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Reverse Mathematics in Bishop's Constructive Mathematics

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Abstract: We will overview the results in an informal approach to constructive reverse mathematics, that is reverse mathematics in Bishop's constructive mathematics, especially focusing on compactness properties and continuous properties.

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1 Introduction

The purpose of *constructive reverse mathematics* is to classify various theorems in intuitionistic, constructive recursive and classical mathematics by logical principles, function existence axioms and their combinations. Classifying mathematical theorems means finding logical principles and/or function existence axioms which are not only sufficient but also necessary to prove the theorems in a base system. We expect that constructive reverse mathematics will contribute to giving some insight into the philosophy of mathematics.

The Friedman-Simpson-program, called (classical) reverse mathematics [Friedman 1975][Simpson 1999][Tanaka 1992], is a formal mathematics using classical logic and assuming, in its base system, a very weak set existence axiom – the Δ^0_1 comprehension scheme. Its main question is "Which set existence axioms are needed to prove the theorems of ordinary mathematics?", and many theorems have been classified by set existence axioms of various strengths. Since classical reverse mathematics is formalized with classical logic, we cannot

- classify theorems in intuitionistic mathematics or in constructive recursive mathematics which are inconsistent with classical mathematics (for example, continuity of mappings from $\mathbf{N}^{\mathbf{N}}$ to $\mathbf{N}$ [Troelstra & van Dalen 1988, 4.6]);
- distinguish theorems from their contrapositions (for example, the fan theorem from weak König's lemma, see [Ishihara To appear, Theorem 16.17 and Theorem 16.21], and [Simpson 1999, IV.1]).

Bishop's constructive mathematics (BISH) [Bishop 1967][Bishop & Bridges 1985][Bridges & Richman 1987][Mines *et al.* 1988][Troelstra & van Dalen 1988] is an informal mathematics using intuitionistic logic and assuming some function existence axioms – the axiom of countable choice, the axiom of dependent choice, and the axiom of unique choice. It is a core of the varieties of mathematics in the sense that it can be extended not only to intuitionistic mathematics (INT) (by adding the principle of continuous choice and the fan theorem) [Bridges & Richman 1987][Brouwer 1981][Dummett 2000][Heyting 1971][Troelstra & van Dalen 1988] and constructive recursive mathematics (RUSS) (by adding Markov's principle and the Church's thesis) [Bridges & Richman 1987][Kushner 1984][Troelstra & van Dalen 1988], but also to classical mathematics (CLASS) practised by most mathematicians today (by adding the principle of the excluded middle and the full axiom of choice).

More on philosophy and practice of Bishop and his followers' constructive mathematics can be found in [Bishop 1970][Richman 1990][Beeson 1985].

Although nonconstructive logical principles such as the limited principle of omniscience (LPO) have been rejected within any constructive framework, some recent proofs in Bishop's (forward) mathematics [Ishihara 1994][Bridges van Dalen Ishihara 2003][Bridges, Ishihara *et al.* 2005][Spitters 2002] have made use of such nonconstructive principles, and many theorems in classical, intuitionistic and constructive recursive mathematics have been classified using such principles within the framework of Bishop's constructive mathematics [Ishihara 2004].

In this paper, we will overview the results in an informal approach to constructive reverse mathematics, that is reverse mathematics in Bishop's constructive mathematics, especially focusing on compactness properties and continuous properties; see also Mandelkern [Mandelkern 1988], Ishihara [Ishihara 1990], Bridges, Ishihara and Schuster [Bridges, Ishihara & Schuster 2002], Ishihara and Schuster [Ishihara & Schuster 2002], and Ishihara [Ishihara 2005] for compactness properties; Ishihara [Ishihara 1991, 1992], Bridges, Ishihara, Schuster and Vîţă [Bridges, Ishihara *et al.* 2005], and Bridges, Ishihara and Schuster [Bridges, Ishihara & Schuster 2002] for continuity properties.

Of course, since Bishop's constructive mathematics is informal, and assumes the function existence axioms, we cannot

- compare those results with the results in classical reverse mathematics (for example, the equivalence in Bishop's constructive mathematics between a function existence axiom, weak König's lemma, and a logical principle, the lesser limited principle of omniscience (LLPO) was proved in [Ishihara 1990]);

- prove neither underivability nor separability of those principles (for example, underivability of LPO and separability of the weak limited principle of omniscience (WLPO) from LPO, see [Akama, Berardi *et al.* 2004] and [Kohlenbach 2004, chapter 6] for such results in **HA** and $\mathbf{HA}^{\omega}$).

For a formal approach to constructive reverse mathematics, see [Ishihara 2005].

2 Bishop's Constructive Mathematics

Bishop's constructive mathematics is an informal mathematics using intuitionistic logic and assuming some function existence axioms: *the axiom of countable choice*

$$\forall n \in \mathbf{N} \exists x \in X A(n,x) \rightarrow \exists f \in X^{\mathbf{N}} \forall n \in \mathbf{N} A(n, f(n)),$$

the axiom of dependent choice

$$\forall x \in X \exists y \in X A(x,y) \rightarrow \forall x \in X \exists f \in X^{\mathbf{N}} (f(0) = x \wedge \forall n \in \mathbf{N} A(f(n), f(n+1))),$$

and *the axiom of unique choice*

$$\forall x \in X \exists ! y \in Y A(x,y) \rightarrow \exists f \in Y^X \forall x \in X A(x, f(x)).$$

Bishop's constructive (forward) mathematicians have been making every effort, for a given classical theorem A, to find its constructive substitute A' such that

$$\mathrm{BISH} \vdash A' \text{ and } \mathrm{CLASS} \vdash A \leftrightarrow A'.$$

In general, we may find more than one such A', say $A'_1, \ldots, A'_n$, and in this case, we try to find A'_k such that BISH $\vdash A'_k \rightarrow A'_i$ for all $i = 1, \ldots, n$. In some cases, we have to be contented ourselves with A' such that BISH $\vdash A'$, CLASS $\vdash A \rightarrow A'$, and it is strong enough for applications. Of course, it happens that sometimes we can take A' as A; for examples we can prove the following classical theorems in BISH.

The completeness of $\mathbf{R}$*: every Cauchy sequence of real numbers converges.*

The constructive compactness of $[0,1]$*: the closed unit interval* $[0,1]$ *is totally bounded and complete.*

When A and A' are not equivalent in BISH, we also try to show that A does not admit a constructive proof by giving a *Brouwerian counterexample* to A such that

$$\mathrm{BISH} \vdash A \rightarrow P \text{ and } \mathrm{BISH} \nvdash P$$

for some principle P. Since BISH is an informal mathematics, BISH $\nvdash P$ does not mean formal unprovability, but unacceptability, or at least high dubitation in BISH. The constructive compactness of $[0,1]$ is classically equivalent to the following special case of *the Bolzano-Weierstraß theorem*:

The sequential compactness of $[0,1]$*: every sequence of* $[0,1]$ *has a convergent subsequence.*

But it is well known that the sequential compactness of $[0,1]$ entails in BISH *the limited principle of omniscience* (LPO):

$$\forall \alpha \in \mathbf{N}^{\mathbf{N}}[\exists n(\alpha(n) \neq 0) \vee \neg \exists n(\alpha(n) \neq 0)],$$

which is an instance of *the Principle of the Excluded Middle* (PEM):

$$P \vee \neg P,$$

and false both in INT and in RUSS [Troelstra & van Dalen 1988, 4.6.4, 4.3.4].

Mandelkern [Mandelkern 1988] showed its converse and proved the equivalence between the Bolzano-Weierstraß theorem and LPO in BISH, which led the subsequent research of reverse mathematics in Bishop's constructive mathematics aiming at finding a logical principle P such that

$$\text{BISH} \vdash A \leftrightarrow P,$$

not only for a theorem A in CLASS but also for a theorem A in INT and in RUSS even if it is inconsistent with CLASS. This is possible because CLASS, INT and RUSS are extensions of BISH.

3 Omniscience Principles

The limited principle of omniscience (LPO) is a strong nonconstructive principle into which the following mathematical theorems are classified. A metric space is *compact* if it is totally bounded and complete, and *sequentially compact* if every sequence of its elements has a convergent subsequence; for further basic notions in metric spaces, see [Bishop 1967][Bishop & Bridges 1985][Bridges & Richman 1987][Troelstra & van Dalen 1988].

Theorem 1 *The following are equivalent in* BISH*.*

1. LPO.
2. $\forall x \in \mathbf{R}(0 < x \vee \neg(0 < x))$.
3. *The monotone convergence theorem [Mandelkern 1988]: every bounded monotone sequence of real numbers converges.*

4. *The Bolzano-Weierstraß theorem [Mandelkern 1988]: every bounded sequence of real numbers has a convergent subsequence.*
5. *The sequential compactness theorem [Ishihara & Schuster 2002]: every compact metric space is sequentially compact.*
6. *The Cantor intersection theorem [Richman 1999]: every sequence of closed sets of a compact metric space with the finite intersection property has nonempty intersection.*

A weaker nonconstructive principle is *the weak limited principle of omniscience* (WLPO):

$$\forall \alpha \in \mathbf{N}^{\mathbf{N}}[\neg\exists n(\alpha(n) \neq 0) \vee \neg\neg\exists n(\alpha(n) \neq 0)].$$

WLPO is an instance of *the weak Principle of the Excluded Middle:*

$$\neg P \vee \neg\neg P,$$

and false both in INT and in RUSS [Troelstra & van Dalen 1988, 4.6.4, 4.3.4].

We can show that the existence of a discontinuous function is equivalent to WLPO. A function f between metric spaces is *discontinuous* if there exist $\delta > 0$ and a sequence $\{x_n\}$ converging to a limit x such that $d(f(x_n), f(x)) \geq \delta$ for all n.

Theorem 2 *The following are equivalent in* BISH.

1. WLPO.
2. $\forall x \in \mathbf{R}(\neg(0 < x) \vee \neg\neg(0 < x))$.
3. *The existence of a discontinuous function [van Atten & van Dalen 2002]: a discontinuous function from* $\mathbf{N}^{\mathbf{N}}$ *into* $\mathbf{N}$ *exists.*

A weak continuity theorem is classified into the logical principle $\neg$WLPO. A function f between metric spaces is *nondiscontinuous* if $x_n \to x$ and $d(f(x_n), f(x)) \geq \delta$ for all n imply $\delta \leq 0$.

Theorem 3 *The following are equivalent in* BISH.

1. $\neg$WLPO.
2. *The nondiscontinuity theorem [Ishihara 1992]: every mapping of a complete metric space into a metric space is nondiscontinuous.*

The lesser limited principle of omniscience (LLPO):

$$\forall\alpha\beta \in \mathbf{N}^{\mathbf{N}}[\neg(\exists n(\alpha(n) \neq 0) \wedge \exists n(\beta(n) \neq 0)) \rightarrow \\ \neg\exists n(\alpha(n) \neq 0) \vee \neg\exists n(\beta(n) \neq 0)]$$

is weaker than WLPO, and is an instance of *De Morgan's law*

$$\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q.$$

It is false both in INT and in RUSS [Troelstra & van Dalen 1988, 4.6.5, 4.3.6].

The following mathematical theorems are classified into LLPO. A (binary) *tree* is a subset T of the set $\{0,1\}^*$ of finite binary sequences such that it is *detachable* from $\{0,1\}^*$ in the sense that for each $a \in \{0,1\}^*$, either $a \in T$ or $a \notin T$, and it is closed under restriction, that is if $a \in T$ and b is an initial segment of a, then $b \in T$. A tree T is *infinite* if for each n there exists $a \in T$ with length n, and an infinite binary sequence α is an *infinite path* in T if the finite initial segment $\overline{\alpha}(n)$ of α with length n belongs to T for all $n \in \mathbf{N}$. A subset S of a metric space X is a *zero set* if there is a pointwise continuous function $f : X \rightarrow \mathbf{R}$ such that $S = \{x \in X \mid f(x) = 0\}$.

Theorem 4 *The following are equivalent in* BISH.

1. LLPO.
2. $\forall x \in \mathbf{R}(\neg(0 < x) \vee \neg(x < 0))$.
3. $\forall xy \in \mathbf{R}(xy = 0 \rightarrow x = 0 \vee y = 0)$.
4. *For all $x, y \in \mathbf{R}$ with $\neg(x < y)$, $\{x, y\}$ is closed subset of $\mathbf{R}$ [Mandelkern 1988a].*
5. *Weak König's lemma (*WKL*) [Ishihara 1990]: every infinite tree has an infinite path.*
6. *The minimum principle [Ishihara 1990][Ishihara & Schuster 2002]: every uniformly continuous real function f on a compact metric space X attains its minimum, that is there exists x in X such that $f(x) \leq f(y)$ for all $y \in X$.*
7. *The Cantor intersection theorem for zero sets [Ishihara 1990][Ishihara & Schuster 2002]: every sequence of zero sets of a compact metric space with the finite intersection property has nonempty intersection.*

Among others, the intermediate value theorem in calculus and the Hahn-Banach theorem in functional analysis are also classified into LLPO [Troelstra & van Dalen 1988][Ishihara 1990].

4 Markov's Principle

Another weaker principle than LPO is *Markov's principle* (MP):

$$\forall \alpha \in \mathbf{N}^{\mathbf{N}}[\neg\neg\exists n(\alpha(n) \neq 0) \rightarrow \exists n(\alpha(n) \neq 0)],$$

which is an instance of *the double negation elimination*:

$$\neg\neg P \rightarrow P.$$

MP is rejected in INT, and accepted in RUSS [Bridges & Richman 1987][Troelstra & van Dalen 1988].

The strong extensionality theorem is classified into MP. A mapping f between metric spaces is *strongly extensional* if $f(x) \neq f(y)$ implies $x \neq y$.

Theorem 5 *The following are equivalent in* BISH.

1. MP.
2. $\forall x \in \mathbf{R}(\neg\neg(0 < x) \rightarrow 0 < x)$.
3. *The strong extensionality theorem [Bridges & Ishihara 1990]: every mapping between metric spaces is strongly extensional.*

A weaker principle than MP is *weak Markov's principle* (WMP):

$$\forall \alpha \in \mathbf{N}^{\mathbf{N}}[\forall \beta \in \mathbf{N}^{\mathbf{N}}(\neg\neg\exists n(\beta(n) \neq 0) \vee \neg\neg\exists n(\alpha(n) \neq 0 \wedge \beta(n) = 0)) \rightarrow \exists n(\alpha(n) \neq 0)],$$

which holds both in INT and in RUSS [Ishihara 1992][Mandelkern 1988a].

A sequential continuity theorem is classified into WMP. A mapping f between metric spaces is *sequentially continuous* if $x_n \rightarrow x$ implies $f(x_n) \rightarrow f(x)$.

Theorem 6 *The following are equivalent in* BISH.

1. WMP.
2. $\forall x \in \mathbf{R}[\forall y \in \mathbf{R}(\neg\neg(0 < y) \vee \neg\neg(y < x)) \rightarrow 0 < x]$.
3. *The strong extensionality theorem for complete spaces [Ishihara 1992]: every mapping from a complete metric space into a metric space is strongly extensional.*
4. *The sequential continuity theorem [Ishihara 1992]: every nondiscontinuous mapping from a complete metric space to a metric space is sequentially continuous.*

The second statement in the theorem was called the almost separating property (ASP) in [Mandelkern 1983], and the weak limited principle of existence (WLPE) in [Mandelkern 1988a].

Anothr principle weaker than MP is *the disjunctive version of Markov's principle* ($\mathrm{MP}^{\vee}$):

$$\forall \alpha\beta \in \mathbf{N}^{\mathbf{N}}[\neg(\neg\exists n(\alpha(n) \neq 0) \wedge \neg\exists n(\beta(n) \neq 0)) \rightarrow \neg\neg\exists n(\alpha(n) \neq 0) \vee \neg\neg\exists n(\beta(n) \neq 0)],$$

which is an instance of De Morgan's law. $\mathrm{MP}^{\vee}$ is rejected in INT, accepted in RUSS.

The following theorems are classified into $\mathrm{MP}^{\vee}$.

Theorem 7 *The following are equivalent in* BISH.

1. $\mathrm{MP}^{\vee}$.
2. $\forall x \in \mathbf{R}[\neg\neg(x \neq 0) \rightarrow \neg\neg(0 < x)) \vee \neg\neg(x < 0)]$.
3. *For all $x, y \in \mathbf{R}$ with $\neg\neg(x < y)$, $\{x, y\}$ is closed subset of $\mathbf{R}$ [Mandelkern 1988a].*

The second statement in the theorem was called LLPE in [Mandelkern 1988a].

5 Intuitionistic Principles

A subset B of $\{0,1\}^*$ is called a *bar* if for each infinite binary sequence α there exists $n \in \mathbf{N}$ such that $\overline{\alpha}n \in B$. A bar B is *uniform* if there exists k such that for each infinite binary sequence α, $\exists i \leq k(\overline{\alpha}(i) \in B)$. Brouwer's fan theorem for detachable bar FAN_{Δ} is stated as:

every detachable bar is uniform.

FAN_Δ is a contrapositive form of WKL, weaker than LLPO [Ishihara 1990] and hence than WKL [Ishihara To appear], accepted in INT, and false in RUSS [Troelstra & van Dalen 1988, 4.7.6].

A pointwise continuous function f from the Cantor space $\mathbf{2^N}$ into $\mathbf{N}$ is *representable* if there exists $\gamma : \{0,1\}^* \to \mathbf{N}$ such that

$$\forall \alpha \in \mathbf{2^N} \exists n [\forall k < n(\gamma(\overline{\alpha}(k)) = 0) \wedge \gamma(\overline{\alpha}(n)) = f(\alpha) + 1].$$

A subset S of a metric space X is a *cozero set* if there is a pointwise continuous function $f : X \to \mathbf{R}$ such that $S = \{x \in X \mid f(x) \neq 0\}$.

Theorem 8 *The following are equivalent in* BISH*.*

1. FAN_Δ.
2. *The uniform continuity theorem for representable functions [Veldman 2005][Loeb 2005]: every representable pointwise continuous function from* $\mathbf{2^N}$ *into* $\mathbf{N}$ *is uniformly continuous.*
3. *The Heine-Borel theorem for cozero sets [Ishihara & Schuster 2002]: every cover of a compact metric space by a sequence of cozero sets has a finite subcover.*

Equivalents of FAN_Δ, including versions of the Heine-Borel theorem, has been extensively studied in [Veldman 2005]. A relation between FAN_Δ and the uniformly continuity theorem for functions without representation, and other equivalents of FAN_Δ can be found in [Berger 2005] and [Julian & Richman 1984] [Berger, Bridges & Schuster 2004] [Berger & Ishihara 2005] [Berger & Schuster To appear], respectively.

Finally, we deal with a pointwise continuity theorem and *the boundedness principle* (BD-N):

every countable pseudobounded subset of $\mathbf{N}$ *is bounded,*

where a subset A of $\mathbf{N}$ is said to be *pseudobounded* if for each sequence $\{a_n\}$ in A, $a_n < n$ for all sufficiently large n. BD-N is weaker than LPO and provable both in INT and in RUSS [Ishihara 1992].

Theorem 9 *The following are equivalent in* BISH.

1. BD-N.
2. *The pointwise continuity theorem [Ishihara 1992]: every sequentially continuous mapping from a separable metric space into a metric space is pointwise continuous.*

Other equivalents of BD-N, including the open mapping theorem in functional analysis, can be found in [Bridges & Ishihara 1998][Ishihara 2001] [Bridges, Ishihara *et al.* 2005] [Ishihara & Yoshida 2002].

6 Relations Between Principles

The following relations hold between principles we have seen so far.

Proposition 10 *The following hold in* BISH.

1. $\mathrm{LPO} \leftrightarrow \mathrm{WLPO} + \mathrm{MP}$.
2. $\mathrm{WLPO} \rightarrow \mathrm{LLPO}$.
3. $\mathrm{MP} \leftrightarrow \mathrm{WMP} + \mathrm{MP}^{\vee}$ *[Mandelkern 1988a][Ishihara 1993].*
4. $\mathrm{LLPO} \rightarrow \mathrm{MP}^{\vee}$ *[Mandelkern 1988a][Ishihara 1993].*
5. $\mathrm{LLPO} \rightarrow \mathrm{FAN}_{\Delta}$ *[Ishihara 1990].*
6. $\mathrm{LPO} \rightarrow \mathrm{BD\text{-}N}$.

In the presence of an intuitionistic continuity principle, *weak continuity for numbers* (WC-N):

$$\forall \alpha \in \mathbf{N}^{\mathbf{N}} \exists n A(\alpha, n) \rightarrow \forall \alpha \in \mathbf{N}^{\mathbf{N}} \exists m n \forall \beta \in \mathbf{N}^{\mathbf{N}} (\overline{\alpha}(m) = \overline{\beta}(m) \rightarrow A(\beta, n)),$$

which is a weak form of the continuous choice, we have the following.

Proposition 11

1. $\mathrm{BISH} + \mathrm{WC\text{-}N} \vdash \neg\mathrm{LLPO}$ *[Troelstra & van Dalen 1988, 4.6.5].*
2. $\mathrm{BISH} + \mathrm{WC\text{-}N} \vdash \mathrm{WMP}$ *[Ishihara 1992].*
3. $\mathrm{BISH} + \mathrm{WC\text{-}N} \vdash \mathrm{BD\text{-}N}$ *[Ishihara 1992].*

On the other hand, with another intuitionistic principle, *Kripke's scheme* (KS):

$$\forall X \subset \mathbf{N} \exists \alpha \in \mathbf{N}^{\mathbf{N}^2} \forall n [n \in X \leftrightarrow \exists m (\alpha(n, m) = 0)],$$

MP entails PEM [Troelstra & van Dalen 1988,4.9.5], and hence LLPO. Thus we have the following.

Proposition 12 *If* BISH + FAN_Δ + WC-N + KS *is consistent, then*

1. BISH $\nvdash$ WMP $\rightarrow$ $\text{MP}^\vee$,
2. BISH $\nvdash$ $\text{FAN}_\Delta \rightarrow \text{MP}^\vee$,
3. BISH $\nvdash$ BD-N $\rightarrow$ $\text{MP}^\vee$.

Note that FAN_Δ + WC-N + KS is consistent relative to elementary analysis [Krol' 1978].

In the presence of Church's thesis (CT_0):

$$\forall n \exists m A(n, m) \rightarrow \exists k \forall n \exists m [A(n, U(m)) \wedge T(k, n, m)],$$

where T is Kleene's T-predicate and U the result-extracting function, we have the following.

Proposition 13

1. BISH + $\text{CT}_0 \vdash \neg\text{FAN}_\Delta$ *[Troelstra & van Dalen 1988, 4.7.6].*
2. BISH + $\text{CT}_0 \vdash$ WMP *[Ishihara 1993].*
3. BISH + CT_0 + MP $\vdash$ BD-N *[Ishihara 1992].*

Thus we also have the following.

Proposition 14 *If* BISH + CT_0 + MP *is consistent, then*

1. BISH $\nvdash$ MP $\rightarrow \text{FAN}_\Delta$,
2. BISH $\nvdash$ BD-N $\rightarrow \text{FAN}_\Delta$.

Note that CT_0 + MP is consistent relative to elementary analysis [Luckhardt 1977].

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