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THE REPRESENTATION CONE OF AN INTERVAL ORDER[1]

Jean-Paul DOIGNON[2], Christophe PAUWELS[3]

RÉSUMÉ – Le cône des représentations d'un ordre d'intervalles.
Un ordre d'intervalles est donné sur un ensemble fini d'éléments. Définies de manière appropriée, ses représentations numériques forment un polyèdre convexe. Nos résultats décrivent la structure géométrique de ce polyèdre. Les facettes correspondent à des objets de quatre types : les éléments minimaux, les éléments contractibles ainsi que les nez et les creux de l'ordre d'intervalles (ces deux dernières notions sont inspirées de Doignon et Falmagne [1997]). Le polyèdre n'a qu'un seul sommet, qui est la représentation minimale de l'ordre d'intervalles (au sens de Doignon [1988a] ; plusieurs nouvelles propriétés sont établies ici). Les représentations forment donc un cône convexe. Nous caractérisons les rayons extrêmes de ce cône. L'unicité du sommet est un résultat surprenant, car Balof, Doignon et Fiorini [2012] ont obtenu, pour le polyèdre des représentations d'un semiordre, de nombreux exemples à sommets multiples.

MOTS CLÉS – Ordre d'intervalles, Polyèdre convexe, Représentation d'un ordre d'intervalles

SUMMARY – *A fixed, interval order is considered on a finite set of elements. When appropriately defined, its representations form a convex polyhedron. Our results describe the geometric structure of the polyhedron. The facets are in a one-to-one correspondence with the objects of one of four types: the minimal elements, the contractible elements as well as the noses and the hollows of the interval order (the latter notions are inferred from Doignon and Falmagne [1997]). The polyhedron has only one vertex, which is the minimal representation (in the meaning of Doignon [1988a]; new properties are established here). All representations thus form a convex cone. We characterize the extreme rays of this cone. The uniqueness of the vertex came as a surprise to us because Balof, Doignon and Fiorini [2012] obtained, for the polyhedron formed by all representations of a semiorder, numerous examples with multiple vertices.*

KEYWORDS – Convex polyhedron, Interval order, Interval order representation

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1. INTRODUCTION

Interval orders are mathematical structures used to encode binary preferences on a set of elements[4] (see for instance [Caspard, Leclerc, Monjardet, 2007; Fishburn, 1985; or Pirlot, Vincke, 1997]). Each element being represented by an interval on the oriented real line, the element i is prefered to the element j exactly if the interval representing i lies completely before the interval representing j. Here, we will assume that the set X of elements is finite, with cardinality n. Without loss of generality, we may assume that all representing intervals lie in $\mathbb{R}_+$ (the set of nonnegative real numbers). Thus, a relation P on X is an *interval order* if there exist two mappings

$$\begin{aligned} f:\ & X \to \mathbb{R}_+ :\ i \mapsto x_i, \\ \rho:\ & X \to \mathbb{R}_+ :\ i \mapsto r_i \end{aligned}$$

such that, for all i, j in X,

$$i\,P\,j \iff x_i + r_i < x_j. \tag{1}$$

In case the mapping ρ can be made constant, say $\rho(i) = r$ for all i in X, the relation P is a *semiorder*.

It will be convenient to take $X = \{1, 2, \ldots, n\}$ and to set $f(i) = x_i$ and $f(i) + \rho(i) = x'_i$. Instead of using f and ρ, we work with the $2n$-tuple

$$(x_1,\, x'_1,\, x_2,\, x'_2,\, \ldots,\, x_n,\, x'_n)$$

that we (abusively but conveniently) denote by the short notation x. Equation (1) is then restated in terms of x as

$$i\,P\,j \iff x'_i < x_j,$$

or, using $\overline{P}$ to denote the complementary relation of P,

$$\begin{cases} i\,P\,j & \Rightarrow \quad x'_i < x_j, \\ i\,\overline{P}\,j & \Rightarrow \quad x'_i \geq x_j. \end{cases} \tag{2}$$

Notice that the set of all $2n$-tuples x satisfying Equation (2) together with $0 \leq x_i \leq x'_i$, for $i \in X$, forms a convex subset of $\mathbb{R}^{2n}$ which is in general neither closed nor open (we require $x_i \leq x'_i$ because of $\rho \geq 0$). On the other hand, the finiteness of X implies, for any fixed $2n$-tuple x satisfying Equation (2), the existence of a strictly positive number ε such that

$$\begin{cases} i\,P\,j & \Rightarrow \quad x'_i + \varepsilon \leq x_j, \\ i\,\overline{P}\,j & \Rightarrow \quad x'_i \geq x_j. \end{cases}$$

In turn, given any interval order P on X and any strictly positive real number ε, there exists (again by the finiteness of X) a $2n$-tuple x satisfying, for all i, j in X,

$$\begin{cases} 0 \leq x_i, & \\ x_i \leq x'_i, & \\ x'_i + \varepsilon \leq x_j, & \text{when } i\,P\,j, \\ x'_i \geq x_j, & \text{when } i\,\overline{P}\,j. \end{cases} \tag{3}$$

[4]Readers who like to see explicit examples are directed to the last section where the concepts and results of the paper are illustrated on two specific cases.

A $2n$-tuple $x = (x_1, x'_1, x_2, x'_2, \ldots, x_n, x'_n)$ satisfying Equation (3) is an *ε-representation (of the interval order P on X)*. Pirlot [1990] raises the general problem of investigating the collection of all ε-representations, while providing motivation (to be precise [Pirlot, 1990] is about semiorders only). Notice that the set of all ε-representations forms a convex polyhedron in $\mathbb{R}^{2n}$; we denote this polyhedron as R^P_ε. (We usually follow Schrijver [1986] or Ziegler [1995] for the terminology on polyhedra). Moreover, it is easily checked that the following holds for $\varepsilon, \varepsilon'$ in $\mathbb{R}^*_+$ (the set of strictly positive real numbers):

$$R^P_{\varepsilon'} \;=\; \frac{\varepsilon'}{\varepsilon}\, R^P_\varepsilon. \tag{4}$$

As a consequence of Equation (4), all polyhedra R^P_ε have (essentially) the same structure. We call them collectively the *representation polyhedron* of the given interval order.

The present paper provides information on the geometric structure of the representation polyhedron of an interval order. First, it describes the facets by relating them to the 'noses' and 'hollows' of the interval order (these notions, implicit in Doignon and Falmagne [1997], are defined in the next section) together with elements of two special kinds, the minimal elements and the 'contractible' elements (with respect to the interval order, see Theorem 4 below). Then it shows that there is exactly one vertex (Theorem 5), which is the 'minimal representation' of the interval order in the sense of Doignon [1988a] (for this concept, see also Section 2.2.). Thus the polyhedron is a cone, the *representation cone* of the interval order P. Finally, our paper establishes a description of the extreme rays of the cone (Theorem 6).

It is worth mentioning here that a similar study was done by Balof, Doignon and Fiorini [2011] for the 'constant-length representations' of a semiorder P on X which, in our previous notation, are $n+1$-tuples $(x_1, x_2, \ldots, x_n, r)$. The geometric findings in the case of semiorders are in some aspects strikingly different from the present ones; for instance the representation polyhedron can have numerous vertices.

Both Balof, Doignon and Fiorini [2011] and the present paper consider only the basic structures of interval orders and semiorders. We do not know of any investigation of similar questions for more general structures. For instance, in multiple interval orders or semiorders the preference relation P decomposes into several degrees that correspond to several thresholds attached to any item (for the concept, see for instance [Doignon, 1988a; Doignon, Monjardet, Roubens, Vincke, 1986; Doignon, 1987; Doignon, 1988b]). Other general structures occur when intervals are replaced with geometric figures, for instance trapezes with bases on two given, parallel lines. Fishburn [1997] reviews such extensions of semiorders and interval orders, as well as the related "tolerance orders". In all cases, as far as we know, even the concept of a minimal representation was not investigated. It would also be interesting to extend our study to representations of semiorders and interval orders (and their generalizations) on an infinite set of elements.

2. INTERVAL ORDERS

2.1. NOSES AND HOLLOWS

Combinatorial characterizations of interval orders and of semiorders are often attributed to Fishburn [1985] and Scott and Suppes [1958], respectively. However, for interval orders, Mirkin [1972] was a precursor with a similar characterization, and Ducamp, Falmagne [1969] establishes Fishburn's characterization in a more general setting; for semiorders, a characterization is formulated in another notation in Luce [1956].

To formulate them as compact formulas, we use P^{-1} to denote the converse relation of P, and $\overline{P}$ to denote the complementary relation.

THEOREM 1. [Fishburn, 1985; Scott, Suppes [1958] *A relation P on X is an interval order if and only if P is irreflexive and satisfies $P\,\overline{P}^{-1}P \subseteq P$. It is a semiorder if and only if it satisfies moreover $\overline{P}^{-1}P\,P \subseteq P$.*

For the rest of the paper, we consider an interval order P on the finite set X. The 'noses' and 'hollows' of P, which we now define, are used in the next section. These terms originally appeared in two papers of Pirlot [1990, 1991] on (reduced) semiorders. The similar notions for interval orders are implicit in Doignon and Falmagne [1997] (they capture the pairs of elements forming respectively the 'inner' and 'outer fringes' of P). We first state the definition, and then provide a characterization easily derived from Theorem 1 (compare with Proposition 7 in Doignon and Falmagne [1997]).

DEFINITION 1. A *nose* of the interval order P is a pair (a, b) in P such that $P\setminus\{(a, b)\}$ is again an interval order. A *hollow* of P is a pair (c, d) in $\overline{P}$ such that $P \cup \{(c, d)\}$ is again an interval order.

THEOREM 2. *The noses of the interval order P form the set $P \setminus P\,\overline{P}^{-1}P$. The hollows form the set $\overline{P} \setminus \big(I \cup \overline{P}\,P^{-1}\overline{P}\big)$, where I is the identity relation on X.*

2.2. THE MINIMAL REPRESENTATION

Let now ε be a fixed real number with $\varepsilon > 0$. As defined in our introduction, an ε-representation of the interval order P is a $2n$-tuple $x = (x_1, x'_1, x_2, x'_2, \ldots, x_n, x'_n)$ such that[5], for i, j in X,

$$\begin{cases} -x_i & \leq & 0, & \\ x'_i - x_j & \leq & -\varepsilon, & \text{when } i\,P\,j, \\ x_j - x'_i & \leq & 0, & \text{when } i\,\overline{P}\,j. \end{cases} \tag{5}$$

Notice that all inequalities $x_i \leq x'_i$ belong to the third form, in view of the irreflexivity of P. The value ε being fixed, we will from now on abbreviate 'ε-representation' into 'representation'. Among all representations, there is one that is minimal in a very strong sense—we sometimes say that it is ubiquitously minimal. Its existence

[5] Obviously, the two "when" in Equation (5) can be safely replaced with two "if and only if".

was established by Doignon [1988a] (as a variant of a similar finding on semiorders, a very nice and much harder result of Pirlot [1990]; an alternative proof for Pirlot's result is in Doignon [1988a]). For completeness and also because we need the setting in order to establish further results, we sketch a proof similar to the one of Doignon [1988a].

THEOREM 3. [Doignon, 1988a] *There exists a representation x^* of the interval order P on X such that for any representation x of P the following holds for all i in X:*

$$x_i^* \le x_i, \qquad x_i'^* \le x_i'.$$

DEFINITION 2. *The representation x^* mentioned in Theorem 3 is called the* minimal representation *of the interval order P.*

The proof in Doignon [1988a] relies on graph potentials, as we will know briefly explain while building the appropriate graph. First, we make a disjoint copy X' of X, formed by new elements i', for i in X. The components of any $2n$-tuple x encoding a representation can be seen as being indexed by $X + X'$ if we identify x_i' with $x_{i'}$. In this way, a representation x has components x_s, with $s \in X + X'$ (thus x_s denotes either an initial or a final extremity of an interval $[x_i, x_i']$ for some i in X).

Define a (weighted, directed) graph G on $X + X'$ with an edge

$$\begin{cases} (j, i') & \text{of weight } -\varepsilon \qquad \text{when } i \, P \, j, \\ (i', j) & \text{of weight } \ 0 \qquad \text{when } i \, \overline{P} \, j. \end{cases}$$

A *potential* for the graph G is a function $p: \ X + X' \to \mathbb{R}_+$ such that

$$\begin{cases} p(i') - p(j) \ \le \ -\varepsilon, & \text{when } i \, P \, j, \\ p(j) - p(i') \ \le \ \ 0, & \text{when } i \, \overline{P} \, j. \end{cases}$$

The representations of P are in a natural one-to-one correspondence with the potentials of G (in the above notations, the correspondence is captured by the equations $x_i = p(i)$ and $x_i' = p(i')$, or shortly $x_s = p(s)$).

As is well-known, the graph G admits a potential if and only if any of its directed circuits has nonnegative weight (cf. Section 8.2 in Schrijver [2003]). Our graph has no circuit because P is irreflexive and $P\,\overline{P}^{-1} P \subseteq P$ (Theorem 1). Moreover a *minimal potential* p^* exists, that is, a potential p^* such that for any potential p we have $p^*(s) \le p(s)$ for all s in $X + X'$. This is easily proved with $p^*(s)$ equal to the maximum weight of a directed path ending at the vertex s (we use the convention that if no such path exists, $p^*(s) = 0$). In our case, the minimal potential corresponds to the minimal representation. This establishes Theorem 3.

A description of p^*, and thus x^*, is a consequence of the previous paragraph. For i in X, we have x_i^* equal to ε times the largest number h such that for some k in X there holds $k \, (P^{-1}\overline{P})^h \, i$ (if the latter holds for no element k and no positive natural number h, then $x_i^* = 0$; also, remember $j \, \overline{P} \, j$). Similarly, $x_i'^*$ equals ε times the largest number h such that $k \, (\overline{P} \, P^{-1})^h \, i$ for some k in X.It is also easily checked that the values taken by x^* form an interval in the chain $\{k \, \varepsilon \mid k \in \mathbb{N}\}$.

Further results on the minimal representation are given in the next subsection as well as in Section 4.

Remark 1. Let us rewrite the system (5) as $A\,x \leq b$, where the matrix A has rows indexed by $X + P + \overline{P}$ and columns by $X + X'$. Thus A results from putting the identity matrix atop the incidence matrix of our (directed) graph G. By a classical result (Theorem 13.9 in Schrijver [2003]), the matrix A is totally unimodular. A nice property of the polyhedron R^P_ε follows: the coordinates of its vertices and the components of well-chosen vectors generating its extreme rays are integral multiples of ε. As a matter of fact, we will prove much more in the sequel (see Theorems 5 and 6).

2.3. THE LEFT, RIGHT AND GLOBAL TRACES

The minimal representation x^* induces a (reflexive) weak order $\preccurlyeq$ on $X + X'$: for s, t in $X + X'$, we set[6]

$$s \preccurlyeq t \quad \text{when} \quad x^*_s \leq x^*_t.$$

The asymmetric part of the weak order $\preccurlyeq$ is denoted as $\prec$ (we also use $\succ$ with its usual meaning). A direct description of $\preccurlyeq$ is easily obtained.

PROPOSITION 1. *The weak order $\preccurlyeq$ satisfies for all i, j in X:*

(a) $i \preccurlyeq j \iff (k\,P\,i \Rightarrow k\,P\,j$, *for all* k *in* $X)$;
(b) $i \preccurlyeq j' \iff i\,\overline{P}^{-1}\,j$;
(c) $i' \preccurlyeq j \iff i\,\overline{P}\,P^{-1}\overline{P}\,j$ *does not hold;*
(d) $i' \preccurlyeq j' \iff (j\,P\,k \Rightarrow i\,P\,k$, *for all* k *in* $X)$.

Proof. All four cases being simple we provide details only in Case (c), working with the contraposition. If $i' \succ j$, we have $x'^*_i > x^*_j$. Because the value x'^*_i of the minimal representation cannot be decreased to x^*_j, there must exist some k in X such that $x'^*_i \geq x^*_k > x^*_j$. Then because x^*_k cannot be decreased to x^*_j, there exists some l in X such that $x^*_k \geq x'^*_l + \varepsilon > x^*_j$. There result $i\,\overline{P}\,k$, $l\,P\,k$ and $l\,\overline{P}\,j$, hence $i\,\overline{P}\,P^{-1}\overline{P}\,j$. To prove the converse, we assume $i\,\overline{P}\,P^{-1}\overline{P}\,j$. There thus exist k, l in X such that $i\,\overline{P}\,k$, $k\,P^{-1}\,l$ and $l\,\overline{P}\,j$, from which follows

$$x'^*_i \;\geq\; x^*_k \;\geq\; x'^*_l + \varepsilon \;>\; x'^*_l \;\geq\; x^*_j$$

and thus $x'^*_i > x^*_j$. □

Proposition 1 is summarized in the following table[7], where $P_\circ$ is the relation from

[6] In geometric language: both of s and t correspond to extremities of intervals of the form $[x^*_i, x'^*_i]$ (an initial extremity if $s \in X$, a final one if $s \in X'$), and $s \preccurlyeq t$ holds if and only if the extremity corresponding to s is less or equal to the extremity corresponding to t.

[7] Comparing with Table 1 in Doignon, Ducamp, Falmagne [1984], we see that $\preccurlyeq$ is the largest quasi order on $X + X'$ having $\overline{P_\circ^{-1}}$ as its intersection with $X \times X'$. The construction of such a quasi order for any given relation goes back to Bouchet [1971], see also Cogis [1982].

X' to X with $i' P_\circ j$ if and only if $i\,P\,j$:

$\preccurlyeq$	X	X'
X	$\overline{P_\circ^{\,-1}\,\overline{P_\circ}}$	$\overline{P_\circ}^{\,-1}$
X'	$\overline{\overline{P_\circ}\,P_\circ^{\,-1}\,\overline{P_\circ}}$	$\overline{\overline{P_\circ}\,P_\circ^{\,-1}}$

(6)

Complementations (indicated by an overbar) are taken within the adequate set product (in first row, $X \times X$ and $X \times X'$). The restrictions of $\preccurlyeq$ to respectively X and X' (as in Cases (a) and (d) of Proposition 1) are called the *left*, resp. *right traces* of P. Thus T_l and T_r are relations on respectively X and X' which are specified by $\overline{T_l} = P^{-1}\overline{P}$ and $\overline{T_r} = \overline{P}\,P^{-1}$. We call the relation $\preccurlyeq$ the *global trace* of P. All these traces (or their asymmetric parts) are often used in the investigation of interval orders (see for instance [Doignon, Ducamp, Falmagne, 1984] or [Fishburn, 1985]). The next proposition offers another useful property of $\preccurlyeq$.

PROPOSITION 2. *For s, t in $X + X'$, the two following assertions are equivalent:*

(i) $s \preccurlyeq t$;

(ii) $x_s \leq x_t$ *for some representation* x.

Proof. *(i)* $\Rightarrow$ *(ii)*. With x^* for the representation in *(ii)*, the implication results from the definition of $\preccurlyeq$.

(ii) $\Rightarrow$ *(i)*. After having contraposed, we need to prove that if $s \succ t$ then $x_s > x_t$ for any representation x. The present assumption amounts to $x^*_s > x^*_t$. We face four possible cases according to the positions of s and t in either X or X'. For instance, consider the case $s = i \in X$ and $t = j \in X$ with thus $x^*_i > x^*_j$. By Proposition 1 (a), there exists k in X such that $k\,P\,i$ and $k\,\overline{P}\,j$. For any representation x there follows $x'_k + \varepsilon \leq x_i$ and $x'_k \geq x_j$, hence also $x_i > x_j$. The three other cases are similarly handled. □

The next proposition is phrased in terms of the minimal representation x^*, but it could have been formulated in terms of the weak order $\preccurlyeq$. Indeed, the second condition in Equation (7) holds exactly if b is in the (equivalence) class of $\preccurlyeq$ which follows the one of a', while the second condition in Equation (8) holds exactly if c' and d are in the same class.

PROPOSITION 3. *For a, b, c, d in X we have:*

$$(a, b) \text{ is a nose} \quad \Longleftrightarrow \quad x'^*_a + \varepsilon = x^*_b, \tag{7}$$

and

$$(c, d) \text{ is a hollow} \quad \Longleftrightarrow \quad x'^*_c = x^*_d. \tag{8}$$

Proof. If (a, b) is a nose, then $a\,P\,b$ and so $x'^*_a + \varepsilon \leq x^*_b$. Proceeding by contradiction, assume $x'^*_a + \varepsilon < x^*_b$. By the ubiquitous minimality of x^*, there must exist some k in X giving $x'^*_a + \varepsilon < x'^*_k + \varepsilon \leq x^*_b$ (because x^*_b cannot be decreased to $x'^*_a + \varepsilon$) and then some l in X with $x'^*_a < x^*_l \leq x'^*_k$ (because x'^*_k cannot be decreased to x'^*_a).

We deduce $a\,P\,l$, $k\,\overline{P}\,l$ and $k\,P\,b$. This gives $a\,P\,\overline{P}^{-1}\,P\,b$, a contradiction in view of Theorem 2. Conversely, if $x_a'^* + \varepsilon = x_b^*$, then $a\,P\,b$. Moreover, we cannot have $(a,b) \in P\,\overline{P}^{-1}\,P$ because otherwise $a\,P\,k$, $k\,\overline{P}^{-1}\,l$ and $l\,P\,b$ for some k, l in X and so

$$x_a'^* + \varepsilon \le x_k^* \le x_l'^* \le x_l'^* + \varepsilon \le x_b^*.$$

Hence (a,b) is a nose.

If (c,d) is a hollow, then $c\,\overline{P}\,d$ and so $x_c'^* \ge x_d^*$. Moreover, by Theorem 2, $(c,d) \notin \overline{P}\,P^{-1}\overline{P}$ and so by the table in Equation (6) $c' \preccurlyeq d$, that is $x_c'^* \le x_d^*$. The converse implication is similarly proved. □

2.4. AN AUXILIARY RESULT

The following result will be used in the next section.

PROPOSITION 4. *Given the interval order P on X and an element i in X, the following three conditions are equivalent:*

(i) $x_i^* = x_i'^*$ *in the minimal representation* x^**;*

(ii) there is some representation x *such that* $x_i = x_i'$*;*

(iii) $j\,P\,k \Rightarrow (j\,P\,i$ *or* $i\,P\,k)$*, for all* j, k *in* X.

An element i satisfying the conditions in Proposition 4 is said to be *contractible* for the interval order P.

Proof. (i) ⇒ (ii). Obvious.

(ii) ⇒ (iii). If some representation x of P satisfies $x_i = x_i'$ and moreover we have $j\,P\,k$ and $j\,\overline{P}\,i$, then $x_j' + \varepsilon \le x_k$ and $x_j' \ge x_i = x_i'$. Therefore, $x_i' + \varepsilon \le x_k$, and we obtain $i\,P\,k$ as desired.

(iii) ⇒ (i). Condition *(iii)* amounts to $(i,i) \notin \overline{P}\,P^{-1}\overline{P}$. By the table in Equation (6), the latter formula is equivalent to $i' \preccurlyeq i$, that is $x_i'^* \le x_i^*$. On the other hand, $x_i^* \le x_i'^*$ holds by the definition of a representation. □

3. THE FACETS

For a given interval order P on $X = \{1, 2, \ldots, n\}$ and a fixed ε in $\mathbb{R}_+^*$, we describe in this section the facets of the representation polyhedron R_ε^P. Remember that R_ε^P is the polyhedron in $\mathbb{R}^{2n}$ defined by the inequalities in Equation (5).

PROPOSITION 5. *The representation polyhedron R_ε^P is of dimension $2n$ and contains no line.*

Proof. Let x be a representation of P satisfying $x_i > 0$ for any i in X. Setting $y_i = 2\,x_i$ and $y_i' = 2\,x_i' + \varepsilon/2$, we get from Equation (5)

$$\begin{cases} 0 & < y_i, & \\ y_i + \varepsilon/2 & \le y_i', & \\ y_i' + \varepsilon + \varepsilon/2 & \le y_j, & \text{when } i\,P\,j, \\ y_i' & \ge y_j + \varepsilon/2, & \text{when } i\,\overline{P}\,j. \end{cases}$$

Thus y is a representation satisfying strictly all the inequalities defining R^P_ε. Hence R^P_ε, having a nonempty interior, is of the same dimension as $\mathbb{R}^{2n}$. The second assertion follows directly from the fact that our representations only take nonnegative values. $\square$

Because the polyhedron R^P_ε is full-dimensional (Proposition 5), we know that there is a unique, minimal system of linear inequalities describing R^P_ε. The minimal system consists of the facet-defining inequalities (see, e.g., [Schrijver, 1986; Ziegler, 1995]). Moreover, an inequality belonging to a system of linear inequalities defining R^P_ε is facet-defining if and only if its deletion results in an increase of the set of solutions. Consequently, a facet-defining inequality cannot be dominated by a positive combination of other inequalities valid for R^P_ε. To describe the facets of the representation polyhedron R^P_ε, we rely on the notions of noses and hollows (see Definition 1).

THEOREM 4. *The facet-defining inequalities for the representation polyhedron of the interval order P are as follows:*

$$\begin{array}{rcll} 0 & \leq & x_i, & \textit{for all } i \textit{ in } X \textit{ which are minimal for } P; \\ x_i & \leq & x'_i, & \textit{for all } i \textit{ in } X \textit{ which are contractible for } P; \\ x'_a + \varepsilon & \leq & x_b, & \textit{for all noses } (a,b) \textit{ of } P; \\ x'_c & \geq & x_d, & \textit{for all hollows } (c,d) \textit{ of } P. \end{array}$$

Proof. The facet-defining inequalities for R^P_ε are for sure among the inequalities given in Equation (5) to define representations, that is, points of R^P_ε. We inspect the four types of inequalities one after the other.

If the inequality $0 \leq x_i$ is facet-defining, there is a point x in R^P_ε, in other words a representation x, with $x_i = 0$. So i must be a minimal element (no j such that $j\,P\,i$ can exist). Conversely, if i is a minimal element, in any representation x we have $x_i \leq x'_j$ for all j in X. Resetting x_i to, say, -1, we get a $2n$-tuple of real numbers which satisfies all the inequalities in Equation (5) except for $0 \leq x_i$. The existence of such a $2n$-tuple implies that the inequality $0 \leq x_i$ is facet-defining.

Next, assume that the inequality $x_i \leq x'_i$ is facet-defining. If i were not contractible, that is, if we had $j\,P\,k$, $j\,\overline{P}\,i$ and $i\,\overline{P}\,k$ for some j, k in X, then the resulting inequalities $x'_j + \varepsilon \leq x_k$, $x_i \leq x'_j$ and $x_k \leq x'_i$ (all valid for R^P_ε) would add up to give $x_i + \varepsilon \leq x'_i$. Thus the inequality $x_i \leq x'_i$ would be dominated by a positive combination of valid inequalities, contradicting the assumption that it is facet-defining. Conversely, assume i is a contractible element, that is, there exists a representation x with $x_i = x'_i$. Let δ be a strictly positive number and consider the two mappings

$$g:\ \mathbb{R} \to \mathbb{R}:\ t \mapsto \begin{cases} t + \delta & \text{if } t > x'_i, \\ t & \text{otherwise,} \end{cases}$$

and

$$h:\ \mathbb{R} \to \mathbb{R}:\ t \mapsto \begin{cases} t + \delta & \text{if } t \geq x'_i, \\ t & \text{otherwise.} \end{cases}$$

Then define a $2n$-tuple y by

$$\begin{cases} y_i & = & x_i + \delta, \\ y'_i & = & x'_i, \\ y_j & = & g(x_j), \quad \forall j \in X \setminus \{i\}, \\ y'_j & = & h(x'_j), \quad \forall j \in X \setminus \{i\}. \end{cases}$$

We leave to the reader to check that all inequalities required in Equation (5) for a representation are satisfied by y except for $x_i \leq x'_i$. In view of the existence of such a y, the inequality $x_i \leq x'_i$ defines a facet of R^P_ε.

The third inequality to investigate is $x'_i + \varepsilon \leq x_j$. If (i, j) is not a nose, then by Theorem 2 either $(i, j) \notin P$ or $(i, j) \in P\,\overline{P}^{-1}P$. In the first case, $x'_i \geq x_j$ for all x in R^P_ε and the inequality under investigation is not valid for R^P_ε. In the second case, there exist elements k and l such that $i\,P\,k$, $k\,\overline{P}^{-1}\,l$ and $l\,P\,j$. This gives for any point x in R^P_ε

$$x'_i + \varepsilon \;\leq\; x_k \;\leq\; x'_l \;\leq\; x'_l + \varepsilon \;\leq\; x_j,$$

and so the inequality under investigation is a consequence of other inequalities valid for R^P_ε, so it cannot be facet-defining. Conversely, if (a, b) is a nose, then $P \setminus \{(a, b)\}$ is an interval order. Take any representation of $P \setminus \{(a, b)\}$, say y. Then y satisfies all the inequalities defining R^P_ε except for $x'_a + \varepsilon \leq x_b$. Hence the latter inequality defines a facet of R^P_ε.

Finally, we consider the inequality $x'_i \geq x_j$. If (i, j) is not a hollow, either $(i, j) \in P$ or $(i, j) \in I \cup \overline{P}\,P^{-1}\overline{P}$ (we use Theorem 2). In the first case, $x'_i + \varepsilon \leq x_j$ is valid for R^P_ε and thus the present inequality $x'_i \geq x_j$ is not valid. In the second case, $i = j$ or there are elements k and l satisfying $i\,\overline{P}\,k$, $k\,P^{-1}\,l$ and $l\,\overline{P}\,j$. The subcase $i = j$ corresponds to our second type of inequalities, so we may assume $i \neq j$ here. We then have

$$x'_i \;\geq\; x_k \;\geq\; x'_l + \varepsilon \;\geq\; x'_l \;\geq\; x_j.$$

Hence the inequality $x'_i \geq x_j$ is a consequence of valid inequalities, so it cannot define a facet. Conversely, if (c, d) is a hollow, then $P \cup \{(c, d)\}$ is an interval order. Any representation of $P \cup \{(c, d)\}$ is a point in $\mathbb{R}^{2n}$ which satisfies all the inequalities that define R^P_ε except for $x'_c + \varepsilon \leq x_d$. Thus the latter inequality defines a facet of R^P_ε. □

4. THE VERTEX

Let again P be an interval order on the finite set X of cardinality n, and ε a strictly positive real number. From previous results, we now easily derive that the representation polyhedron R^P_ε has a unique vertex. This vertex is the minimal representation $x^* = (x^*_1, x'^*_1, x^*_2, x'^*_2, \ldots, x^*_n, x'^*_n)$ of P, which we defined in Subsection 2.2.

THEOREM 5. *The representation polyhedron R^P_ε of an interval order P has exactly one vertex, formed by the minimal representation x^* of P.*

Proof. By its ubiquitous minimality, the minimal representation x^* clearly forms a vertex of R^P_ε. Next we check that x^* belongs to all the facets of R^P_ε (this establishes that x^* is the only vertex of R^P_ε). To this aim, we consider in turn each of the four types of facet-defining inequalities listed in Theorem 4.

The inequality $0 \le x_i$ is facet-defining if and only if i is an element which is minimal for P. For such an element i we must have $0 = x^*_i$, thus the inequality is satisfied with equality by x^*.

The inequality $x_i \le x'_i$ defines a facet if and only if the element i satisfies, for all j, k in X,

$$j\,P\,k \Rightarrow (j\,P\,i \text{ or } i\,P\,k).$$

By Proposition 4, the latter condition implies $x^*_i = x'^*_i$.

The inequality $x'_a + \varepsilon \le x_b$ defines a facet exactly when (a, b) is a nose. On the other hand, if (a, b) is a nose Proposition 3 states $x'^*_a + \varepsilon = x^*_b$.

Finally, if (c, d) is a hollow, we have by the same proposition $x'^*_c = x^*_d$. This entails equality in the facet-defining inequality $x'_c \ge x_d$ of the fourth and last type. □

Remark 2. Theorem 5 has the following consequence. Given any representation x of P, the minimal representation x^* satisfies the two following properties, for all i, j in X,

$$i\,P\,j \quad \Rightarrow \quad x^*_j - x'^*_i \le x_j - x'_i$$

and

$$i\,\overline{P}\,j \quad \Rightarrow \quad x'^*_i - x^*_j \le x'_i - x_j.$$

To derive the first implication, notice that the linear form $\mathbb{R}^{2n} \to \mathbb{R} :\ x \mapsto x_j - x'_i - \varepsilon$, which is nonnegative on R^P_ε, must take its minimum value on R^P_ε at the vertex of R^P_ε. For the second implication, use similarly the form $\mathbb{R}^{2n} \to \mathbb{R} :\ x \mapsto x'_i - x_j$. In turn, the two last properties imply the uniqueness of the vertex. Another proof we designed for Theorem 5 started by establishing the two properties.

5. THE EXTREME RAYS

The representation polyhedron R^P_ε of an interval order P on X is a pointed, convex cone: indeed, R^P_ε has only one vertex (Theorem 5), and moreover it is contained in $\mathbb{R}^{2n}_+$. We know that the vertex of R^P_ε is the minimal representation x^*. In this Section, the extreme rays of the representation cone R^P_ε are characterized. As we will easily see, any such ray is generated by a binary vector $v = (v_1,\, v'_1,\, v_2,\, v'_2,\, \ldots, v_n,\, v'_n)$ located at x^* (we may write $v_{i'}$ for v'_i). This fact has a direct interpretation for the extreme rays of R^P_ε. Any such ray corresponds to an 'admissible' subset Y of $X + X'$, and its points are generated from the minimal representation x^* by fixing the components of x^* with indices outside Y, and increasing the components with indices in Y by the same varying positive quantity. The next theorem provides a characterization of the admissible subsets Y of $X + X'$. It relies on a variant $\sqsubseteq$ of the weak order $\preccurlyeq$ which we defined in Subsection 2.2.

For s, t in $X + X'$, we set $s \sqsubseteq t$ when either $x^*_s < x^*_t$, or $\bigl(x^*_s = x^*_t$ together with $s \in X$ or $t \in X'\bigr)$. Accordingly, the relation $\sqsubseteq$ is a weak order contained in the weak order $\preccurlyeq$—some of the classes of $\preccurlyeq$ are split into two classes in order to get the classes of $\sqsubseteq$. It can be combinatorially characterized as shown in the following table, which uses the same conventions as those for the table in Equation (6).

$\sqsubseteq$	X	X'
X	$\overline{P_\circ^{-1}\,\overline{P}_\circ}$	$\overline{P}_\circ^{-1}$
X'	$\overline{\overline{P}_\circ\, P_\circ^{-1}\,\overline{P}_\circ} \setminus \overline{P}_\circ$	$\overline{\overline{P}_\circ\, P_\circ^{-1}}$

For the asymmetric part $\sqsubset$ of $\sqsubseteq$ and for s, t in $X + X'$, we have $s \sqsubset t$ if and only if $x^*_s < x^*_t$, or $\big(x^*_s = x^*_t$ together with $s \in X$ and $t \in X'\big)$. This time, $\prec$ is a subset of $\sqsubset$.

PROPOSITION 6. *For s, t in $X + X'$, the two following conditions are equivalent:*

(i) $s \sqsubseteq t$;

(ii) $x_s < x_t$ *for some representation* x.

Proof. *(i)* $\Rightarrow$ *(ii)*. Our assumption $s \sqsubseteq t$ implies $x^*_s \leq x^*_t$. In case $x^*_s < x^*_t$ the representation x^* makes *(ii)* true. In the remaining case, we have $x^*_s = x^*_t$ together with $s \in X$ or $t \in X'$. Assume first $s = i$ for some i in X. In the $2n$-tuple $2\,x^*$, subtract ε from the component indexed by i. The result forms a representation (as easily checked), with $x_s < x_t$. Second, assume $t = j'$ for some j in X. This time add ε to the component indexed by j' in $2\,x^*$. The result forms again a representation with $x_s < x_t$.

(ii) $\Rightarrow$ *(i)*. Proceding by contraposition, we assume $t \sqsubset s$. The latter implies $x^*_t \leq x^*_s$. In case $x^*_t < x^*_s$, Proposition 2 implies $x_t \leq x_s$ for any representation x as desired. In the remaining case, $x^*_t = x^*_s$ together with $t = i$ and $s = j'$ for some i, j in X. Then $j\,\overline{P}\,i$, from which follows $x_t \leq x_s$ for any representation. □

Here is the main result of this Section.

THEOREM 6. *The extreme rays of the representation cone R^P_ε of the interval order P are the rays generated by the binary vectors $v = (v_1,\, v'_1,\, v_2,\, v'_2,\, \ldots,\, v_n,\, v'_n)$ of $\mathbb{R}^{2n}$, located at the minimal representation x^*, which satisfy two properties:*

1. *if $v_s = 1$ and $s \sqsubset t$ for some elements s, t of $X + X'$, then $v_t = 1$;*
2. *if the indices of the components of v equal to 1 form a subset S of the last class of $\sqsubseteq$, then $|S| = 1$.*

The proof is delayed after Propositions 7 and 8 are established. It will be convenient to translate the cone R^P_ε in $\mathbb{R}^{2n}$ so that its vertex x^* becomes the origin o of $\mathbb{R}^{2n}$. The resulting cone, denoted as $\big(R^P_\varepsilon\big)_o$, is the set of solutions in $\mathbb{R}^{2n}$ of the system

$$\left\{\begin{array}{rcll} 0 & \leq & x_i, & \text{for } i \in X, \\ x'_i & \leq & x_j, & \text{for } (i,j) \in P, \\ x'_i & \geq & x_j, & \text{for } (i,j) \in \overline{P}. \end{array}\right. \tag{9}$$

The special form of the latter system has a direct consequence for the extreme rays.

PROPOSITION 7. *Any extreme ray of the cone R^P_ε is generated by a binary vector located at the vertex x^*.*

Proof. We work with the cone $\left(R^P_\varepsilon\right)_o$, formed by all the solutions of Equation (9). Given an extreme ray of $\left(R^P_\varepsilon\right)_o$, we may assume it is generated by a vector v whose nonzero components are all strictly larger than 1. Define a vector w by $w_s = 1$ if $v_s > 0$, and $w_s = 0$ otherwise, and then another vector u with $u = v - w$. It is easily checked that both w and u are also solutions of Equation (9), thus vectors in $\left(R^P_\varepsilon\right)_o$. Because $v = u + w$ and v generates an extreme ray, there must exist some positive real number λ such that $v = \lambda w$, in other words v is a positive multiple of a binary vector. □

Remark 3. The conclusion of Proposition 7 remains valid for any cone defined in $\mathbb{R}^d$ by $0 \leq x_i$, for $i = 1, 2, \ldots, d$, and any number of inequalities of the form $x_i \leq x_j$, where $i, j \in \{1, 2, \ldots, d\}$. We suspect that the result is well known (it can be proved in other ways, for instance in terms of the inequalities satisfied with equality by the extreme ray).

PROPOSITION 8. *For all s, t in $X + X'$, we have $s \sqsubset t$ if and only if $v_s \leq v_t$ holds for any vector v in the cone $\left(R^P_\varepsilon\right)_o$.*

Proof. Use Proposition 6: $s \sqsubset t$ if and only if any point of the representation cone R^P_ε satisfies the homogeneous inequality $x_s \leq x_t$, and thus if and only if $v_s \leq v_t$ holds for any vector v of $\left(R^P_\varepsilon\right)_o$. □

Proof. [Proof of Theorem 6] In view of Proposition 7, we need only show that a binary vector v generates an extreme ray of R^P_ε if and only if it satisfies both Properties *1* and *2*.

Assume that the binary vector v belongs to $\left(R^P_\varepsilon\right)_o$. Then v satisfies Property *1*, because $v_s = 1$ and $s \sqsubset t$ for some s, t in $X + X'$ imply $1 \leq v_t$ by Proposition 8. Next we show by contradiction that if v generates an extreme ray of $\left(R^P_\varepsilon\right)_o$, then v satisfies Property *2*. Setting $S = \{s \in X + X' \,|\, v_s = 1\}$, we assume that S is contained in the last class of $\sqsubseteq$ and also $|S| \geq 2$. Clearly $S \subseteq X'$. Take an element k in S and form the vector u having $u_k = 1$ and all other components set to zero. Form also the vector $w = v - u$. Then u, w belong to $\left(R^P_\varepsilon\right)_o$ and $v = u + w$, with u and w not proportional, contradicting the assumption that v generates an extreme ray of R^P_ε. This shows $|S| = 1$.

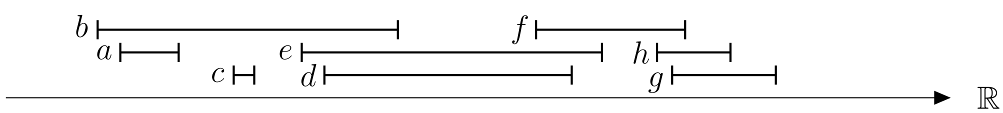
Conversely, let the binary vector v satisfy Properties *1* and *2* and set again $S = \{s \in X + X' \,|\, v_s = 1\}$. To show that the ray generated by v is an extreme ray of $\left(R^P_\varepsilon\right)_o$, we first derive $v \in \left(R^P_\varepsilon\right)_o$ from Property *1* and Equation (9). Second, we assume $v = u + w$ for some vectors u, w in $\left(R^P_\varepsilon\right)_o$ and prove that u and w are proportional. Let s be in S. If $v_s = 0$, then $u_s = 0 = w_0$. If $v_s = 1$, then for all t in $X + X'$ such that $s \sqsubset t$ we have $v_t = 1$ (Property *1*). Except if s is an element of the last class of $\sqsubseteq$, we deduce $u_s = u_t$ and $w_s = w_t$ (because, by Proposition 8, $u_s \leq u_t$ and $w_s \leq w_t$). Consequently, if S is not included in the last class of $\sqsubseteq$, we derive that the nonzero components of u are equal one to the other, that a similar property holds for w and in fact that u and w are proportional vectors. On the other hand, if S is a subset of the last class of $\sqsubseteq$, Property *2* implies $|S| = 1$. As a trivial consequence, u and w are then also proportional vectors. □

6. TWO EXAMPLES

In this section, the main concepts and results of the paper are illustrated on two examples. After having selected an interval order, we collected data by running the software `porta` [Christof, Loebel, 2010] on the system of inequalities which define representations (Equation (5)). Of course, we could have used the results of the previous sections to derive by hand all the information required on the two examples.

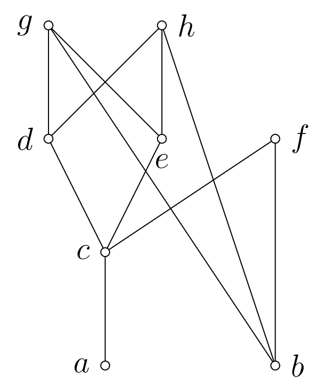
6.1. EXAMPLE 1

Consider eight intervals on the oriented real line, which represent the elements in the set $X = \{a, b, \dots, h\}$ (here $n = 8$).

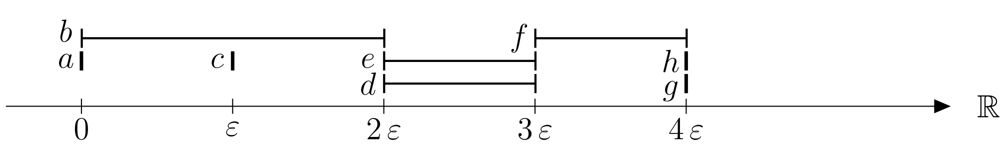


The resulting interval order P is described just below by both its 'table' and its 'Hasse diagram'. Notice that the two elements d and e are 'twins', in the sense that each of them has exactly the same comparisons as the other one with all the elements in X; the same holds for elements g and h.

P	a	b	c	d	e	f	g	h
a			1	1	1	1	1	1
b						1	1	1
c				1	1	1	1	1
d							1	1
e							1	1
f								
g								
h								



The minimal representation is as follows.



The twenty-three facet-defining inequalities for R_ε^P are indicated by the marked entries in the next 0/1-table. According to Theorem 4, there are four types of such inequalities. The first one, $0 \leq x_i$, obtains when the element i is minimal for P; in our example a and b are the two minimal elements (indicated in the table by an underline). Combining Proposition 4 and Theorem 4, we see that the inequality $x_i \leq x'_i$ (where $i \in X$) defines a facet of R_ε^P if and only if $x_i^* = x_i'^*$; this occurs here for a, c, g and h (shown in the table by an overline). Facet-defining inequalities of the third type, that is $x'_a + \varepsilon \leq x_b$, come from the noses (a, b) of P; the latter are shown by a “ $\llcorner\underline{1}$ ” in the table. Finally, facet-defining inequalities of the fourth type, that is $x'_c \geq x_d$, correspond to the hollows (c, d) of P; the hollows are indicated by a “ $\overline{0}\!\urcorner$ ”.

P	a	b	c	d	e	f	g	h
a	$\overline{\underline{0}}$	$\overline{0}\!\urcorner$	$\llcorner\underline{1}$	1	1	1	1	1
b	0	$\underline{0}$	0	$\overline{0}\!\urcorner$	$\overline{0}\!\urcorner$	$\llcorner\underline{1}$	1	1
c	0	0	$\overline{0}$	$\llcorner\underline{1}$	$\llcorner\underline{1}$	1	1	1
d	0	0	0	0	0	$\overline{0}\!\urcorner$	$\llcorner\underline{1}$	$\llcorner\underline{1}$
e	0	0	0	0	0	$\overline{0}\!\urcorner$	$\llcorner\underline{1}$	$\llcorner\underline{1}$
f	0	0	0	0	0	0	$\overline{0}\!\urcorner$	$\overline{0}\!\urcorner$
g	0	0	0	0	0	0	$\overline{0}$	$\overline{0}\!\urcorner$
h	0	0	0	0	0	0	$\overline{0}\!\urcorner$	$\overline{0}$

The global trace $\precsim$ (a weak order) is summarized below by its ordered list of classes

$$\{a, a', b\} \prec \{c, c'\} \prec \{b', d, e\} \prec \{d', e', f\} \prec \{f', g, g', h, h'\}$$

and similarly for the weak order $\sqsubseteq$ (here, one-element sets lose their enclosing braces)

$$\{a, b\} \sqsubset a' \sqsubset c \sqsubset c' \sqsubset \{d, e\} \sqsubset b' \sqsubset f \sqsubset \{d', e'\} \sqsubset \{g, h\} \sqsubset \{f', g', h'\}.$$

The vertex of the representation cone R^P_ε, which corresponds to the minimal representation, is

$$\varepsilon\,(0,0,\ 0,2,\ 1,1,\ 2,3,\ 2,3,\ 3,4,\ 4,4,\ 4,4).$$

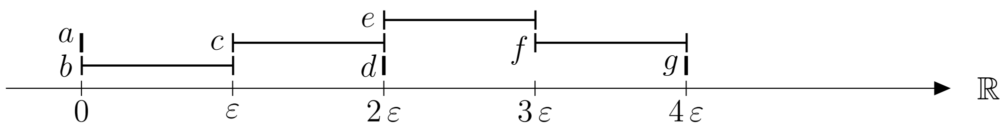
The representation cone R^P_ε has twenty extremal rays, each one generated by a binary vertex v located at x^*. To specify the rays, we provide the twenty sets $S = \{s \in X + X' \mid v_s = 1\}$ by listing their elements together with the relation induced on them by $\sqsubset$:

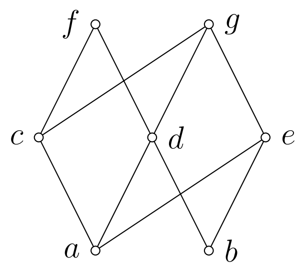
(1) $\{a, b\} \sqsubset a' \sqsubset c \sqsubset c' \sqsubset \{d, e\} \sqsubset b' \sqsubset f \sqsubset \{d', e'\} \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(2) $a \sqsubset a' \sqsubset c \sqsubset c' \sqsubset \{d, e\} \sqsubset b' \sqsubset f \sqsubset \{d', e'\} \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(3) $b \sqsubset a' \sqsubset c \sqsubset c' \sqsubset \{d, e\} \sqsubset b' \sqsubset f \sqsubset \{d', e'\} \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(4) $a' \sqsubset c \sqsubset c' \sqsubset \{d, e\} \sqsubset b' \sqsubset f \sqsubset \{d', e'\} \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(5) $c \sqsubset c' \sqsubset \{d, e\} \sqsubset b' \sqsubset f \sqsubset \{d', e'\} \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(6) $c' \sqsubset \{d, e\} \sqsubset b' \sqsubset f \sqsubset \{d', e'\} \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(7) $\{d, e\} \sqsubset b' \sqsubset f \sqsubset \{d', e'\} \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(8) $d \sqsubset b' \sqsubset f \sqsubset \{d', e'\} \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(9) $e \sqsubset b' \sqsubset f \sqsubset \{d', e'\} \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(10) $b' \sqsubset f \sqsubset \{d', e'\} \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(11) $f \sqsubset \{d', e'\} \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(12) $\{d', e'\} \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(13) $d' \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(14) $e' \sqsubset \{g, h\} \sqsubset \{f', g', h'\}$,
(15) $\{g, h\} \sqsubset \{f', g', h'\}$,
(16) $g \sqsubset \{f', g', h'\}$,
(17) $h \sqsubset \{f', g', h'\}$,
(18) f',
(19) g',
(20) h'.

This provides a good illustration of Theorem 6.

6.2. Example 2

We provide the essential data of a second example, starting with the minimal representation and then the Hasse diagram together with the table pointing to the facets.





P	a	b	c	d	e	f	g
a	$\underline{\overline{0}}$	$0^{\urcorner}$	$_{\llcorner}1$	1	1	1	1
b	0	$\underline{0}$	$0^{\urcorner}$	$_{\llcorner}1$	$_{\llcorner}1$	1	1
c	0	0	0	$0^{\urcorner}$	$0^{\urcorner}$	$_{\llcorner}1$	1
d	0	0	0	$\overline{0}$	$0^{\urcorner}$	$_{\llcorner}1$	1
e	0	0	0	0	0	$0^{\urcorner}$	$_{\llcorner}1$
f	0	0	0	0	0	0	$0^{\urcorner}$
g	0	0	0	0	0	0	$\overline{0}$

Here are descriptions of the weak orders $\preccurlyeq$ and $\sqsubseteq$:

$$\{a, a', b\} \prec \{b', c\} \prec \{c', d, d', e\} \prec \{e', f\} \prec \{f', g, g'\},$$
$$\{a, b\} \sqsubset a' \sqsubset c \sqsubset b' \sqsubset \{d, e\} \sqsubset \{c', d'\} \sqsubset f \sqsubset e' \sqsubset g \sqsubset \{f', g'\}.$$

The vertex of the representation cone R^P_ε is

$$\varepsilon\,(0,0,\ 0,1,\ 1,2,\ 2,2,\ 2,3,\ 3,4,\ 4,4).$$

We specify the rays following the same convention as in Example 1:

$$\begin{array}{lr}
(1) & \{a, b\} \sqsubset a' \sqsubset c \sqsubset b' \sqsubset \{d, e\} \sqsubset \{c', d'\} \sqsubset f \sqsubset e' \sqsubset g \sqsubset \{f', g'\},\\
(2) & a \sqsubset a' \sqsubset c \sqsubset b' \sqsubset \{d, e\} \sqsubset \{c', d'\} \sqsubset f \sqsubset e' \sqsubset g \sqsubset \{f', g'\},\\
(3) & b \sqsubset a' \sqsubset c \sqsubset b' \sqsubset \{d, e\} \sqsubset \{c', d'\} \sqsubset f \sqsubset e' \sqsubset g \sqsubset \{f', g'\},\\
(4) & a' \sqsubset c \sqsubset b' \sqsubset \{d, e\} \sqsubset \{c', d'\} \sqsubset f \sqsubset e' \sqsubset g \sqsubset \{f', g'\},\\
(5) & c \sqsubset b' \sqsubset \{d, e\} \sqsubset \{c', d'\} \sqsubset f \sqsubset e' \sqsubset g \sqsubset \{f', g'\},\\
(6) & b' \sqsubset \{d, e\} \sqsubset \{c', d'\} \sqsubset f \sqsubset e' \sqsubset g \sqsubset \{f', g'\},\\
(8) & d \sqsubset \{c', d'\} \sqsubset f \sqsubset e' \sqsubset g \sqsubset \{f', g'\},\\
(9) & e \sqsubset \{c', d'\} \sqsubset f \sqsubset e' \sqsubset g \sqsubset \{f', g'\},\\
(10) & \{c', d'\} \sqsubset f \sqsubset e' \sqsubset g \sqsubset \{f', g'\},\\
(11) & c' \sqsubset f \sqsubset e' \sqsubset g \sqsubset \{f', g'\},\\
(12) & d' \sqsubset f \sqsubset e' \sqsubset g \sqsubset \{f', g'\},\\
(13) & f \sqsubset e' \sqsubset g \sqsubset \{f', g'\},\\
(14) & e' \sqsubset g \sqsubset \{f', g'\},\\
(15) & g \sqsubset \{f', g'\},\\
(16) & f',\\
(17) & g'.
\end{array}$$

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