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Brouwer's Real Thesis on Bars

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ΣΩ. ...Νῦν δὲ ἤτοι οὐκ ἔστι ψευδὴς δόξα, ἤ ἅ τις οἶδεν, οἷόν τε μὴ εἰδέναι. Καὶ τούτων πότερα αἱρῆι;
ΘΕΑΙ. Ἄπορον αἵρεσιν προτίθης, ὦ Σώκρατες.

SOC. ...But as the matter now stands, either there is no such thing as false judgement; or a man may not know what he knows. Which do you choose?
THEAET. You are offering me an impossible choice, Socrates.
Plato, Theaetetus 196c

Abstract: L.E.J. Brouwer made a mistake in the formulation of his famous bar theorem, as was pointed out by S.C. Kleene. By repeating this mistake several times, Brouwer has caused confusion. We consider the assumption underlying his bar theorem, calling it *Brouwer's Thesis.* This assumption is not refuted by Kleene's example and we use it to obtain a conclusion different from Brouwer's. Thus we come to support a view first expressed and defended by E. Martino and P. Giaretta in [Martino 1981]. We also indicate that Brouwer's Thesis has many more applications than Brouwer dreamt of.

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1 Brouwer's Basic Assumption on Bars and its Consequences

We let $\mathbb{N}$ be the set of the natural numbers $0, 1, 2, \dots$ and use variables m, n for elements of this set. We let *Baire space* $\mathcal{N}$ be the set of all infinite sequences of natural numbers and use variables α, $\beta, \dots$ for elements of this set. We consider an infinite sequence of natural numbers as a function from $\mathbb{N}$ to $\mathbb{N}$. For every α in $\mathcal{N}$, for every n in $\mathbb{N}$, we denote the value of the function α in the argument n by $\alpha(n)$.

Like the sequence $0, 1, 2, \dots$ of the natural numbers itself, every infinite sequence $\alpha = \alpha(0), \alpha(1), \alpha(2), \dots$ of natural numbers is a growing and incomplete object, that is always under construction and never finished. It might be better to call infinite sequences *projects* rather than *objects*.

The sequence with the constant value 0 and the decimal expansion of π are examples of infinite sequences of natural numbers but the intuitionistic mathematician does not expect or require that every infinite sequence of natural numbers is given by a finite description or an algorithm that determines in advance what the values of the sequence will be. He admits the possibility of sequences $\alpha = \alpha(0), \alpha(1), \alpha(2), \dots$ that are created *step-by-step* and thus, in some sense, are given by a black box. He is very much aware that he is unable to make any kind of survey of the totality of all infinite sequences of natural numbers.

We suppose that some primitive recursive coding of finite sequences of natural numbers has been defined, that makes every natural number code exactly one finite sequence of natural numbers. For every k in $\mathbb{N}$, for every finite sequence $(n_0, n_1, \dots, n_{k-1})$ of natural numbers of length k, we denote the number coding this finite sequence by $\langle n_0, n_1, \dots, n_{k-1} \rangle$. For every s, we let $length(s)$ be the length of the finite sequence coded by s. For every s, for every $i < length(s)$, we let $s(i)$ be the value of the finite sequence coded by s at i. We let $\langle\ \rangle$ denote the *empty sequence*, the unique finite sequence of length 0.

We let $*$ denote the function corresponding to concatenation of finite sequences: for all s, t in $\mathbb{N}$, $s * t$ codes the finite sequence one obtains by putting the finite sequence coded by t behind the finite sequence coded by s.

For all s, t, we define: *s is an initial part of t* if and only if, for some u, $s * u = t$.

For all s, t, we define: s is *the immediate shortening of t* or: *t is* *an immediate prolongation of* s, if and only if there exists n such that $s * \langle n \rangle = t$.

For every α, for every n, we define $\overline{\alpha}n := \langle \alpha(0), \alpha(1), \ldots, \alpha(n-1) \rangle$.

For every α, for every s, we define: α *passes through* s, or: s *is an initial part of* α, if and only if, for some n, $s = \overline{\alpha}n$.

For every s, we let $\mathcal{N} \cap s$ be the set of all elements of $\mathcal{N}$ passing through s.

Now let B be a subset of $\mathbb{N}$. B is called a *bar in* $\mathcal{N}$ if and only if every infinite sequence α has an initial part in B.

More generally, let B be a subset of $\mathbb{N}$ and X a subset of $\mathcal{N}$. B is called a *bar in* X if and only if every infinite sequence α in X has an initial part in B.

Let s belong to $\mathbb{N}$. We say: "B *bars* s", or, in Brouwer's words: "s *is securable with respect to* B", if and only if B is a bar in $\mathcal{N} \cap s$. S.C. Kleene, in [Kleene 1965, 50], proposes to use the expression "*securable (with respect to B) in the explicit sense*" for this notion.

Suppose that B is a subset of $\mathbb{N}$ and a bar in $\mathcal{N}$. Brouwer reflected on the question how one may be sure that B is indeed a bar in $\mathcal{N}$. Note that we must convince ourselves that for *every possible* infinite sequence we can *effectively find* an initial part in B. He came to the following conclusion that we want to call *Brouwer's basic assumption* or *Brouwer's thesis on bars*:

> If B is a subset of $\mathbb{N}$ and a bar in $\mathcal{N}$, there exist a *proof* with the following properties:
>
> (i) the **conclusion** of the proof is the statement: B bars $\langle\ \rangle$,
>
> (ii) the **starting points** of the proof are steps of the following form, *immediate conclusions* from certain *basic facts*:
>
> s belongs to B, *therefore:* B bars s, and
>
> (iii) the proof uses only two kinds of reasoning steps:
>
> the **reasoning steps of the first kind** are of the following form:
>
> B bars $s * \langle 0 \rangle$, B bars $s * \langle 1 \rangle$, B bars $s * \langle 2 \rangle$, ..., *therefore:* B bars s, and
>
> the **reasoning steps of the second kind** are of the following form:
>
> B bars s, *therefore:* B bars $s * \langle 17 \rangle$.

In reasoning steps of the second kind 17 of course may be replaced by any other natural number. Note that reasoning steps of the first kind have infinitely many premisses. The proof thus has infinitary character, and, as Brouwer emphasizes, one should not think of the proof as a (finite) text on paper, but as a mental construction.

We shall call a proof with the above-mentioned properties a *canonical proof of the fact that B is a bar in $\mathcal{N}$.*

Brouwer seems to have been brought to his basic assumption by the conviction that, somehow, the truth of the statement: "*B is a bar in $\mathcal{N}$*" must be founded upon a well-ordered organization of the information available in the form of basic facts. (Basic facts are statements of the form: "*s belongs to B*".)

What use does Brouwer make of his basic assumption? Pictorially speaking, he takes *the skeleton* of the canonical proof, and employs it as a structure on which to append other proofs, isomorphic to the given one. Replacing the statements occurring in the canonical proof by other statements, obeying the same logical dependencies, he obtains new proofs and establishes new facts, as follows.

> Suppose that B is a subset of $\mathbb{N}$ and a bar in $\mathcal{N}$. Also assume that Q is a subset of $\mathbb{N}$ with the following properties:
>
> (i) B is a subset of Q,
> (ii) for every s, if, for all n, $s*\langle n\rangle$ belongs to Q, then s belongs to Q, and
> (iii) for every s, if s belongs to Q, then, for all n, $s*\langle n\rangle$ belongs to Q.
>
> We may conclude: $\langle\ \rangle$ belongs to Q.

The idea would be to replace, in a canonical proof of the fact that B is a bar in $\mathcal{N}$, every statement: "*B bars s*" by the statement: "*s belongs to Q*". The result would be another valid proof, this time with conclusion: "*$\langle\ \rangle$ belongs to Q*".

We may state Brouwer's idea more succinctly, as follows.

Let Q be a subset of $\mathbb{N}$. Q is called a *monotone* subset of $\mathbb{N}$ if and only if, for each s, for each n, if s belongs to Q, then $s*\langle n\rangle$ belongs to Q. Q is called an *inductive* subset of $\mathbb{N}$ if and only if, for each s, if, for each n, $s*\langle n\rangle$ belongs to Q, then s belongs to Q.

Let us introduce, for every subset B of $\mathbb{N}$, the set $Sec(B)$ as *the least monotone and inductive set Q containing B*, that is $Sec(B)$ itself contains B and is monotone and inductive, and for every subset Q of $\mathbb{N}$, if Q contains B and is monotone and inductive, then $Sec(B)$ is a subset of Q.

Brouwer probably would not have accepted the *impredicative* description of $Sec(B)$ we just gave as a definition, but, in an indirect way, he obviously accepts the existence of such a set as it may be defined as the set of all s for which we have a canonical proof that B is a bar in $\mathcal{N} \cap s$.

For practical purposes, therefore, Brouwer's basic assumption may be formulated as follows:

> For every subset B of $\mathbb{N}$, if B is a bar in $\mathcal{N}$, then $\langle\ \rangle$ belongs to $Sec(B)$.

Let us also introduce, for every subset B of $\mathbb{N}$, the set $Ind(B)$ as *the least inductive set containing* B. We may accept this definition for reasons similar to the ones that made us accept the definition of the set $Sec(B)$. Kleene, in [Kleene 1965, 50], uses the expression: "*s is securable (with respect to B) in the inductive sense*" for: "*s belongs to $Ind(B)$*". Kleene did so because of Brouwer's Footnote 7 in [Brouwer 1927]. As will become clear in the sequel of this paper, it would perhaps have been wiser to reserve the expression "*s is securable (with respect to B) in the inductive sense*" for "*s belongs to $Sec(B)$*".

We want to apply Brouwer's idea and draw a conclusion from his basic assumption. Our conclusion, however, is slightly different from Brouwer's own first conclusion.

We first have to give some further definitions.

We let *Cantor space* $\mathcal{C}$ be the set of all sequences α in $\mathcal{N}$ that assume no other value than 0 or 1.

For every α in $\mathcal{C}$, we let D_α, the set *decided by* α, be the set of all n in $\mathbb{N}$ such that $\alpha(n) = 1$.

For every α in $\mathcal{N}$, for every n, we let α^n be the element of $\mathcal{N}$ satisfying: for each m, $\alpha^n(m) = \alpha(\langle n \rangle * m)$.

We let $\underline{0}$ be the element of $\mathcal{N}$ with the constant value 0.

We now introduce the set of *stumps*. We borrowed the word "*stump*" from [Brouwer 1954] but are giving it a different meaning. Stumps are the characteristic functions of certain decidable subsets of $\mathbb{N}$, that is, they are elements of *Cantor space* $\mathcal{C}$. We define the set of the stumps inductively, as follows:

(i) $\underline{0}$ is a stump. Note that $\underline{0}$ is the characteristic function of the empty set. We sometimes call $\underline{0}$ the *empty stump*.
(ii) For every α in $\mathcal{C}$, if $\alpha(\langle\ \rangle) = 1$ and, for each n, α^n is a stump, then α is a stump.
(iii) Every stump is obtained from $\underline{0}$ by the repeated application of (ii).

We shall use $\sigma, \tau, \ldots$ as variables over stumps.

Note that we may decide, for every stump σ, if σ is the empty stump or not.

For every non-empty stump σ, we call the stumps $\sigma^0, \sigma^1, \ldots$ the *immediate substumps of* σ.

The following *principle of induction on the set of stumps* should be accepted as an axiom, once one decides to admit the set of stumps as a sensible totality:

Let P be a subset of the set of stumps. If the empty stump belongs to P and every non-empty stump σ with the property that each one of its immediate substumps $\sigma^0, \sigma^1, \ldots$ belongs to P, belongs itself to P, then every stump belongs to P.

Note that this principle of induction may be read as an impredicative characterization of the set of stumps: the set of stumps is the least subset P of $\mathcal{C}$ such that $\underline{0}$ belongs to P and, for every α in $\mathcal{C}$, if $\alpha(\langle\ \rangle) = 1$ and, for each n, α^n belongs to P, then α belongs to P.

We let 1^* be the element of $\mathcal{C}$ satisfying: for each s, $1^*(s) = 1$ if and only if $s = \langle\ \rangle$. Note that 1^* is a stump and that $D_{1^*} = \{\langle\ \rangle\}$.

For every subset X of $\mathbb{N}$, for every s, we let $s * X$ be the set of all numbers $s * t$, where t belongs to X.

Note that, for every non-empty stump σ, $D_\sigma = \{\langle\ \rangle\} \cup \bigcup\limits_{n \in \mathbb{N}} \langle n \rangle * D_{\sigma^n}$.

One may prove the following facts by transfinite induction on the set of stumps:

(i) For every stump σ, for every s, for every n, if $\sigma(s * \langle n \rangle) = 1$, then $\sigma(s) = 1$.
(ii) For every stump σ, for every α, there exists n such that $\sigma(\overline{\alpha} n) = 0$, that is, $\mathbb{N} \setminus D_\sigma$ is a bar in $\mathcal{N}$.

It is very useful, in the development of intuitionistic analysis, to have stumps available. Brouwer seems to have hesitated to take the decision to accept the set of stumps, or some other set, given by a similar inductive definition, as a sensible totality and to take the corresponding principle of induction as an axiom, but, in fact, handling notions like "*(inductively constructed) well-ordered sets*", or "*securable/versicherbar*" in [Brouwer 1927], he implicitly did so. The role of stumps in intuitionistic analysis is comparable to the rôle fulfilled by countable ordinals in classical analysis.

The *Second Axiom of Countable Choice*, an axiom that we want to use in the proof of the next theorem, is an accepted principle of intuitionistic mathematics, as it seems to follow naturally from the idea that infinite sequences of natural numbers are growing step by step. This axiom claims:

> For every subset R of $\mathbb{N}\times\mathcal{N}$, if, for each n, there exists α such that $nR\alpha$, then there exists α such that, for each n, $nR\alpha^n$.

In the formal systems developed in [Kleene 1965] and [Howard 1966], stumps do not occur. In these systems, a direct formulation of the following theorem is impossible.

Theorem 1.1: *(Improved Bar Theorem)*
Let B be a subset of $\mathbb{N}$ and a bar in $\mathcal{N}$.
There exists a stump σ such that $B \cap D_\sigma$ is a bar in $\mathcal{N}$.

Proof: Suppose that B is a subset of $\mathbb{N}$ and a bar in $\mathcal{N}$. Let Q be the set of all s in $\mathbb{N}$ such that, for some stump σ, for every α, there exists n such that $s * \overline{\alpha}n$ has an initial part in B and $\overline{\alpha}n$ belongs to D_σ.

Note that B is a subset of Q.

Note that Q is monotone: assume that s belongs to Q. Find a stump σ such that, for every α, there exists n such that $s * \overline{\alpha}n$ belongs to B and $\overline{\alpha}n$ belongs to D_σ. Observe that $\sigma(\langle\ \rangle) = 1$, that is, σ is not the empty stump. Let n be a natural number and consider σ^n. We have to distinguish two cases.

Case (i). σ^n is the empty stump. Note that s itself has an initial part in B. Recall that 1^* is a stump and that $D_{1^*} = \{\langle\ \rangle\}$. Also note that, for every α, there exists p such that $s * \langle n\rangle * \overline{\alpha}p$ has an initial part in B and $\overline{\alpha}p$ belongs to D_{1^*}.

Case (ii). σ^n is not the empty stump. Observe that, for every α, there exists p such that $s * \langle n\rangle * \overline{\alpha}p$ has an initial part in B and $\overline{\alpha}p$ belongs to D_{σ^n}.

Note that Q is inductive: assume that s belongs to $\mathbb{N}$ and, for each n, $s * \langle n\rangle$ belongs to Q. Using the Second Axiom of Countable Choice, we find σ in $\mathcal{C}$ such that $\sigma(\langle\ \rangle) = 1$ and, for each n, σ^n is a stump, and for every α, there exists p such that $s * \langle n \rangle * \overline{\alpha}p$ has an initial part in B and $\overline{\alpha}n$ belongs to D_{σ^n}. Note that σ is a stump and that, for every α, there exists n such that $s * \overline{\alpha}n$ has an initial part in B and $\overline{\alpha}n$ belongs to D_σ.

Using Brouwer's basic asumption, we conclude that $\langle\ \rangle$ belongs to Q, that is, there exists a stump σ such that $B \cap D_\sigma$ is a bar in $\mathcal{N}$. QED

Let σ be an element of $\mathcal{C}$ and a stump. We let σ', the *border of* σ, be the element of $\mathcal{C}$ such that $\sigma'(\langle\ \rangle) = 1$ if and only if $\sigma = \underline{0}$ and, for every s, for every n, $\sigma'(s * \langle n \rangle) = 1$ if and only if $\sigma(s) = 1$ and $\sigma(s * \langle n \rangle) = 0$.

Note that $D_{\sigma'}$ consists of the numbers coding a finite sequence that is *just outside* D_σ, that is, the sequence itself does not belong to D_σ, but its immediate shortening does.

We may give the following inductive characterization of all elements of $\mathcal{C}$ that are the border of a stump.

(i) 1^* is the border of a stump. Note that 1^* is the border of $\underline{0}$.
(ii) For every α in $\mathcal{C}$, if $\alpha(\langle\ \rangle) = 0$ and, for each n, α^n is the border of a stump, then α is the border of a stump.
(iii) Every element of $\mathcal{C}$ that is the border of a stump is obtained from 1^* by the repeated application of (ii).

Let X be a subset of $\mathbb{N}$. We say that X *coincides with the border of a stump* if and only if there exists a stump σ such that $X = D_{\sigma'}$. We say that X *contains the border of a stump* if and only if there exists a stump σ such that $D_{\sigma'}$ is a subset of X.

Lemma 1.2: *(On induction with respect to the border of a stump) For every stump σ, the code number of the empty sequence, $\langle\ \rangle$, belongs to $Ind(D_{\sigma'})$, that is, $\langle\ \rangle$ belongs to the least inductive set containing the border of σ.*

Proof: The proof is by straightforward transfinite induction on the set of stumps and left to the reader. QED

Theorem 1.3: *(On decidable and monotone bar induction)*
Let B be a subset of $\mathbb{N}$.
(i) For every stump σ, for every decidable subset B of $\mathbb{N}$, if $B \cap D_\sigma$ is a bar in $\mathcal{N}$, then the set of all s in B with the property that every proper initial part of s does not belong to B coincides with the border of a stump.

(ii) If B is a decidable subset of $\mathbb{N}$ and a bar in $\mathcal{N}$, then $\langle\ \rangle$ belongs to $Ind(B)$.
(iii) If B is a monotone subset of $\mathbb{N}$ and a bar in $\mathcal{N}$, then $\langle\ \rangle$ belongs to $Ind(B)$.

Proof: (i) is proved by transfinite induction on the set of stumps, as follows. Clearly, the statement is true if σ is the empty stump. Now assume that σ is a non-empty stump and that the statement holds for every one of its immediate substumps σ^n. We distinguish two cases.

Case (1). $\langle\ \rangle$ belongs to B. Note that the set of all s in B with the property that every proper initial part of s does not belong to B coincides with $\{\langle\ \rangle\}$, that is, with D_{1^*}.

Case (2). $\langle\ \rangle$ does not belong to B. Using the induction hypothesis we conclude that, for each n, the set of all s such that every proper initial part of $\langle n\rangle * s$ does not belong to B coincides with the border of a stump. It follows that the set of all s in B with the property that every proper initial part of s does not belong to B coincides with the border of a stump.
(ii) follows from (i), Theorem 1.1 and Lemma 1.2.
(iii) Using Theorem 1.1, find a stump σ such that $B \cap D_\sigma$ is a bar in $\mathcal{N}$. As B is monotone, the border $D_{\sigma'}$ of D_σ is a subset of B. Now use Lemma 1.2. QED

We want to give a second proof of Theorem 1.1. We want to show how one may avoid the use of the Second Axiom of Countable Choice, using instead the weaker *Second Axiom of Countable Unique Choice:*

> For every subset R of $\mathbb{N}\times\mathcal{N}$, if, for each n, there exists *exactly one* α such that $nR\alpha$, then there exists α such that, for each n, $nR\alpha^n$.

Let γ be an element of $\mathcal{N}$. γ *codes a continuous function from $\mathcal{N}$ to $\mathcal{N}$* if and only if, for each α, for each n, there exists p such that $\gamma(\langle n\rangle * \overline{\alpha}p) \neq 0$.

Suppose that γ codes a continuous function from $\mathcal{N}$ to $\mathcal{N}$. For every α we let $\gamma|\alpha$ be the element β of $\mathcal{N}$ such that, for each n, there exists p with the property $\gamma(\langle n\rangle*\overline{\alpha}p) = \beta(n)+1$ and, for all $i < p$, $\gamma(\langle n\rangle*\overline{\alpha}i) = 0$.

Lemma 1.4: *(On defining functions on a stump by recursion)*
For every stump σ, for every α, for every γ coding a continuous function from $\mathcal{N}$ to $\mathcal{N}$, there exists exactly one δ such that

(1) for every s in $D_{\sigma'}$, $\delta^s = \alpha^s$,

(2) for every s in D_σ, there exists β with the property $\delta^s = \gamma|\beta$ and, for each n, $\beta^n = \delta^{s*\langle n\rangle}$ and $\beta(\langle\ \rangle) = 0$, and

(3) for every s, if s does not belong to $D_\sigma \cup D_{\sigma'}$, then $\delta^s = \underline{0}$, and $\delta(\langle\ \rangle) = 0$.

Proof: We use induction on the set of stumps. Note that the statement holds for the empty stump. Assume that σ is a non-empty stump and that the statement holds for every one of its immediate substumps σ^n. Let γ, α belong to $\mathcal{N}$ and suppose that γ codes a continuous function from $\mathcal{N}$ to $\mathcal{N}$. Using the assumption, we conclude that, for every n, there exists exactly one δ such that

(1) for every s in $D_{(\sigma^n)'}$, $\delta^s = \alpha^{\langle n\rangle * s}$,

(2) for every s in D_{σ^n}, there exists β with the property $\delta^s = \gamma|\beta$ and, for each p, $\beta^p = \delta^{s*\langle p\rangle}$ and $\beta(\langle\ \rangle) = 0$, and

(3) for every s, if s does not belong to $D_{\sigma^n} \cup D_{(\sigma^n)'}$, then $\delta^s = \underline{0}$, and $\delta(\langle\ \rangle) = 0$.

Using the Second Axiom of Countable Unique Choice, conclude that there exists exactly one ε in $\mathcal{N}$ such that, for each n,

(1) for every s in $D_{(\sigma^n)'}$, $(\varepsilon^n)^s = \alpha^{\langle n\rangle * s}$,

(2) for every s in D_{σ^n}, there exists β with the property $(\varepsilon^n)^s = \gamma|\beta$ and, for each p, $\beta^p = (\varepsilon^n)^{s*\langle p\rangle}$ and $\beta(\langle\ \rangle) = 0$, and

(3) for every s, if s does not belong to $D_{\sigma^n} \cup D_{(\sigma^n)'}$, then $(\varepsilon^n)^s = \underline{0}$, and $\varepsilon^n(\langle\ \rangle) = 0$.

(4) $\varepsilon(\langle\ \rangle) = 0$.

We now let δ be the element of $\mathcal{N}$ satisfying

(1) for every s in $D_{(\sigma)'}$, $\delta^s = \alpha^s$,

(2) for every n, for every s, if $\langle n\rangle * s$ belongs to D_σ, then $\delta^{\langle n\rangle * s} = (\varepsilon^n)^s$, and there exists β such that $\delta^{\langle\ \rangle} = \gamma|\beta$, and, for each n, $\beta^n = (\varepsilon^n)^{\langle\ \rangle}$ and $\beta(\langle\ \rangle) = 0$, and

(3) for every s, if s does not belong to $D_\sigma \cup D_{\sigma'}$, then $\delta^s = \underline{0}$, and $\delta(\langle\ \rangle) = 0$.

It is easily seen that δ is the unique element of $\mathcal{N}$ satisfying the requirements. QED

We now formulate Brouwer's basic assumption more explicitly, by introducing canonical proofs as objects, in the following way.

Let B be a subset of $\mathbb{N}$. Let σ be a stump and let γ be an element of $\mathcal{N}$. We say that the pair (σ, γ) is a *canonical proof of the fact that B is a bar* if and only if the following conditions are fulfilled:

(i) $\gamma(\langle\ \rangle) = \langle\ \rangle$,
(ii) for every s, if s belongs to $D_{\sigma'}$, then $\gamma(s)$ belongs to B, and
(iii) for every s, *either:* for every n, $\gamma(s * \langle n \rangle) = \gamma(s) * \langle n \rangle$, *or:* $\gamma(s)$ has positive length and, for every n, $\gamma(s * \langle n \rangle)$ is the immediate shortening of s.

Alternative proof of Theorem 1.1: Let B be a subset of $\mathbb{N}$ and a bar in $\mathcal{N}$. Find a stump σ and a sequence γ such that (σ, γ) is a canonical proof of the fact that B is a bar. Using Lemma 1.4 we find δ in $\mathcal{N}$ such that (1) for every s in $D_{\sigma'}$, $\delta^s = 1^*$, and (2) for every s in D_σ, for all n, if $\gamma(s) = \gamma(s * \langle 0 \rangle) * \langle n \rangle$, then *either* $(\delta^{s*\langle 0 \rangle})^n$ is the empty stump and $\delta^s = 1^*$, *or* $(\delta^{s*\langle 0 \rangle})^n$ is not the empty stump and $\delta^s = (\delta^{s*\langle 0 \rangle})^n$, and (3) for every s in D_σ, if $\gamma(s * \langle 0 \rangle) = \gamma(s) * \langle 0 \rangle$, then $\delta^s(\langle\ \rangle) = 1$ and, for each n, $(\delta^s)^n = \delta^{s*\langle n \rangle}$.

Let Q be the set of all s such that, if s belongs to $D_\sigma \cup D_{\sigma'}$, then δ^s is a stump and, for every α, there exists n such that $\gamma(s) * \overline{\alpha}n$ has an initial part in B and $\overline{\alpha}n$ belongs to D_{δ^s}. Note that Q is inductive and contains $D_{\sigma'}$. By Lemma 1.2, it follows that $\langle\ \rangle$ belongs to Q, that is, $B \cap D_{\delta\langle\ \rangle}$ is a bar in $\mathcal{N}$. QED

For every subset B of $\mathbb{N}$, we let $Mon(B)$ the set of all s in $\mathbb{N}$ such that some initial part of s belongs to B.
The following theorem has been added upon a suggestion by the referee.

Theorem 1.5:
(i) For every subset B of $\mathbb{N}$, the sets $Sec(B)$ and $Ind(Mon(B))$ coincide.
(ii) For every monotone subset B of $\mathbb{N}$, the sets $Sec(B)$ and $Ind(B)$ coincide.
(iii) (E. Martino, P. Giaretta, see [Martino 1981]) Brouwer's basic assumption, that is, the statement: "For every subset B of $\mathbb{N}$, if B is a bar in $\mathcal{N}$, then $\langle\ \rangle$ belongs to $Sec(B)$" is equivalent to the principle of Monotone Bar Induction, that is, the statement: "For every monotone

subset B of $\mathbb{N}$, if B is a bar in $\mathcal{N}$, then $\langle\ \rangle$ belongs to $Ind(B)$".
(iv) For every decidable subset B of $\mathbb{N}$, the sets $Sec(B)$ and $Ind(B)$ coincide.

Proof: (i) Let B be a subset of $\mathbb{N}$.
As $Sec(B)$ is inductive and contains $Mon(B)$, and $Ind(Mon(B))$ is the least inductive subset of $\mathbb{N}$ containing *Mon(B)*, the set $Ind(Mon(B))$ is a subset of the set $Sec(B)$.
As to the converse, we claim that $Ind(Mon(B))$ is not only inductive but also monotone. In order to see this, let C be the set of all s in $\mathbb{N}$ such that, for every t, if s is an initial part of t, then t belongs to $Ind(Mon(B))$. Note that C contains $Mon(B)$ and is inductive and monotone, therefore $Ind(Mon(B))$ is a subset of C. It follows that the sets C and $Ind(Mon(B))$ coincide, and that $Ind(Mon(B))$ is both inductive and monotone. Therefore, $Sec(B)$ is a subset of $Ind(Mon(B))$.
(ii) and (iii) are easy consequences of (i).
(iv) Let B be a decidable subset of $\mathbb{N}$. Let C be the set of all s in $Ind(B)$ such that *either* s belongs to $Mon(B)$ *or* for all n, $s * \langle n \rangle$ belongs to $Ind(B)$. Note that C is inductive. (If s belongs to $\mathbb{N}$, and, for all n, $s * \langle n \rangle$ belongs to C, then we may decide: *either* s belongs to $Mon(B)$ *or*, for all n, either $s * \langle n \rangle$ belongs to B, or $s * \langle n \rangle$ belongs to $Ind(B)$, therefore, for all n, $s * \langle n \rangle$ belongs to $Ind(B)$, and s itself belongs to $Ind(B)$.) As C contains B, and $Ind(B)$ is the least inductive set containing B, the sets $Ind(B)$ and C coincide. Now observe that C is also monotone. Therefore, as $Sec(B)$ is the least monotone and inductive set containing B, $Sec(B)$ is a subset of $C = Ind(B)$. As $Sec(B)$ is inductive, the converse is also true. QED

Note that Theorem 1.5(ii) and (iv) offer an alternative proof of Theorem 1.3 (iii) and (ii).

Corollary 1.6:
Let B be a subset of $\mathbb{N}$ and a bar in $\mathcal{N}$. There exists a canonical proof of the fact that B is a bar in $\mathcal{N}$ with the property that the conclusion of a reasoning step of the first kind never functions as the premiss of a reasoning step of the second kind.

Proof: Let B be a subset of $\mathbb{N}$ and a bar in $\mathcal{N}$. Find a stump σ such that $D_\sigma \cap B$ is a bar in $\mathcal{N}$. Now construct a canonical proof that first proves, for every s in the border $D_{\sigma'}$ of D_σ, that s is securable with respect to B, and then concludes, using these facts as starting points and only reasoning steps of the first kind that $\langle\ \rangle$ is securable with respect to B.

One may also argue as follows. If B is a bar in $\mathcal{N}$, then $\langle\ \rangle$ belongs to $Sec(B)$ and thus, according to Theorem 1.5, to $Ind(Mon(B))$. QED

We might call a canonical proof with the property mentioned in Corollary 1.6 a *normal* canonical proof.

Let γ belong to $\mathcal{N}$. We let E_γ, the *set enumerated by* γ, be the set of all m in $\mathbb{N}$ such that, for some n, $\gamma(n) = m + 1$.

Let A be a subset of $\mathbb{N}$. A is called *enumerable* if and only if, for some γ, A coincides with E_γ.

The *First Axiom of Countable Choice* is a consequence of the Second Axiom of Countable Choice, and claims:

> For every subset R of $\mathbb{N} \times \mathbb{N}$, if, for each n, there exists m such that nRm, then there exists α such that, for each n, $nR\alpha(n)$.

Note that, in a constructive context, this axiom is less evident than in a classical context. We can not *define* $\alpha(n)$ as "*the least* m *such that* nRm" as, sometimes, we are unable to decide if $nR0$ is true or not.

Corollary 1.7:
Every bar in $\mathcal{N}$ *contains an enumerable subbar.*

Proof: Let B be a bar in $\mathcal{N}$. Using Theorem 1.1, find a stump σ such that $B \cap D_\sigma$ is a bar in $\mathcal{N}$. Using the First Axiom of Countable Choice, find α such that, for each s, if s does not belong to $D_{\sigma'}$, then $\alpha(s) = 0$, and, if s does belong to $D_{\sigma'}$, then there exists t in B such that t is an initial part of s and $\alpha(s) = t + 1$. Clearly, E_α is a subset of B and a bar in $\mathcal{N}$. QED

It is not true that every bar in $\mathcal{N}$ contains a decidable subbar, although Brouwer seems to have thought so, see Section 2.

2 Brouwer's Bar Theorem and Kleene's Counterexample

Intent upon proving his Uniform Continuity Theorem, Brouwer came to assert the following, see [Brouwer 1924], Theorem 1:

> Let R be a subset of $\mathcal{N} \times \mathbb{N}$. Suppose I am able to find, for every α in $\mathcal{N}$, a natural number n such that $\alpha R n$, that is, such that (α, n) belongs to R.
>
> If I am really able to do so, there must exist a subset B of $\mathbb{N}$ such that:
> (i) B is a decidable subset of $\mathbb{N}$,
> (ii) every s in B *is of sufficient information with respect to* R, that is, there exists n such that for every α passing through s, $\alpha R n$,
> (iii) B *is a bar in* $\mathcal{N}$, and
> (iv) B is *well-ordered*, that is, if we translate Brouwer's terminology into the terminology used in Section 1: B is the border of a stump.

A finite sequence should be seen as the beginning of an infinite sequence that is created step-by-step. If the code number of a finite sequence belongs to B, then the finite sequence contains sufficient information for determining a natural number that will suit every infinite sequence that is continuation of this finite beginning. Apparently, it is important one realizes that every infinite sequence, also an algorithmically given sequence, might be the result of a free step-by-step-construction.

Brouwer proves his assertion in two steps:

> *Step (1).* There exist a subset B of $\mathbb{N}$ with the properties (i), (ii), (iii). This statement has become known as *Brouwer's principle for numbers*, in [Kleene 1965], or *the strong axiom of continuity*, in [Howard 1966], or *the First Axiom of Continuous Choice*, in [Veldman 2001b].
> *Step (2).* Every subset B of $\mathbb{N}$ satisfying (iii) (*or should we understand: satisfying (iii)* ***and*** *(i)?*) contains the border of a stump.

Note that Theorem 1.3 implies that every set satisfying (i) and (iii) contains a set satisfying (i), (iii) and (iv).

It has been rightly emphasized by Mark van Atten, see [Atten 2004], that Brouwer, in [Brouwer 1924] and [Brouwer 1927], although dealing with a bar that he assumes to be decidable, does not make it clear where he uses this condition in his argument. In [Brouwer 1924] he seems to argue that reasoning steps of the second kind may be eliminated from the canonical proof, and indeed, in [Brouwer 1992], page 144, he explicitly says so. In [Brouwer 1927] he has a longer argument, using a more complicated terminology, but, at none of these places, he ever makes an appeal to the fact that B is a decidable subset of $\mathbb{N}$. Brouwer therefore may be suspected of believing that every set satisfying (iii) will have a subset satisfying (i), (iii) and (iv). And indeed, in [Brouwer 1954] and [Brouwer 1981], he explicitly claims, in his *bar theorem*, that every bar contains the border of a stump. Unfortunately, this is false, as has been observed by S.C. Kleene.

Theorem 2.1: *(Kleene's counterexample)*
There exists an enumerable subset B of $\mathbb{N}$ such that:
(i) B is a bar in $\mathcal{N}$,
(ii) we can not prove that there is a decidable subset C of $\mathbb{N}$ that is a subset of B and a bar in $\mathcal{N}$,
(iii) we can not prove that B contains the border of a stump, and
(iv) we can not prove that $\langle\ \rangle$ belongs to $Ind(B)$.

Proof: Let d be the decimal expansion of π. Let B be the set of all s such that, *either*, there exists n such that $s = \langle n \rangle$ and there is no $i \leq n$ such that for all $j < 99$, $d(i+j) = 9$, *or* $s = \langle\ \rangle$ and there exists i such that for all $j < 99$, $d(i+j) = 9$.
Let γ be an element of $\mathcal{N}$ such that for each n, if there is no $i \leq n$ such that for all $j < 99$, $d(i+j) = 9$, then $\gamma(n) = \langle n \rangle + 1$ and, if there exists i such that for all $j < 99$, $d(i+j) = 9$, then $\gamma(n) = \langle\ \rangle + 1$. Note that B coincides with E_γ, and thus is an enumerable subset of $\mathbb{N}$.
(i) B is a bar in $\mathcal{N}$: for every α, either there is no $i \leq \alpha(0)$ such that for all $j < 99$, $d(i+j) = 9$, and $\langle \alpha(0) \rangle$ belongs to B, or there exists such i and $\overline{\alpha}0 = \langle\ \rangle$ belongs to B.
(ii) Assume that C is a decidable subset of $\mathbb{N}$ and a subset of B and a bar in $\mathcal{N}$. We may distinguish two cases.

Case (i). $\langle\ \rangle$ belongs to C. Then $\langle\ \rangle$ also belongs to B and there exists i such that for all $j < 99$, $d(i+j) = 9$.

Case (ii). $\langle\ \rangle$ does not belong to C. Then, for each n, $\langle n \rangle$ belongs to C, and therefore to B, and there is no i such that for all $j < 99$, $d(i+j) = 9$.

In both cases, we end up with a statement for which we have no proof.
(iii) Note that every border of a stump is a decidable subset of $\mathbb{N}$ and a bar in $\mathcal{N}$ and use (ii).
(iv) Let Q be the set of all s such that, *either*, there exists n such that $s = \langle n \rangle$ and there is no $i \leq n$ such that for all $j < 99$, $d(i+j) = 9$, *or* $s = \langle\ \rangle$ and *either* there exists i such that for all $j < 99$, $d(i+j) = 9$, *or* there is no such i.

Q is an inductive set containing B but we have no proof that $\langle\ \rangle$ belongs to Q as the statement: "$\langle\ \rangle$ *belongs to* Q" is equivalent to the statement: "*either there exists* i *such that for all* $j < 99$, $d(i + j) = 9$, *or there is no such* i." QED

Generalizing the example given in the proof of Theorem 2.1, we see that the bar theorem, as formulated by Brouwer, entails the so-called *limited principle of omniscience:*

> For all α, either there exists i such that $\alpha(i) \neq 0$ or, for all i, $\alpha(i) = 0$.

It is well-known that the limited principle of omniscience, together with Brouwer's Continuity Principle, leads to a contradiction, see [Veldman 2001a], or [Kleene 1965, 87 (Theorem *27.23)].

Faced with Brouwer's apparent mistake, Kleene, in [Kleene 1965], decided to believe Brouwer's argument only for *effective, decidable* bars. He thus obtains the principle of induction on decidable bars, and derives the principle of induction on monotone bars by an application of *Brouwer's principle for numbers*, that is, the *strong axiom of continuity* or *First Axiom of Continuous Choice* mentioned before.

As appears from Section 1, and, indeed, already from [Martino 1981], it is possible to argue the plausibility of the principle of induction on monotone bars without invoking this axiom. This may be of some importance, also because of the fact that the continuity principles flatly contradict classical mathematics, and the principle of induction on monotone bars does not.

The view on Brouwer's Thesis on bars sketched in Section 1 has been advocated in [Veldman 1981] and [Veldman 1985].

Kleene's example does not refute Brouwer's basic assumption. Although Brouwer has been a bit clumsy in his attempts to exploit his basic assumption, this basic assumption remains, as Michael Dummett says, in [Dummett 2000], *of great interest*, not only from the philosophical, but also, as we want to explain in the next Section, from the mathematical point of view. We should observe here that Brouwer's mistake has brought Dummett into great trouble. In the first edition of [Dummett 2000] he made an attempt to explain the mistake which subsequently was shown to be unsuccessful in [Martino 1981]. In the second edition of [Dummett 2000], the necessary improvements have been made.
We also have to observe that the discussion of the bar theorem in the existing literature, for instance, in [Dalen 1999, 380 ff.], or in [Troelstra

1988], chapter IV, Section 8, or in [Atten 2004], Chapter 4, is, from the perspective sketched in this paper, not completely satisfying.

3 Some Applications

Brouwer's Thesis on bars has many applications in intuitionistic analysis, see [Veldman In-prep.]. We mention some examples.

(i) The Fan Theorem

A subset X of $\mathcal{N}$ is a *fan* or a *finitary spread* if and only if there exists β in $\mathcal{C}$ such that (i) for every s, $\beta(s) = 1$ if and only if, for some n, $\beta(s * \langle n \rangle) = 1$, and (ii) for every α, α belongs to X if and only if, for all n, $\beta(\overline{\alpha}n) = 1$, and (iii) for every s, the set of all n such that $\beta(s * \langle n \rangle) = 1$ is finite.

The Fan Theorem is the statement:

> *For every subset F of $\mathcal{N}$, for every subset B of $\mathbb{N}$, if F is a fan and B is a bar in F, then there exists a finite subset B' of B that is a bar in F.*

This is the so-called *unextended* Fan Theorem, Brouwer's first and only application of his bar theorem. In [Brouwer 1924], Brouwer is combining this result with the Second Axiom of Continuous Choice and he obtains a more general statement, the *Extended Fan Theorem.*

Our version does not put any restriction on the bar B. Kleene derives the case that B is a decidable subset of $\mathbb{N}$ from his principle of induction on decidable bars. He then has to apply a strong axiom of continuity in order to derive the unrestricted case from the decidable case.

Brouwer uses the Fan Theorem in order to prove that every continuous function from $[0, 1]$ to $\mathbb{R}$ is uniformly continuous. More information on the Fan Theorem may be found in [Veldman 2005b].

(ii) The Almost-Fan-Theorem

A decidable subset A of $\mathbb{N}$ is called *almost-finite* if and only if, for every strictly increasing sequence in $\mathcal{N}$ there exists n such that $\gamma(n)$ does not belong to A. Brouwer's Thesis plays a rôle in the study of this notion, see [Veldman 2005a].

A subset X of $\mathcal{N}$ is an *almost-fan* or an *almost-finitary spread* if and only if there exists β in $\mathcal{C}$ such that (i) for every s, $\beta(s) = 1$ if and only if, for some n, $\beta(s * \langle n \rangle) = 1$, and (ii) for every α, α belongs to X if and only if, for all n, $\beta(\overline{\alpha}n) = 1$, and (iii) for every s, the set of all n such that $\beta(s * \langle n \rangle) = 1$ is almost-finite.

The Almost-Fan-Theorem is the statement:

> *For every subset F of $\mathcal{N}$, for every subset B of $\mathbb{N}$, if F is an almost-fan and B is a bar in F, then there exists an almost-finite subset B' of B that is a bar in F.*

The Almost-Fan-Theorem implies the Fan Theorem and seems to be a stronger statement, see [Veldman 2001c], [Veldman 2001d] and [Veldman 2005b].

(iii) A characterization of the well-founded enumerable subsets of the set $\mathbb{Q}$ of the rational numbers.

Let $\mathbb{Q}$ be the set of the rational numbers and let $<$ denote the usual ordering.

For all subsets X, Y of $\mathbb{Q}$, we define: $X < Y$ if and only if, for all p in X, for all q in Y, $p < q$.

We define the class of $\mathcal{EWO}$ of the *explicitly well-ordered subsets of* $\mathbb{Q}$ by means of the following inductive definition:

> (i) The empty set $\emptyset$ belongs to $\mathcal{EWO}$ and, for every p in $\mathbb{Q}$, the singleton $\{p\}$ belongs to $\mathcal{EWO}$.
> (ii) For every sequence $X_0, X_1, X_2, \ldots$ of subsets of $\mathbb{Q}$, if, for each n, $X_n < X_{n+1}$ and X_n belongs to $\mathcal{EWO}$, then $\bigcup_{n \in \mathbb{N}} X_n$ belongs to $\mathcal{EWO}$.
> (iii) Every element of $\mathcal{EWO}$ is obtained from sets mentioned under (i) by the repeated application of the operation mentioned under (ii).

Using Brouwer's Thesis on bars one may prove, see [Veldman In-prep.],

> *For every enumerable subset A of $\mathbb{Q}$ the following two statements are equivalent:*
> *(i) For every function γ from $\mathbb{N}$ to A there exists n such that $\gamma(n) \leq \gamma(n+1)$.*
> *(ii) A belongs to $\mathcal{EWO}$.*

(iv) Some results in intuitionistic descriptive set theory.

A subset X of Baire space $\mathcal{N}$ is called *positively Borel* if it may be obtained from basic open subsets of $\mathcal{N}$ by means of the operations of countable union and countable intersection. A subset X of Baire space $\mathcal{N}$ is called *strictly analytic* if and only if there is a continuous function ϕ from $\mathcal{N}$ to $\mathcal{N}$ such that X coincides with the range of ϕ. A subset X of $\mathcal{N}$ is *co-analytic* if and only if there is an open subset Y of $\mathcal{N}$ with the property that X is the set of all α such that, for all β, $\langle \alpha, \beta \rangle$ belongs to Y. Here $\langle\ \rangle$ denotes a continuous one-to-one mapping of $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$. Using Brouwer's Thesis one may prove intuitionistic versions of some famous classical results, see [Lusin 1972], [Moschovakis 1980], [Kechris 1996], [Veldman 1981], and [Veldman 2001b].

> *(i) Suslin's Theorem: every subset of* $\mathbb{N}$ *that is both strictly analytic and co-analytic is positively Borel.*
> *(ii) The Lusin Separation Theorem: if X, Y are strictly analytic subsets of* $\mathbb{N}$ *such that every member of X is apart from every member of Y, then there are positively Borel subsets A, B of* $\mathbb{N}$ *such that every member of A is apart from every member of B, and X is a subset of A and Y is a subset of B.*
> *(iii) Every subset of* $\mathbb{N}$ *that is the range of a strictly one-to-one continuous function from* $\mathbb{N}$ *to* $\mathbb{N}$ *is positively Borel.*

A constructive version of the Lusin separation theorem is also studied in [Aczel To appear]. The importance of Brouwer's thinking on bars for some developments in (classical) descriptive set theory has been emphasized by Charles Parsons, see [Parsons 1967].

(v) Some Combinatorial Theorems.

Using Brouwer's Thesis one may prove intuitionistic versions of the Infinite Ramsey Theorem, and of Kruskal's Theorem, see [Veldman 1993] and [Veldman 2004].

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