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Review of Graßmann, Robert, *Theory of Number or Arithmetic in Strict Scientific Presentation by Strict Use of Formulas* (1891)

Otto Hölder (1859-1937)
Translation by Mircea Radu

Graßmann, Robert, *Theory of Number or Arithmetic in Strict Scientific Presentation by Strict Use of Formulas.* Stettin, Robert Graßmann Pub. Co. 1891, XII and 242 pages. Price: 5 Marks.

The author of this book pursues the objective of treating the whole of pure mathematics [*die ganze reine Mathematik*] in four sections [*Abtheilungen*].[a] One half of the first of these sections is dedicated to arithmetic and is already available. The other half of the first section "A heuristic treatise on number [*Zahlenlehre in freier Gedankenentwicklung*]" which treats the same discipline is supposed to follow.[b] The author may have opted for such an unusual separation [of the treatment of arithmetic – M. R.] in the assumption that this

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a. Robert Graßmann opens the preface of his book with the following words: "The theory of forms or mathematics will be presented (...) free of any logical fallacy [*Trugschluss*]. It will cover all branches [*Zweige*] of pure mathematics: the theory of numbers or arithmetic [*Zahlenlehre*], the theory of functions [*Funktionenlehre*], the calculus of extension [*Ausdehnungslehre*], and the *Erweiterungslehre*", [Graßmann 1891, Preface]. I found no adequate translation for Graßmann's *"Erweiterungslehre*". Robert Graßmann explains that the *"Erweiterungslehre*" is a development of ideas put forward by Hermann in his *"Ausdehnungslehre*" under No. 410-527. This description only fits the second edition of Hermann Graßmann's *Calculus of Extension* of 1862. There Hermann Graßmann treats fundamental subjects belonging to analysis. Kannenberg translates Hermann Graßmann's title of the corresponding chapter by "Theory of Functions". These dificulties need not concern us here, since Hölder's *Review* only covers arithmetic.

b. Graßmann emphasizes the need to provide two complementary presentations [*die Zahlenlehre in zwei Formen darstellen*] of arithmetic. He calls the first presentation a "strict development by formulas" [*strenge Formelentwicklung*]. The second is called "freie Gedankenentwicklung", a rather uncommon expression, not so easy to translate appropriately. One should perhaps translate this by "a speculative conceptual development" or perhaps a "heuristic treatment" of arithmetic. Graßmann's distinction, is the consequence of his general concept of method. His main methodological concern was to accomplish an abstract and logically stringent development of arithmetic by formulas, the only possible strictly scientific [*wissenschaftliche*] prewould underpin his strong commitment to rigor [*Strenge*], something of great importance to him. [c]

The *Theory of Number* [d] begins with an extensive introductory chapter [*Einleitung in die Zahlenlehre*]. It contains a theory of operations with abstract magnitudes in the most general sense of the term [*Theorie der Größenoperationen im weitesten Sinne des Worts*]. Even at this early stage of the presentation, great care is taken to accommodate multiplication as an operation, which, in addition to not satisfying the commutative law, does not even satisfy the associative law. The introduction also provides, right from the beginning, a discussion of the various forms of proof [*Beweisformen*], particularly of the so-called mathematical induction [*vollständige Induction*]. [e]

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sentation of arithmetic (in his opinion). Graßmann claims that the "Formulas" (they are special aggregations of signs) used by him are direct expressions of the thinking operations [*Denkoperationen*], icons of the mental operations involved in arithmetic. In Graßmann's view, such a strictly formal treatment of arithmetic is elementary and scientific at the same time, and is therefore also appropriate and, indeed, necessary for teaching. Graßmann's understanding of his own account of the mathematical inscriptions is reminiscent of Leibniz's famous ideal of developing a *Calculus Ratiocinator*, an ideal that Hölder considered impossible to achieve. The second "speculative" or "heuristic" treatment of arithmetic mentioned by Graßmann seems to have a genuine pedagogical function. It is supposed to deal with an aspect of arithmetic that the strict scientific treatment cannot deal with, namely, additional explanations and descriptions of alternative paths of proving the same result [Graßmann 1891, Preface, v ff.]. Hölder, who also emphasizes conceptual clarity, regards Graßmann's identification of symbolic expression and conceptual content as a blind alley and Graßmann's obsession with it a methodological evil. Splitting the treatment of arithmetic in a scientific and a separate didactic part only becomes necessary because of Graßmann's belief that in mathematics abstract sign aggregates are the only direct and therefore the only appropriate expression of thinking. As far as I know, Graßmann never wrote the second book.

c. The word "Strenge" occurs twice in the title of Graßmann's book: "The Theory of Number or Arithmetic in strict scientific presentation by strict use of formulas [*Die Zahlenlehre oder Arithmetik streng wissenschaftlich in strenger Formelnentwicklung*]."

d. Due to the meaning that the expressions "number theory" and "theory of numbers" have today, it would be misleading to use any of them as a translation of Graßmann's "*Zahlenlehre*". In Graßmann's work, this term refers to the general construction of the number systems beginning with the system of the natural through to that of the complex numbers. Graßmann does not use set-theory in his work, so that it would also be misleading to use the word "set" in describing his work. The same difficulties hold for the term "Arithmetik". To avoid misunterstandings I have translated Graßmann's term "Zahlenlehre" by "Theory of Number".

e. Graßmann calls the theory presented in his introductory chapter "Theory of Magnitudes [*Größenlehre*]". This theory of magnitudes includes, among other things, a general discussion of algebraic operations and structures. It extends over fifty (!) pages. To the modern reader, the "Größenlehre" looks like a general theory of algebraic structures. In it Graßmann harshly criticizes Paul du Bois-Reymond, "who fails to recognize the purely formal nature of mathematics" [Graßmann 1891, 2]. More importantly Graßmann writes "Mr. Paul du Bois-Reymond (...) regards

The introduction is followed by the treatment of arithmetic in four chapters. The first chapter contains the four fundamental operations with whole—positive and negative—numbers and with fractions. Results concerning prime-factor decomposition of integers are also included. Beside that, this chapter also contains a detailed exposition of practical calculations with decimal fractions and with named numbers [*benannte Zahlen*]. The second chapter deals with powers, roots and logarithms; one also finds the binomial and the geometrical series, as well as the arithmetical series here. The third chapter contains a combined treatment of trigonometry and of the complex numbers, which also includes solving algebraic equations up to the fourth degree. Approximation methods for determining higher order roots [of polynomials – M. R.] are also presented.

Selection of the material covered is determined by the author's aim of writing a book for teachers [*Lehrer*] that at the same time meets the highest standards of scientific rigor. One can accept that the elementary teaching of arithmetic should be pursued in a more rigorous way. It is often dealt with mechanically, with insufficient attention dedicated to the justification of the methods presented. One would, however, not deny that it is possible to go too far in the pursuit of rigor.

I would not recommend Graßmann's treatment of arithmetic for teaching purposes. The treatment of the operations [*Rechnungsoperationen*] in the most general terms possible is too abstract for the student. The concepts dealt with [in schools – M. R.] should rather be introduced starting with the positive whole numbers [*die positiven ganzen Zahlen*]. I do not attach great weight to the objection already raised by Mr. Graßmann in his book that, in this case, one would have to repeat the same steps over and over again. Transposing a proof [*Beweis*] from a special case to a more general one is a useful exercise. In the case of frequent repetition, one can invoke the analogy and proof may be omitted. A presentation which begins with concrete examples also has the advantage that certain objects and operations satisfying certain computation laws are known beforehand, so that one has a firm ground under one's feet, right from the start.

I now come to my fundamental criticism of the author. He holds his approach to be a milestone on the path towards a rigorous treatment of arithmetic and scorns alternative treatments. It is thus only reasonable to judge his work

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the pure theory of number [*Zahlenlehre*], which generates its numbers and number-magnitudes by itself, as nothing more than a symbolic game, comparable to chess, not a science" [Graßmann 1891, 3]. This reminds of Hermann Weyl's ciritque of formal axiomatics. In his *Review*, Hölder never explicitly mentions Graßmann's critique of du Bois-Reymond. Much of Hölder's criticism of Graßmann and indeed many of Hölder's subsequent writings, however, attach great importance to the distinction between the symbolic statement of mathematical results and methods (their mere form, as it were), on the one hand, and their true conceptual origin, on the other [Radu 2003, 365 ff.]. In a sense, Hölder's *Review* can be seen as a reinforcement of the position attributed to du Bois-Reymond by Graßmann.

by the same high standards. I do not, however, believe that his work can pass such a test.

It is perhaps not so easy to do justice to the author, because his conception differs so strongly from the commonly held views. His conception is, so it seems, underpinned by a peculiar philosophical principle.

By magnitude [*Größe*] he understands (No. 2) "everything that is or can become an object of thinking [*Gegenstand des Denkens*], in that it only possesses one rather than multiple values [*Werthe*]". He calls equal (No. 10) "two magnitudes which can be replaced the one for the other in the connections [*Knüpfungen*] of the theory of magnitudes [*Größenlehre*] without, however, changing the value of the connection". [f]

I do not wish to claim that the term "magnitude" was defined by replacing it with the synonymous term "value". Requiring of a magnitude to have a unique value is, I think, simply the expression of the determination of the principle used when comparing magnitudes. But should there be no objects of thinking that can be compared based on criteria not involving value? And how are we supposed to apply the above criterion of equality? Conceptual formulations of this kind remind us of Scholastic philosophy, which is not yet fully extinct in Germany. [g]

In my view, one should begin by describing the objects to be dealt with. One should afterwards explicitly name the criteria to be used for comparing objects, for deciding if two objects are declared equal or unequal. [h] Then,

f. These formulations are an interesting expression of the difficulties still being encountered in 1892 with describing concepts and results that, from our point of view, belong to the standard description of algebraic structures. They also represent a historic relic insofar as they repeat a terminology originally developed by Hermann Graßmann and published in the 1844 edition of his famous *Calculus of Extension*. In this book Hermann Graßmann makes a great effort to create a language suitable for a presentation of vector-algebra. Hermann Graßmann's terminology was developed further in the writings of his brother Robert.

g. Graßmann is defining magnitudes, value, and operation. His definitions are not easy to use. Could Graßmann have introduced all these notions as primitive ones, as Hölder seems to be suggesting here? The term "magnitude" might perhaps have been taken as a primitive notion. What about "value" and "operation"? Graßmann's definition of his operation concept expresses a perfectly legitimate requirement also found in modern definitions of the concept. Today we state this simply by saying that an algebraic operation is a function. Graßmann does not think of operations in these terms, and states his definition in alternative ones. I find Graßmann's definition of operation adequate.

h. As will be seen, Hölder returns several times to this issue. A better understanding of Hölder's intentions can be reached by taking his subsequent writings into account. It concerns the relationship between equality as a general logical concept and equivalence. Hölder explains that a general concept of equality, defined by means of general axioms is useless in mathematics because, in his view, such a concept would be completely vacuous. Hölder argues that equality makes sense only as equality between well defined equivalence-classes in respect to some explicitly defined equivalence relation. Thus the general axioms of equality only make sense if they are proved as theorems based on equivalence relations defined in advance.

after having clarified the operations involved, one can move on to proving, say, that equals added to equals yields equals. In this manner, the proposition according to which equals can be substituted for equals [*daß man Gleiches für Gleiches setzen kann*] would become a non-tautological proposition [Lehrsatz], and only such propositions allow fruitful applications. My remarks concerning the treatment of fractions [*Bruchlehre*] provides an illustration of this.

Still, I do not wish to attach too much importance to the definitions mentioned above. They are only an external robe [*äußeres Gewand*]. In fact, there is, despite these definitions, little to object to with regard to the consistency [*Folgerichtigkeit*] of the inferences given in the general introductory part. Solely, under this kind of derivation [*Herleitung*], it is only possible to assign a hypothetical validity [*eine hypothetische Giltigkeit*] to the results thus derived.[i] If, for instance, an addition operation is defined on a domain of magnitudes, and this operation admits an inverse operation defined without restrictions (that is, if it leads to a subtraction that can be always carried out, i.e., the outcome of which is uniquely defined), Mr. Graßmann calls this addition a separable connection [*trennbare Knüpfung*].[j] However, the possibility of a separable connection should have been proved beforehand. In the case of a treatment such as that of Mr. Graßmann, which takes only a single operation as its starting point, this result may almost be seen as obvious, and the requirement of proving it overlooked. The matter would look rather different if two interconnected operations were considered. This happens when a multiplication or a separable multiplication is added to the separable addition. In this case, multiple relations between the two operations are required, and it is not immediately obvious, whether an iteration of these operations would not lead to contradictions. And even if the system as a whole were to be proved consistent [*widerspruchslos*] one would have to require a justification of its applicability.[k]

Such a justification should be given in the first subsection right after the general introduction. Here, however, something seems to be missing in the introduction of negative numbers and of fractions (No. 165): "division is the name of the separation corresponding to the multiplication of the numbers" and immediately afterwards the term "dividing-magnitude" [*Theilgröße*] is introduced without any critical examination.[l] Concerns about the existence of such a magnitude are simply overlooked. It goes without saying that, according to the spirit of Mr. Graßmann's own approach, a foundation of the theory of fractions must be laid independently of any considerations of external intu-

i. Here Hölder aknowledges that Graßmann's "Einleitung" provides a general axiomatic treatment of the theory of operations.

j. A "separable connection" is Graßmann's phrase for an invertible operation.

k. I have discussed these and other important claims in greater detail in [Radu 2003].

l. Graßmann defines "Teilgröße" as any magnitude that is preceded by the division sign ":". The term stands for the inverse element of an element a in respect to an algebraic operation, something now commonly denoted by a^{-1}. Graßmann writes $\frac{1}{a}$ instead, [Graßmann 1891, 71].

ition [*äußere Anschauung*].[1] Thus, our difficulty cannot be put to rest simply, say, by calling upon the fact that it is always possible to divide continuous extended magnitudes.

It would, however, be possible to hold another position, which, even though nowhere explicitly stated in the book discussed here, may express the actual conception of its author, namely, that the *peculiar* basic formulas [*besondere Grundformeln*] [m] posited at the beginning of the treatment are so chosen as to define both the numbers and the operations at the same time. [n] This is to some extent correct. To clarify this, I must emphasize two distinct concepts standing behind the expression whole number [*ganze Zahl*]. [o]

The concept of cardinal number [*Cardinalzahl oder Anzahlbegriff*] emerges based on comparing aggregates of discrete objects. This is done by associating one individual of one of the aggregates to an individual of the other aggregate. While doing this, one examines whether, while performing this procedure of comparing the two aggregates, both aggregates are simultaneously exhausted or not. As Mr. v. Helmholtz pointed out,[2] Mr. Schröder[3] was the first to recognize that here an additional assumption is tacitly made, namely, that the outcome of this comparison is independent of the manner in which it is carried out. [p] Because this fact can be easily proved,[4] a treatment of arithmetic taking the concept of cardinal number as its starting point, looks unproblematic to me. I also do not accept the idea that the cardinal number concept requires external experience, for I am able to count things given just in thinking—names stored in my memory, or things like that. In doing this, one only needs such psychological and logical actions as are required by any presentation of arithmetic and by any mathematical deduction.

Another possibility is to take the ordinal number concept as a starting point. It is possible to carry out the addition, subtraction, and multiplication of the whole numbers without previously introducing the cardinal number concept.[5] In this case one regards the number sequence

$$1, 2, 3, 4, 5, \ldots$$

as made of arbitrary signs, which gain their specific meaning only from their fixed order within the sequence. To add 1 to a number a thus means nothing

m. Hölder calls certain equations—such as $a + (b + 1) = (a + b) + 1$—that are used by Graßmann "basic formulas", "formulas", and also "equations". This will become clearer in the sequel.

n. In Hölder's view, such an interpretation of the basic formulas used by Graßmann would make it possible to regard the formulas as constructive procedures and not as descriptive axioms. Hölder is right, when emphasizing that Graßmann does not discuss this distinction. It is indeed not easy to establish whether Graßmann is clearly aware of this distinction and of its significance. Reading Robert Graßmann's writings, one gets the impression that the genetic and the axiomatic standpoint are conflated.

o. Hölder has positive natural numbers in mind.

p. It is independent of the order in which the elements of the two sets are associated to each other.

other than moving on to the next member of the sequence, which is then denoted by $a+1$ in addition to its original notation.[6] It now becomes possible to deduce all the laws of addition based on the formula:

$$a + (b + 1) = (a + b) + 1. \tag{1}$$

This formula can be used as a definition of addition, because it explains what it means to add $b + 1$, assuming that one already knows what it means to add b. Since adding 1 is fully defined, everything is uniquely determined. In this sense, it is possible to say that taken together, the formulas

$$\begin{aligned} &1 \cdot a = a \\ &(b + 1) \cdot a = b \cdot a + a \end{aligned} \tag{2}$$

define multiplication (in $b \cdot a$ the number b should be seen as the multiplying factor), and all the laws of multiplication can be deduced [*deducieren*] based on these formulas.

The formulas I have labeled by 1) and 2) were introduced as a foundation for arithmetic by the Graßmann brothers a long time ago,[7] and founding arithmetic on such simple principles is a great accomplishment of the Graßmann brothers. It is possible, as pointed out by Mr. v. Helmholtz, to move on directly to the negative numbers, by pursuing the number sequence backwards. One then gets the bi-directional infinite sequence

$$\ldots 5', 4', 3', 2', 1', 0, 1, 2, 3, 4, 5, \ldots$$

Every sign introduced to the left or to the right of the sequence is taken to be distinct from all the others. By holding the condition governing the addition of 1 to be generally true, and by requiring equation 1) to hold successively for $b = 0, 1', 2', 3', \ldots$ one obtains, one after the other, the definitions [*Erklärungen*] ruling the addition of 0, of $1'$, of $2'$ and so forth. It then becomes possible to take a as a positive or as a negative number in formulas 2). By positing $b = 0, 1', 2', 3', \ldots$ one gets conditions, which are sufficient for defining multiplication for negative multiplication-factors. Developing subtraction then raises not the slightest difficulty. One must define $a - b$ as a fully determined number satisfying the equation [*Gleichung*]

$$(a - b) + b = a \tag{3}$$

and, with this, subtraction becomes possible in all cases.

Frequently negative numbers are introduced by directly allowing symbols of the form $a - b$, which are supposed to satisfy equation 3). In this respect, the treatment of Arithmetic discussed here follows this practice. Equivalence relations hold between symbols of the form $a - b$. These equivalences are fully determined by equations 1) and 3), as soon as we also add that the equation

$$x + b = a$$

always has a unique solution. Equation 3), for instance, implies that

$$(3-5)+5=3.$$

By adding 1 we next get

$$((3-5)+5)+1=4.$$

Then, by applying equation 1) on the left side of the former equation, we obtain

$$(3-5)+6=4.$$

This proves that $3-5$ is the solution of the equation $x+6=4$, and therefore, one must posit

$$3-5=4-6.$$

It is, however, not entirely obvious that this way of introducing the symbols $a-b$ is justified. It seems conceivable that, taken together, equations 1), 2), and 3) may also lead to equivalences of a quite different kind. It would then no longer be permissible simply to posit that (Graßmann No. 114) "Numbers generated by successively adjoining of 1, are all taken as distinct from each other".

As soon as this difficulty is eliminated in the way proposed above [q] or in any other way, it becomes possible to regard equations 1), 2), and 3) as definitory relations [*definierende Relationen*][8] for the positive numbers, for the negative numbers, and for the operations of addition, subtraction, and multiplication to be carried out on these numbers.

The path adopted in the book under scrutiny here is in essence the path just described, which is rooted in the ordinal number concept. [r] With respect to multiplication, we must, however, emphasize that once the formula

$$(b+1)\cdot a=b\cdot a+a.$$

has been postulated, it is inappropriate to adopt the equation

$$a\cdot(b+1)=a\cdot b+a$$

q. By defining natural number taking the cardinal-number concept as a starting point.

r. Here Hölder seems to be oscillating between his sharp criticism of Graßmann's approach as something fundamentally based on, as it were, abusive use of the permanence principle and an alternative interpretation according to which "in essence" Graßmann's approach is simply a genetic approach similar to that advocated by Hölder himself. This uncertainty is rooted in the way in which Graßmann's book reviewed by Hölder is written. The two tendencies seem to be competing with each other. This also holds for all the other books written by the Graßmann brothers, and it is responsible for the ongoing debate concerning the place of axiomatics in the writings of the Graßmann brothers. Compare [Radu 2003]; [Radu 2011].

as an additional postulate. I hope I have managed to show that the former equation suffices for establishing the definition of multiplication, and, for that reason, the latter equation can no longer be introduced by arbitrary stipulation. Such a practice cannot exclude the emergence of contradictions. If the former equation is regarded as the definition of multiplication, it then becomes necessary to regard the latter as a proposition [*Lehrsatz*]. I have reached the conviction that it can be proved.

The difficulties facing introducing fractions [*gebrochene Zahlen*] without relying on geometric premises can be easily overcome based on an idea generally used by Mr. Stolz [*einen Gedanken, der in allgemeiner Weise von Herrn Stolz durchgeführt worden ist*]. The theory of integers and its first three operations can thereby be taken as accomplished. The propositions ruling over the relations "greater than" and "smaller than" are subsequently introduced without difficulty. Expressions of the form $\frac{a}{b}$ are considered next, a and b being positive or negative integers, b being different from 0. Such a symbol has no meaning attached to it yet. It is simply a mere form inside which we only can distinguish two numerical values: a and b. We are obviously able to call the numbers a and b numerator and denominator. Two such symbols seem, at a first glance, different, if their numerators and denominators are not respectively identical. We then, however, explicitly stipulate that two such symbols $\frac{a}{b}$ and $\frac{a'}{b'}$ should be seen as equivalent only if $ab' - a'b = 0$ holds. It is now possible to prove that, if two such symbols are equivalent to a third, then they are equivalent to each other. If all symbols equivalent to each other are united to form a single category, it follows that all symbols of one category are equivalent to each other. We have thus created a new concept, for which we use the term "value". We attribute the same value to all mutually equivalent symbols. Addition and multiplication are next defined directly through the formulas

$$\frac{a}{b} + \frac{a'}{b'} = \frac{ab' + a'b}{bb'}$$

$$\frac{a}{b} \cdot \frac{a'}{b'} = \frac{aa'}{bb'},$$

and the only thing left to prove (and this is an easy exercise) is that the value of the sum depends only on the value, not on the form, of the magnitudes added.

Using this approach, propositions such as "if two magnitudes are equal to a third, then they are equal to each other" and "equals can always be substituted for equals" are not tautological [*tautologisch*]. These propositions were made redundant by the ordinal construction of the whole numbers [*ganze Zahlen*], because, in that case, distinct objects of the same value were not available.

The earlier whole number a is now equivalent with the symbol $\frac{a}{1}$. The associative and commutative laws for addition and multiplication of fractions can now be easily proved. This also holds for the law connecting the two operations—namely, the distributive law. There is no difficulty in moving on to proving the laws governing subtraction and division. The relations "greater than" and "smaller than" can be easily defined as well. By definition it is stipulated that $\frac{a}{b} > \frac{a'}{b'}$, if $ab' > a'b$, whereby the numbers a, a', b, b' are positive integers. Propositions such as the following then become provable: of two different numbers, one is greater than the other; if a number is greater than a second one, and the latter is greater than a third, then the first is also greater than the third; each number can be repeatedly multiplied by another until it becomes greater than some given other number;[9] adding greater numbers to equal or greater numbers leads to greater numbers, etc. [s]

What was previously said regarding the introduction of the negative integers and of the fractions in the new book of Mr. Graßmann applies, of course, even more so, with respect to the introduction of the roots and the logarithms in the second chapter of the book. Graßmann however introduces even the irrationals without any justification. He simply provides a definition (No. § 82): "Irrational numbers are magnitudes that are not rational numbers [*nicht Endzahlen sind*]. [t] The laws ruling the comparison of rational numbers also hold for the irrational numbers." This definition is followed by a proposition (No. 383): "All propositions found in arithmetic, which hold for arbitrary integers and for fractions, also hold for the irrational numbers." If the existence theorems are taken for granted,[10] then Mr. Graßmann's treatment is basically consistent. [u]

I do not wish to discuss Graßmann's introduction of $\sqrt{-1}$ in the third subsection. Nor do I wish to raise the question of whether the straight forward introduction of the concept of "oriented angle" [*Winkel der Richteinheit*] in No. 435 is consistent with the point of view initially adopted by its author. It seems that, in this case, considerations of angle-measurement were taken into

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s. The way in which Hölder expresses these well-known laws is a bit surprising. His formulations are rather cumbersome. Indeed they sound very much like the formulations of these laws found in Euclid's books. It would have been a great deal easier to express these laws symbolically. Contrary to Graßmann, Hölder, however, is not happy with symbolically expressed laws. In his subsequent methodological writings, he constantly emphasizes the "conceptual" nature of these laws and the fact that symbolic abbreviations are nothing more than a more economic expression of laws originally stated in conceptual terms. (Compare [Radu 2003, 345–359]).

t. Graßmann often uses terms no longer common today. Rational numbers are called "Rationalzahlen" and also "Endzahlen". Irrational numbers are called "Irrationalzahlen" and also "Unzahlen". Graßmann defines irrational numbers as numbers that are not rational. Rational numbers are defined in the usual way, [Graßmann 1891, 182].

u. Hölder's comments on Graßmann's treatment are surprisingly mild. Graßmann's "definition" 382 is anything but harmless. Graßmann does not just define irrational number, but also explains that certain comparison laws hold for the newly introduced numbers as well. These laws are those already quoted by Hölder in the present *Review* in his discussion of the comparison of rational numbers. Such a claim, however, should be a theorem rather than an axiom. It certainly cannot be a definition. Also, Graßmann's proposition 383 reproduced by Hölder is adopted without any proof by Graßmann.

account. I wish, however, to use another example, to show that the author, who is so severe in criticizing others for their "fallacies" [*Trugschlüsse*], is not himself fully free of them. In No. 451 the formula

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

is proved as follows: Calculating the left side of the equation leads to

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).$$

This magnitude should also be a fundamental unit, or, as it could be expressed, its module should be 1. It therefore can also be brought to the form

$$\cos\gamma + i\sin\gamma \tag{4}$$

in which γ depends on α and β. The author, therefore, posits

$$\gamma = \alpha \circ \beta \tag{5}$$

and calls this a connection [*Verknüpfung*] of α and β. After showing that $\alpha \circ \beta$ becomes β for $\alpha = 0$ and α for $\beta = 0$, the deduction is pursued in this way: "The connection $\alpha \circ \beta$ is, therefore, the connection having zero [*die Null*] as its non-changing magnitude, that is, the connection is the addition." No. 71 is invoked as a basis for this inference. There, however, one only finds the following:

> Definition. The non-changing magnitude [*nicht ändernde Größe*] of addition [*Fügung*] is called zero [*Null*]. The sign of zero is 0. [v]
>
> Zero is thus that magnitude which can be connected to any other, without thereby changing the value of the latter magnitude; or
>
> The connection having zero as its non-changing magnitude is the *Fügung* or addition. [w]

To this I emphasize that according to Mr. Graßmann's terminology a connection [*Knüpfung*] is the most general term used to describe a combination of magnitudes. In No. 5 we read:

> A connection of magnitudes is any combination or union of magnitudes that is in the reach of the human mind, in as far as its outcome has just one not multiple values.

Basically, this means that, every single valued function of two variables $F(\alpha,\beta)$, with $F(0,\alpha) = F(\alpha,0) = \alpha$, must coincide with $\alpha+\beta$. I leave

v. An alternative translation of Graßmann's term "nicht ändernde Größe" would be non-value-changing magnitude.

w. Obviously, Graßmann's term "non-value-changing magnitude [*nicht ändernde Größe*]" stands for the modern term "neutral element" with respect to an internal algebraic operation.

the criticism of such a claim to the reader. However, although the author did not indicate this, it is possible that

$$F(\alpha, \beta) = F(\beta, \alpha),$$
$$F(\alpha, F(\beta, \gamma)) = F(F(\alpha, \beta), \gamma)$$

can be deduced. Even so, one would have yet to prove that $F(\alpha, \beta) = \alpha + \beta$.

Tübingen Otto Hölder

Göttingische gelehrte Anzeigen, Nr. 15, 1892, 585–595.

Notes

[1] One should compare pages 2 and 3 of the introduction, where reference to measuring lengths is explicitly rejected.

[2] v. Helmholtz, Zählen und Messen, *Philosophische Aufsätze, Eduard Zeller zu seinem fünfzigsten Doctorjubiläum gewidmet*, Leipzig, 1887, [Helmholtz 1887, 19].

[3] *Lehrbuch der Arithmetik und Algebra,* Leipzig, 1873 [Schröder 1873].

[4] Compare O. Stolz, *Vorlesungen über allgemeine Arithmetik,* Leipzig 1885, [Stolz 1885, 9–10].

[5] Compare the already mentioned work of Mr. v. Helmholtz and the differing presentation given by Kronecker in the same collection of papers.

[6] In essence, this view had already been advocated by Leibniz. He used to define $a + 2$ through $(a + 1) + 1$ and could then provide his famous proof of the formula $2 + 2 = 4$.

[7] Hermann Graßmann, *Arithmetik,* Stettin 1860, Berlin 1861 [Graßmann 1860], Robert Graßmann, *Die Formenlehre oder Mathematik,* Stettin 1872 [Graßmann 1872].

[8] A similar view has been developed in the group-theory of Mr. Dyck: *Mathematische Annalen,* Bd. 20 [Dyck 1882]. In this case, one would, however, have to add the relation expressed by the associative law, which is generally valid, to the definitory relations explicitly stated in group-theory (and which only contain special symbols) in order to preserve the analogy to the text discussed here.

I do not wish to call these generating relations axioms. It seems to me that here the situation is different from that in geometry. The primitive concepts [*Grundbegriffe*] and first principles [*Grundsätze*] of geometry do not have their origin in sequential processes [*fortschreitenden Parcessen* (sic!)], of the kind used in geometric and arithmetic deductions alike. One may regard the first principles of geometry as rooted in intuitive evidence, or, perhaps more appropriately, as principles abstracted from external experience. In both cases, the first principles and the primitive concepts of geometry come from a foreign realm [a realm foreign to pure reason – M. R.]. If one wishes to regard the above mentioned formulas as axioms, one would then have to introduce similar axioms for innumerable concepts of arithmetic and analysis, and the number of arithmetical axioms would become infinite.

[9] This proposition may look fully obvious here. I emphasize it because it plays a great role when we treat limit-processes. In order to justify using arithmetical

propositions in geometry, it is necessary to introduce in geometry results analogous to some of the fundamental arithmetical theorems as geometric axioms. This holds for the proposition mentioned here. The importance of this proposition was strongly emphasized by Mr. Stolz.

[10] I think that the foundations of the theory of irrational numbers has been completed thanks to the works of Weierstraß, Dedekind, Lipschitz, and Cantor. It is possible to define such numbers on a purely arithmetical basis, introduce the concepts equal and greater than, define the operations and prove all propositions. The introduction of an irrational magnitude requires, of course, a specific law. I therefore wish to express my doubt with respect to the legitimacy of bringing all rational and all irrational numbers under a single general concept. I am not so sure that it is possible to construct the continuum, on which certain chapters of the theory of functions rely, in a purely arithmetical manner.

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