Philosophia Scientiæ
Travaux d'histoire et de philosophie des sciences

15-1 | 2011
Hugh MacColl after One Hundred Years

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Electronic version
URL: http://journals.openedition.org/philosophiascientiae/362
DOI: 10.4000/philosophiascientiae.362
ISSN: 1775-4283

Publisher
Éditions Kimé

Printed version
Date of publication: 1 April 2011
Number of pages: 55-76
ISBN: 978-2-84174-551-7
ISSN: 1281-2463

Electronic reference
Francine F. Abeles and Amirouche Moktefi, « Hugh MacColl and Lewis Carroll: Crosscurrents in geometry and logic », *Philosophia Scientiæ* [Online], 15-1 | 2011, Online since 01 April 2014, connection on 19 April 2019. URL : http://journals.openedition.org/philosophiascientiae/362 ; DOI : 10.4000/philosophiascientiae.362

Hugh MacColl and Lewis Carroll: Crosscurrents in geometry and logic

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Résumé : Dans une lettre adressée à Bertrand Russell, le 17 mai 1905, Hugh MacColl raconte avoir abandonné l'étude de la logique après 1884, pendant près de treize ans, et explique que ce fut la lecture de l'ouvrage de Lewis Carroll, *Symbolic Logic* (1896), qui "ralluma le vieux feu qu'il croyait éteint". Dès lors, il publie de nombreux articles contenant certaines de ses innovations majeures en logique. L'objet de cet article est de discuter la familiarité de MacColl et son appréciation du travail de Carroll, et de comprendre comment cela l'amena à réinvestir le domaine de la logique. Rien n'indique que les deux hommes se soient jamais rencontrés ou aient échangé une correspondance personnelle. Cependant, MacColl recensa dans le journal *The Athenaeum* plusieurs ouvrages de Carroll, essentiellement en géométrie et en logique. Souvent, Carroll répondait aux critiques de MacColl dans les éditions suivantes de ses ouvrages. Une discussion de ces sources fournit des éclaircissements intéressants sur les investigations mathématiques et logiques de MacColl et Carroll.

Abstract: In a letter to Bertrand Russell, dated 17 May 1905, Hugh MacColl tells how he abandoned the study of logic after 1884 for about thirteen years and how it was the reading of Lewis Carroll's *Symbolic Logic* (1896) that "rekindled the old fire which [he] thought extinct." From then onwards, he published several papers containing some of his major logical innovations. The aim of this paper is to discuss MacColl's acquaintance with and appreciation of Carroll's work, and how that reading convinced him to come back into the realm of logic. There is no evidence that the two men ever met or exchanged any personal correspondence. However, MacColl reviewed in *The Athenaeum* several of Carroll's books, mainly on geometry and logic. Often, Carroll replied to MacColl's criticisms in the following editions of his works. A discussion of these sources provides interesting insights to MacColl's (and Carroll's) mathematical and logical investigations.

Philosophia Scientiæ, 15 (1), 2011, 55–76.

It is well known that Hugh MacColl (1837-1909) interrupted his logical investigations around 1884 for about thirteen years, a period which he mainly devoted to literature. In a letter to Bertrand Russell (1872-1970), dated 17 May 1905, MacColl recalls this break and tells how it was the reading of Lewis Carroll's *Symbolic Logic* in 1896 that led him to reinstate his logical studies:

> When, more than twenty-eight years ago, I discovered my Calculus of Limits [...], I regarded it at first as a purely mathematical system restricted to purely mathematical questions. [...] When I found that my method could be applied to purely logical questions unconnected with the integral calculus or with probability, I sent a second and a third paper to the *Mathematical Society,* which were both accepted, and also a paper to *Mind* (published January 1880). [...] I sent a fourth paper (in 1884) to the *Math. Soc.,* on the "Limits of Multiple Integrals," which was also accepted. This I thought would be my final contribution to logic or mathematics, and, for the next twelve or thirteen years, I devoted my leisure hours to general literature. Then a friend sent me Mr. Dodgson's ("Lewis Carroll's"[1]) *Symbolic Logic,* a perusal of which rekindled the old fire which I thought extinct. My articles since then I believe to be far more important from the point of view of general logic than my earlier ones. [Astroh, Grattan-Guinness & Read 2001, 93–94]

The aim of this paper is to discuss MacColl's acquaintance with and appreciation of Carroll's work, and how that reading convinced him to come back into the realm of logic. There is no evidence that the two men ever met or exchanged any personal correspondence. Both mathematicians worked in relative isolation on the margins of the mathematical scene. They shared an interest in educational issues and solving mathematical problems.[2] Their activities made them interact face to face, however, through the London journal, *The Athenaeum.* Indeed, MacColl acted as the main reviewer of mathematical books for that journal in the 1880s and 1890s.[3] As such he reviewed anony-

1. "Lewis Carroll" is the pen name of Charles L. Dodgson (1832-1898). To avoid confusion, in this paper we will use the pseudonym, Carroll, even when we are concerned with works signed with his real name. However, the works signed with the real name (Dodgson) and those signed with the pen name (Carroll) are listed separately in the bibliography.

2. Both were active contributors to the "mathematical questions and solutions" section of the *Educational Times* [Grattan-Guinness 1992]. At least once, they were concerned with the very same question in probability [Abeles 1994, 213–218]. Later on, MacColl replied to a logical problem published posthumously by Carroll [Abeles 2010, 184–185].

3. Reviews in *The Athenaeum* are anonymous. However, the editor's marked copies (now at City University Library, London) reveal the name of the reviewers. No survey of MacColl's reviews in this journal has yet been published. A look at the marked copies shows, however, that MacColl reviewed most mathematical books from the mid-1880s onwards. It might look surprising to see a minor mathematics

mously most of Carroll's mathematical books during that period, mainly on geometry and logic. Often, Carroll replied to MacColl's criticisms in the following editions of his works. A discussion of these connected sources provides interesting insights to both Carroll's and MacColl's mathematical and logical investigations. They also reveal influences on each other's thinking in several areas that they wrote about subsequently.

Carroll was the mathematical lecturer at Christ Church, Oxford. As such, he was greatly interested in geometrical teaching and participated actively in the debates of the time on the adequacy of Euclid's *Elements* as a manual for teaching purposes in schools and Colleges [Moktefi 2007b]. He argued for Euclid against the new rival textbooks, but didn't object to some changes in Euclid's text when appropriate, as long as the order of the theorems was not violated. In 1888, however, he published *A New Theory of Parallels,* a book where, as its title clearly indicates, he suggested a completely new Euclidean parallel axiom [Dodgson 1888]. Two more editions of the book appeared in the immediately subsequent years [Dodgson 1889 & 1891], and a fourth appeared later in 1895 [Dodgson 1895a]. MacColl reviewed the three first editions in *The Athenaeum.* In the following sections, we will first explore the dominant themes that attracted his attention when he read *A New Theory of Parallels*: the nature of axioms, the order in which axioms and theorems are used, the meaning and use of the terms 'infinity' and 'infinitesimal', the role of Archimedes's axiom in a geometry, and the relationship between Euclidean and non-Euclidean geometry.

It is important to keep in mind the time period when MacColl was reviewing Carroll's work on geometry (1888-1891). The non-Euclidean geometry of Nikolai Lobachevsky (1792-1856) had begun to be widely known soon after Jules Hoüel (1823-1886) published the first edition of his French translation of it in 1863. Beginning in 1882 with the work of Moritz Pasch (1843-1930) in his *Vorlesungen über neuere Geometrie,* and continuing with the work of David Hilbert (1862-1943) in his *Grundlagen der Geometrie* (1899) which was translated into French in 1900, and into English in 1902, the transition began to occur from the older view of an axiom as a generally accepted principle that is a self-evident truth to the modern view that an axiom is arbitrarily chosen and deals with indefinable concepts about arbitrarily chosen objects of thought. Thus, mathematics and formal logic each can be described as a science of pure thought.

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figure living in a foreign country act as the main mathematical reviewer for such an established journal, a task that was for instance previously undertaken by the London Professor of mathematics Augustus de Morgan (1806-1871) for over twenty years (1850s-1860s). Hugh MacColl might have been recommended by his brother Malcolm (1831-1907), who had previously reviewed some theological books for the journal. He might also have benefitted from a "natural" sympathy by the editor of the journal with whom he shared Scottish origins and even the name (although we know of no direct family relation between them), for Norman MacColl (1843-1904) was the editor of *The Athenaeum* from 1869 to 1901.

The nature of axioms and how they are used

We commonly consider that the theorems of a mathematical system are understood from its axioms and the logical machinery employed to deduce them. Beginning in the second half of the 19^{th} century the question arose of how an axiom is to be understood. One possibility was that we believe the axioms are true because some of the obvious theorems of the system can be deduced from them. Each axiom is expressing something that is true about its concepts and terms. We also trust the integrity of the mathematical system itself because the less obvious and non-obvious theorems can then be deduced from the axioms and from the obvious theorems. This is not really circular thinking, but more an assertion that a psychological element is involved in understanding an axiom. The second possibility follows the earlier and stricter Euclidean approach that only the terms and concepts in the axiom matter, so that we understand an axiom only when, after suitable reflection, we have a clear idea of the meaning of those terms and concepts, i. e. the axiom expresses a self-evident truth. How then are axioms selected? In any mathematical system there are some true theorems that are not dependent on others. Because these theorems involve fundamental truths that do not need to be proved, they become the candidates of choice for the axioms of the mathematical system. These fundamental truths are not necessarily self-evident except in the Euclidean approach to understanding an axiom, where they would have to be. [4]

In *A New Theory of Parallels,* Carroll chose as an equivalent to the Euclidean parallel axiom, an axiom, number 6, the last in his set of axioms, that states:

> In every Circle, the inscribed equilateral Hexagon is greater [in area] than any one of the Segments which lie outside it. [Dodgson 1888, Frontispiece]

Carroll used this axiom to prove Euc. I. 32 [Abeles 1995]. MacColl objected to this axiom on several grounds and continued to criticize it even after Carroll had modified it in response to his criticisms. In his review of the first edition, MacColl questioned Carroll's "implied assumption of the *possibility* of the inscribed equilateral hexagon," that he asserts Euclid did not demonstrate until Book IV, Proposition 15 of the *Elements* [MacColl 1888, 557].

MacColl clarified his objection to the axiom in his review of the second edition of this book a year later, after Carroll had responded specifically, but not by name, to his objection in the preface to this new edition. In that preface, in response to another reviewer, Carroll had added at the bottom of the page containing his offending axiom that "it is assumed to be *theoretically* possible to inscribe an equilateral Hexagon in a Circle" [Dodgson 1889, xi]. Then answering MacColl, Carroll questioned whether the possibility *needs* to be demonstrated:

4. For a penetrating analysis of self-evidence, see [Shapiro 2009].

> May we not assume (1) that the Magnitude 'four right angles' contains 6-6$^{\text{ths}}$ of itself; (2) that it is *theoretically* possible to draw radii dividing it into these 6-6$^{\text{ths}}$? Once grant me this, and I ask no more. I have then the logical *right* to join the ends of these radii, and to prove (by Euc. I. 4) that the chords are equal. [Dodgson 1889, xi]

But MacColl still was not satisfied with Carroll's statements in the preface to this second edition. He rejoined that Carroll's assumption conflicts with Euclidean practice and asks him to keep within Euclid's restrictions:

> We objected that it was not consistent with the spirit or practice of Euclid's reasoning to assume the "theoretical possibility" of a regular hexagon inscribed in a circle without first proving that such a figure could be actually constructed from his three postulates. Euclid's restrictions may be arbitrary, unnecessary, cramping, vexatious, absurd—indeed, we think they deserve these and many other epithets—but there they are, and if Mr. Dodgson accepts them he is bound to keep his assumptions within the boundaries which they prescribe. [MacColl 1889, 457]

The third edition of *A New Theory of Parallels* was published in 1890 and it was really an expanded and revised edition of the book. Carroll modified his problematic axiom, replacing the hexagon with a tetragon. In the preface to the new edition, he responded to MacColl quite sarcastically, when he described for the hexagon, the construction of an angle one sixth of four right angles in size, using Euc. I. 1 to construct an equilateral triangle; then using Euc. I. 9 to draw the bisectors of two of its angles; then post. 1 to join the points of intersection to the third vertex; then Euc. I. 4 to prove the three angles at the third vertex are equal; finally to conclude that each of these angles is one third, (and that therefore its half is one sixth) of four right angles [Dodgson 1890, xiii].

Besides objecting to Carroll's unnecessary sarcasm, MacColl remained unsatisfied. In his much longer review of the third edition he criticized Carroll's use of propositions, in addition to postulates, to construct the new figure, the square. He wrote:

> An *axiom* founded on *propositions*, however few and elementary, strikes us as somewhat of an anomaly. [MacColl 1891, 196]

Then he gives his own development "to find a simple substitute for Euclid's twelfth axiom" by exchanging its order with I. 30, then proving I. 29. "The twelfth axiom of Euclid could then be proved in place of his thirtieth proposition so that there would be hardly any disturbance in the accepted arrangement of matter " [MacColl 1891, 196–197]. By way of explanation, Euclid's fifth postulate, the parallel postulate, came to be known in England as the twelfth axiom from the well regarded and popular edition of the *Elements* by the Scottish mathematician, Robert Simson (1687-1768), which had first appeared in 1756. Simson's text contains three postulates and twelve axioms. Of the first ten axioms in the edition reproduced in the 19^{th} century by Isaac Todhunter (1820-1884), only five can be attributed to Euclid. And of the last two of the twelve axioms, Simson's twelfth axiom was Euclid's fifth postulate.

Alluding to Carroll's sarcastic comments about his criticisms, MacColl went on to say that, "Postulates and axioms are *permissions* to do or to assume certain things, and are therefore, the opposite of restrictions. The restrictions of Euclid arise from the paucity of his postulates and axioms, a paucity which [he thinks] regrettable" [MacColl 1891, 197]. It seems quite clear that MacColl is a strict Euclidean in that he believes that an axiom should express a fundamental self-evident truth. Carroll, on the other hand, believes that a psychological element is involved in knowing an axiom as he explained in the preface to the first edition of his book:

> I shall be told, no doubt, that this is too *bizarre* and unprecedented an Axiom—that it is an appeal to the *eye* and not to the reason. That it is somewhat *bizarre* I am willing to admit—and am by no means sure that this is not rather a *merit* than a defect. But, as to its being an appeal to the *eye,* what is 'two straight Lines cannot enclose a space' [a fact included in the meaning of postulate 1: to draw a straight line from any point to any point] but an appeal to the eye? What is 'all right angles are equal' [postulate 4] but an appeal to the *eye*? [Dodgson 1888, xv]

What exactly did Carroll have in mind when he created his axiom 6? First, it appears that he selected a hexagon because of all the Euclidean polygonal constructions that of a regular hexagon is the simplest. [5] But there appears to be another reason. In hyperbolic geometry there is an upper limit to the area of a triangle, but there is no upper limit to the lengths of its sides. Since a polygon can be triangulated and its area equals the sum of the areas of these triangles, eventually the sides of the hexagon will become very long while its area will reach a maximum. In a sufficiently large circle the area of a segment will exceed the area of the inscribed hexagon. From the center of the circle circumscribing the hexagon, the hexagon has a unique triangulation while for an inscribed square (which does not exist in hyperbolic geometry) there are two such triangulations. As he indicated in a diary entry dated 8 April 1888, he was initially considering a different formulation for his axiom:

> My Axiom has taken many forms, the last a discovery of this evening, "if an equilateral triangle be inscribed in a circle, it cannot be made to lie partly within it, with each vertex outside, and the circle cutting each side in two points." [Wakeling 2004, 390]

5. The construction of a square does not occur until Euc. 1. 46. And unlike the square, the hexagon construction does not require the parallel postulate.

He alluded to this argument in Appendix IV of the first edition of *A New Theory of Parallels* where he provided greater detail, calling it a Will-o'-the-Wisp (but omitted it in the third edition). In the end he settled on the inscribed hexagon formulation but concluded that:

> *[A]ny* polygon will do, and *any* ratio between it and the out-lying Segment, so long as it is a *finite* ratio. [Dodgson 1888, 61; 1890, 74–75]

MacColl may have been influenced by his exchange with Carroll on the nature of axioms. This becomes apparent in his late writings on geometry, particularly an article written at the end of 1909 and published posthumously in *Mind* the following year. In section II, devoted to "Axioms, Inference, Implication", MacColl wrote:

> What is an axiom? No clear line of demarcation can be drawn between an axiom and any other general proposition or formula that is known and admitted to be true [...] A proposition that may appear axiomatic to one person may appear doubtful to another, until he has obtained a satisfactory proof of it; after which he treats it as an axiom in all subsequent researches [...] To an omniscient mind would not all true propositions be equally axiomatic? Would it not be absurd to speak of such a mind as *inferring* B from A? This, of course, is an extreme case, but such cases are precisely those that most effectively test the validity of a principle. [MacColl 1910, 193]

This passage suggests that MacColl has given up his belief that an axiom must express a fundamental self-evident truth, and he has accepted the idea that people differ in what they view as axioms as opposed to propositions. MacColl goes on to say that, "strict Euclideans consider no proof valid [...] if it takes anything for granted that is not founded on Euclid's twelve axioms, although as a matter of fact, Euclid himself, in several of his formal proofs, tacitly assumes axioms which are absent from his given list" [MacColl 1910, 193]. These are the *restrictions* we referred to earlier (that MacColl had alluded to in his last review of *A New Theory of Parallels*). Further evidence that MacColl may have been influenced by Carroll's views is that in the same paper he describes axioms as "fundamental formulae of appeal" [MacColl 1910, 196], a term difficult to understand except when cast in the light of Carroll's statement that his axiom is an appeal to the eye rather than to the mind. These are the *permissions* that MacColl had also alluded to in this same last review.

Infinity, infinitesimal, the Archimedean axiom, and non-Euclidean geometry

It is interesting to note, however, that when MacColl reviewed the three editions of *A New Theory of Parallels* he did not analyze Carroll's unusual formulation of the Euclidean parallel axiom nor did he include any mention of the appendices in that book where Carroll dealt with infinities and infinitesimals. For the superiority of his axiom, Carroll claimed:

> While *every* other Theory (that I have seen), which attempts to supersede Euclid's 12$^{\text{th}}$ [parallel] Axiom, introduces the ideas of Infinities and Infinitesimals, *mine* dispenses *wholly* with their aid, and deals with nothing but what is, by universal consent, absolutely *within* the field of Human Reason. [Dodgson 1890, xiv]

Although he was silent here, MacColl discussed the issue of infinity and infinitesimal in a subsequent review of Russell's *Essay on the Foundations of Geometry* (1897) that appeared in *The Athenaeum* in January 1898. Russell discussed the validity of non-Euclidean geometries whose principal writers he named as Lobachevsky, Carl Friedrich Gauss (1777-1855), Bernhard Riemann (1826-1866), Hermann von Helmholtz (1821-1894) and Arthur Cayley (1821-1895) [Russell 1897]. MacColl identified as the key issue of the book, Russell's belief that certain axioms, particularly the parallel axiom, should be "regarded as empirical laws, derived from the investigation and measurement of our actual space, and true only [...] within the limits set by errors of observation" [MacColl 1898, 25]. MacColl, asserting that Russell's view is based on his erroneous notion of an infinitesimal, writes:

> Those who assert the possibility of discovering error in Euclid's axioms by a conceivably more refined method of measurement forget (as Mr. Russell appears to do) [...] that the infinitesimal dx is never synonymous with *zero,* that it is wholly relative, and has no fixed and absolute magnitude. The infinitesimal dx has simply any magnitude, great or small, we choose to assign to it, consistently with the limitation that the variable x, with which we compare it is (when not zero) *infinite in comparison.* The infinitesimal dx may, in fact [...] have any value, from the millionth part of the diameter of an hypothetical atom to a million times the distance of the farthest star that will ever become visible through the most powerful telescope. [MacColl 1898, 25]

MacColl goes on to say that Russell's "thoughts may be correct, but the words in which he expresses them border upon self-contradiction. [...] Whence come these contradictions? [...] they arise solely from the unnecessary ambiguity created by confounding the 'infinitesimal' with 'zero"' [MacColl 1898, 26]. However, MacColl's notions about infinitesimals and infinities were somewhat odd and sometimes incorrect. He had discussed these ideas in "The calculus of limits" [MacColl 1904], and again in "Symbolic Reasoning (VIII)" (1906) where, for example, he asserted that "the symbol 0 represents [...] *the death of a real infinitesimal ratio h* in passing from the positive to the negative state" [MacColl 1906b, 506].

In contrast to Russell, Carroll's view of axioms is quite different. In the *Supplement to Euclid and His Modern Rivals* from 1885, Carroll gave what amounts to his definition of an axiom: " [It] always requests the *voluntary* assent of the reader to some truth, for which no proof is offered, and which he is not logically compelled to grant" [Dodgson 1885, 351]. He goes on to say that accepting any axiom that may be proposed "largely depends on the amount of truth he [the reader] has already grasped in connection with it" [Dodgson 1885, 351]. So, Carroll does not subscribe to the idea that they are empirical laws subject to verification, but rather that they must be acceptable to the faculties of reason which, by its nature, cannot apprehend infinities and infinitesimals. MacColl seems to be more in agreement with Carroll's view. It was primarily for this purpose that Carroll chose his axiom 6 as a substitute for Euclid's parallel axiom. It is a closed form that deals with finite area and not with lines, and is unique in this respect.

In view of MacColl's and Carroll's lack of clear and correct ideas about the nature of infinitesimals, it is remarkable that Carroll arrived at the role the Archimedean axiom plays in a geometry. Euclid stated this axiom as the first proposition in Book X. Using Euclid's definitions of ratio, i. e. relative magnitude, which are numbers 3 and 4 in Book V, Carroll reasoned correctly that definition 4 was meant to exclude the relation of a finite magnitude to an infinitely great or infinitely small one, where both are magnitudes of the same kind. And he saw that Euclid employed this definition in his proof of X. 1. So, Carroll was one of the few mathematicians of his time to understand that the Archimedean axiom excludes infinitesimals, and by implication that hyperbolic geometry, because it admits infinitesimals, is necessarily non-Archimedean. Carroll wrote:

> My conclusion, then, is that, in Book X. Prop. 1, Euclid is limiting his view to the case of two homogeneous Magnitudes *which are such that neither of them is infinitely greater than the other* [...] that he is contemplating *Finite Magnitudes only.* [Dodgson 1890, 42]

The importance of Carroll's discovery became apparent in the independent work of Giuseppe Veronese (1854-1917). Just as one can exclude the parallel axiom from the rest of the axioms of Euclidean geometry to investigate what other geometries are possible, so one can exclude the Archimedean axiom and obtain the non-Archimedean geometries of which Veronese's 1890 geometry was the first one. [6]

However, Carroll's discussion of area in the context of non-finite objects like strips and sectors was largely incorrect. He made comparisons between finite and infinite areas that led to paradoxical, actually fallacious results, because he handled infinities as if they were magnitudes having a definite size. He based his understanding of the infinite and the infinitesimal primarily on

6. Pasch was probably the first mathematician to fully understand the significance of Archimedes's axiom [Schlimm 2010]. Russell had discussed Pasch's book in his *Principles of Mathematics* [Russell 1903, 393–403], but it is doubtful that MacColl read this book.

the work of the Swiss mathematician Louis Bertrand (1731-1812) who used the notion of infinite area [Bertrand 1778]. Carroll came to know Bertrand's work because Adrien-Marie Legendre (1752-1833) used it in an attempt to prove Euclid's parallel axiom. In an unpublished letter, dated 2 April 1878, to the Oxford mathematician Wallis Hay Laverty (1847-1928), Carroll discussed Legendre's proof [Dodgson 1878].

In spite of their disagreements, Carroll and MacColl were both Euclideans. MacColl stated clearly his strong preference for Euclidean geometry with an implicit rejection of hyperbolic non-Euclidean geometry when he wrote in his 1910 paper that:

> The Euclidean geometry seems to me to be the only true one, not merely because it is admittedly the simplest and most convenient, but also, and chiefly, because it is the only system that frankly accepts the customary conventions of ordinary language. [MacColl 1910, 188]

MacColl had already alluded to this belief in his review of Russell's *Essay on the Foundations of Geometry* when he wrote that the non-Euclideans' "so-called straight lines are not straight lines in the ordinary sense, but two great circles of an infinite sphere, or some other curves of infinitesimal (but not quite zero) curvature [...] his familiarity [...] with their peculiar ways of thinking leads him occasionally [...] into inconsistency of language, if not into positive error" [MacColl 1898, 26]. Carroll's rejection of hyperbolic non-Euclidean geometry was consistent with his conception of the nature of human reasoning, i. e. the impossibility of picturing two parallel lines produced to infinity. So we see that both Carroll and MacColl agreed in their negative assessments of non-Euclidean geometry, but not entirely for the same reasons.

The barbershop problem

Despite their exchanges in print on geometry, it seems that Carroll and MacColl didn't know each other. Indeed, the barbershop controversy that occurred in the British logical scene in the subsequent years, with no participation by MacColl at its beginning, suggests that MacColl was not personally acquainted at all with Carroll or with the British logicians in the early 1890s, although he was familiar with John Venn (1834-1923), William S. Jevons (1835-1882) and Charles S. Peirce (1839-1914) in the early 1880s and with Russell in the early 1900s.

The barbershop problem was Carroll's first publication in the journal *Mind.* It is the transcription of a dispute which opposed him to the Oxford Professor of Logic John Cook Wilson (1849-1915). The two men debated regularly on various topics in geometry, chances, and logic. The dispute which led to the barbershop problem began around the end of 1892. Despite their exchanges during the next year, the two men never agreed. Carroll asked Wilson for an accepted transcript of the problem, which he obtained in February 1894. Then, he distributed copies of the problem to his colleagues and friends, in Oxford and elsewhere, and asked them for their opinion, in order to collect and compare their solutions.[7] Recipients included: Venn, Henry Sidgwick (1838-1900), James Welton (1854-1942) and Francis H. Bradley (1846-1924). The contradictory opinions that Carroll collected from his correspondence encouraged him to publish the problem, as he explains in a note annexed to his *Mind* paper:

> The paradox, of which the foregoing paper is an ornamental presentment, is, I have reason to believe, a very real difficulty in the Theory of Hypotheticals. The disputed point has been for some time under discussion by several practised logicians, to whom I have submitted it; and the various and conflicting opinions, which my correspondence with them has elicited, convince me that the subject needs further consideration, in order that logical teachers and writers may come to some agreement as to what Hypotheticals *are,* and how they ought to be treated. [Carroll 1894, 438]

The problem was as follows: there are three men (Allen, Brown and Carr) working in a barbershop. It is known that at least one man should be in the shop, hence:

(1) If Carr is out, then (if Allen is out, then Brown is in).

Also, it is known that Allen can't leave the shop without being accompanied by Brown, hence:

(2) If Allen is out, then Brown is out.

Suppose that Carr is out. From (1), it follows that:

(3) If Allen is out, then Brown is in.

The proposition (3) is considered as incompatible with (2) which is known to be true. Hence proposition (3) is false. From (1), by contraposition, it follows that Carr is in the shop (which contradicts the initial hypothesis that Carr was out). From this absurdity, one concludes that Carr can't leave the shop and should necessarily be inside. This result is paradoxical because it is easy to show that Carr can leave the shop without contradicting any of the rules (1) and (2), as long as Allen is in the shop.

The debate that followed the publication of this problem focuses on the assumption that the two propositions "If Allen is out, then Brown is in" and "If Allen is out, then Brown is out" were incompatible. The modern logician will clearly see here what is known to him as the paradoxes of material implication. Indeed, Russell used the barbershop problem in his *Principles of Mathematics* to illustrate his principle that a false proposition implies all others [Russell

7. On the genesis of the barbershop problem, see [Moktefi 2007a]. Different versions of the problem are reproduced in [Bartley 1986, 449–465] & [Abeles 2010, 111–128].

1903, 18]. We do not aim to discuss the problem here. The interesting point is that the barbershop problem continued, after its publication, to be widely commented about by logicians. [8] Venn was one of the first to discuss it in print, in the second edition of his *Symbolic Logic,* where he writes:

> This particular aspect of the question will very likely be familiar to some of my readers from a problem recently circulated, for comparison of opinions, amongst logicians. As the proposer is, to the general reader, better known in a very different branch of literature, I will call it *Alice's* Problem. [Venn 1894, 442]

Venn confirms in this passage that the problem circulated between the British logicians of the time. It seems, however, that MacColl did not participate in this wide debate. He probably discovered it through Venn, as suggested by his first reference to the problem in 1897:

> To show the working of this logical calculus of three dimensions I may take the following problem, which Dr. Venn in his *Symbolic Logic* (second edition, p. 442) calls "Alice's Problem," and which I understand has (under another form) been already discussed by logicians, but with varying conclusions. [MacColl 1897, 501–502]

By 1897, the barbershop problem had already been discussed by various authors in the journal *Mind,* but MacColl was not yet among them. This late interest is comprehensible: MacColl was not an active reader of the mathematical and philosophical journals of his time, although he continued to review books for *The Athenaeum.* He certainly had access to some British (and foreign) periodicals at the Merridew Library, founded in Boulogne-sur-Mer by the British librarian Henry Melville-Merridew (1826-1879) who lived there. But MacColl's reading was not regular as he explained in a letter to Bradley on 14 December 1904 [Keene 1999, 308]. Moreover, when the Barbershop debate occurred, MacColl was no longer involved in the study of logic and was more interested in fiction, as he told Russell in the letter quoted at the opening of this paper. Of course, the historian has to deal with MacColl's testimony with caution. Still, MacColl's publications record confirms that he returned to the study of logic at the end of 1896, the year Carroll published his *Symbolic Logic.* Indeed, that year MacColl published his first logical writings since the mid-1880s and continued in the following years to contribute actively to various mathematical and philosophical journals. Ultimately, he published in 1906 his only logic book: *Symbolic Logic and its Applications,* where he gave a large overview of his logical theory [MacColl 1906a]. [9]

In the 1890s, MacColl was the main reviewer of mathematical books in *The Athenaeum.* Until 1896, however, he didn't review books on logic. These

8. All of the responses to the barbershop problem, published between 1894 and 1905 in *Mind,* are reproduced in [Abeles 2010, 129–183].

9. The book is reproduced in [Rahman & Redmond 2007, 79–226] together with most of MacColl's other logical writings. The editor's introduction provides an interesting overview of MacColl's contributions to logic. On this, see also [Astroh & Read 1998].

were customarily attributed to other regular reviewers and were usually noticed in the 'Literature' section of the journal, contrary to the mathematics books reviewed by MacColl in the 'Science' section. [10] It is therefore a kind of "historical accident" to see MacColl reviewing Lewis Carroll's *Symbolic Logic* in 1896, especially as Carroll signed it with his literary pseudonym, not with the real name (Dodgson) that he usually used for his mathematical works. Carroll tried his best to keep secret his real identity and denied in print any connection between the two names. However, it was probably a public secret in the 1890s that the author of the Alice tales was Dodgson, the Oxford mathematician. MacColl himself alluded to Alice in his 1891 review of the third edition of *A New Theory of Parallels,* although the book was signed 'Dodgson'. It is certain that Carroll did not appreciate this allusion:

> [W]e strongly recommend non-mathematicians as well as mathematicians to read his witty and ingenious 'Curiosa,' which (if their experience agrees with ours) they will find as entertaining as little Alice found the curiosa of Wonderland. [MacColl 1891, 197]

It seems likely that MacColl was asked to review Carroll's logic book mainly because of his acquaintance with the author's previous works. In addition to his reviews of Carroll's successive editions of *A New Theory of Parallels,* discussed earlier in this paper, MacColl reviewed in 1893 another mathematical book by Carroll, *Pillow Problems,* a collection of seventy-two problems "solved, in the head, while lying awake at night" [Dodgson 1895b, xiii]. The last problem is particularly intriguing and was assumed by MacColl to be "more intended as a joke than as a real question admitting a definite answer" [MacColl 1893, 557]. It has since given rise to several interpretations and commentaries. [11] Both *A New Theory of Parallels* and *Pillow Problems* reached a fourth edition in 1895, but apparently neither was reviewed by MacColl.

Symbolic logic

MacColl's review of *Symbolic Logic* appeared in October 1896. It was based on the first edition of the book (having a preface dated January 1896). In the meantime, Carroll published two further editions (the preface of the third edition is dated 20 July 1896). A fourth and final edition appeared later at the beginning of 1897 (but the preface is dated Christmas 1896) and contained some changes which might have been motivated by MacColl's review, although Carroll never refers to him explicitly. It is, however, certain that Carroll read MacColl's review carefully as he wrote a long response which unfortunately

10. There is one exception: in 1892, MacColl reviewed a logic book by Edward T. Dixon (1862-1935): *An Essay on Reasoning* [MacColl 1892b], together with a mathematical treatise on *The Foundations of Geometry* by the same author [MacColl 1892a]. Both reviews appeared in the 'Science' section however.

11. On Carroll's pillow problems, and particularly the 72^{th} problem, see [Seneta 1993] & [Dale 1999, 447–464].

has not been published and did not survive [Anonymous 1898]. In his review, MacColl says that he appreciated the exposition of the book but considered that its author didn't go beyond the existing logical theories of the time:

> Before offering any detailed criticism of Lewis Carroll's methods we may state certain favourable points which his book undoubtedly possesses. It is well arranged, its expositions are lucid, it has an excellent stock of examples—many of them worked out, and not a few witty and amusing; and its arguments, even when wrong, are always acute and well worth weighing. [...] On the whole, though the author has unquestionably written an interesting and useful little work, his proposed symbolic method does not appear to possess any advantages over some other methods that have preceded it; but, of course, we cannot say what important surprises parts ii. and iii. of his system may have in store for us when they make their appearance. [MacColl 1896, 520–521]

MacColl's general judgment is comprehensible as he was reviewing a book that Carroll wrote in such a way as to be understood by a wide audience. Carroll himself considered the content of the book as elementary, but warned that the second and third parts of his logic trilogy will address advanced and transcendental logic respectively. MacColl is, however, severe when he says that Carroll's method has no advantages over existing methods. In fact, in the bulk of the review, MacColl himself recognized that Carroll developed elegant and effective symbolism and diagrams to represent relations of classes. [12] What MacColl really did reproach Carroll for is that he followed his contemporaries in working with a logic of classes, which differs from MacColl's own propositional approach.

It is well known that MacColl developed a propositional notation, which he believed would improve (and thus should supersede) the mainstream symbolic approach inherited from George Boole (1815-1864). Peirce and Ernst Schröder (1841-1902) were sympathetic with MacColl's work. In Britain however, it had been virulently criticized by Jevons and Venn who both defended an equational notation where letters represent terms or classes. There is no evidence that Carroll ever knew about MacColl's early work on logic, although he might have seen some references to him in logical treatises by other authors. Carroll's notation was different from those of his predecessors as it requires the use of subscripts to express the existence (or not) of classes. Carroll did occasionally use a propositional notation, but his book expresses primarily a logic where letters represent classes, not propositions as MacColl would have wished. Indeed, in his review of *Symbolic Logic,* MacColl takes advantage of his anonymity and maintains that propositional calculus (and particularly his own notation for it) was better and more convenient:

12. For an overview of Carroll's contributions to logic, see [Moktefi 2008] & [Abeles 2010].

> Of the various proposed notations in symbolic logic, the most convenient and symmetrical in our view is that adopted by Mr. MacColl, who (contrary, we believe, to all his predecessors in this field) insists that the simplest as well as the most effective symbolic system is that in which each single letter denotes not a class and not a property but a complete proposition, i. e. an assertion or denial. [...] Nothing so well illustrates the disadvantages of a system of symbolic logic in which single letters represent classes or properties (instead of propositions) as the way in which logicians are sometimes obliged to twist and torture simple sentences in order to adapt them for syllogistic reasoning. For example, Lewis Carroll paraphrases the simple proposition "John is not well" into the astounding assertion that *all Johns are men who are not well,* as if the illness of one member of that numerous and widely scattered family must necessarily involve the illness of all! Among the many Johns of Lewis Carroll's acquaintance is there really *not one* who enjoys good health? [MacColl 1896, 520]

MacColl refers here to an individual proposition used by Carroll to explain how to write a given proposition in a normal form. The proposition is: "John is not well." In the first edition of his book, Carroll explains that the subject of this proposition is the class "Johns" which includes all persons referred to by the name "John", and then writes it in the following normal form:

> All Johns are men who are not well. [Carroll 1896a, 17]

In the second edition however, Carroll changed his mind and followed a suggestion given to him by his colleague George Osborn (1864-1932) [Dodgson 1896]. He explains that the subject is a one Member Class shaped by the "John" referred to in the proposition. The normal form becomes:

> All Johns are not-well. [Carroll 1896b, 16]

Carroll maintained this form in the third edition of the book [Carroll 1896c, 12]. However, in the fourth edition (which followed the publication of MacColl's review), Carroll again changed his normal form, in accordance with MacColl's above criticism, to make it unambiguous as to what is the subject of the proposition:

> Let us take, as an example, the Proposition "John is not well." This of course implies that there is an *Individual,* to whom the speaker refers when he mentions "John," and whom the listener *knows* to be referred to. Hence the Class "men referred to by the speaker when he mentions 'John"' is a one-Member Class, and the Proposition is equivalent to "*All* the men, who are referred to by the speaker when he mentions 'John', are not well." [Carroll 1897, 10]

Carroll's difficulties here come partly from his failure to distinguish between a single thing and a one-member-class [Carroll 1897, 2–3]. A class in Carroll's logic is a collection of things, not a set (as modern logicians understand it). As such, a single thing coincides with the one-member-class, and more importantly there are no classes without individuals. This conception leaves no room for a null class properly. Venn avoids the difficulty by pointing out that logic symbols represent strictly compartments, in which classes may be put, and are not classes themselves [Venn 1894, 119–120]. Consequently, there are no null classes, but merely empty compartments. Carroll and MacColl handled the problem otherwise.

Both Carroll and MacColl assumed the universe to contain things which do really exist and things which do not really exist. The latter are said to be imaginary by Carroll [Carroll 1897, 11]. Hence, the classes "Things" (which is the Universe) and "Things that exist" denote different extensions. Carroll insists on the mental nature of the processes (division and classification) he uses to form classes. As such, they do not ask classes to contain existing things. All they need is to contain things, whatever be their real existence. MacColl tackles the problem on similar grounds. He might have been inspired by Carroll's theory, but there is no way to establish that with certainty. MacColl considers too that the Universe contains both existing and not existing things:

> Let e_1, e_2, e_3, etc. (up to any number of individuals mentioned in our argument or investigation) denote the universe of *real existences.* Let 0_1, 0_2, 0_3, etc., denote our universe of *non-existences,* that is to say, of unrealities, such as *centaurs, nectar, ambrosia, fairies,* with self-contradictions, such as *round squares, square circles, flat spheres,* etc., including, I fear, the non-Euclidean Geometry of four dimensions and other hyper-spatial geometries. Finally, let $S1$, $S2$, $S3$, etc., denote our *Symbolic Universe,* or "Universe of Discourse," composed of all things real or unreal that are named or expressed by words or other symbols in our argument or investigation. By this definition we assume our Symbolic Universe (or "Universe of discourse") to consist of our Universe of realities, e_1, e_2, e_3, etc., together with our universe of unrealities, 0_1, 0_2, 0_3, etc., *when both these enter into our argument.* [MacColl 1905, 74]

Russell objected to this approach on the ground that MacColl was confusing two meanings of existence: in philosophy and life, existence means real existence. So that: Aristotle exists while Hamlet doesn't. In mathematics and symbolic logic, however, existence is applied only to classes which are said to exist if they contain members [Russell 1905, 398]. It follows that only the null class is considered as non-existent in logic because it contains no individuals. Carroll defined the existence of classes too, in addition to that of individuals. Contrary to Russell, however, his definitions were related to each other. A class is said to be real if it contains things existing really, and is said to be imaginary if it contains only imaginary things [Carroll 1897, 2]. Consequently, an imaginary class is not a null class.

In spite of their analogous conceptions of existence, Carroll and MacColl had opposite views on the existential import of propositions. This issue forms MacColl's second important criticism in his review of Carroll's *Symbolic Logic.* Indeed, Carroll considered that universal affirmative propositions have existential import, a position severely attacked by MacColl:

> Hence, according to Lewis Carroll's ruling, the assertion "All S is P," if correct, implies that S really exists. Now there is a certain theorem, generally (we will not rashly say universally) accepted as valid, which does not seem to accept this ruling with the meekness that it ought. The theorem is that "A is A." It will generally be admitted, we think, that "All non-existent things are non-existent"; yet, according to the author, this proposition would imply that non-existent things really exist: a rather staggering assertion in the prosaic world of our experience, though the most fundamental of all axioms in Wonderland. [MacColl 1896 , 520]

Carroll was conscious of the difficulties raised by his theory. Indeed, he limited it in the third edition of his book to propositions of relation where the terms are "two Species of the same Genus, such that each of the two Names conveys the idea of some Attribute *not* conveyed by the other" [Carroll 1896c, 14]. This restriction eliminates the propositions of the kind used by MacColl in his criticism, from the scope of his logic treatise. Still, Carroll had always been in trouble with the issue of existential import, even if he had maintained the same position since his early symbolic investigations in the 1880s. In his first logic book, *The Game of Logic* (1887), he already attributed existential import to I propositions (of the form "Some x are y") but none to E propositions (of the form "No x are y"). Propositions of the form "All x are y" are double however: they assert that "Some x are y" and "No x are not-y". Consequently, A propositions (of the form "All x are y") do assert the existence of their subject [Carroll 1887, 19]. In *Symbolic Logic* (1897), Carroll insists at some length on the purely conventional nature of any choice regarding the existential import of propositions:

> [W]ith regard to the question whether a Proposition is or is not to be understood as asserting the existence of its Subject, I maintain that every writer may adopt his own rule, provided of course that it is consistent with itself and with the accepted facts of Logic.
>
> Let us consider certain views that may *logically* be held, and thus settle which of them may *conveniently* be held; after which I shall hold myself free to declare which of them I intend to hold. [Carroll 1897, 166]

MacColl would certainly agree on this conventional feature, but he disagreed on what ought to be considered as convenient. Carroll paid more attention to what he called "the facts of life" and did his best to keep his theory faithful to common use in real life rather than looking for symmetry and simplicity in his notation and calculus. The fourth edition of *Symbolic Logic* (which followed the publication of MacColl's review) contains, however, several minor changes and warnings which suggest that Carroll might have been on the way to changing his mind. For instance, he opens the section on existential import with the following notice: "Note that the rules, here laid down, are arbitrary, and only apply to Part I of my *Symbolic Logic*" [Carroll 1897, 19]. We cannot tell, however, whether Carroll followed MacColl's advice, as the few surviving fragments of the second and third parts of *Symbolic Logic* are silent on this issue.

Conclusion

Lewis Carroll and Hugh MacColl shared an interest in Euclidean geometry and symbolic logic. In spite of some fundamental differences and opposed views, each benefitted from the reading of the other's writings. From the evidence provided in this paper, it is clear that each man's work influenced that of the other, leading both of them to adjust their arguments to make them more precise. Ultimately, MacColl's reading of Carroll's *Symbolic Logic* "rekindled the old fire which [he] thought extinct" and encouraged him to work again in the domain of logic, as he told Russell. Carroll's literary skills certainly intrigued MacColl who enjoyed reading Carroll's prefaces! In addition to clear exposition and the unusual style that characterize his books, there seems to be one more essential affinity that supports MacColl's attraction to Carroll's work. Indeed, their exchanges show that both had a deep interest in the precise use of words. And both see no harm in attributing arbitrary meanings to words, as long as the meaning is precise and the attribution agreed upon. Carroll promoted this license in most of his writings, and MacColl made thorough use of it in his later mathematical work and second-stage logical theory.

Acknowledgments

The authors express grateful acknowledgement to the librarians of City University Library (London) who helped the second author to consult *The Athenaeum* marked copies collection. The authors acknowledge also the two anonymous referees for their remarks and suggestions.

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