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Neural networks and contagion

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NEURAL NETWORKS AND CONTAGION (*)

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Key words : Local Interaction, Logit Model, Potential Games, Stochastic, Stability.

I. — INTRODUCTION

The appeal of evolutionary game theory to social scientists in general and economists in particular rests on the fact that it allows to investigate the dynamics and long-run properties of a population of interacting boundedly rational players. Methods borrowed from biology have helped to adresse questions of equilibrium selection and stability as documented in several monographs : van Damme (1991), Weibull (1995), Vega-Redondo (1996), Samuelson (1997). In the young tradition of Blume (1993, 1995), Berninghaus and Schwalbe (1996a,b), Kandori, Mailath, and Rob (1993), abbreviated KMR in the sequel, Ellison (1993), Rhode and Stegeman (1996), Young (1998, Ch. 6), and Baron *et al.* (2002a,b), we consider best response dynamics where at each time, one or every player plays a (static) best response against the empirical distribution of the last strategies played by his neighbors. This constitutes rational behavior

(*) This paper considers contagion in the context of Haller and Outkin (1999). We thank Richard Baron, Jacques Durieu, Philippe Solal, Mark Stegeman, Jean-Marc Tallon, Peyton Young, and a referee for valuable comments. Financial support by the German Science Foundation (DFG), through SFB 504, is gratefully acknowledged.

impaired by myopia. Myopia in the temporal sense means that the player is not forward looking, does not take into account that other players might be changing their strategies. This trait is shared, for example, by naive Bayesian learners [Eichberger *et al.* (1993)]. Myopia in the spatial sense, if applicable, means that the player is influenced only by his local environment.

In the sequel, an « interaction structure » is modelled as an undirected finite graph whose vertices or nodes are the members of the player population. Two players are neighbors, if they form an edge of the graph. We assume that the graph and *a fortiori* the interaction structure is regular, *i.e.* all players have the same number of neighbors. We finally assume that direct interaction is only possible between neighbors. We are going to analyze local as well as global interaction in population games, using the formalism of neural networks. More specifically, we put forward a modelling approach to best response dynamics that (a) allows for rather general interaction structures; (b) exhibits spatial patterns of play; (c) exhibits non-uniform noise; (d) links stochastic and deterministic dynamics; (e) involves asynchronous updating; (f) encompasses majority imitation.

Blume (1995) studies and compares local and global interaction for specific interaction structures (infinite and finite two-dimensional lattices). In an otherwise deterministic model, Blume assumes asynchronous updating where each period, a player is selected at random and plays a myopic best response against his neighbors' previous actions. Berninghaus and Schwalbe (1996a,b), in a model with simultaneous updating, were the first to demonstrate that the theory of neural networks can be successfully applied to analyze deterministic best response dynamics with global or local interaction. Here we go beyond Blume (1995) and Berninghaus and Schwalbe (1996a,b) and introduce noise, random non-best responses to be precise, into the system. This has been done before, notably by Blume (1993) who pioneered the use of statistical mechanics methods for population games on infinite lattices. KMR and Ellison allow for noise in what they call « best reply » and other dynamiques (1). We use a « Boltzmann machine », a particular kind of stochastic neural network, to model noise. The noise operates on the thresholds of the threshold automata (neurons) constituting the network. This produces two major innovations. First, a state of the system describes a spatial pattern of play, not merely a summary statistics as in KMR and Ellison. Second, the probability of a « flip », *i.e.* of non-best-response play is a continuous and decreasing function of the payoff loss caused by the flip whereas in KMR and Ellison the probability is independent of state and player.

(1) See also the comment on KMR by Rhode and Stegeman (1996). See further Foster and Young (1990), Young and Foster (1991), Fudenberg and Harris (1992), Young (1993), and Binmore *et al.* (1995). Our approach, while independently conceived and developed, is closely related to Chapter 6 of Young (1998).

Running the « Boltzmann machine », we obtain an explicit formula for the stochastic steady state (invariant distribution) in terms of the parameters of the model for any regular interaction structure. In fact, we obtain an invariant distribution of the Gibbs-Boltzmann type (2). We are able to relate the long-run equilibria à la KMR (introduced as *stochastically stable states* in Foster and Young (1990)) to the deterministic steady states. We can, in principle, compute all long-run equilibria for any symmetric 2×2 game and any regular interaction structure, by solving a discrete optimization problem without resorting to a root counting procedure. We not only determine how often a strategy occurs, but also detect the spatial pattern of play in a long-run equilibrium. The explicit determination of long-run equilibria in Haller and Outkin (1999) shows that by and large, the properties of long-run equilibria reported by KMR and Ellison are confirmed – which implies a certain robustness of the model. Notice, however, that Rhode and Stegeman have amended and corrected some of the KMR results, ending up with a richer taxonomy of games. Notice further that according to Bergin and Lipman (1996), different types of noise can give rise to different sets of long-run equilibria. Baron *et al.* (2002a) provide an explicit example that yields different long-run equilibria in our model and in a model with Bernoulli or uniform trembles.

The distinction between global and local interaction is only interesting, if it makes a difference. Apart from purely descriptive reasons, the comparison of interaction structures has been motivated by the pioneering work of Novak and May (1993) whose simulations of deterministic best performance imitation have generated significant differences across interaction structures. The theoretical analysis of deterministic best response dynamics by Berninghaus and Schwalbe (1996b) shows that the size and shape of neighborhoods can affect the nature, number, stability, and attractiveness of limit cycles and steady states. Also notice L. Blume's lucid comment on the impact of the interaction structure on the rate of convergence [Blume (1995, p. 130)]. In the case of differentiation or anti-coordination games (games with no symmetric pure strategy equilibrium), our analysis of stochastic best response dynamics with asynchronous updating demonstrates a significant difference of long-run equilibria across interaction structures, an important new discovery : With an even number of players and a circular graph, in each long-run equilibrium, the two actions of the constituent game are alternating along the graph ; hence the ratio of their frequencies is 50:50. In a global inter-action game, the ratio approximately equals the ratio of the probabilities in a mixed equilibrium strategy of the constituent game and can be quite different from 1.

Careful inspection reveals a close relation between our analysis and Chapter 6 of Young (1998). More generally, we establish a link between the neural net-

(2) This kind of distribution has been derived or postulated before; see in particular Blume (1997) and Young (1998). Baron *et al.* (2000a,b) extend the analysis to the case of non-binary choices, among other variations.

work approach and the stochastic discrete choice approach. In the specific case at hand, we show that if the thresholds of the neural network are perturbed by logistic noise, then state transitions of the system are governed by the log-linear response rule (6.3) of Young (1998). The neural network approach leads to a convenient explicit formula for a potential of the « spatial game ». This finding allows us to determine long-run equilibria without resorting to the more cumbersome tree counting method introduced into game theory by Foster and Young (1990).

Applications in Industrial Economics: The model lends itself to the study of various coordination problems in industrial economics, for exemple the choice of industry standards and norms. In Section 8, we elaborate on a specific application, a model of formation of user networks.

In the next three sections, we redesign and augment the basic deterministic model of Berninghaus and Schwalbe (1996b). We proceed to the more elaborate stochastic model in Section 5. Section 6 is a digression on invariant Gibbs-Boltzmann distributions. Section 7 specializes, introducing logistic noise into the model which gives rise to a log-linear response model and invariant Gibbs-Boltzmann distributions. Section 9 concludes.

II. — NETWORKED MODELS

In the sequel we consider dynamics that can be represented by means of neural networks, a special case of automata networks. The use of automata networks is quite common in computer science. For instance, the interaction within a computer network falls into this category. Parallel processing is another example. The approach also suggests itself for decentralized models of robots interacting in a production process. Maes (1989) aims at « the building of an intelligent system as a society of interacting mindless agents, each having their own specific competence ». Similarly, in the area of industrial organization, automata networks appear well suited for the description of information flows between units or divisions of an organization (firm, industry). For extensive reading on automata networks, see Coughlin and Baran (1995), Goles and Martinez (1990, 1992), Haykin (1994). We commence with a few formal definitions, first of an Automata Network (AN) and then of a Neural Network (NN). Throughout, $I = \{1, \dots, N\}$ with $N \geq 2$ is a non-empty finite set. Let us denote $\mathcal{J} \equiv \{J \subseteq I : |J| = 2\}$, the set of two-member subsets of I.

An **Automata Network** *on I is a triple* $A = (G, \Sigma, (f_i, i \in I))$ *where:*

$G = (I, V)$ *is a (undirected) graph.*

Σ *is the set of states of any vertex of the graph G.*

f_i *is the transition function associated with the vertex* $i \in I$.

First, the graph $G = (I, V)$ comprises I, its set of vertices (points, nodes), and $V \subseteq \mathcal{J}$, the set of edges (arcs, links) of the graph. I will be interpreted as the player population. I can also be interpreted at a set of locations where each location i is occupied by exactly one player whom we conveniently label i as well. We interpret G or simply V as « interaction structure ». Namely, given the player population I, the edges of the graph define the opportunities for direct interaction. If $i, j \in I$ and $i \neq j$, we interpret $\{i, j\} \in V$ as i and j being adjacent or neighbors. A player $i \in I$ directly interacts with and only with players in his neighborhood $V_i = \{j \in I : \{i, j\} \in V\}$ (3). We assume the interaction structure to be regular, *i.e.* there is a number $n \in \mathbb{N}$ with $|V_i| = n$ for all $i \in I$, and to be connected. Second, for every vertex $i \in I$, the set Σ represents the potential states of vertex i – or of the player located at vertex i. A state of vertex or player i is denoted by s_i. In our context, s_i is a strategy played by player i. Third, to each vertex or player i, we associate the transition function $f_i : \Sigma^{V_i} \to \Sigma$. This means that the current strategic choices of his neighbors determine a player's next strategic choice. The evolution in discrete time t of the global state $s(t) \in \Sigma^I$ is governed by the global transition function $F : \Sigma^I \to \Sigma^I$ obtained as composition of all the local ones: $F(s) = (f_i((s_j)_{j \in V_i}))_{i \in I}$ for $s = (s_1, \ldots, s_N) \in \Sigma^I$. Each f_i is also called a (memory-less) automaton.

Every stationary population game dynamics with finite strategy sets and finite neighborhoods corresponds to an automata network. This by itself does not allow strong conclusions. To arrive at interesting results, more structure has to be imposed. For our purposes, the more restrictive structure of a neural network is imposed:

A **Neural Network (Threshold Automata Network)** *is a particular type of automata network. Its individual state space is binary. Here we assume for convenience that* $\Sigma = \{0, 1\}$. *The network's transition function and, implicitly, its graph are based on a weight structure, given by a symmetric* $N \times N$*-matrix* $W = (w_{ij})$. *Namely: every arc* $(i, j) \in V$ *is assigned a real number* $w_{ij} \in \mathbb{R}$ *which represents its weight. If i and j are not neighbors, we put* $w_{ij} = 0$. *The transition function takes on the following form:*

$$s_i(t+1) = L\left(\sum_{j \in V_i} w_{ij} s_j(t) - b_i\right) \quad (1)$$

where b_i *is a threshold and the function* $L(\cdot)$ *is given by* $L(x) = 0$ *if* $x \leq 0$ *and* $L(x) = 1$ *if* $x > 0$. *An automaton described by* (1) *is called a* neuron *or* threshold automaton.

As the term suggests, neural networks can be used to model the interaction between brain or nerve cells: If and only if a receptor-transmitter receives a strong enough stimulus, it will emit a signal of its own. In a similar vein, in a social environment, an individual's decision to be violent or not may depend on the amount of violence in the neighborhood. Our current interest in neural

(3) Notice that by definition, $i \notin V_i$ which is the prevalent convention in graph theory. See for instance Chartrand (1985). The convention $i \in V_i$ coexists.

networks stems from the fact that simple best response dynamics, among others, can be described in terms of a neural network.

III. — DETERMINISTIC BEST RESPONSE MODELS

Consider a symmetric two person game:

	0	1
0	a,a	b,c
1	c,b	d,d

(2)

Like Berninghaus and Schwalbe (1996b), we are going to represent the best response dynamics of an associated population game as an NN. The player population is I, endowed with an interaction structure V. We shall assume that all players have the same number of neighbors $n \geq 1$: $|V_i| = n$ for all $i \in I$. In other words, the interaction structure (graph) is n-regular. We further assume that only pure strategies are played. The set of pure strategies available to each player is $\Sigma = \{0, 1\}$, and the state space of the entire system is $\Sigma^I = \{0, 1\}^I$. A state records the action taken by each of the players.

3.1. Nash Configurations

Let us first consider the static spatial game associated with the interaction structure V and introduce the concept of Nash configuration. Player $i \in I$ directly interacts only with his neighbors and has information only about the strategies played in his neighborhood, V_i. For a state $s = (s_1, \dots, s_N)$ and a player $i \in I$, let $s_{-i} = (s_k)_{k \in V_i}$ denote the profile of strategies played in V_i. Let $\pi_i(s_i, s_{-i})$ denote the aggregate payoff to player i from playing strategy s_i once against each neighbor, if s_{-i} is the profile of strategies played in his neighborhood. The static spatial game assumes the normal form

$$\Gamma = (I, (S_i)_{i \in I}, (v_i)_{i \in I})$$

where $S_i = \Sigma$ for all $i \in I$ and $v_i(s) = \pi_i(s_i, s_{-i})$ for all $i \in I$ and $s = (s_1, \dots, s_N) \in S \equiv S_1 \times \dots \times S_N = \Sigma^I$. Obviously, global states and strategy profiles for Γ are the same. Following Blume (1993, 1995), the Nash equilibria in pure strategies of Γ will be called *Nash configurations*.

3.2. A Potential

A *potential* of the spatial game is a function $H: S \to \mathbb{R}$ such that if $i \in I$ and $s, s' \in S$ differ only in the i'th component, then

$$v_i(s) - v_i(s') = H(s) - H(s'). \quad (3)$$

In the sequel, we denote the differences in (3) by Δv_i and ΔH, respectively. Young (1998, Ch. 6) demonstrates that the spatial game has a potential. Here we derive an explicit formula for a potential that will prove very useful later on. To this end, let us determine Δv_i for some player $i \in I$. Given a profile s_{-i} of strategies played in V_i, let z_i equal the number of players using strategy 1 in V_i and $n - z_i$ equal the number of players using strategy 0 in V_i. Then i's aggregate payoffs are

$$\pi_i(0, s_{-i}) = (n - z_i)a + z_i b;$$
$$\pi_i(1, s_{-i}) = (n - z_i)c + z_i d.$$

Hence the payoff difference $\Delta\pi_i = \pi_i(1, s_{-i}(t)) - \pi_i(0, s_{-i}(t))$ satisfies

$$\begin{aligned} \Delta\pi_i &= z_i(a - c + d - b) + n(c - a) \\ &= \sum_{j \in V_i} (a - c + d - b)s_j - n(a - c) \\ &= w \sum_{j \in V_i} s_j - n\beta \end{aligned} \tag{4}$$

where $\beta = a - c$ and $w = a - c + d - b$. Now define $H: S \to \mathbb{R}$ by

$$H(s) = \sum_{k \in I} s_k \left[\tfrac{1}{2} w \sum_{j \in V_k} s_j - n\beta \right]. \tag{5}$$

Next consider two states $s, s' \in S$ which differ only in the i'th coordinate. Without loss of generality, assume $s_i = 1$ and $s'_i = 0$. Then

$$H(s) - \mathrm{H}(s') = w \sum_{j \in V_i} s_j - n\beta$$

whereas by (4), $v_i(s) - v_i(s') = \Delta\pi_i = w \sum_{j \in V_i} s_j - \mathrm{n}\beta$. Thus we have shown:

Proposition 1 - *The function H defined by (5) is a potential function for the spatial game.*

IV. — DETERMINISTIC NEURAL NETWORK DYNAMICS AND CONTAGION

Now we model the dynamics of myopic best-response play in terms of a neural network. Let $s_i(t)$ denote the strategy played by player i in period $t = 0, 1, \ldots$. *Contagion (with respect to action 1) occurs from an initial subset I_0 of I*, if $s_i(0) = 1$ for $i \in I_0$, $s_i(0) = 0$ for $i \notin I_0$, and there exists $T \in \mathbb{N}$ such that $s_i(t) = 1$ for all $i \in I$ and $t \geq T$. We say that contagion is *optimal*, if contagion does not occur from any proper subset I'_0 of I_0. We call contagion *monotone* if $s_i(t + 1) \geq s_i(t)$ for all i and t.

Again, player $i \in I$ directly interacts only with his neighbors and has information only about the strategies played in his neighborhood, V_i. We assume that a player (when he has a choice) employs a myopic best response, *i.e.* the strategy he chooses for the period $t + 1$ is a best response against the empirical distribution of strategies played in his neighborhood at time t. One can find a discussion of validity of the myopia assumption in KMR and Ellison (1993).

We denote by $s_{-i}(t)$ the profile of strategies played in V_i at time t and by $\pi_i(s_i, s_{-i}(t))$ the aggregate payoff to player i at time t from playing strategy s_i once against each neighbor. This means that all neighbors are equally important. Alternatively, one could work with average payoffs.

Because of (4), the decision rule for a population of best response players can be written in a threshold form :

$$s_i(t + 1) = L\left[w \sum_{j \in V_i} s_j(t) - n\beta \right]. \quad (6)$$

Thus deterministic best response dynamics is modelled as a neural network.

4.1. Simultaneous or Synchronous Updating

Simultaneous or synchronous updating means that the updating rule (6) is applied to all i and t. The model has been analyzed in depth by Berninghaus and Schwalbe (1996b) who follow the logic of Goles and Martinez (1990) and find, using a potential like (5), that with simultaneous updating the only cycles are fixed points (steady states) and two-cycles. Their analysis also shows that both the size and shape of neighborhoods can affect the nature, number, stability, and attractiveness of limit cycles and steady states. Durieu *et al.* (2005) investigate and characterize contagion for the dynamical system with simultaneous updating when $w \geq \beta \geq 0$. In view of the result of Goles and Martinez, there are three possibilities : (a) contagion ; (b) convergence to a fixed point without contagion ; (c) convergence to a two-cycle. The analysis of Durieu *et al.* (2005) indicates that, as a rule, there is no fast way to verify (a), whereas there exists several fast routines (easy-to-check necessary conditions) to rule out (a).

Clearly, contagion cannot occur if the constituent bi-matrix game is a differentiation or anti-coordination game. These are the « games with no symmetric pure strategy equilibrium » where it is in both players' interest to choose strategies different from each other. Examples are fashion games among non-conformists and hawk-dove games. These games are characterized by $a - c < 0$ and $d - b < 0$, hence $w < \beta < 0$. Obviously, contagion cannot occur either if 0 is a strictly dominant strategy of the constituent bi-matrix game, like in a Prisoner's Dilemma game, which amounts to $a - c > 0$ and $d - b < 0$. In the opposite case, when $a - c < 0$ and $d - b > 0$ and 1 is the strictly dominant strategy, contagion does occur. In the case of coordination games, where $a - c > 0$ and $d - b > 0$, the outcome depends on the interaction structure, I_0, and which action is risk dominant. Consider for example $N \geq 2$, V the circular graph given by $V_i = \{j \in I | j = i \pm 1 \text{modulo } N\}$ for $i \in I$, and $I_0 = \{k\}$ for some $k \in I$. If action 0 is risk dominant – which is the case if $a - c > d - b > 0$ – then in one step, convergence to the fixed point where all play 0 occurs. If N is even and action 1 is risk dominant – which is the case if $d - b > a - c > 0$ – the convergence to a two-cycle occurs. If N is odd and action 1 is risk dominant, then optimal contagion occurs.

We further observe that larger neighborhoods (larger interaction windows) and, hence, more connectivity may but need not favor contagion.

4.2. Asynchronous Updating

Asynchronous updating means that at any time only one player will be updating his state. This is modelled by means of a sequence $K(t)$, $t = 0, 1, 2, \dots$, of I-valued random variables. $K(t)$ determines which of the players will have a chance to alter his strategy at time t. Accordingly, the updating rule (6) is applied if $i = K(t)$; otherwise, $s_i(t + 1) = s_i(t)$.

We assume that the stochastic process $K(t)$; $t = 0, 1, \dots$, is *recurrent*, that is for each $i \in I$ and $t \geq 0$, almost certainly the process returns to i after time t:

$$Prob\,(\{K(t') = i \text{ for some } t' > t\}) = 1.$$

Recurrence is satisfied, for instance, by the periodic deterministic sequence $0, 1, \dots, N-1, 0, 1, \dots$ given by $K(t) = t$ modulo N at one extreme and by a sequence of independent and identically distributed (i.i.d.) random variables with full support I at the other extreme. As an immediate consequence of Proposition 1, we obtain

Proposition 2 - *Under asynchronous updating governed by a recurrent process* $K(t)$, $t = 0, 1, \dots$, *convergence to a Nash configuration of* Γ *occurs with probability 1.*

Like with simultaneous updating, contagion cannot occur if the constituent bi-matrix game is an anti-coordination game or if 0 is a strictly dominant strategy. In case that 1 is the strictly dominant strategy, contagion does occur with probability 1. In the case of coordination games, consider again the example $N \geq 2$, V the circular graph given by $V_i = \{j \in I | j = i \pm 1 \text{ modulo } N\}$ for $i \in I$, and $I_0 = \{k\}$ for some $k \in I$. Moreover, assume that the random variables $K(t)$ are independent and identically distributed (i.i.d.) with $Prob(\{K(0) = i\}) = 1/N$ for all $i \in I$. If action 0 is risk dominant then with probability 1, there is convergence to the Nash configuration where all play 0. If action 1 is risk dominant, then with probability 1/3, convergence to the Nash configuration where all play 0 occurs and with probability 2/3 contagion occurs. In contrast to simultaneous updating, the latter result holds irrespective of the odd- or evenness of N.

In the following sections, we shall go beyond Berninghaus and Schwalbe by adding noise to the basic model.

V. — NOISY BEST RESPONSE DYNAMICS

In many cases it is beneficial to introduce randomness into the model. For example, deterministic models may have cycles or multiple equilibrium points. Frequently, under reasonable assumptions, a random Markov process on the same system has a unique stationary distribution. Also, we know from computer science that the learning capabilities of a stochastic network can be substantially better than those of a deterministic one. Whenever applicable, we keep the previous notation. Again, two polar updating rules are possible: synchronous and asynchronous iteration. In the synchronous or simultaneous mode we assume that all agents update their strategies at the same time, and in the asynchronous mode we assume that at any given time only one agent can update his strategy. We shall adopt the latter assumption in the sequel. However, we first present a model of synchronous updating to exhibit the difference.

Synchronous Updating. A model of synchronous updating is described by (6) with noise added. The noise shifts the thresholds $n\beta$. It is given by a family of random variables $\varepsilon_i(t)$, $i = 1, \dots, N$; $t = 0, 1, 2, \dots$, so that

$$s_i(t+1) = L\left[w \sum_{j \in V_i} s_j(t) - \beta n + \varepsilon_i(t) \right] \qquad (7)$$

for all i and t. The noise can come from several sources: noise in the level of the threshold, in the strategy played or perhaps in the payoff parameters; in Ellison's eloquent words, noise in the form of deliberate experimentation, trembles in strategy choices and the play of new players unfamiliar with the history of the game. If the noise distribution $\varepsilon_i(t)$ is continuous and has sufficiently large support, (7) means that the probability of a « flip », *i.e.* of non-best-response play is a continuous and decreasing function of the payoff loss caused by the flip whereas in KMR and Ellison the probability is independent of state and player.

Asynchronous Updating. Our subsequent analysis is based on asynchronous updating, *i.e.* only one player will be updating his state at a time. We can express this by means of an updating rule similar, but not identical to (7): The updating rule (7) is applied if $i = K(t)$ whereas $s_i(t+1) = s_i(t)$ if $i \neq K(t)$ where as before, $K(t)$, $t = 0, 1, 2, \dots$, is a sequence of I-valued random variables. $K(t)$ determines which of the players will have a chance to alter his strategy at time t. Again, the noise is modelled by means of a family $\varepsilon_i(t)$, $i = 1, \dots, N$; $t = 0, 1, 2, \dots$, of real-valued random variables. We shall proceed under the following further.

ASSUMPTIONS :

(I) The random variables $K(t)$ are independent and identically distributed (i.i.d.) with full support I.

(II) The random variables $\varepsilon_i(t)$ are independent (across i and t) and given any i, identically distributed.

(III) The player-picking process $\{K(t)\}$ and the noise process $\{\varepsilon_i(t)\}$ are independent.

(IV) The event $\{\varepsilon_i(t) > n \cdot (|w| + |\beta|)\}$ has positive probability for each pair (i, t).

(V) The event $\{\varepsilon_i(t) < -n \cdot (|w| + |\beta|)\}$ has positive probability for each pair (i, t).

Assumptions (I) – (III) guarantee that the dynamic process with asynchronous updating is a stationary Markov process on Σ^I whose transition matrix we denote by P. A generic entry $P(s'|s)$ of the transition matrix P is the probability that the next state is s', if the current state is s.

Assumptions (IV) and (V) guarantee that « flips » occur with positive probability regardless of the state of the system. Thus players are payoff-sensitive even when making mistakes. Consequently, the Markov chain generated by P is aperiodic and irreducible. Moreover, if the distributions $\varepsilon_i(t)$ are continuous, then the probability of a « flip », *i.e.* of non-best-response play is a continuous and decreasing function of the payoff loss caused by the flip.

Let us represent probability distributions on S by means of $|S|$-dimensional probability vectors $\rho = (\rho(s))_{s\in S}$. If ρ_t is the distribution of states at some time t, then the distribution at time $t + 1$ is given by

$$\rho_{t+1} = P \cdot \rho_t. \tag{8}$$

A distribution ρ is called an *invariant distribution* or a *stochastic steady state*, if

$$\rho = P \cdot \rho. \tag{9}$$

Now well known results for discrete time Markov processes with finite state space yield :

Proposition 3 - *If (I)-(V) hold, then the Markov chain has a unique invariant distribution ρ which has full support. Moreover, the distributions ρ_t defined recursively by (8) converge in distribution to ρ regardless of the initial distribution of states, ρ_0.*

Among the classical references for this sort of result are Doob (1953), Feller (1968), and Loève (1960). Grimmet and Stirzaker (1982) and Seneta (1981) provide easy access to the material.

VI. — DIGRESSION ON GIBBS-BOLTZMANN DISTRIBUTIONS

The invariant distribution, *i.e.* the solution of (9) is an eigenvector of P with eigenvalue 1. It can be found by means of routine, albeit cumbersome techniques. If, however, P is of a particular form, then the unique invariant distribution assumes the Gibbs-Boltzmann form (12) below whose explicit formula proves quite useful. Most of the literature on Gibbs-Boltzmann distributions is instructive, but rather sketchy and/or preoccupied with deriving certain properties from first principles of statistical mechanics. Therefore, we provide a brief, yet self-contained treatment of our own, following Haykin (1994, section 8.12) to some degree.

To begin with, let us assume that the transition process P on the finite state space S, specifically $S = \Sigma^I$ in our case, can be factored into two steps:

$$P(s'|s) = r(s'|s) \cdot q(s'|s) \text{ for } s \neq s' \qquad (10)$$

where $r(s'|s)$ is the probability of an opportunity for a transition from state s to state s' and $q(s'|s)$ is the probability of a transition conditional on the event that an opportunity arises. Further restrictions are:

- **Symmetry:** $r(s'|s) = r(s|s')$ for all $s \neq s'$
- **Normalization:** $\Sigma_{s' \neq s}\, r(s'|s) = 1$
- **Complementarity:** $q(s'|s) + q(s|s') = 1$ for $s \neq s'$

The key result is

Proposition 4 - Suppose that:

(a) *There is a unique invariant measure* ρ *that has full support.*

(b) *There is a function* $G : \Sigma^I \to \mathbb{R}$ *such that*

$$q(s'|s) = \frac{1}{1 + \exp(G(s) - G(s'))} = \frac{\exp(G(s'))}{\exp(G(s)) + \exp(G(s'))}\,. \qquad (11)$$

Then ρ *assumes the Gibbs-Boltzmann form:*

$$\rho(s) = \frac{\exp(G(s))}{\Sigma_{s'} \exp(G(s'))}\,. \qquad (12)$$

Proof. Suppose (a), (b) and that ρ is given by (12). Then by (11), a strong « detailed balance principle » holds:

$$q(s'|s)\rho(s) = q(s|s')\rho(s') \text{ for all } s \neq s'. \qquad (13)$$

As an immediate consequence of (13), (10), and symmetry, we obtain the usual « detailed balance principle » :

$$P(s'|s)\rho(s) = P(s\,|s')\rho(s') \text{ for all } s \neq s'. \tag{14}$$

But (14) combined with the fact that P is a stochastic matrix and the normalization condition on the probabilities of opportunity implies (9) : For any s', $\Sigma_s\, P(s'|s)\rho(s) = \Sigma_s\, P(s\,|s')\rho(s') = \rho(s')\; \Sigma_s\, P(s\,|s') = \rho(s')$, that is (9). By (a), the assertion follows.

Notice that because of (11), complementarity has been used implicitly in the proof. Also notice that (14) implies (13), in case $r(s'|s) > 0$ for all $s \neq s'$. The latter will not be the case in our application. But suppose that it is the case and that (14) can be assumed on *a priori* grounds. Then like in the literature, the order of crucial arguments can be reversed. In particular, (11) does not need to be assumed any longer, but is rather a consequence. Namely, first (14) implies (13). But then, by complementarity, (11) holds with $G(s) \equiv \ln(\rho(s))$:

$$q(s'|s) = \frac{1}{1 + \rho(s)\,/\rho(s')} = \frac{1}{1 + \exp(G(s) - G(s'))}$$

VII. — LOGISTIC NOISE

We now are ready to apply Proposition 4 in the game-theoretic context of Section 4. Throughout, we assume noisy best response dynamics with asynchronous updating and make the above assumptions (I) – (V). Moreover, we specialize and assume here that

(VI) each noise variable $\varepsilon_i(t)$ is a logistic random variable with zero mean and common scaling parameter T, *i.e.* the cumulative distribution function (c.d.f.) is given by :

$$\Pr[\varepsilon_i(t) \leq \varepsilon] = \frac{1}{1 + \exp[-\varepsilon/T\,]}\ , \forall\ i, t \tag{15}$$

One can prove :

Proposition 5 - *In a population of best response players, with the noise being distributed according to (15), the invariant distribution of strategies has the Gibbs-Boltzmann form (12) where $G(s) = H(s)/T$ and H is the potential function of the spatial game defined by (5).*

Proof. Let us verify that the system has the two-step factorization property. Let $s \neq s'$. A direct transition from s to s' requires that s and s' differ in exactly one coordinate, say the i-th. Then an « opportunity » for a transition from s to s' arises if and only if it is i's turn to move. Also, an « opportunity » for a transition from s' to s arises if and only if it is i's turn to move. Thus $r(s'|s) = r(s|s') = \text{Prob}(K(0) = i)$. In case s and s' differ in more than one coordinate, let us set $r(s'|s) = r(s|s') = 0$. Suppose that at time $t + 1$, the current state is s and player i has the opportunity to change s_i from 1 to 0, resulting in the new state s'. This transition only happens if

$$w \sum_{j \in V_i} s_j - n\beta + \varepsilon_i(t) \leq 0 \text{ or } \varepsilon_i(t) \leq n\beta - w \sum_{j \in V_i} s_j .$$

The probability of the latter event is

$$q(s'|s) = \frac{1}{1 + \exp[(w \sum_{j \in V_i} s_j - n\beta)/T]} = \frac{1}{1 + \exp[(H(s) - H(s'))/T]} .$$

Clearly, complementarity applies: If the current state is s' and player i has the opportunity to change s'_i from 0 to 1, then the probability of this happening is $q(s|s') = 1 - q(s'|s)$ and, therefore,

$$q(s|s') = \frac{1}{1 + \exp[(H(s') - H(s))/T]} . \tag{16}$$

Since s and s' were arbitrary, this covers all relevant contingencies. For the sake of completeness, we may extend the formula for $q(s'|s)$ to the case where s and s' differ in more than one coordinate. By Proposition 4, the assertion follows.

7.1. The Log-Linear Response Model

Here we uncover the connection between the neural network model with logistic noise and the log-linear response model. Young (1998, ch. 6) assumes $\Pr(\{K(0) = i\}) = 1/N$ for all i and shows that for the log-linear response model with parameter $1/T$, the invariant distribution is of the form

$$\rho(s) = \exp[R(s)/T] / \sum_{s'} \exp[R(s')/T] \tag{17}$$

where R is a particular potential function of the spatial game. Obviously, R in (17) can be replaced by any other potential function of the spatial game, for example H, but by none other than a potential function. Indeed, if R is any potential function of the spatial game and the noise is logistic with zero mean and scale parameter T, then (12) holds true for $G = R/T$. Moreover, the neural network model then gives rise to the log-linear response model with parameter $1/T$, as given by (6.4) in Young (1998). From Proposition 4, we know that (12) and, hence, (17) holds for arbitrary $\Pr(\{K(0) = i\}) > 0$. The explicit form (5) of the potential function H, which we found via the neural network approach, proves very useful for the determination of long-run equilibria.

Remark. Recent work by Baron *et al.* (2002a) establishes the Gibbs-Boltzmann form (12) with $G(s) = H(s)/T$ on the basis of a different set of first principles: The logit adjustment rule (11) is the solution of a maximization problem involving a trade-off between the magnitudes of trembles and control costs. The approach encompasses constituent games with more than two strategies. Baron *et al.* (2002b) extend some of the analysis to games which are not necessarily spatial games or potential games.

7.2. Long-Run Equilibria

The scaling parameter T that is controlling the noise distribution is frequently interpreted as temperature in a physical context. In a socio-economic context, one can interpret T as a macroeconomic parameter which determines the level of exogenous noise at the microeconomic level. This interpretation presumes that the neural network operates in a larger unspecified economic environment. A similar interpretation is less compelling for uniform noise models à la KMR.

In case $T \to \infty$, the c.d.f. (15) becomes flat and half of the mass is moved towards either tail. The noise becomes the sole driving force of the dynamics. Accordingly, $H(s)/T \to 0$ and in the limit, the invariant measure assignes equal probability to all states. The case $T \to 0$ shifts all the mass towards the mean of the noise distribution and, thus, constitutes a gradual removal of noise. The support of the resulting *limit distribution* ρ^* consists of the states $\bar{s}$ at which H is maximized, to which ρ^* assigns equal probabilities. The points in the support (carrier) of ρ^* have been called *stochastically stable states* by Foster and Young (1990) and *long-run equilibria* by KMR. It turns out that in general, the long-run equilibria form a subset of the steady states of the associated deterministic dynamics. For a concise formulation of the result, let us say that there are *ties*, if $w \sum_{j \in V_i} s_j - n\beta = 0$ for some $i \in I$ and $s \in \Sigma^N$. In such a case, the argument of $L(\cdot)$ in (6) is 0. Player i is indifferent between $s_i = 0$ and $s_i = 1$. The convention $L(0) = 0$ breaks the tie in favor of $s_i = 0$. We say further that H attains a *local maximum* at state s, if $H(s) \geq H(s')$ for all s' that differ from s in only one component.

Proposition 6 - *Suppose that there are no ties. Then for any state* $\bar{s} \in \Sigma^I$, *properties (i) and (ii) are equivalent and properties (iii) and (iv) are equivalent.*

(i) $\bar{s}$ *is a long-run equilibrium.*

(ii) $H(\bar{s})$ *is a maximum of H.*

(iii) $H(\bar{s})$ *is a local maximum of H.*

(iv) $\bar{s}$ *is a steady state of the deterministic dynamics.*

Remark. Statement (iv) allows for both synchronous and asynchronous updating. $\bar{s}$ is a steady state of the (synchronous or asynchronous) deterministic dynamics, if and only if none of the players wishes to deviate unilaterally from $\bar{s}$ when the opportunity arises. The only difference is that in the synchronous case the opportunity to deviate from $\bar{s}$ occurs, with certainty, for all players simultaneously whereas in the asynchronous case an opportunity to deviate from $\bar{s}$ occurs with positive probability for each of the players. As observed in Section 4, the coincidence of steady states does not mean that the dynamics are identical. With or without ties, the property that nobody wants to deviate unilaterally defines a Nash configuration in the sense of Blume (1993, 1995).

Proof. The equivalence of (i) and (ii) has already been established. $\bar{s}$ is a steady state of the (synchronous or asynchronous) deterministic dynamics, if and only if none of the players wishes to deviate unilaterally from $\bar{s}$ when the opportunity arises. If there are no ties, this is equivalent to $\bar{s}$ being a Nash configuration. Since *H* is a potential function of the spatial game, the latter is equivalent to *H* attaining a local maximum at $\bar{s}$. It follows that the conditions for a local maximum and a steady state coincide. This shows the equivalence of (iii) and (iv).

Corollary 1 - *Suppose that there are no ties. Then each long-run equilibrium is a steady state of the associated deterministic dynamics.*

Corollary 2 - *There exists a deterministic steady state.*

If there are ties, then not every local maximum of *H* is a deterministic steady state. Namely, the condition $L(0) = 0$ breaks ties in favor of 0. It therefore can happen that a transition from $s_i = 1$ to $s_i = 0$ does not affect the value of *H*. We can conclude, however, that every long-run equilibrium is a Nash configuration and a deterministic steady state with respect to some, possibly personalized, tie-breaking rule. Notice that the choice of tie-breaking rule has no impact on the stochastic dynamics, since there ties are zero probability events.

Connectivity of the interaction structure is not required for our results. Suppose for example that *N* is even, $N = 2Z$ for some large *Z*. Then a 1-regular interaction structure consists of *Z* pairs of players with interaction only within each pair. The long-run behavior of the system is the same as if we had tracked each pairwise interaction separately. Hence in a sense, it is a matter of convenience whether one studies the population game with 2*Z* players or *Z* population games with 2 players each. However, there is a short-run difference between the two models. The two dynamic processes progress at different speeds. Namely, for any pair $\{i, j\} \in V$, the chance that *i* or *j* is picked as the next mover from the entire population is less than one – in fact much less than that for most pairs. In contrast, in the 2-player population game with population $\{i, j\}$, one of them is picked with certainty. Hence *Z* parallel 2-player population games tend to move faster than the corresponding 2*Z*-player population game.

7.3. Long-Run Equilibria and Contagion

Haller and Outkin (1999) and Baron *et al.* (2000a) determine the long-run equilibria for various games and interaction structures. In particular, if the constituent bi-matrix game is a coordination game and has a risk dominant equilibrium (r, r), then the stochastic best-reply dynamics has a unique long-run equilibrium where all players choose action r. Now suppose that $r = 1$ is the risk dominant strategy. Does this mean that contagion occurs ? Not exactly. For the long-run equilibrium is the state in which the system stays most of the time (but not necessarily all the time) when very little, but still some noise remains. Therefore, very likely contagion occurs. But whenever contagion has occurred, there is a small chance that at some point a deviation happens, then very likely contagion occurs again, etc.

VIII. — INDUSTRIAL ECONOMICS : NETWORK FORMATION

Network formation can be analyzed within our formal setting or modifications thereof. Network formation means either creation of a graph (network) or formation of user networks. The first case, creation of a graph (network) is considered in Baron *et al.* (2006). In the second case, each user has to adopt one of a finite number of technologies or network goods, for instance computer systems, word processors, or internet providers, and the adopters of the same good constitute a user network. The community of Linux users is a specific example, where each user has to choose from among several partially compatible « distribution packages » : Debian, Fedora Core, Gentoo Linux, Knoppix, Mandriva, Slackware Linux, SUSE Linux, Ubuntu, etc. The individual choices give rise to separate sub-communities or networks of Debian users, Ubuntu users, and so on. With perfect incompatibility of technologies and identical users, the value of such a network is merely a function of its size, that is the number of its users. Kandori and Rob (1998) allow for the more realistic case of partial compatibility so that the value of a network is affected not only by its size. Otherwise, their evolutionary model is cast in the framework of population and spatial games and takes the « uniform error » approach like KMR and Ellison (1993).

Using the « logit error » approach, Baron *et al.* (2000b) also examine a model of user network formation. They find, among other things, that all long-run equilibria can be asymmetric (different technology choices by different people) even though the game has symmetric equilibria. In that case, occurrence of contagion is rather unlikely.

In principle, a little bit of persistent noise – the premise underlying the concept of long-run equilibrium – can break up path dependence and overcome lock-ins in technology choice. It suffices that the perturbed dynamical system is an irreducible Markov process with finite state space. Irreducibility means that the system passes in finitely many steps from any given state to any other state, with positive, perhaps very small probability. Almost certainly such a system will not remain in an inferior state forever nor will it stay in any other state. Rather it tends to visit every state from time to time. But some states may be visited much more frequently than others. While the system no longer gets locked into a particular state in a deterministic sense, it may be hooked to certain states, in a statistical sense. These are the stochastically stable states or long-run equilibria. Stochastic evolution (persistent noise) overcomes path dependence and dependence on initial conditions. But the run-long equilibria, the states where the system resides most of the time, can be inferior states. For instance, suppose the choice is between two partially compatible technologies, 0 and 1, represented by a coordination game with $a = 5$, $c = 4$, $d = 3$, $b = 1$. Then coordination on action 0 is payoff dominant whereas coordination on action 1 is risk dominant, and, therefore, constitutes the long-run equilibrium behavior.

IX. — CONCLUSION

The paper demonstrates both the power and the limitations of neural network theory when applied to best response dynamics. Propositions 4 and 5 hold for any regular interaction structure. A potential shortcoming shared with some of the most prominent alternative approaches, e.g. KMR's, is that the neural network theory applied here seems to require that each player makes binary choices. This theory also seems to require asynchronous updating (4). The homogeneity of the population assumed in the paper and the literature is convenient, but of limited appeal in socioeconomic contextes and, perhaps, not absolutely necessary. Conceivably each pair of player might have a pair-specific symmetric payoff matrix. Neighborhood membership can be fuzzy like in Young (1998, ch. 6) and Baron *et al.* (2000a) – which would allow to distinguish between strong and weak links, near and distant neighbors.

There are several, more or less related literatures we have barely touched upon. The potential fruitfulness of statistical mechanics approaches to socio-economic interaction has been demonstrated further by Föllmer (1974), Blume (1997), Durlauf (1993, 1997), among others. Automata have been used before for different game-theoretical modelling purposes: A small strand of literature, pioneered by Neyman (1985) and Rubinstein (1986) has used finite automata to model the complexity of strategies and bounded rationality in repeated games. Finally, instead of being rational, a player can follow other principles. Imitation is but one possibility. Imitation among humans is the premise underlying social learning theory [Bandura (1977)]. Define the « reference group » of a player as his neighborhood plus himself. Then imitation broadly defined means that tomorrow the player plays one of the strategies played in his reference group today. Following a rôle model is a very special case of imi-

(4) With respect to simulations, Huberman and Glance (1993) suggest that the order (synchronous versus asynchronous) of updating matters. See also Blume (1995).

tation. Berninghaus and Schwalbe have observed that majority imitation (following the majority in the reference group) can be modelled by means of neural networks. In contrast, best performance imitation cannot be modelled by means of neural networks. Best performance imitation, akin to fitness criteria in biology, has been studied by, among others, Nowak and May (1993) who find through simulation of Prisoner's Dilemma games that local interactions are dramatically different from global ones. Notice, however, the caveat by Huberman and Glance (1993) that the differences might disappear if simulations adhered to asynchronous rater than simultaneous updating. Subsequent theoretical analysis has been performed by Eshel, Samuelson, and Shaked (1998), Kirchkamp (2000), and Outkin (2003).

RÉFÉRENCES

BANDURA A. (1977) : « Social Learning Theory ». Englewood Cliffs, NJ : Prentice Hall.

BARON R., DURIEU J., HALLER H., and P. SOLAL (2002a) : « Control Costs and Potential Functions for Spatial Games », *International Journal of Game Theory*, 31 , 541-561.

BARON R., DURIEU J., HALLER H., and P. SOLAL (2002b) : « A Note on Control Costs and Logit Rules for Strategic Games », *Journal of Evolutionary Economics*, 12 , 563-575.

BARON R., DURIEU J., HALLER H., and P. SOLAL (2006) : « Complexity and Stochastic Evolution of Dyadic Networks », *Computers and Operations Research*, 33, 312-327.

BERGIN J., and B.-L. LIPMAN (1996), « Evolution with State-Dependent Mutations », *Econometrica*, 64, 943-956.

BERNINGHAUS S.-K., and U. SCHWALBE (1996a) : « Evolution, Interaction, and Nash Equilibria », *Journal of Economic Behavior and Organization*, 29, 57-85.

BERNINGHAUS S.-K., and U. SCHWALBE (1996b) : « Conventions, Local Interaction, and Automata Networks », *Journal of Evolutionary Economics*, 6, 297-312.

BINMORE K.-G., SAMUELSON L., and R. VAUGHAN (1995) : « Musical Chairs : Modeling Noisy Evolution », *Games and Economic Behavior*, 11, 1-35.

BLUME L.-E. (1993) : « The Statistical Mechanics of Strategic Interaction », *Games and Economic Behavior*, 5, 387-424.

BLUME L.-E. (1995) : « The Statistical Mechanics of Best-Response Strategy Revisions », *Games and Economic Behavior*, 11, 111-145.

BLUME L.-E. (1997) : « Population Games », in W.-B. Arthur, S.-N. Durlauf, and D.-A. Lane (eds.), «Economics as an Evolving Complex System II». Santa Fe Institute Proceedings Volume XXVII. Reading, MA : Addison-Wesley, pp. 425-460.

COUGHLIN J.-P., and R.-H. BARAN (1995) : « Neural Computation in Hopfield Networks and Boltzmann Machines ». Newark : University of Delaware Press.

DOOB J.-L. (1953) : « Stochastic Processes ». New York : John Wiley & Sons, Inc.

DURIEU J., HALLER H., and P. SOLAL (2005) : « Contagion and Dominating Sets », Mimeo.

DURLAUF S.-N. (1993) : « Nonergodic Economic Growth », *Review of Economic Studies*, 60, 349-366.

DURLAUF S.-N. (1997) : « Statistical Mechanics Approaches to Socioeconomic Behavior », *in* W.B. Arthur, S.N. Durlauf, D.A. Lane (eds.) : «The Economy as an Evolving Complex System II». A Proceedings Volume in the Santa Fe Institute Studies in the Sciences of Complexity. Reading, MA : Addison Wesley, pp. 81-104.

EICHBERGER J., HALLER H., and F. MILNE (1993) : « Naive Bayesian Learning *in* 2×2 Matrix Games », *Journal of Economic Behavior and Organization*, 22, 69-90.

ELLISON E. (1993) : « Learning, Local Interaction, and Coordination », *Econometrica*, 61, 1047-71.

ESHEL I., SAMUELSON L., and A. SHAKED (1998) : « Altruists, Egoists, and Hooligans in a Local Interaction Model », *American Economic Review*, 88, 157-179.

FELLER W. (1968): « An Introduction of Probability Theory and its Applications ». Vol. 1, third edition, revised printing. New York: John Wiley & Sons, Inc.
FÖLLMER H. (1974): « Random Economies with Many Interacting Agents », *Journal of Mathematical Economics*, 1, 51-62.
FOSTER D., and P. YOUNG (1990): « Stochastic Evolutionary Game Dynamics », *Theoretical Population Biology*, 38, 219-232.
FUDENBERG D., and C. HARRIS (1992): « Evolutionary Dynamics with Aggregate Shocks », *Journal of Economic Theory*, 57, 420-441.
GOLES E., and S. MARTINEZ (1990): « Neural and Automata Networks ». Dordrecht: Kluwer.
GOLES E., and S. MARTINEZ (eds.) (1992): « Statistical Physics, Automata Networks and Dynamical Systems ». Dordrecht: Kluwer.
GRIMMETT G.-R., and D.-R. STIRZAKER (1982): « Probability and Random Processes ». Oxford: Clarendon Press.
HALLER H.-H., and A.-V. OUTKIN (1999): « Best Response Dynamics and Neural Networks », Department of Economics, Virginia Polytechnic Institute and State University, Blacksburg, VA.
HAYKIN S. (1994): « Neural Networks: A Comprehensive Foundation ». New York: Macmillan College Publishing Company.
HUBERMAN B.-A., and N.-S. GLANCE (1993): « Evolutionary Games and Computer Simulations », *Proceedings of the National Academy of Sciences*, 90, 7716-7718.
KANDORI M., MAILATH G., and R. ROB (1993): « Learning, Mutation, and Long Run Equilibria in Games », *Econometrica*, 61, 29-56.
KANDORI M., and R. ROB (1998): « Bandwagons Effects and Long Run Technology Choice », *Games and Economic Behavior*, 22, 30-60.
KIRCHKAMP O. (2000): « Spatial Evolution of Automata in the Prisoners' Dilemma », *Journal of Economic Behavior and Organization*, 43, 239-262.
LOÈVE M. (1960): « Probability Theory ». Princeton, NJ: D. Van Nostrand Company, Inc.
LUCE R.-D., and P. SUPPES (1965): « Preference, Utility, and Subjective Utility », *in* R.-D. Luce, R.-R. Bush, and E. Galanter (eds.): *Handbook of Mathematical Psychology, III*, New York: Wiley, pp. 249-409.
MAES P. (1989): « How to do the Right Thing », *Connection Science*, 1, 291-323.
NEYMAN A. (1985): « Bounded Complexity Justifies Cooperation in the Finitely Repeated Prisoner's Dilemma », *Economics Letters*, 19, 227-229.
NOWAK M.-A., and R.-M. MAY (1992): « Evolutionary Games and Spatial Chaos », *Nature*, 359, 826-829.
NOWAK M.-A., and R.-M. MAY (1993): « The Spatial Dilemmas of Evolution », *International Journal of Bifurcation and Chaos*, 3, 35-78.
OUTKIN A.-V. (2003): « Cooperation and Local Interactions in the Prisoners' Dilemma Game », *Journal of Economic Behavior and Organization*, 52, 481-503.
RHODE P., and M. STEGEMAN (1966): « A Comment on `Learning, Evolution, and Long-Run Equilibria in Games'», *Econometrica*, 64, 443-449.
RUBINSTEIN A. (1986): « Finite Automata Play Repeated Prisoner's Dilemma », *Journal of Economic Theory*, 39, 83-96.
SAMUELSON L. (1997): « Evolutionary Games and Equilibrium Selection ». Cambridge and London: MIT Press.
SENATA E. (1981): « Non-Negative Matrices and Markov Chains ». Second edition. New York *et al.*: Springer-Verlag.
Van DAMME E. (1991): « Stability and Perfection of Nash Equilibria ». Second edition. New York: Springer-Verlag.
VEGA-REDONDO F. (1996): «Evolution, Games, and Economic Behaviour». Oxford: Oxford University Press.
WEIBULL J.-W. (1995): «Evolutionary Game Theory». Cambridge and London: MIT Press.
YOUNG H.-P. (1993): « The Evolution of Conventions », *Econometrica*, 61, 57-84.
YOUNG H.-P. (1998): «Individual Strategy and Social Institutions: An Evolutionary Theory of Institutions». Princeton: Princeton University Press.
YOUNG H.-P., and D. FOSTER (1991): « Cooperation in the Long-Run », *Games and Economic Behavior*, 3, 145-156.