auction mechanism* is a mechanism (M, s, t) where for every i, $M_i = A_i$. For each individual the possible bids are identified to the set of possible valuations: A message consists in announcing a valuation. The outcome function s and the payment scheme t are now functions of the announced valuations. By the revelation principle, one can always associate to any chosen auction mechanism and equilibrium, respectively a payoff-equivalent direct mechanism and the corresponding truthtelling equilibrium. The equilibrium conditions in the direct mechanisms are specified by the following *Bayesian incentive compatibility* (BIC) constraints, inducing the bidders to truthful revelation.

$$\begin{aligned}&\forall\ \ i\in N,\quad \forall\ \ \alpha_i\in A_i,\quad \forall\ \ a_i\in A_i,\\ &\sum_{\alpha_{-1}}(u_i(s(\alpha),\alpha)+t_i(\alpha))\,p(\alpha_{-i}\mid\alpha_i)\geq\\ &\quad\sum_{\alpha_i}(u_i(s(a_i,\alpha_{-i}),\alpha)+t_i(a_i,\alpha_{-i}))\,p(\alpha_{-i}\mid\alpha_i)\end{aligned}\tag{4}$$

For any BIC mechanism (A, s, t), another auction mechanism $(\mathcal{M}, \sigma, \tau)$ is said to *implement uniquely* the truthtelling equilibrium of (A, s, t) if and only if, for all equilibria $\tilde{\mu}$ of $(\mathcal{M}, \sigma, \tau)$, we have: $\sigma(\widetilde{\mu}(\alpha)) = s(\alpha)$ and $\tau(\widetilde{\mu}(\alpha)) = t(\alpha)$ for all α.

For an auction mechanism to be BIC and to give maximal surplus to the seller, there are known conditions imposed on the beliefs alone (see Crémer and McLean, 1988). The purpose is now to find additional conditions, also imposed on the beliefs alone, in order to ensure unique implementation.

3.2. Conditions for Unique Implementation

In this section, for simplicity, we impose the following additional assumption (weaker ones could be imposed):

$$\sum_{\{\alpha_k:\alpha_k\in A_k, i\neq k\neq j\}} p(\alpha_{-i}\mid\alpha_i) \equiv p(\alpha_j\mid\alpha_i) > 0 \quad \text{for all } \alpha_i \text{ and } \alpha_j \tag{5}$$

This assumption, which holds for nearly all information structures, will allow us to prove the following result.

Proposition 3.1 – *For any BIC mechanism* (A,s,t)*, there exists an auction mechanism* $(\mathcal{M},\sigma,\tau)$ *implementing uniquely the truthtelling equilibrium of* (A,s,t)*, whenever*

$$p(\alpha_{-i}\mid\alpha_i) \neq p(\alpha'_{-i}\mid\alpha'_i) \qquad \textit{for all } i\in N, \textit{ all } \alpha,\ \alpha'\in A,\ \alpha\neq\alpha'. \tag{6}$$

Condition (6) holds for nearly all information structures.

The inequalities (6) plays a role [2] similar to that of condition NCD (No Consistent Deceptions), introduced by Matsushima (1990). The proof of proposition 1 goes through the following steps:

- we introduce a new condition *ACCUI* (A Condition Concerning Unique Implementation), which is of independent interest and, as proved in lemma 1, rather transparently ensures unique implementation for any BIC mechanism;
- we show in lemma 2 that if condition (6) holds, condition *ACCUI* also holds;
- lemma 3 shows that condition (6) holds generically;

Let us start by stating the new condition:

2. Such conditions are indispensable if we are to find conditions on information structures alone that guarantee unique implementation. To see this, consider the case where the same utility function is attached to two different types. We can only guarantee unique implementation if the types generate different probability distributions over the types of the other agents.

Condition 1 (Condition ACCUI) *A belief (A,p) structure satisfies condition ACCUI if and only if for all i and all bijections*[3] $\gamma : A_{-i} \to A_{-i}$, *not equal to the identity mapping, there exists an α_i' such that the system*

$$\begin{cases} \sum_{\alpha_{-i}\in A_{-i}} \widetilde{t}_i(\alpha_{-i})p(\gamma(\alpha_{-i}) \mid \alpha_i') > 0 \\ \sum_{\alpha_{-i}\in A_{-i}} \widetilde{t}_i(\alpha_{-i})p(\alpha_{-i} \mid \alpha_i) < 0 \qquad \textit{for all } \alpha_i \neq \alpha_i' \end{cases}$$

has a solution $\widetilde{t}_i : A_{-i} \to \mathfrak{R}$.

Notice that Condition ACCUI has an equivalent dual version (by standard results on linear systems):

Condition 2 (Condition ACCUI*) *An information structure satisfies condition ACCUI* if and only if for all i and all bijections $c : A_{-i} \to A_{-i}$, not equal to the identity mapping, there exists an α_i' such that the system*

$$p(c(\alpha_{-i}) \mid \alpha_i') = \sum_{\alpha_i \in A_i} \lambda(\alpha_i) p(\alpha_{-i} \mid \alpha_i), \textit{ for all } \alpha_{-i}, \tag{7}$$

$$\lambda(\alpha_i) \geq 0, \textit{ for all } \alpha_i. \tag{8}$$

does not have a solution in $\lambda : A_i \to \mathfrak{R}$.

The first step of the proof can now be performed.

Lemma 3.1 – Consider any BIC mechanism (A,s,t). If condition *ACCUI* holds, there exists another mechanism $(\mathcal{M},\sigma,\tau)$ implementing uniquely the truthtelling equilibrium of (A,s,t).

Proof. To build the new auction mechanism let us define, as in the proof of Proposition 1, the set Θ_i of functions $\theta_i : A_{-i} \to \mathfrak{R}$ such that either

$$\theta_i(\alpha_{-i}) = 0 \quad \text{for all } \alpha_{-i} \in A_{-i}$$

or

$$\sum_{\alpha_{-i}} \theta_i(\alpha_{-i})\, p(\alpha_{-i} \mid \alpha_i) < 0 \quad \text{for all } \alpha_i \in A_i. \tag{9}$$

Agent i announces a type and a function in Θ_i, therefore $\mathcal{M}_i \equiv A_i \times \Theta_i$. If for all $i \in N$ the message is equal to $(\alpha_i,\theta_i) \in \mathcal{M}_i$ we have

$$x((\alpha_1,\theta_1),\ldots,(\alpha_n,\theta_n)) = s(\alpha),$$

$$\tau_i((\alpha_1,\theta_1),\ldots,(\alpha_n,\theta_n)) = t_i(\alpha) + \theta_i(\alpha_{-i}) \quad \text{for all } i.$$

As before, in the augmented mechanism, it is an equilibrium for every bidder i of type α_i to announce the message $(\alpha_i,0)$, because truthtelling is an equilibrium of

3. This is somewhat stronger than we need. Only the bijections $\gamma = \prod_{j\neq i}\gamma_j$ where γ_j is a bijection from A_j into A_j need to be considered. This is true for all the bijections we consider.

the original mechanism, and because the "extra" transfers can only yield negative expected payoffs when the other bidders tell the truth. We now show that it is the only equilibrium when *ACCUI* holds.

An equilibrium strategy $\tilde{\mu}_i$ of agent i will be written $(\widetilde{\alpha}_i, \widetilde{\theta}_i)$ with $\widetilde{\alpha}_i : A_i \to A_i$ and $\widetilde{\theta}_i : A_i \to \Theta_i$. The reasoning of the preceding paragraph shows that there exists no equilibrium in which all bidders announce their true types ($\widetilde{\alpha}_i(\alpha_i) = \alpha_i$ for all i and all α_i) and in which we have $\widetilde{\theta}_i$ not identically zero for some bidder. Therefore in any candidate equilibrium $(\widetilde{\alpha}, \widetilde{\theta})$,with $\widetilde{\theta}$ not identically zero, at least one bidder must lie about his type.

Assume first that there exists such a bidder $j \neq i$ such that $\widetilde{\alpha}_j$ is not a bijection. Then some $\alpha''_j \in A_j$ is never announced by agent j. By (5), for K large enough, the function defined by

$$\theta_i(\alpha_{-i}) = \begin{cases} 1 & \text{if } \alpha_j \neq \alpha''_j, \\ -K & \text{if } \alpha_j = \alpha''_j, \end{cases}$$

belongs to Θ_i and is a better second component of the message of i than {0}. Because bidder i will want to announce as large a multiple of θ_i as possible, there is no equilibrium where the $\widetilde{\alpha}_j$'s are not all bijections.

Assume now that bidder j does not use the truthtelling strategy ($\widetilde{\alpha}_j(\alpha_j) \neq \alpha_j$ for some α_j), and for any $i \neq j$, and let γ be the inverse function of $\widetilde{\alpha}_{-i}$. Then $p(\gamma(\alpha'_{-i}) \mid \alpha'_i)$ is the probability that agent i assigns to the annoncement α'_{-i} when he is of type α'_i. Let θ_i be equal to $\lambda \widetilde{t}_i$ where λ is a very large real and $\widetilde{t}_i : A_i \to \mathfrak{R}$ is the function whose existence is guaranteed by condition *ACCUI*; θ_i belongs to Θ_i. Agent i will find it a profitable second component of his message, and because the greater the λ the better the response, we have eliminated all non-truthtelling equilibria.

We have therefore shown that, for the augmented mechanism, truthtelling is an equilibrium, and that there is no non-truthtelling equilibrium. The lemma is proved. ∎

The second step of the proof of Proposition 1 is given by the following lemma:

Lemma 3.2 – If condition (6) holds, then *ACCUI* holds.

Proof. Without loss of generality, choose a bijection $\gamma : A_{-1} \to A_{-1}$, not equal to the identity. Then there exists a state of nature α', such that

$$\gamma(\alpha'_{-1}) \neq \alpha'_{-1} \tag{10}$$

and, by condition (6), such that

$$p(\alpha'_{-1} \mid \alpha'_1) > p(\alpha_{-1} \mid \alpha_1)$$
$$\text{for all } \alpha \neq \alpha' \text{ such that } \gamma(\alpha_{-1}) \neq \alpha_{-1}. \tag{11}$$

From (10), and because γ and therefore γ^{-1} are bijections

$$\alpha'_{-1} \neq \gamma^{-1}(\alpha'_{-1}). \tag{12}$$

From (11) and (12) we obtain

$$p(\alpha'_{-1} \mid \alpha'_1) > p(\gamma^{-1}(\alpha'_{-1}) \mid \alpha_1) \quad \text{for all } \alpha_1. \tag{13}$$

The lemma will be proved when we will have shown that there exists an $\eta > 0$ such that the transfer function $\widetilde{t}_1$ defined by

$$\widetilde{t}_1(\gamma^{-1}(\alpha'_{-1})) = 1 \tag{14}$$

$$\widetilde{t}_1(\alpha_{-1}) = -\eta \qquad \text{for all } \alpha_{-1} \neq \gamma^{-1}(\alpha'_{-1}) \tag{15}$$

satisfies the conditions of the definition of *ACCUI*. To see this note that

$$\begin{aligned}
&\sum_{\alpha_{-1}\in A_{-1}} \widetilde{t}_1(\alpha_{-1})p(\gamma(\alpha_{-1}) \mid \alpha'_1) \\
&\quad = \widetilde{t}_1(\gamma^{-1}(\alpha'_{-1}))p(\alpha'_{-1} \mid \alpha'_1) - \eta(1 - p(\alpha'_{-1} \mid \alpha'_{-1})) \\
&\quad = (1+\eta)p(\alpha'_{-1} \mid \alpha'_1) - \eta
\end{aligned} \tag{16}$$

and that for $\alpha_1 \neq \alpha'_1$

$$\begin{aligned}
&\sum_{\alpha_{-1}\in A_{-1}} \widetilde{t}_1(\alpha_{-1})p(\alpha_{-1} \mid \alpha_1) \\
&\quad = \widetilde{t}_1(\gamma^{-1}(\alpha'_{-1}))\, p(\gamma^{-1}(\alpha'_{-1}) \mid \alpha_1) - \eta\,(1 - p(\gamma^{-1}(\alpha 4_{-1}) \mid \alpha_1)) \\
&\quad = (1+\eta)\, p(\gamma^{-1}(\alpha'_{-1}) \mid \alpha_1) - \eta.
\end{aligned} \tag{17}$$

Equations (13), (16) and (17) imply

$$\sum_{\alpha_{-1}\in A_{-1}} \widetilde{t}_1(\alpha_1)p(\gamma(\alpha_{-1}) \mid \alpha'_1) > \sum_{\alpha_{-1}\in A_{-1}} \widetilde{t}_1(\alpha_1)p(\alpha_{-1} \mid \alpha'_1),$$

and it is clear that if we take η just large enough that the left hand side of this inequality is positive while the right hand side is negative, we will find the transfers that we are looking for. ■

Lemma 3.3 – Condition (6) holds for nearly all belief structures.

Proof. It is sufficient to show that condition (6) holds for nearly all information structures such that $p(\alpha) > 0$ for all α. First, because the conditional probabilities are continuous functions of the $p(\alpha)$'s it is straightforward that the set of belief structures that satisfy (6) contains an open neighborhhood of any of its elements. Second, if some belief structure does not satisfy (6), we can find another probability structure arbitrarily close that satisfies this property, by a proof similar to that used in d'Aspremont, Crémer, and Gérard-Varet (1990). The proof begins by showing that we can modify slightly any information structure that does not satisfy (6) and reduce the number of equalities between conditional probabilities. A sequence of such reductions will lead to a belief structure that satisfies (6). ■

This completes the proof of proposition 1.

3.3. The Case of Free Beliefs

Finally we turn to the case (implied by independence) of *free beliefs*:

$$p(\alpha_{-i} \mid \alpha_i) = p(\alpha_{-i} \mid \alpha_i') = p(\alpha_{-i}), \text{ for all } \alpha \in A,\ \alpha_i \in A_i,\ i \in N.$$

In such a case the preceding result can be strenghtened:

Lemma 3.4 – Under free beliefs, condition (6) is equivalent to *ACCUI*.

Proof. It is sufficient to show that condition $ACCUI^*$ implies condition (6).

With free beliefs, the equalities in (7) become

$$p(\gamma(\alpha_{-i})) = \left[\sum_{\alpha_i \in A_i} \lambda(\alpha_i)\right] p(\alpha_{-i}).$$

Summing both sides of this equation over all $\alpha_{-i} \in A_{-i}$ shows that $\sum_{\alpha_i \in A_i} \lambda(\alpha_i)$ is equal to 1, and therefore $ACCUI^*$ is equivalent to the statement: for all i and for all γ we do not have $p(\gamma(\alpha_{-i}) = p(\alpha_{-i})$, which proves the result. ■

4. Extension to Public Good Problems

The techniques that we have presented so far have been derived to solve auction problems. However, it is possible to use them in many other contexts, including the design by a public planner of mechanisms ensuring the efficient provision of some public good or service. The main difference in such a context is the way in which the transfers are affected. In auctions, we have assumed that the transfers were payments made by the buyers to the seller, who designs the mechanism so as to maximize the expected revenue. In the provision of a public good the transfers are taken to cover the cost of the public good and to realise redistributions among the consumers. The objective is to achieve efficiency and, may be, some redistributive objective.

Formally the problem $(A, p, X, (u_i)_{i \in N})$ can be viewed as being an abstract framework and reinterpreted as a public good problem simply by taking N to be the set of agents in the economy and X, the set of outcomes, to be states of the economy that include the level of public goods. Still assuming that the utilities are measured in money (perfect transferability), a mechanism (M, s, t) and a direct mechanism (A, s, t) are defined as before. The outcome function s now associates to every vector of announced messages (which, in a direct mechanism, are announced types) a state x in X. The transfer scheme t, which includes the required payments for the production of public goods, have to satisfy a budget-balance equation:

$$\sum_{i \in N} t_i(m) = 0, \text{ for all } m \in M.$$

To any mechanism (M, s, t) can be associated a game of incomplete information, and the concept of Bayesian equilibrium is still defined by (2). For direct mechanisms, Bayesian incentive compatibility is defined accordingly, as in (4).

Unique implementation can be obtained by adding an assumption, imposed on the beliefs only, such as condition (6). Indeed, Proposition 2 is straightforwardly adapted to give

Proposition 4.1 – *For any BIC mechanism (A, s, t), there exists a mechanism $(\mathcal{M}, \sigma, \tau)$ implementing uniquely the truthtelling equilibrium of (A, s, t), whenever*

$$p(\alpha_{-i} \mid \alpha_i) \neq p(\alpha'_{-i} \mid \alpha'_i) \quad \text{for all } i \text{ in } N, \text{ and } \alpha, \alpha' \text{ in } A. \tag{6}$$

Condition (6) holds for nearly all information structures.

Proof. The proof repeats the arguments used in the proof of Proposition 2, adapted to ensure that the constructed transfer scheme is budget-balanced. The augmented mechanism is constructed as follows. For all agents $i \in N$ a message is a vector $(\alpha_i, \theta_i) \in A_i \times \Theta_i$ such that

$$\sigma((\alpha_1, \theta_1), \ldots, (\alpha_n, \theta_n)) = s(\alpha), \text{ (as before)}$$

$$\tau_i((\alpha_1, \theta_1), \ldots, (\alpha_n, \theta_n)) = t_i(\alpha) + \theta_i(\alpha_{-i}) - \sum_{j \in N_{-i}} \frac{1}{n-1} \theta_j(\alpha_{-j}).$$

Hence the modified transfers are budget-balanced. ■

In d'Aspremont, Crémer and Gérard-Varet (1997), we show that any outcome function s can be implemented for generic beliefs, as long as there are at least three agents. This yields the following corollary:

Corollary 4.1 – *Assume that there are at least three agents. For any utility functions u_i of the agents, any outcome function s and nearly all information structures, it is possible to find a mechanism that uniquely implement s.*

Note, finally, that we have not required, as is often done, that the outcome function s be *(ex post) efficient* in the sense that

$$s(\tilde{m}(\alpha)) \in \arg\max_{x \in X} \sum_{i \in N} u_i(\tilde{m}(\alpha)).$$

In the present framework, efficiency can also be ensured by conditions imposed on the beliefs only. One such (weak) condition is the "compatibility condition" introduced in d'Aspremont and Gérard-Varet (1979). [4]

4. D'Aspremont, Crémer and Gérard-Varet (1997) discuss the issue of the existence of efficient mechanisms and introduce new conditions.

5. Conclusion

In implementation theory, the position of the "mechanism designer" remains fuzzy, somewhat inside somewhat outside the game, the rules of which have to be fixed and imposed. In the design of optimal auctions, the seller can be viewed as the mechanism designer, since the rules of the auction are usually taken to be in the seller's best advantage. In the public good problem, the mechanism designer is the group of all players acting collectively through a "planner".

For unique implementation under incomplete information via augmented revelation mechanisms – the problem with have dealt with here –, a strenghtening of the role of the mechanism designer is required. During the play of the game, the players may strategically propose additional side-payments. These have to be guaranteed. In the augmented auction mechanism, the seller might even have to reward a deviating player, "acting as a stool pigeon" in order to destroy equilibria that are bad from the seller's point of view. In the public good context, the point of view is collective and the planner has to require that all side-payments balance.

In this paper, we have fully exploited the power given to the mechanism designer in order to coordinate the equilibrium selection. An alternative approach would be to introduce, with the purpose of coordinating on a good equilibrium, a pre-play communication stage between players (as done in Palfrey and Srivastava (1991)). The organization of such pre-play communication may be seen as a supplementary instrument available to the mechanism designer. Maintaining the tranferable utility assumption, we have concentrated our attention on the instrument provided by transfers. In many given contexts with nontransferable utility, more instruments are to be considered, exploiting the specific features of the situation.

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