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Inaugural Academic Lecture held on July 22, 1899. With supplements and notes

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Intuition and Reasoning in Geometry

Inaugural Academic Lecture held on July 22, 1899. With supplements and notes

Otto Hölder (1859-1937)

Translation by Paola Cantù and Oliver Schlaudt *

The way in which geometrical knowledge has been obtained has always attracted the attention of philosophers. The fact that there is a science that concerns things outside our thinking and that proceeds inferentially appeared striking, and gave rise to specific theories of experience and space. Nonetheless, the geometrical method has not yet been sufficiently investigated. Philosophers who investigate the theory of knowledge discuss the question of whether geometry is an empirical science, but do not investigate in detail the technical concepts used by geometers. Yet, this would be appropriate. Mathematicians involved in the study of the foundations of geometry, and in particular of the so-called non-Euclidean geometry, which has often been rejected from a philosophical perspective, investigate what follows from certain assumptions. Even if geometers are aware of the fact that they did not make those assumptions arbitrarily, and that they could not make them completely arbitrarily, it is quite natural for them not to worry about the sources from which those assumptions flew out of. Geometers are only concerned with inferring consequences, and do not pay attention to the different mental activities involved in the exercise of this. A combination of different skills acting in unison would most likely solve the problem posed for the theory of knowledge by the existence of geometry.

Surveying the concepts used by geometers, one might notice a fundamental difference. Some concepts are defined by means of a construction, others are basically taken as given. The concept of a square is never introduced without a construction, no matter how well-known the figure is.[1] You start from an arbitrary side of the square, add at a right angle a second side that is equal in length to the first, then a third, and finally you close the figure. In such a

* The translation is a work of joint authorship, but Paola Cantù mainly contributed to the translation of the first half (up to page 23, endnote 42 included), while Oliver Schlaudt mainly contributed to the second half of Hölder's inaugural lecture.

construction one presupposes the notions of straight line, right angle, and equal lengths. The concept of a straight line is not defined by this construction.[2] The idea of a straight line appears thus as primary. Like the straight line, the point is also a given concept in geometry. The definition given by Euclid, i.e., that which has no parts, is not appropriate to give the idea of a point to someone who does not know what it is; indeed, this definition is not used by Euclid afterwards.[3]

Beside these concepts, which are available to the geometer as a given material, some assertions of a more general kind (which hold or should hold for these concepts) are also established from the beginning. These fundamental facts, being presupposed without proof, are called axioms or postulates.[4] For example, one uses all the time the axiom that a straight line can be traced between any two points and be thus fully determined.[5]

Where do the concepts that are taken for granted in geometry come from? And where do the axioms come from? These were the first questions raised as one began to investigate the theory of geometrical knowledge. Some answered: "From the intuition of space", others answered: "From experience". As is well-known, Kant saw the source of our geometrical knowledge in intuition. He tried to explain that this knowledge is—as he said—necessary,[6] by assuming a "pure" intuition, independent from experience. To explain that the laws of this pure intuition—that is proper to us—can be applied to the world of phenomena, Kant claimed that this intuition of ours is the condition set by our own nature for having a sensory experience of spatially extended objects. For this reason, some inner laws impose themselves on the form of our perception. So, according to Kant, intuition makes external experience possible and thus can not be explained as deriving from experience.[7]

Unlike Kant, other authors considered the intuitive representations of spatial objects that we come across in our imagination just as mnemonical representations of visual and tactile sense perceptions. We can arrange these representations in succession, and maybe also partially modify them, yet, they derive from sense perception. Intuition is considered as the source of geometrical knowledge but it appears here, conversely, as a result of experience.[8]

Geometry is immediately related to experience by those who consider axioms as the straightforward results of observations or measurements made on physical bodies. Thus the primitive concepts are also explained in a more physical way; for example, the straight line is for them the line of sight, or the position taken by a thread in a state of tension. This view was defended by outstanding natural scientists: it has recently been made prominent by Helmholtz, but it had already been fostered by Newton.[9] This same view has been attacked fervidly by Kant's followers, who considered it a vicious circle to ground on experience facts that are contained in intuition, because they saw intuition as a condition of external experience. If one wants to hold the above-mentioned view without running into fallacies, one should at any rate limit oneself to the observations that are not themselves grounded in geometrical considerations.[10] One should use facts of immediate perception.

An example of an immediate perception of this kind could be that a succession of smaller objects might overlap from a given point of view, whereas they do not overlap from another viewer's standpoint, and that these objects that overlap with respect to a given standpoint do not overlap partially from any other viewpoint.[11]

From such perceptions one can draw[12] the concept of the straight line and also the fact that there is one and only one straight line through two points. I have used here the facts related to the line of sight. We could also consider the taut thread. We must assume that the visual images which characterize the taut thread have a certain immediate[13] similarity to one another, and differ from those of a thread which is not taut. The simultaneous occurrence of those first visual images with some tactile sensations, and with muscular sensations provoked in us by the effort of making the thread taut, the experiences that we have when we pluck the taut thread, or when we look along it, build up a factual basis from which the concept of the straight line, and also the mechanical concept of tension can be drawn. Since the repeated attempt to stretch a thread always reaches the same position, experience teaches us that between two points there can be one and only one straight line.

Another concept that should be explained in a similar way is the concept of equality of line segments and angles. Whether two line segments are equal in size can be empirically tested by repeated application of a mathematical compass or a ruler. The compass, or the ruler, should not be modified in the meanwhile, but should be rigid. For this reason, according to many empiricists, the concept of equality of lengths is based on the physical concept of a rigid body, or on something similar.[14] Similarly, the comparison of angles presupposes that one moves a rigid body. One would fall into a vicious circle, if one wanted to define at the same time a rigid body as a body whose points have invariable distance from each other—as happens when one presupposes geometry. To understand how one could get to the pure concept of a rigid body independently from geometry, and in full accordance with experience, we should imagine for ourselves the perceptions that we experience when on the one hand we moved back and forth a so-called rigid object and when on the other hand we worked a piece of wax or clay. In both cases some variable visual representations, which are accompanied by tactile sensations and muscular feelings letting us perceive the effort of our action, follow one another. At the same time, it occurs to us that we govern the modifications by means of our will, and so we immediately become aware of the impulses of our will. One will remark that in the first case the same visual representations can always be reproduced easily, whereas in the second case we manage to do so only through very hard efforts. The two cases of rigid versus flexible bodies can be distinguished by the ease with which the visual representations can be reproduced.

One could similarly arrive, in pure accordance with experience, to the concept of an invariable, rigid body. If one admits that a comparison of two distances by means of a compass is an observation that needs no developed intuition nor any developed geometrical concepts, then one will also concede that there is what Helmholtz called physical geometry,[15] namely a geometry that is in pure accordance with experience, because one can obtain theorems just from the comparison of distances. One could, for example, imagine two physical models, each one with five prominent points that could be distinguished in both models by the same five colors. Now, let's compare the distance between two points on one model and the distance between two points having the same color in the other model. A similar comparison might be carried out ten times: given that one finds a correspondence between the distances nine times, then it follows that one finds the correspondence the tenth time too. This experience repeats itself whenever one repeats the test with other models. So, a theorem would follow inductively from measurement.[16]

Certainly, a strict follower of Kant would still suspect that any of those results obtained through measurement were already influenced by ready and exact intuition taken as a condition of any experience. Maybe, if he wants to hold on to his point of view, he could not be refuted. On the other hand, it cannot be denied that his standpoint is artificial, and it seems legitimate to drop that standpoint, if one believes one can do without it. It seems to me that the empiricist view already has an advantage in the fact that it allows a detailed explanation of intuition, while the hypothesis assumed by the follower of Kant cuts off all the rest. Indeed, it is always an advantage to set up new problems.

On the nature of intuition and its relation to geometry one might want to take into account the history of geometry, or even the development of intuition in a single individual. If the history of geometry could nowadays establish in a more precise way how the fundamental concepts of this science developed in mankind, this would surely shed light on these questions. Probably, in ancient times geometry had the character of an experimental science in the strict sense of the word, because at that time it had mainly practical ends. The geometry of Egyptians was probably limited to some empirical rules applied by master builders and land measurers, while only the Greeks gradually learned to prove all truths of geometry from a small number of axioms.[17] But this ancient development of geometry is so cloaked in darkness that we cannot draw completely certain conclusions.

Given also that the development of the intuition of space in a single person can hardly be observed, there remains just one way to further explore the nature of intuition: an analysis of the simplest[18] facts of perception, which might in some cases be supported by the knowledge of the construction of our sense organs.[19]

Maybe such a psychological and physiological analysis will lead to results that will be later acknowledged once and for all, or maybe the problem in question will always receive different answers depending on the individual philosophical standpoint. Anyway, it is possible to consider the whole construction of geometry independently from that question, investigating what assumptions are effectively used by geometry, no matter where they come from, and observing how other knowledge is derived from these assumptions by a succession of smaller and more certain steps, namely, how geometers work *deductively*. To this purpose, one should go through all proofs of geometry, decompose them into their smaller steps, and pay attention to all the assumptions that are thereby made, explicitly or tacitly. It is sufficient to do this for elementary geometry, because higher geometry would yield no essentially new results with respect to the mentioned purpose. A series of important investigations has been made by mathematicians on the foundations of geometry, with the purpose of establishing all the axioms that are used, reducing them to the smallest number possible, and showing that the remaining axioms are independent from each other. There has also been a successful attempt to modify one of the axioms, and to build a non-Euclidean geometry such that the usual geometry is in some sense a special case of it or, more precisely, a borderline case.[20] Less complete is the investigation of geometrical deduction itself.

I will study the latter by means of a simple example, recalling the proof of the theorem of the sum of the angles in a triangle. This proof rests on two facts that hold for parallel lines, and that can be formulated as follows:[21]

1. When two straight lines lying on a plane do not intersect, no matter how far they are extended—i.e. when they are parallel lines—then they form equal angles with a straight line that intersects them in some way, and in particular the angles that lie between the parallel lines on opposite sides of the intersecting straight line are equal.
2. For any point, there is a line that is parallel to a given straight line.

The first fact, though slightly differently formulated, corresponds to the axiom by Euclid that we simply call the axiom of parallel lines and that is dropped in non-Euclidean geometries.[22] The second fact appears in Euclid as a theorem that can be proved on the basis of certain axioms; it corresponds to Proposition 31 of Book I. For the present purpose, we can consider this fact as an axiom.

Now, to prove the theorem of the sum of the angles, I will—for the sake of simplicity—consider a triangle which is upright and whose surface is directly in front of me. Let's consider that one side is horizontal and let's take it as the base line. From the apex let's trace a line parallel to the base: according to the assumed representation, this line is also an horizontal straight line. There appear now three angles at the apex: the median is an angle of the triangle, while the other two have both been generated just now. From the first of the aforementioned propositions about parallel lines it results that these two angles are equal to the angles situated on the triangle's base line, and in particular the angle situated to the left of the apex is equal to the angle situated on the left-hand side of the triangle's base, while the angle situated to the right of the apex is equal to the angle situated on the right-hand side of the triangle's base. Now the three angles of the triangle are represented by means of three angles situated at the apex; these angles together fill up the flat space on the underside of the line we traced parallel to the base line, and thus their sum equals two right angles.

The parallel line traced through the apex of the triangle is an essential step of this proof. It has already been remarked many times that in a geometrical proof the figure is almost always expanded through auxiliary lines.[23] The possibility to trace the auxiliary line rests here on the second proposition about parallel lines, saying that through any point there is a parallel to any straight line. This is an existential proposition. It has already been suggested with insistence that existential propositions play a role in geometry.[24]

Concerning the conclusion inferred here, it seems to me that it does not fit the usual forms of scholastic logic.[25] I would rather describe it in the following way. We imagine a series of geometrical elements that should be built in a determinate sequence according to certain axioms; relations between these elements are thereby also given. Now, we have several axioms at our disposal saying that if certain relations occur between several elements, then further relations must subsist between these elements alone, or between these elements and other elements. By means of these rules we can derive further properties of our figure. Besides, since existential propositions allow the introduction of an unlimited number of new elements in the figure, we can find many other relations from the known relations in the figure. Even if the execution of this procedure follows certain rules, its results cannot be foreseen in full detail, unless one carries out the procedure in a given way. We are thus obliged to seek and fumble: we make some kind of experiment on whose account we finally predict how the measurements that we carried out on a precisely drawn figure, or on a model, should turn out to be. We have thus replaced a real experiment by a thought experiment: this is what deduction consists in.[26]

Now, what role does intuition play in deduction itself? Are the elements being combined in the deduction, and the rules according to which they are combined, all what is taken from intuition, or respectively from experience? Or does intuition co-operate also in the single steps of deduction? The latter point of view has been mainly defended by philosophers and seems to have been also Kant's point of view, as he saw in intuition the essential principle for obtaining geometrical knowledge.[27] This is also the reason why most philosophers reject non-Euclidean geometries. They consider Euclid's axiom of parallel lines as necessarily given within intuition,[28] and consider any geometry that operates intuitively with some other assumption as an absurdity.

To clarify the role of intuition in a proof, I'll now go back to the previous example. One must concede that we have here taken from intuition that those three angles on the triangle's apex fill up the angular space of two right angles, and that the angles near the parallel lines are located in such a way that they are equal, and do not complement each other so as to form two right angles. Something else is often taken to be extracted immediately from intuition. For example, one will readily concede, on the basis of intuition, that the bi-secants of two of the angles of the triangle intersect at a point inside the triangle, whereas one would not assume by a mere appeal to intuition the fact that the bi-secant of the third angle goes through the same point.

It has already been suggested that we judge the intuitive picture somehow immediately, and that these immediate judgments play a role in a geometrical proof, although only rough judgements are endorsed in that way, whereas we deal with finer judgements by the artificial method of deduction.[29] A rough judgement on the mental image of a triangle tells us that the bi-secant lines of two angles of the triangle intersect inside the triangle, but it cannot be proved with certainty on the basis of our intuitive picture that the third bi-secant goes through the same intersection point, because our intuitive picture always has a certain indetermination.

As a matter of fact, we usually behave as described above. The procedure remains valid also in practice, but from a theoretical point of view it has a double disadvantage. Firstly, we might be unsure as to the distinction between rough judgments and fine judgments. Secondly, the intuitive picture that we have before us, is just one single picture, but it is taken to be typical for all configurations that the figure of the theorem to be proved could produce. Thus, the geometrical proof in *this form* actually contains an inference by analogy.

On the other hand, the latter remark has also been considered wrong, because we imagine the figure on which we execute a proof, as in motion, and we disregard all the configurations that it might assume. It would thus be a complete induction rather than an inference by analogy, and we would be certain of the general validity of our observation. Those who consider intuition as a very special principle of knowledge attach great weight to the power of our inner intuition to set its own pictures in motion.[30] But if one actually tries to represent a similar process of movement case by case, one discovers that it works only with very simple figures. If one considers two straight lines that intersect, one in a fixed position, and the other rotating around the intersection point, one would easily see that in all cases four angles are generated, namely two pairs of equal angles. In complicated cases, a similar thing would be nearly impossible: one never actually conjures up a complex figure in its entirety as in motion, but only singular parts of it—and only in certain kinds of proofs.

One would have to assume the cooperation of an inference by analogy in a geometrical proof, were it not possible to give a different form to this proof. But this is indeed the case.[31] The use that we make of intuition in the proof can be reduced to certain rules: this amounts to the formulation of some axioms that are commonly used, and that were already tacitly used by Euclid. For example, one needs the following axiom: if A, B, C, D are points of a straight line, and B lies between A and D, and C lies between B and D, then B lies between A and C, and C lies between A and D. Axioms of this kind have recently been called "axioms of order" by Hilbert.[32]

Euclid makes a particular use of intuition in the proof of the first proposition on congruence. This proposition says that if two sides and the included angle of two triangles are congruent, then the other parts of the triangles are congruent too. This is proved by hinting at the fact that if a triangle is moved forward without alteration of its form and size, it overlaps the other. This proof does not correspond to the previous description of deduction. On the contrary, it consists in an immediate hint to intuition, or, if we prefer, to the experiences made through the movement of rigid bodies.[33] The theorem of congruence under discussion expresses thus a primarily intuitive content, and must then be considered as axiomatic.[34]

By means of such a formulation of all intuitive assumptions, one can divest geometrical deduction itself of intuition. If one denotes the geometrical elements and the operations on these elements through symbols and through successions of symbols—just as in algebra one symbolically represents the numerical magnitudes and the operations to be carried out on them—so, in the simplest cases, it is possible to entirely reduce the geometrical deduction into a calculus that is carried out on symbols, just as in algebra the inferences concerning numerical magnitudes are carried out by means of the so-called letter-calculus.[35] Such a calculus has already been accomplished for single domains of geometry; Peano has assembled these symbolical procedures.[36] This is a realization of an idea that Leibniz had first enunciated in 1679 in a letter to the famous physicist Huyghens. In this passage Leibniz expresses himself as follows:[37] "I have found some principles of a new language of symbols that is completely different from algebra, and that allows us to advantageously represent in thought, in a precise and adequate manner, and without figures, anything that depends on intuition." In a following passage of the same letter,[38] Leibniz suggests a much broader application of his language of symbols, showing his characteristic overestimation of conceptual operations.[39] Huyghens proves to be, from the very beginning, very skeptical about the new method.[40] But it cannot be denied that such a purely formal inference method, which is dressed-up as a calculus, can be adequately applied only to very limited domains.

If, in the aforementioned manner, we divest the geometrical inferences of intuition, we can say with full certainty that something follows from given assumptions, and we can do it even if these assumptions contradict intuition. So, it follows with certainty from Lobachevsky's assumptions that in no triangle can the sum of the angles be greater than two right angles, and that if it is smaller than two right angles in one triangle, then it must really be smaller than two right angles in any triangle.[41]

The mentioned form of deduction, to some extent its pure representation, is not the usual way of geometrical discovery. The latter is mainly guided by intuitive pictures, and sometimes by observations that are made in a series of cases, i.e., guided by analogy and induction, just as often in all mathematical domains some results obtained inductively show the direction along which the deductive discovery will be made.[42]

We distinguished in geometry between primary concepts (taken from intuition or experience) on the one hand, and geometrically constructed concepts on the other hand. But in fact it also happens that concepts which were clearly immediately abstracted from intuition, can later be deduced. The different treatments of the theory of proportions provide an instructive example for that. The concept of the proportion of lengths is closely related to the concept of similarity of bodies and plane figures. It may be taken as an immediate intuitive fact that we can reproduce a body with its proper form but on a different scale. In this case the edges of the reproduction have the same ratio to each other as the edges of the original body; also all angles remained unchanged. If one admits from the outset the possibility of constructing for any body a similar body at any scale,[43] all propositions of the theory of proportions can easily be found. We might even have formed the concept of proportion according to the fact that there are similar bodies.[44] On the contrary, Euclid bases the existence of geometrically similar bodies on the theory of proportions.[45] He however reduced even the concept of the proportion to more simple concepts. I shall now enlarge on the concept of the measure of length in order to appreciate these ideas developed by Euclid[46] which formerly were not understood for a long time and were held to be dispensable.

In geometry one sometimes starts with the concept of the measure of length without properly having established this concept. In doing so it is taken for granted that all the lengths occurring in a figure have a determinate numerical ratio to one of them chosen by us. The length chosen is then called the unit of length. The number indicating how often the unit is contained in a line segment is called the measure-number of this line segment. The line segment to be measured can be either commensurable or incommensurable to the unit; in the latter case the measure-number will be irrational. The measure-number of a square's diagonal for example is irrational if the side of the square is chosen as the unit.

A closer look however reveals a difficulty in the concept of measure. For sake of simplicity let us consider two commensurable line segments. In order to conceive their length ratio as a numerical ratio we have to imagine an additional line segment both given line segments are multiples of. It lies in commensurability that such a third line segment exists; there is however more than one such line segment. What would we do if we could yield different numerical ratios as measuring results when choosing a different third line segment? In order to *establish* the concept of measure, we have to prove that this is not possible. In fact we could admit this concept without foundation; this is equivalent to the assumption of new and basically complicated axioms.

The proof mentioned can nevertheless be done by means of the two facts that equals added to equals yield equals and that the greater added to the greater yields the greater.[47] These facts may be regarded as axioms.[48] Adding the Archimedean axiom, according to which we can exceed any line segment by means of multiplication of any given line,[49] and taking into account also the propositions on parallels and congruence allows us to construct in the style of Euclid the whole theory of proportions and similar figures.[50] The definition of proportion used by Euclid[51] and by means of which he reduces, as mentioned above, this concept to simpler ones can be stated as follows: Two line segments a and b have the same ratio as a' and b' if any multiple μa of a is, in comparison, smaller than, equal to, or greater than any multiple νb of b, depending on whether the according multiple $\mu a'$ of a' is smaller than, equal to, or greater than the according multiple $\nu b'$ of b'.[52] This definition applies to two incommensurable line segments as well as to commensurable ones.

Galilei put forward a different treatment of proportions, which aimed at simplifying the Euclidean account. In doing so, however, he had to postulate the concept of proportion along with several axioms referring to it.[53] The way he carries out the proof does not satisfy the claim of presupposing as little as possible. For this reason, from a scientific point of view, Euclid's account has to be considered as the more complete one.

Recently Hilbert presented a completely new derivation of the theory of proportions.[54] In this account even the Archimedean axiom is avoided.

In this derivation—as in the Euclidean one—the concept "equal" appears as a primary concept that several axioms in addition are held to apply to. The concept of the ratio of lengths of two line segments as well as the concept of the proportion of four line segments are constructed;[55] the propositions about proportions appear to be provable.[56]

The concept of content of plane figures and bodies has not been constructed deductively by Euclid, but it is possible to do so.[a] In the case of bodies, one may regard the content as an empirical concept extracted from the usage of measures of capacity in measuring fluids, and in this way one may also get to the content of plane figures. Euclid takes it for granted that figures have a content, and that to them also the axioms apply that equals added to equals yield equals and the greater added to the greater yields the greater. If however one does not want to presuppose the concept of content, but to establish it geometrically, one first and foremost has to show that figures, *in particular figures of different form*, can be compared as to their magnitude. As a first step congruent figures will be declared to be equal; the figure that completely comprises the other figure will be called the greater one. In order

a. As regards the concept of content [*Inhalt*], Joel Michell remarks in a note to his and Catherine Ernst's translation of Hölder's *The Axioms of Quantity*: "Hölder's term *Inhalt* literally means contents and, so, is more abstract than either of the English terms area or volume (*Flächeninhalt* and *Rauminhalt* in German, respectively). Since contents is not a specifically quantitative concept in English, translating *Inhalt* as *contents* would cause Hölder's term to lose an essential ingredient. *Spatial magnitude* might be a better choice, but there is evidence [...] that he may have meant *area*", [Hölder 1996, 251].

to compare any two figures, they have however to be cut through, and it must be proved that the result of the comparison does not depend on the actual way this is done.[57]

The construction of the proportions of line segments as well as of the concept of content presupposes that certain geometrical operations are repeated an indefinite number of times. Indeed, in the consideration of proportion, we had to take the nth multiple of a line segment, n being an indeterminate number. Into such considerations general concepts of arithmetic enter and such questions cannot always be solved in a mechanical calculus.[58] Together with arithmetical, or in general combinatorical, concepts, more complicated modes of inference occur within the ambit of geometry.

The method of exhaustion which permits us to evaluate figures of curvilinear boundary provides an example for this kind of inference.[59] Let me take the surface of a circle as an example. Its content equals the content of a triangle having the circle's circumference as its base and the radius as its height. The comparison of the contents, required in the proof, cannot be done by means of a number of decompositions. Rather certain comparative figures have to be found which serve as an intermediary and only approximately represent the circular surface. An inscribed regular octagon for instance can be compared first to the circular surface and then to the triangle, by which it is found that the triangle and the circular surface in any case do not differ by more than the half of the latter. The difficulties involved in this comparison here might be passed over. Hereafter a regular hexadecagon might be inscribed in the circle, and thus it can be shown that the triangle and the circular surface differ by less than a quarter of the circular surface. This goes on in the same manner. By means of the regular inscribed polygon of 2^{n+2} edges, it is shown that the triangle differs from the circular surface by less than a 2^nth of the latter. Everyone will understand instinctively that from this the exact equality of the circular and the triangular surfaces can be concluded.

For the sake of logical rigor it should be noted that we are actually dealing with an infinity of results. The difference between the figures in question is less than 1/2, than 1/4, than 1/8 of the circle. We are sure of all of these results, notwithstanding their unlimited number, for they arise in a uniform manner; the series of considerations they result from proceeds according to a law and thus can be covered by a single view. All of these infinite results are only approximative. The result concluded from them consists in a perfectly exact proposition. In order to get to this proposition, things have to be put otherwise. The fact we make use of here is that the difference of the contents of two surfaces—the surface of the triangle and the surface of the circle—can be represented as a determinate surface. The consideration of this surface leads into a contradiction. The proof is thus indirect.[60]

I shall now elaborate on this proof. The surface mentioned is duplicated in thought, the surface thus obtained duplicated again and so forth. By a sufficient number of repetitions of duplications any given piece of a surface will necessarily be exceeded. This follows from the Archimedean axiom as applied to surfaces. The surface in question shall now be duplicated n times with n chosen such that the surface of the circle will be exceeded. Consequently, the 2^nth multiple of the surface is greater than the circle, and, accordingly, the surface—i.e., the difference between the surface of the circle and the surface of the triangle—is greater than a 2^nth of the surface of the circle. This however contradicts the result obtained before by means of the inscribed polygon of 2^{n+2} edges. From this contradiction follows the incorrectness of the only uncertain assumption introduced, the assumption namely that the surface of the circle and the surface of the triangle differ in their content. I think that this example at once shows that geometry cannot do without indirect proofs.[61]

Wherever it can be applied in the exact sciences, deduction appears in a way quite similar to how I characterized its role in geometry. This is particularly clear in mechanics. I hold most of the proofs appearing in mechanics to be as good proofs as any proof of geometry, even though they require suppositions which are generally held to originate from experience.[62] Empiricists indeed also hold the geometrical axioms to originate from experience, without rejecting the deductive method by virtue of which science augments its knowledge. Such an empirical fact which plays in mechanics the role of a postulate (axiom) is, e. g., that the point where a force acts on a rigid body can be shifted in the line of the force without altering its effect. Galilei's principle of inertia as well as Newton's "leges"[63] are such postulates. These postulates and the concepts—force (attractive and repulsive force, pressure, tension), rigid connection, mass, time, place—between which the postulates establish relations, result by way of a certain generalization and idealization[64] from empirical observations. They may count as evident insofar as most people let themselves be determined by everyday life alone to recognize the concepts and connections of concepts as valid.[65]

Neither the postulates (axioms) nor the primitive concepts of mechanics are of logical necessity. All attempts to define satisfactorily the primitive concepts failed. They can be explained only by reference to the simple facts they were extracted from.[66]

We make use of the primitive concepts of mechanics in quite a similar way to the primitive concepts of geometry.[67] In deductions only a formal use is made of these concepts. The real application of the concepts according to their content would be the application to the objects that the concepts refer to. In spatial geometry, in order to prove the mutual relations of points, straight lines and planes a purely formal use is made of the fact that a straight line which has two points in common with a plane completely lies in this plane. This fact is applied in the real world by the stone-cutter who applies his rod to the block he is working on in order to achieve a flat surface. In the same sense we make a formal use of the concept of force in the theory of the equilibrium, whereas we really apply the concept in any material device involving the production of pressure or tension. In deduction we thus do not operate with the objects themselves; but rather we operate theoretically with the mutual relations of the objects.

It may be that we cannot present deduction in mechanics in as a pure way as in geometry. It may be that in mechanics—having a rather material content—we unconsciously use certain empirical analogies in addition to the axioms. In any case it is correct that in mechanical deduction, too, we completely abandon—at least temporarily—the object investigated and perform independently from this object a thought experiment, relying on the fact that its result must finally be in accordance with the object. That we can successfully apply such a procedure rests on the exact—and in a certain sense verified, though not properly provable[68]—validity of the laws which we found by virtue of experience and which we considered as generally true. This is the reason why this kind of deduction can be applied in the exact sciences; nonetheless it would only be in vain to try to apply to the forms of organic life or even to the historical sciences a procedure which can be applied to mathematics, mechanics, and certain parts of astronomy and physics.[69]

The procedure of deduction consists in inferences of a peculiar form as has been shown in the preceding examples. The mathematical sciences thus indeed have a particular method. Reasoning in itself of course is always the same and always consists in the very same simple mental activities. The particularity of the mathematical sciences rests in the object which permits us to perform a long series of thought operations in characteristic connections so that in this way particular forms of conclusions emerge. In this sense it can be said that mathematics and the exact sciences have a logic of their own.

Notes

[1] Euclid constructs the square in Prop. 46 of Book I (cf. [Euclid & Heiberg 1883]).

[2] Euclid's definition of a straight line does not contain any construction and is to be conceived as a mere nominal definition (Book I, Definition 4); indeed, no use is made of this definition afterwards.

[3] In his *Lectures on Modern Geometry,* Pasch explains the notion of a point as follows: "The bodies whose division is incompatible with the limits of observation are always called points, whereas in geometry the word 'body' is reserved for a different use." Pasch alludes here to the abstraction by means of which we are able to think of the concept of a point [Pasch 1882]. Anyway, in the construction of geometry this definition has been used as rarely as any other definition of a point that has ever been given.

[4] As a matter of fact, we hardly distinguish between axioms and postulates; when we do, we simply want to express a nuance in our conception: whereas by the word 'axiom' we want to emphasize the fact that it is a very certain assumption, by the word 'postulate' we emphasize the fact that the generality of the rule that has been put forward is actually something that is only assumed by us. In Euclid there are two groups of principles: the *aitémata* (*postulata*) and the *koinaí ennoiaí* (*communes animi conceptiones*), the latter being called *axiómata* by Proclus. According to the

latter, Euclid considered the former principles as postulates, and the latter as self-evident truths. This seems to be quite plausible also with respect to the content of the principles, if one considers the repartition of the principles to be found in Heiberg's edition of the axioms, which is based on a rigorous analysis of the sources [Euclid & Heiberg 1883]. In other editions the repartition is different (cf. e.g., the 1781 German translation by Lorenz based on the 1703 Oxford edition [Lorenz 1781].)

5 The first of the mentioned facts is included by Euclid in the first postulate, the second is missing in Heiberg's edition, but it appears as the 12th axiom in other editions. This axiom, which might thus constitute a later addition, says that two straight lines cannot encompass any space. If one could join two points in a plane by means of two different straight line segments, these would encompass a surface; thus, it follows from axiom 12 that there can be only one straight line that joins two points. Facts of the kind here considered are called by Hilbert axioms of connection [Hilbert 1899, 5].

6 One could contest that the axioms of geometry are necessary in the Kantian sense; they might be necessary only in the same sense in which one calls natural laws necessary. But if one wants to infer some kind of inner necessity of the axioms—from the mutual agreement of different consequences that can be derived from them—then one could reply that the non-Euclidean geometry is in itself as consistent as the Euclidean geometry (compare note 20). In this regard, mathematical investigations on non-Euclidean geometry have supported an empirical conception in the theory of knowledge. For example Benno Erdmann claims that the new geometrical knowledge has a negative impact on the rationalist theory [Erdmann 1877, 116].

According to the empirical theory, the axioms cannot be called strictly necessary; something that is strictly necessary might of course follow from them, but what is required is the universal validity of the axioms (compare also [Mill 1869, 254]).

7 In the *Kritik der reinen Vernunft* [Critique of Pure Reason], 1st ed., 1781, Doctrine of Elements, Part 1 [The Transcendental Aesthetic], Section 1 [On Space], § 2 [Metaphysical Exposition of this Concept, A 24] Kant says that space is a "necessary and *a priori*" idea that grounds all external intuitions and should be regarded "as the condition for the possibility of phenomena, and not as a determination depending on them" [Kant 1781-1787, 175]. In an add-on from the 2nd edition of 1787 (Doctrine of Elements, Part 1, Section 1, § 3 [Transcendental Exposition of the Concept of Space, B 41]) it is stated even more clearly that the idea of space is "a pure, non empirical intuition", and that this intuition "has its place merely in the subject, as its formal quality of being affected by objects and thereby receive immediate representation, i.e., intuition, of them" [Kant 1781-1787, 176]. Then, on this necessity of the idea of space is grounded the "apodeictic certainty of all geometrical principles" [Kant 1781-1787, 1st ed., nr. 3–4]. It is fully clear here, that Kant takes the single axioms to actually arise from pure intuition, because he says in the mentioned passage: "Thus also all geometric principles, e.g., that in a triangle two sides taken together are always greater than the third, are never derived from the general concepts of line and triangle, but rather derived from intuition, and indeed derived *a priori* and with apodeictic certainty." (This is a fact that I would not even rate among the principles.)

8 It is not my aim to go into all the conceptions that the inventors of different philosophical systems held of space and spatial intuition. Compare here [Baumann 1869], and [Wundt 1883, 85–96].

9 Compare [Newton 1687, Preface], and [Helmholtz 1884, vol. 2, 217]. Of course, the advocates of this empirical viewpoint will not deny that any elaboration of experience comes out from assumptions, at least from the assumption of a certain conformity to laws of the investigated object, which we could not otherwise grasp

conceptually (compare in particular [Helmholtz 1884, vol. 2, 266ff.]). Indeed, any single fact of experience, if expressed by means of concepts—and how could one want to express it otherwise—is the result of a mental elaboration of experience. But unlike Kant's conception, empiricism emphasizes the fact that according to the empirical viewpoint, no law referring to external objects comes about independently from external experience. Kant says on the contrary that geometrical knowledge comes from intuition and that intuition is independent of experience.

[10] If one uses for example a micrometer screw in measurement, then one infers the displacement of the longitudinal axis of the screw from the rotations and their fractional parts. In measurements of this kind one already uses a set of relations from the science of geometry. So, the observations on which one would like to ground geometry should anyway have a much simpler nature than a measurement of that kind.

[11] When the observer steps in between the objects, he will certainly be able to see only one part at a time; but certainly, if from his standpoint two of them are overlapping, then all the objects he can see will be overlapping.

[12] For a Kantian, the fact that, from a determinate viewer's standpoint, certain small bodies overlap, is the reason why the viewer—following the intuition that he has independently from experience and by means of which the essence of the straight line is fully given to him—designs the places of those bodies as rectilinear. For the empiricist, the straight line is nothing else but a concept that is abstracted from the occurrence of the overlapping of small bodies, or maybe from other similar occurrences.

When, following a well-known linguistic use, I designate the process of concept construction as an "abstracting" or "extracting" process, I do not want to be associated with a theory that claims that the concept arises from a succession of ideas when we "leave out" the features that differentiate ideas and keep the features they have in common. When we build a concept from a succession of ideas, the ground will always be that we perceive those ideas as immediately similar, and that, in accordance with the linguistic use of other people, we learned the use of a word that expresses what is typical in those ideas. We say that we have mastered the concept, when we believe that we will be able, in future cases, to apply it with confidence and in agreement with other people. Basically, we can never prove but only claim that we are able to do this (compare [Volkelt 1886, 181ff.]). In particular, we require an absolutely safe use of the concepts in the scientific treatment. For example, in geometry we act not only as if we could pinpoint exact points and exact lines, but also as if we could always distinguish, given a straight line and a point, whether the point lies on the straight line or not, and in doing so the concept of this difference is also considered as unconditionally applicable.

If considered from the empiricist point of view, the geometrical concepts appear at first as hardly distinguishable from other empirical concepts obtained by abstraction. The difference between the former and the latter concepts, which was strongly accentuated by Wundt [Wundt 1883, vol. 2, 95], emerges only afterwards in the scientific use of geometrical concepts, which influenced also their practical use (cf. note 64).

As we can extract or abstract a concept from a succession of ideas, we can also abstract a *rule* from a succession of combinations of ideas, when we find that these combinations are not only similar one to the other but also to a complex process of consciousness, which thereby represents their connection and thus exhibits their rule. For example, we might have noticed, in a series of cases, that a thread stretched between two small rings takes a fully determinate position. We compare these observations to a conscious process, when we distinguish three elements and consider one of them as determined by the other two. Besides, if we postulate that our thoughts are universally valid for all cases of the observed kind, we get to the rule: "Through two points there always passes one and only one straight line." Since the common concept appears unitary in thought, and accordingly occurs mostly in association with a single word, a rule of that kind is a "concept of higher order", that displays a logical structure [Volkelt 1886, 384].

[13] 'Immediate' should simply mean here that we immediately judge to be similar the successions of sensory perceptions, on which are based the visual images that first appear as unitary to our consciousness. This does not contradict a conception of physiology, according to which those comparisons of visual images are possible only on the basis of eye-movements.

[14] [Helmholtz 1884, vol. 2, 260]. One can also use the thread to compare distances. An objection has been raised to this however, that a comparison of distances is possible only by means of an inelastic thread, and that there is indeed no feature that can distinguish an inelastic from an elastic thread apart from the invariability of the length of the former: so one would end up moving in a circle. Against this objection, I would like to remark, that in the case of a piece of inelastic thread kept stretched between two hands, we cannot generate any visible variation by merely intensifying our stretching effort, whereas in the case of an elastic thread we can do that. So, we have here a feature of the inelastic thread that does not presuppose the given concept of length.

One could probably try to define the concept of equality of line segments also in many other ways (cf. Helmholtz's Bemerkungen über 'physische Gleichwerthigkeit' von Raumgrössen, [Helmholtz 1884, vol. 2]). The above explanation just aims at showing that one can indeed consider the concepts 'equal', 'bigger', 'smaller' as extracted from certain empirical activities, without having to concede a claim that has already been advanced [Zindler 1889, 11], that any comparative activity of that kind already contains the concepts of length and equality. Basically, the following appear thus as facts of experience: that two line segments being equal to a third are equal to each other; and that given three line segments such that the second is smaller than the first, and the third is smaller than the second, then the third has also to be smaller than the first.

The fact that two magnitudes being equal to one and the same third magnitude are equal to each other is described by Sigwart [Sigwart 1889, vol. 1, 414] as an analytical fact, i.e., it is described as a proposition that follows from the concept of equality. In some sense, this will be conceded by everybody, insofar as the mentioned fact is apparently inextricably associated in our consciousness with the concept of 'equal'. However, one should not forget that a different intuitive content or, if one prefers, empirical content can be attributed to this proposition, depending on the use we make of it. The equality of line segments is something different from the equality of angles, or even from the equality of surface portions of different form. Besides, the mentioned proposition does not appear to me to be a consequence of the so-called principle of identity, because objects that are declared to be equal do always differ in some other respect, and are thus not logically identical. We could most correctly say, with Helmholtz [Helmholtz 1887, vol. 3, 356ff., 375–376] that by the term 'equal' we designate a relation between two things such that, whenever the relation subsists between a and b, one can infer that the same relation subsists between b and a, and whenever the relation subsists at the same time between a and b and between b and c, one can infer that it subsists between a and c. However, whether a relation is of that kind, might result, in the mentioned case, only from the nature of the concerned objects.

In certain circumstances, in particular, one could as well say that a fact is abstracted from experience, or that it arises immediately from a concept, inasmuch as the concept is taken, together with the fact associated to it, from experience or intuition. However, there can never be any disagreement whether a fact can or cannot, in the proper sense of the word, be deduced from others, i.e., whether an already obtained deduction of that kind is binding or not.

15 [Helmholtz 1896, vol. 2, 260ff.], where the author gives a convincing example.

16 This would be an induction that does not rest on any deduction. On the relation between induction and deduction one finds apparently opposite views. Some say that deduction is always grounded in facts that are discovered inductively [Wundt 1883, vol. 2, 27], others say that induction relies on the results of demonstrative logic, i.e., on deduction [Lotze 1874, 340], [Sigwart 1893, vol. 2, 384–385]. The two theses can be combined, up to a certain degree. The first one, which anyway seems to me inappropriate for arithmetic, might be correct for all sciences that concern objects of experience in the strict sense of the word. It is correct, insofar as certain facts of a general kind (laws) are inductively obtained from experience, and exactly these facts provide the rules along which deduction proceeds. On the other hand, there are complicated inductions that become possible, only after one has obtained certain concepts that arise from a deductive elaboration of a domain of knowledge. When, for example, we induce from many observations that in the case of a falling body the spaces traversed from the beginning of the fall are proportional to the squares of the falling times, then we must possess not only a general concept of the process that we call 'to fall', but also the concept of a square number, which has its source in the deductively elaborated domain of arithmetic.

On the contrary, it seems to me that the intellectual activity that from the beginning goes hand in hand with experience, when we build concepts of experience, is a preliminary activity that should not be assigned neither to deduction nor to induction.

17 [Cantor 1880, vol. 1, 46ff.], [Hankel 1874, 88]. Compare also the interesting remarks by Kant in the Preface to the 2nd edition of the *Critique of Pure Reason*.

18 That is, of the facts that are most simple for our consciousness.

19 Localization is a problem whose solution, requiring psychology and physiology, should greatly clarify the nature of our intuition of space. The importance of the localization problem will certainly be denied by those who consider the pure intuition of space as a condition of any perception. But they must certainly admit that the empirical part of my perception, i.e., also my sensory perceptions, must contain something that allows me, in a single case, to locate an object in this way, and not otherwise, namely to see the object in this determinate position and not in some other position with respect to my body (cf. the theory of 'local signs' by Lotze, [Lotze 1884, 547ff.]). Similarly, there must be an empirical reason why I see an object, in a given instant, exactly in the form in which it appears to me, and not in one of the other forms that would be possible according to the laws of 'pure intuition'. This concession could already be considered as conflicting with Kant's conception, according to which the form of the perceptions should have its source in the intuition of space, which is independent from experience. One could at any rate be tempted to understand Kant in such a way that pure intuition, i.e., the form of perception, has nothing to do with the empirical content of perception. But if, as Kant says, the geometrical 'knowledge' (laws)—which, according to him, must have its origin *a priori*—"perhaps serve[s] only to establish a connection among our sensory representations" [Kant 1781-1787, 1st edition, Introduction, 128], then one should undoubtedly understand Kant as saying that in a singular case we obtain from the senses certain empirical data,

and that from these data, under the action of the laws of intuition, we delineate the form of the object that stands before us, and we determine its position. Anyway, since the observation of the object can be arbitrarily extended, it cannot be ignored that then we should be able to avoid an arbitrary increase of those empirical data, or we would obtain many more data than those that are necessary to determine the intuitive picture of the object. Anyway, all these empirical data, with the help of the laws of intuition, produce a corresponding picture of the object: this can only be explained, according to me, by the assumption that the laws of intuition have at the same time the function of exhibiting a valid order for the empirical part of perception. But then we would basically be back to the empirical point of view.

[20] The task of building a simple and complete system of independent axioms for Euclidean geometry has recently been solved by Hilbert [Hilbert 1899]. The geometries that differ from the usual Euclidean geometry had their point of departure in the efforts to prove Euclid's axiom of parallel lines. The latter is the 11^{th} axiom in Heiberg's edition, and the 5^{th} axiom in other editions, cf. [Lobachevsky & Engel 1899, 373–383]. For the purpose of that proof, it was at first tentatively assumed that the axiom of parallel lines was not satisfied, originally with the intention to derive a contradiction from this assumption. But the contradiction did not ensue, and non-Euclidean geometries were developed from such considerations.

Lobachevsky was the first (1829) to publish a fully accomplished geometry without the mentioned axiom (this geometry, which was also discovered by Gauß, was discovered independently by other authors; for further details, cf. [Lobachevsky & Engel 1899]). Lobachevsky used the so-called synthetic method, i.e., the usual geometrical method. One could also use the 'analytic' method. This rests on the possibility—which can be proved on the basis of Euclidean geometry—to establish the position of a point in space through three measurements (this is the property of space that we express by saying that space has three dimensions). One can thus relate the totality of points in space to the totality of number triplets, which is expressed in words as follows: the space can be conceived as a threefold infinite numerical manifold. If one lets space correspond in such a way to a numerical manifold, then the relations of position, and so on, that subsist in space between points, straight lines, and surfaces are reflected into relations between numerical structures, and the same laws hold in both cases. This conception opens up a new way to answer the question of possible geometries—a way that was first developed by Riemann ([Riemann 1867]; [Riemann 1876, 254]), and afterwards further pursued especially by Helmholtz ([Helmholtz 1883, vol. 2, 610, 618]; [Helmholtz 1884, vol. 2, 3]). These inquiries presuppose that the relations of space might be exactly represented through certain dominant relations in a numerical manifold (this requirement has been emphasized especially by Sophus Lie, who also corrected the mentioned inquiries on several issues; cf. [Lie & Engel 1893, 393]), and are therefore only concerned with the structures of the numerical manifold. As a result, in other similar threefold numerical multiplicities structures have been defined whose reciprocal relations show laws that are both analogous to, and different from, the laws into which usual geometry is reflected: one such new case—discovered in that way—corresponds precisely to Lobachevsky's geometry.

Anyway, these inquiries ensured that Lobachevsky's geometry is in itself consistent, while the original, pure synthetic treatment of this geometry did not at first exclude the possibility that further additional considerations might later lead to contradictions. It is sufficient to consider, for the moment, that by the words 'point', 'straight line', and so on, we intend to designate just those numerical structures in the new numerical multiplicities. Thus, the axioms postulated for the new system are satisfied, and therefore they cannot, at any rate, contradict one another.

The relations of Lobachevsky's geometry can also be represented with the help of certain complicated conceptual constructions that are tied to Euclidean geometry ([Beltrami 1868, vol. 6, 284], [Klein 1871, vol. 4, 573ff.], [Klein 1873, vol. 6, 112ff.], [Cayley 1872, vol. 5, 630ff.]). So, if one assumes that Euclidean geometry is in itself consistent, the consistency of the new geometry is proved. However, the mentioned assumption—generally and tacitly assumed before the discovery of non-Euclidean geometries—primarily rested merely on the belief that the Euclidean geometry is the true expression of certain objective relations. Of course, one could also prove the consistency of Euclidean geometry by pointing to the numerical manifold that represents it, as was done by Hilbert [Hilbert 1899, 19–21].

The consistency of the non-Euclidean geometry, in which the Euclidean postulate does not hold but the other axioms are satisfied, ensues that no axiom can be inferentially derived from the others. This result is maybe the most important advantage obtained through the inquiries into non-Euclidean geometries.

[21] Some people try to elude the difficulties that lie in the parallel theory by introducing from the very beginning the concept of 'direction' (cf. [Zindler 1889, 7]), and by defining the angle as 'difference of direction'. In this case, certain axioms are set by the way the concepts of 'direction' and 'difference of direction' are used. Firstly, in this conception one tacitly postulates the fact that is highlighted in note 25, namely that two straight lines lying on a surface and forming equal angles with an intersecting straight line, form equal angles with any other intersecting line. Besides, in the mentioned conception one also tends to tacitly presuppose the fact that straight lines that lie on a surface and do no not intersect have the same direction, i.e., that they form equal angles with a third straight line; but then one already presupposes the axiom of parallel lines. However, with the the help of assumptions that are anyway used, Euclid showed that the first fact follows from the second. Thus, if one assumes the here-delineated conception, which is legitimate, provided one explicitly assumes the axiom (cf. the remarks by Clebsch-Lindemann, [Clebsch 1891, vol. 2, part 1, 543, note]), then one has not only deprived himself of the non-Euclidean geometry, but also built the Euclidean geometry on a larger-than-necessary number of axioms.

Mill has suggested to modify the Euclidean definition of parallel straight lines ([Mill 1869, vol. 2, 156]). He wants to define two straight lines a and b as parallel, if and only if they everywhere have the same distance one from the other. Given that the distance between the parallel lines is measured by means of a line segment that is perpendicular to both, Mill thus basically requires two properties: any straight line that is perpendicular to a should also be perpendicular to b, and vice versa; and the parts of all these perpendiculars that lie between a and b should be equal to one another. Yet (in the usual plane geometry) it is also the case that two straight line are parallel, when a third straight line is perpendicular to both. But this fact cannot be derived from Mill's definition, so it would have to be assumed axiomatically. Anyway, from this fact and from Mill's definition result two facts that can be expressed even without the concept of parallel lines: 1) given two straight lines a and b, if a third straight line is perpendicular to them, then any straight line that is perpendicular to a is perpendicular to b; 2) in the supposed case, all the straight lines that lie between a and b, and that are perpendicular to them, are equal. It seems now appropriate to let these two facts come first, and add the definition of parallel lines afterwards. Of the two facts 1) and 2), the second follows from the first with the help of the axioms that are used in Euclidean geometry, with the exception of the axiom of parallel lines. I do not doubt that if Mill had known this state of affairs, he would have carried out a reduction of the second fact to the first, because he acknowledges [Mill 1869, vol. 2, 156, note, 146, note] the procedure by means of which one selects some properties of a concept so as to derive its other properties. However fact number 1) remains axiomatic.

No matter how one approaches the issue, the usual theory of parallel lines (apart from some inessential changes) must be derived just as Euclid did it. It is not sufficient to assume a definition of parallel lines; one also needs an axiom. If one does not assume an axiom of parallel lines, then one obtains, beside the usual geometry, the non-Euclidean geometry too.

[22] Let's imagine one traces all possible rays through a point—i.e., straight lines that begin in the point and that go to infinity on one side—, and let g be a straight line that does not go through the point, and that goes to infinity on both sides. All the rays that intersect g thus fill an angle whose sides do not intersect the straight line g. Now, whereas the Euclidean geometry consists in the assumption that this angle be equal to two right angles, Lobachevsky assumes it to be smaller than two right angles (cf. also note 41). It is also evident that the Euclidean geometry is a borderline case of Lobachevsky's geometry. Now, introducing without proof the fact that those rays actually fill up an angle is equivalent to assuming an axiom. Lobachevsky gives a proof of this fact that cannot be considered as fully rigorous, at least not in the given version [Lobachevsky 1840, Th. 16, 7ff.]. Nonetheless, there is no doubt that the fact can be proved rigorously from more simple axioms, which must include the axioms of order (note 32) and the continuity axioms (note 49).

[23] [Sigwart 1893, vol. 2, 275, 280].

[24] [Zindler 1889, 33]. The demonstrable existential proposition introduced in the text rests, amongst others, on an axiom that establishes existence. Geometrical existence is of course something different from the physical existence of an object, yet I think it can be expressed more clearly by the word 'existence' rather than by the word 'possibility'.

[25] The one-sidedness of the usual 'syllogistic' has been pointed out especially by Lotze [Lotze 1874, 133]; it is not suitable neither for the various kinds of inferences that are drawn in a science such as mathematics, nor for the various kinds of assertions that occur here. A geometrical assertion, properly, always refers to reciprocal properties of several geometrical elements (points, straight lines, and so on). I think that mathematical inferences, should they be once represented in a purely formal way, can be better conceived from the point of view adopted by English authors in their "Logic of relatives" (cf. [Jevons 1877, 122]), rather than from the point of view of the Aristotelian [syllogistic] figures.

According to me, the so-called principle of identity does not clarify mathematical inferences either. For example, in many cases one could conceive the inferences that are drawn from the propositions on parallel lines in the following way. From the propositions on the parallel lines mentioned in the text together with the theorem it follows that if two straight lines in a plane form—in the previously indicated way—equal angles with a given straight line, then they form equal angles with any straight line that intersects them. Here, we can also substitute in a given assertion a straight line for another straight line. The mentioned theorem expresses thus a principle of substitution that constitutes the rule according to which certain inferences proceed. However, the inferences that can be drawn from this principle of substitution are not analogous to the inferences that can be drawn from the logical principle of identity, because here we do not substitute identical for identical, but rather different things for different things.

[26] [Kroman 1883, 26, 139].

[27] [Kant 1781-1787, 1st edition 1781, 2nd part, 1st section, 2nd book, 2nd chapter, § 3.1, 288]: "On this successive synthesis of the productive imagination, in the

generation of shapes, is grounded the mathematics of extension (geometry) with its axioms..." Cf. also [Kant 1781-1787, part 1, § 1.4]. By the way, from the mentioned passages it cannot be derived with full clarity, whether Kant admits that only the principles have their source in intuition, or rather assumes that intuition is at stake in any step of a proof.

28 Sigwart says that the metric relations of space "are grounded on a necessity of our spatial intuition that cannot be further analyzed", whereas he understands by 'metric relations of space' the metric relations that Euclid has assumed as laws of space [Sigwart 1893, vol. 2, 82, note].

29 [Kroman 1883, 92ff.].

30 [Kroman 1883, 74–79]. Sigwart too finds ([Sigwart 1893, vol. 2, 226]) that the movement of points and straight lines in space is operant in all syntheses (constructions), and when he speaks about the multiplicity of constructions that extends itself into the inconceivable, he says: "but everywhere there is the same requirement: to run through the whole extension of the assumed possibilities by means of a conceptual formula (what is meant here is a prescription of the construction), obtaining at the same time the borderline cases, and tracing the limits of the extension for the purpose of a classification" [Sigwart 1893, 227].

31 Cf. the examples given in notes 47 and 48.

32 Which facts concerning order one wants to assume as axioms, so as to prove the others from them, is up to a certain degree arbitrary. Cf. [Pasch 1882, 5–7], where similar axioms are established for the first time, and [Hilbert 1899, 6].

To the facts concerning order belongs another fact that has often been considered as an axiom (cf. [Klein 1873, vol. 6, 113]): the infinite length of straight lines. This infinite length distinguishes in quite an essential way the straight line from the circle, which returns upon itself and has a finite extension. In exact terms, the fact that holds for the straight line should be expressed as follows. Given two points A_0 and A_1 on a straight line, we can imagine an unlimited succession of points $A_2, A_3, A_4, \ldots$ constructed in such a way that A_1 lies between A_0 and A_2, A_2 lies between A_1 and A_3, A_3 lies between A_2 and A_4 and so on, generally A_ν lies between $A_{\nu-1}$ and $A_{\nu+1}$. We further imagine that the line segments A_0A_1, A_1A_2, A_2A_3, $A_3A_4, \ldots$ are all equal. Then, one can generally assert for any value 2, 3, 4,... of the number ν that neither A_ν nor any point situated between $A_{\nu-1}$ and A_ν coincides with A_0, nor with A_1, nor with a point situated between A_0 and A_1. The first part of this assertion, expressing the possibility of that construction, follows from the axioms of equality (see note 48); the second part can be proved from the axioms of order, as formulated by Pasch and Hilbert.

33 Lobachevsky too presupposes the facts of congruence in his geometry. Apart from that geometry, the facts of congruence also hold in another non-Euclidean geometry, which can be obtained when, besides abandoning the axiom of parallel lines, one also modifies the axioms of order (note 32, cf. also note 41). According to the philosophical perspective, the determinate facts of congruence are sometimes inserted in certain—quite vaguely formulated—properties of space. For example, Lotze says that the usual conception can only imagine space as homogeneous in its whole extension [Lotze 1884, 261]; to this regard, he uses the words: "On the contrary, space itself, as an impartial arena that features all these events, cannot have local differences in its own nature, because they would prevent that everything that exists or happens around one of its points, could be repeated without variation around any other point."

It is remarkable that Mill does not acknowledge in his analysis of Prop. 5 of Book I of Euclid's *Elements* ([Mill 1869, vol. 1, 242ff.]), that he himself used the first theorem of congruence (Prop. 4 of the Book I by Euclid) in his presentation, without substituting it with a deduction. However, this is concealed by the vague way in which he expresses himself in this passage. He already used the principle that equal line segments overlap in the widened sense that a pair of unequal line segments must overlap with another pair of line segments that are respectively equal to the former. Thereby, he obviously imagines a pair of line segments in motion, so rigidly tied to each other that the included angle, which was initially identical to the angle between the other line segments, remains equal during the motion. He is able to draw his conclusion only because he assumes that the pair in motion overlaps the other pair; here one has to be guided by some representations of the relative position, and these representations used by Mill are fully equivalent to the application of the theorem of congruence.

[34] Strictly speaking, one needs, as Hilbert [Hilbert 1899, 12–14] has shown, to assume as an axiom just a part of the assertion of the congruence's theorem; the remaining part can be proved afterwards. The second theorem of congruence, which says that two triangles coincide in all their parts if they coincide in one side and two angles, had already been proved—in the proper sense of the word—by Euclid: it is Prop. 26 of Book I. Among the axioms of congruence Hilbert counts also some facts that ground the addition of line segments and angles, and that are tacitly used in the customary and most common way to carry out geometrical proofs (cf. note 48).

[35] Sometimes one experiences—and the first algebra lesson, being often too formalistic, is partly responsible for this—that people readily operate with the rules of the letter-calculus but encounter difficulties when they have to substitute numbers in the formulas. In such a case, there is a lack of knowledge of the meaning of the algebraic symbols. But, more generally, what is missing is an understanding of why the algebraic transformations are legitimate. So, the general formulas on which the transformations are grounded, such as $a \cdot b = b \cdot a$ or $a(b+c) = ab+ac$, have a content that is fundamentally non self-evident. These are theorems, the first of which says that on the whole I obtain as many objects—let's say, for example, spheres—when I form a heaps of b spheres or vice versa b heaps of a spheres. Algebra consists in the fact that we have such theorems, and use them to make inferences; the letters thereby only provide an adequate denomination, and the form of the calculus given to inferences has only the function of clarifying the overall view. Each formula and each transformation of formulas can be expressed in words, and there is no fundamental difference between tasks that, as they say, can be solved by 'raisonnement', and tasks whose solution necessarily requires calculus.

One can thus designate the procedure of the letter-calculus as an inference process that is ruled by formulas, such as $ab = ba$, $a(b + c) = ab + ac$ used to derive inferences. This is again an inference process that corresponds to the "Logic of relatives" (note 25). If one considers those rules as axioms, then the analogy with geometry is complete. Anyway, an essential difference with respect to geometry consists in the fact that we can prove, inasmuch we go back from the algebraic formalism to the number concept, that $ab = ba$, and precisely, we can prove it not simply by a confirmation of the given formula by examples, as may have been the case when it was originally discovered by induction, but by a deductive procedure (note 58).

[36] [Peano 1888].

[37] [Leibniz 1899, 570].

[38] [Leibniz 1899, 575].

[39] On this cf. [Baumann 1869, vol. 2, 56–63].

[40] Cf. the above-mentioned *Briefwechsel* [Correspondence], [Leibniz 1899, 577].

[41] [Lobachevsky 1840, 13–17]. I think that among the axioms of Lobachevsky one should also count the axioms of order (note 32), tacitly used by him in a similar form to the one adopted by Hilbert [Hilbert 1899, 6–7]. One can then rigorously prove that the sum of the angles of the triangle is not bigger than two right angles, and that the angle mentioned in note 22 must be equal or smaller than two right angles.

[42] Even in arithmetic, which is, in my opinion, purely deductive (see note 58), the results are usually first found by induction. An example is provided by the arithmetical theorem, according to which a prime number that is divided by 4 gives a remainder of 1, can always be represented as the sum of two squares. This theorem was found by Fermat, undoubtedly by induction, and afterwards first proved by Euler.

[43] It roughly amounts to the same as saying that from the beginning geometry is only concerned with relative ratios of magnitudes [Zindler 1889, 5]; for this claim could only be justified by the preceding acquaintance with the phenomenon of similarity which Euclid rightly deduces from the more simple facts of equality, congruence, etc. In non-Euclidean geometry, there are no similar non-congruent figures; there is a length proper to any of these geometries which takes a particular position in relation to all other lengths, as the right angle does in Euclid in relation to all other angles.

[44] Cf. Helmholtz's remark on the intuitive knowledge of the typical behavior of bodies, obtained in the observation of spatial relations [Helmholtz 1884, vol. 2, 30].

[45] Book VI, Def. 1, propositions 4–7.

[46] In Book V.

[47] Euclid asserts all these axioms as applying to any kind of magnitudes (Book I). From the point of view of geometry, the content of these axioms differs according to the kind of magnitude taken into consideration; in the present paper I will restrict myself to line segments.

The proof mentioned additionally requires the properties of the whole numbers (which by the way can be deduced) as well as some commonly known axioms. This is how the proof is set down: take a straight line g and two points A_0 and A_1 on g. Now there is exactly one point A_2 on g so that A_1 lies between A_0 and A_2 and the line segment A_1A_2 is equal to a given line segment; this fact has to be considered as an axiom ([Hilbert 1899, 10]). We take A_1A_2 as being equal to A_0A_1 and then assume a point A_3 on g such that A_2 lies between A_1 and A_3 and such that A_2A_3 is equal to A_1A_2 and thus—according to the former as well as to the known axiom (note 14)—to A_0A_1. Moreover we choose A_4 such that A_3 lies between A_2 and A_4 and at the same time the line segment A_3A_4 is equal to the line segments A_2A_3, A_1A_2, and A_0A_1. For our next step we choose A_5 in the same manner and so on. As a consequence of the axioms of order, e.g., not only does A_2 lie between A_1 and A_3—as it does according to the construction—, but also between A_0 and A_4 and so on. Given that the line segments between any two consecutive points are equal, we can conclude the equality of the line segments A_0A_m and A_mA_2, m being a whole number, by virtue of the axiom which states that equals added to equals yield equals (this axiom being asserted here only for the addition of line segments and the latter being conceived purely geometrically as concatenation [Hilbert 1899, 11]). Now it is possible—by virtue of the definition of the concatenation of line segments and by virtue of the axioms of order—to conceive the line segment A_0A_{nm}, n and m being whole numbers, both as the sum of the line segments A_0A_m, A_mA_{2m}, $A_{2m}A_{3m}$ and so on, and as the sum of the line segments A_0A_1, A_1A_2, A_2A_3, $\ldots$ $A_{nm-1}A_{nm}$, from what follows that for the line segment A_0A_1, i.e., for any line segment, the proposition holds that its nmth multiple is equal to the nth multiple of its mth multiple.

These considerations had to be stated before. If one takes now two given line segments AB and $A'B'$—being respectively the mth and m'th multiple of the line segment CD—it follows from what was said before that the m'th multiple of AB is equal to mth multiple of $A'B'$; for the former is the $m'm$th, the latter the mm'th multiple of CD, and we know that for numbers (see note 58) it holds that $m'm = mm'$. Now if, at the same time, AB and $A'B'$ equal respectively the nth and the n'th multiple of one and the same line segment EF, the n'th multiple of AB also equals the nth multiple of $A'B'$. In the next step one has to show that using CD or using EF in measuring AB and $A'B'$ yields the same numerical ratio, i.e., it has to be shown that $\frac{m}{m'} = \frac{n}{n'}$, viz. that $mn' = nm'$.

If one takes now the nm'th multiple of AB, this equals the nth multiple of the m'th multiple of AB, i.e., it equals the nth multiple of the mth multiple of $A'B'$, thus it is equal to the nmth multiple, i.e., to the mnth multiple of $A'B'$. This in turn is the mth multiple of the nth multiple of $A'B'$, i.e., it is equal to the mth multiple of the n'th multiple of AB, thus to the mn'th multiple of AB. We have now shown that the nm'th multiple of AB equals the mn'th multiple of the same line segment. From this follows that $mn' = nm'$. The rigorous proof for that has to be set out indirectly. If it were true e.g., that $mn' > nm'$, one would take the $(nm' + 1)$th and the $(nm' + 2)$th multiple of AB and so forth. The axiom that the greater added to the greater yields the greater—which I hold to contain that adding something to the greater or to the equal yields something greater than adding nothing to the smaller or the the equal—permits us to conclude that the $(nm' + 1)$th multiple, the $(nm' + 2)$th multiple and so forth and finally the mn'th multiple of AB would be greater than the nm'th multiple of this line segment, which contradicts the result found above.

In this somewhat circumstantial proof it becomes clear that nowhere has any use of the geometric intuition been made in an immediate way, but only in a mediate way in implementing a couple of axioms which can be stated in a perfectly precise manner.

[48] In the case of line segments, the axioms of equality can be put in such a way ([Hilbert 1899, 10–11]) that from them together with the axioms of order follows the fact concerning the addition of the greater to the greater. This of course presupposes the reduction of the concept "greater" to the concepts "between" and "equal" by giving the following definition: The line segment AB is greater than CD if there is on the first line segment a point E between A and B such that AE equals CD.

The difficulty of such a proof for the untrained is that one has to put out of one's mind in an artificial way certain ideas which are taken for granted due to a long process of familiarization. That it is nevertheless possible to unobjectionably set down the proof might be shown by the following considerations.

The axioms concerning the equality of line segments posited by Hilbert can be stated in the following way:

(1) The line segment AB always equals the line segment BA (to this axiom corresponds the fact that if two rods, one having the ends A and B, the other A' and B', can be superposed in such a way that A' coincides with A and B' with B, it is also possible to superpose them inversely such that A' coincides with B and B' with A).

(2) If the line segment AB equals the line segment $A'B'$, and if the latter equals the line segment $A''B''$, it is also true that $AB = A''B''$ (see note 14).

(3) If A, B, and C are points on a straight line, and if A', B', and C' also are points on a straight line, and if furthermore B lies between A and C and B' between A' and C', it follows from $AB = A'B'$ and $BC = B'C'$ together that always $AC = A'C'$ (axiom of the addition of equals to equals).

(4) Given a point A on a straight line g and a second point B on the same line, and furthermore somewhere a line segment CD; then there will be *one and only one* point E with the twofold property that the line segment AE equals CD and that E is placed on the same side of A as B (i.e., such that A does not lie between B and E); moreover there is also *one and only one* point E' with the twofold property that $AE' = CD$ and that B and E' are placed on different sides with respect to A (i.e., such that A lies between B and E').

Having given the definition of "greater", I firstly note that the proposition $AB >$ (greater) CD, corresponding to the way we put it, excludes the proposition $AB = CD$. Indeed according to the definition there is a point E *between* A and B (and, thus, different from A and B) such that $AE = CD$. If it were true in addition that $AB = CD$, it would follow from axiom (2) that also $AE = AB$; but since E and B lie on the same side with regard to A, E and B must coincide according to axiom (4), in contradiction to what we said before.

Next we suppose that $A'B' = AB$ and $AB > CD$ and show that from this it follows that $A'B' > CD$. In order to do so we take the point E as before; in addition we take a point E' on the straight line g' given by A' and B' (axiom (4)) such that $A'E' = CD$ and such that E' lies on the same side of A as B'. According to the axioms of order E' either lies between A' and B' or coincides with B' or it takes such a position that B' lies between A' and E'. It has to be proven that the first case is true. Suppose that there is a second point B'' on g' such that $E'B'' = EB$ and that E' lies between A' and B''. Now from $AE = CD = A'E'$ (cf. axiom (2)) and from $EB = E'B''$ follows that according to axiom (3) AB also equals $A'B''$. The points B' and B'' thus have the same properties; it holds that $AB = A'B'$, that $AB = A'B''$, and by virtue of the assumptions as well as the axioms of order both B' and B'' lie on the same side with regard to A'. Therefore B' and B'' must coincide according to (4). Thus E', being located between A' and B'', also lies between A' and B'; according to the definition $A'B'$ therefore is greater than CD being equal to the line segment $A'E'$.

If $AB > CD$, it follows from axiom (1) and from what was proved above that $BA > CD$ (this result states—put in an intuitive way—that, if the line segment CD can be cut off from the line segment AB at its end A, this can also be done at its end B; and from the preceding consideration it also follows that both ways of cutting off yield an equal line segment).

Now one might assume that AB is greater than CD and that CD is greater than FG; it is to be proved that also $AB > FG$. As a result of the assumptions just made there is a point E between A and B with $AE = CD$. Given that also $CD > FG$, and that according to what has already been proved $AE > FG$, there is a point H between A and E with $AH = FG$. From the axioms of order now follows that H also lies between A and B which, together with the last equation, means that $AB > FG$.

From this it results that, according to the way we put the concepts in this note, the propositions $AB > CD$ and $CD > AB$ exclude each other; for from both taken together would follow that $AB > AB$, whereas the concepts "greater" and "equal" are incompatible (*vide supra*).

On the basis of these preliminaries the proof we are interested in can be set down. The proof can immediately be reduced to the following arrangement. Given a point B between A and C, a point B' both between A and B and between A and C'; if furthermore $BC > B'C'$, it is to be proved that C' lies between A and C. From the assumptions together with the axioms of order it follows that C' and C are located on the same side in regard to B'; hence C' either lies between B' and C, or it coincides with C, or it is located in such a way that C lies between B' and C'. In the second case, it follows from the arrangement of the points together with the definition of the concept "greater" that $B'C' > BC$ (or, actually, first of all that $C'B' > CB$), which contradicts the original assumption that $BC > B'C'$. In the third case, the same reasoning yields that both $B'C' > B'C$ and $B'C > BC$, for which reason according to what has been proved before $B'C' > BC$ would hold, which again contradicts the original assumption. The first case hence applies, i.e., C' lies between B' and C and thus, according to the axioms of order, between A and C, which was to be demonstrated.

For the sake of simplicity and clarity I restricted myself to line segments, the axioms concerning the order of points being at our disposal in the proofs. Euclid always asserts the relevant facts in a general manner for all magnitudes. The concept of a magnitude or a quantum could indeed be conceived as given; in this case it would be possible to show that, insofar as we postulate some of the facts commonly attached to this concept, we can prove the remaining facts. This is what Mill aims to do and what to some extent he indeed carries out [Mill 1869, 146, 147, note]. He is however not aware of all the assumptions he actually makes use of; he thus assumes that adding b to $(a-b)+c$ yields $a+c$, i.e., he presupposes that $(a-b)+c = (a+c)-b$. It only appears as if Mill only refers to numbers, and for numbers this assumption can indeed be proved (see note 58). But if it is to be shown deductively that any objects can be conceived numerically, i. e., that they are in numerical relations to each other, certain facts have to be presupposed as axioms in order to deduce the other required facts from them.

Considerations of this kind are also of importance as a preliminary for the measurement of a physical state. They make it clear which experiments suffice to prove that the physical state—e.g., the charge of an electrical body—can be conceived quantitatively, cf. [Maxwell 1873, 35].

It is also helpful to note that we cannot speak of quantity properly in all cases where we are aware of graduation. We can for example construct a hardness scale by indicating whether two given bodies are equally hard and, in case they are not, which of them is the harder one. Given three bodies we consequently are also able to indicate which one lies between the two others, but we cannot say that one body is twice as hard as a different one, or that two bodies taken together are as hard as a third one. We thus cannot quantitatively conceive hardness.

[49] If one adds the "axiom of continuity", the Archimedean axiom can be proved on the basis of the axioms of order and the axioms of equality. The proof can take the following form. If all points located between A and B are distributed in any way on two categories such that any point belongs to a determinate category, that there are points in both categories, that furthermore, given a point X from the first category and an point Y from the second, X always lies between A and Y, there is in any case between A and B a point C such that between A and C there are only points of the first category and between C and B there are only points of the second category. The point C itself might belong to one category or the other, as the case might be (cf. the different versions of this axiom in [Hilbert 1895, 92]).

The axiom of continuity provides a number of additional conclusions. If we divide the points between A and B into two categories, e. g. by assigning a point X to the first category if the double of AX is smaller than AB, and by assigning a point Y to the second category if the double of AY is greater than or equal to AB, it is easy to show that both categories bear the properties which are necessary for the application of the axiom of continuity. This axiom then leads to the existence of a point C. The assumption that the double of AC is smaller than AB can be reduced *ad absurdum* by virtue of the propositions mentioned; the same holds for the assumption that the double of AC is greater than AB. From this it follows finally, that the double of AC is exactly equal to AB.

It is thus proved that for a given line segment there is a second one which represents exactly half of the former. In the same way the existence of the third, the quarter and so on can be proved, and thus a great many facts have been reduced to a single axiom.

50 Beside the axioms of order of course the "axioms of concatenation" have to be presupposed.

51 Book V, definition 5.

52 Both μ and ν are *whole* numbers and I reemphasize that the concept of multiplication of a line segment presupposes the concept of equality, but not the concept of measure. E. g. AB equals the threefold of CD if there are two points M and N between A and B such that M lies between A and N, N between M and B, and $AM = MN = NM = CD$.

53 [Galilei 1638, Fifth Day, 22ff.]. The different facts admitted as immediately clear by the people in the "Discorsi" imply the assumption of as many axioms.

54 [Hilbert 1899, 26ff.].

55 The measure of length established in such a way leads to assigning the positive and negative numerical magnitudes to the points of a straight line, fixing arbitrarily the points the numbers 0 and 1 are to correspond to. The so called "projective" geometry leads to a different manner of distributing the positive and negative numerical magnitudes to the points of a straight line, fixing arbitrarily the points three determinate numbers are to correspond to. As one repeatedly concatenates equal line segments in the treatment concerning length, so in this projective treatment an operation has to be repeated several times, which is merely based on the connection of points and the partition of straight lines, without ever equalizing two distances or comparing them as to their equality. It is thus possible to establish a sort of measure in geometry without making use of the concept of length or even of the the concept of equality of line segments (cf. [Staudt 1857, 166ff.]; [Klein 1873, 112ff.]).

56 It was already remarked that Euclid proves too much when sometimes deriving from evident facts other facts which are also said to be evident [Zindler 1889, 36ff.]. Because of the fact, however, that we can avoid intuition itself in the proper proof, the attempt at deducing the whole geometry from a minimum of intuitive facts—or empirical facts—assumed as axiomatic, gains the highest importance. All deductions of this kind are of interest for the interrelation of knowledge—even in the case that we are able to deduce both of two intuitive facts one from the other, using the axioms which have been introduced anyway (in this case it is in fact an arbitrary decision which of the intuitive facts should be added to the axioms).

In any case there will be quarrels about what self-evident means, while there is normally no dissent in mathematics whether a given proof is stringent or not.

57 [Schur 1892, section 5, 5], [Killing 1898], [Hilbert 1899, 40ff.]. According to the approach indicated in the text, we do not start with a definition of the content that is common to equal figures, but with a definition of a procedure the result of which is taken to be decisive in determining whether two figures (or bodies) can be said to be "equal" or different. Since the definition chosen by us can be proved to be such that two figures which are equal to a third one also have to be said to be equal to each other, it holds that bringing together into one totality all figures equal to a given one will provide a totality of figures equal to each other. This collective concept replaces effectively the concept of content. This method is often applied in mathematics; in

our case we can easily pass over to the definition of the *measure of area*, i.e., to the proper concept of content.

The approach set out here can well be compared with Euclid's theory of proportions. As we did not define content, but merely declared under which conditions two figures are to be said to be equal, so Euclid did not define the ratio of two line segments as a number, but merely indicated under which conditions two line segments are to be said to be in the same ratio as two different line segments.

[58] Arithmetical concepts can be used in deduction—also in geometrical deduction—in a way that geometrical concepts are never used *within deduction.* The geometrical concepts of point, line segment, angle, and so forth are handled in proofs in the purely "formal" manner already explained in the text. We imagine a plurality of geometrical elements bearing certain relative properties and can derive from this new relative properties of the elements, operating not really with the geometrical concepts, but only with the axioms relating them. If one has to prove a proposition concerning a square or a hexagon, one will achieve this by repeating four times or six times a certain thought operation. But if one has to prove a proposition concerning the general n-sided polygon or the general polygon with an even number of sides, one has to *conceive* a thought operation as repeated an indeterminate number of times. In doing so general concepts have to be associated with the repetition of this thought operation as well as with their order. These general concepts of a secondary kind are number-concepts or combinatorical concepts, and in the proof these concepts are used not only "formally", but "contentually".

The difference between the formal and the contentual use of concepts can be made clear by comparing algebra—in the ordinary sense of the word—to arithmetics. In showing e.g., that $(a-b)(a^2+ab+b^2) = a^3-b^3$, one operates merely with algebraic symbols according to external rules comprising the rule that $ab = ba$. But if one wants to prove that the content expressed by the formula $ab = ba$ is true, one has to take into account the nature of the concepts of number and multiplication. We have to show that we are dealing with the same number of objects, e.g., spheres, whether there are a collections of b spheres or b collections of a spheres. As is well known, the proof goes like this: one begins with conceiving a collections of b spheres. Next one takes away one sphere from each collection, which results in a collections of $b-1$ spheres and, by bringing together the removed spheres, a new collection of a spheres. Now once again one takes away one sphere from each of the a collections, what results in a collections of $b-2$ spheres and, together with the collection previously formed, two collections of a spheres. By proceeding in such a manner finally all of the a collections will be exhausted at the same time, and the spheres originally given are now parted in b collections of a spheres.

We thus provided a proof of $ab = ba$ in terms of rearranging spheres. It is difficult to say on which ground we conceive the *general* possibility of this rearrangement; one cause may be seen in the fact that we actually already carried out operations of this kind.

As was noticed for the first time by Schröder ([Schröder 1873]) and as then was stressed by Helmholtz ([Helmholtz 1895, 358]), the concept of the cardinal number which the preceding consideration is based on presupposes a certain fact which I nevertheless hold to be provable, cf. [Stolz 1885, 9, 10].

Helmholtz, following the example of H. Graßmann ([Graßmann 1860]), presented a different proof of the propositions concerning numbers (cf. also [Kronecker 1887, 263ff.]). He makes use amongst others of the formula $ab = (a-1)b + b$ which he considers to be an "axiom of arithmetics". This axiom (of Graßmann) however is—as Helmholtz himself put it in an analogous case (mentioned in the above quoted

"Philosophische Aufsätze" ([Helmholtz 1887, 24] and [Helmholtz 1895, 363])—merely a description of the procedure of multiplication. The formula can indeed be conceived as the *definition* of multiplication, presupposing that the concept of addition is known. For given the case that someone does not know what a times b is, we could explain to him: 1 times b is b, 2 times b is 1 times b increased by b, 3 times b is 2 times b increased by b, and so forth; that is to say that in general a times b equals $(a-1)$ times b increased by b, or, as a formula, $ab = (a-1)b + b$.

In the same manner any procedure that can be repeated indefinitely yields a general concept as well as a rule for the application of this concept, the rightness of which we understand together with the reality of the concept. For example, we are able to multiply 1 by 2, the result by 3, the result obtained by 4, and so forth. The result obtained in the multiplication by a will be designated by $a!$, and to this concept applies the rule that $a!$ equals $(a-1)! \cdot a$. Such rules may appear as axioms, since they cannot be formally proved, but are immediately abstracted from the procedure in question in order to form the basis of formal deductions. But it is in fact clear that in arithmetic we would get an infinite number of axioms, because any procedure that can be indefinitely continued yields such an axiom. For this reason I would not like to call the formulas thus obtained axioms; but there is still another reason for this: we frequently introduce into arithmetic a preceding or a combinatorical procedure in order to find, with the help of the new concepts and rules yielded in such a way, certain hidden properties of already known concepts or, insofar as these properties have already been inductively found, to prove them. We can prove for instance the theorem of arithmetics mentioned in note 42 by means of a procedure of "reduction" of the so called "quadratic forms". In arithmetic the abstraction of new general concepts and rules thus forms in a way a constituent of deduction, and the procedure providing the basis of abstraction is a process we do not count among experience in a narrower sense of the word, that is to say in opposition to reasoning.

Having said this I want to assert that arithmetic—at least in so far as it concerns whole and rational numbers (as regards rational numbers cf. [Stolz 1885, 25ff.] and a review by the author in *Göttinger gelehrte Anzeigen*, [Hölder 1892, 592ff.] [b])—has no axioms in the proper sense, and that the arithmetical proof process is not a purely formal process, but a mixed and extremely complex process.

[59] The method of exhaustion, usually attributed to Archimedes, has already been used by Euclid in the proof that two circular surfaces stand in the same ratio as the squares of their radii (Book XII, no. 2).

[60] The content of the circular surface can also be determined as follows. The circle is said to be a regular polygon of an infinite number of infinitesimally small sides which can be divided into an infinite number of infinitely narrow triangles; the proposition in question follows from composition of these triangles, having the radius as their height. Inferences of this type are called "method of infinity".

The ideas involved in this method of infinity basically have to be considered as auxiliary ideas used for the sake of brevity. Such considerations get a precise sense when interpreted—as done in the text—in terms of the method of exhaustion (i.e., the method of inclusion in successively narrower limits). In this way strength is given to these considerations, and it goes without saying that in mathematics the method of infinity does not contain a new (transcendental) principle. Moreover, in the method of infinity the indirect procedure only seems to be avoided.

[61] There are a couple of mathematical proofs which have never successfully been set down in a direct manner, and it might be possible to show that they can actually

b. Cf. Mircea Radu's translation of this text in this volume, pp. 57–70 only be set down as an indirect (apagogic) proof. The proof given in the text for instance has been reached only by virtue of a theoretical twist rendering the proof indirect. The concerns raised by some authors thus seem to me unjustified. The indirect proof rests on the principle that the contrary of an assumption from which a contradiction can be derived must be true. The principle is quite a useful version of the principle of contradiction, whereas the so-called identity principle which is said to comprise the principle of contradiction is definitely unfruitful.

[62] I hold the Archimedean proof of the Law of the Lever to be among these proofs in mechanics. In this regard I disagree with the remarks of Mach in [Mach 1883]. Mach advances the view that in Archimedes it is already presupposed that the effect that a weight exerts on the lever depends on the product of the magnitudes of the weight and the arm of the lever. This is not the case. Archimedes ([Archimedes 1881, 142]) indeed presupposes that there is no equilibrium in the case of equal weights and unequal arms of the lever, and that an existing equilibrium will be disturbed when one of the weights is increased; it is implied therein that the equilibrium depends on the arm of lever on the one hand and on the weight itself on the other hand, but it is not implied that an existing equilibrium is preserved in the case that for instance one of the weights is triplicated and at the same time the fulcrum is shifted in such a way that the arm of the lever is reduced to a third.

In addition to these precisely stated assumptions, Archimedes makes use of another assumption. He assumes that the apparent use he makes of the center of gravity amounts to nothing else—and this also seems to be Mach's opinion—than that in an arrangement of any number of weights two equal weights suspended at different points can always be replaced by a single one of double weight which is suspended at the mid-point between the points of suspension of the two weights. Archimedes additionally presupposes that the inverse substitution also is feasible. The assumption can be thought of as being motivated on the one hand by the consideration that equal weights at equal arms of the lever are in equilibrium and exert exactly the same force on the point of suspension as these two weights would do if conjoined below the point of suspension; and on the other hand by the assumption that the observed equivalence also holds under different circumstances. The latter assumption can again be reduced to the following one, that a system of forces acting on a rigid body is in equilibrium, if a part of the forces considered separately is in equilibrium and if the same also holds of the other part of the forces. This is a postulate (an axiomatic assumption)—motivated, of course, by experience. This assumption can be said to contain a wholly general idea of the fact that effects of forces can be composed, and hence one may be led to the opinion that the assumption is equivalent to the physical principle of "superposition" (on superposition, cf. [Volkmann 1894, 21] and [Volkmann 1896, 69ff.]). An essential difference however consists in the fact that the application of the principle of superposition presupposes a particular *metrical* law according to which the effects of *any two forces* can be composed. The assumption made here is easier.

Whatever one might think about Archimedes' suppositions, it cannot be denied that it is not possible to immediately, i.e., by means of the mere power of judgment, perceive in them the Law of the Lever, though this metrical law necessarily follows from these suppositions in a mathematical reasoning. Such proofs (deductions) are always of value for the deeper knowledge of the scientific interrelation they provide; this is true even if the result to be deduced has previously been found inductively—a claim we cannot assert in the case of the Law of the Lever since the history of its discovery is unknown to us. In any case, in order to immediately establish in an inductive way the Law of the Lever, numerous and more precise measurements would have to precede, whereas already the experiences of every day life might lead us to postulate the Archimedean assumptions, which recommends these assumptions as the simpler foundations.

These foundations nevertheless are inductive; but this fact does not prevent us from using them deductively, indeed just as we also use empirical laws of nature deductively. The real method of mathematical physics consists of a conjunction of the inductive and the deductive procedure (cf. [Wundt 1883, 66]).

[63] Cf. *Philosophiae naturalis principia mathematica* [Newton 1687].

[64] It is of utmost interest that the mechanical concepts, though they are widely considered as extracted from experience, show in their scientific use the same ideal perfection as geometrical concepts. As regards the latter, the circumstance in question was exploited for the benefit of a philosophical theory, in particular for the benefit of Kant's conception of space. It is usually said, then, that the empirical objects never perfectly represent the geometrical concepts, for in experience there are merely little bodies instead of points, whereas the concepts and propositions of geometry are held to be exactly valid for the perfect laws of the intuition proper to us and thus basically apply to intuition rather than to experience.

In this regard I have to object that I cannot credit intuition neither with perfection nor with exact correspondence to geometry insofar as this intuition is not merely postulated, but is actually to be observed in us. In fact we are only acquainted with the faculty of our imagination, to represent images of spatial objects with a vividness close to that of actually seeing these objects. On the contrary, I cannot pretend to be able to imagine two completely equal line segments. I am however completely able to picture two approximately equal images of line segments and, by doing so, to decide that I will consider them as equal, i. e., I am able to conceive them as equal. Nor I could pretend to imagine with sensual vividness a straight line as infinitely extended. I rather imagine a limited straight line augmented by a segment, and the thought that the same segment can be added to the straight line again and again makes me conceiving of it as infinite. Similarly in mechanics the thought of a "material point" results from the idea of a body whose mass is taken into consideration whereas its extension is so little or can be made so little by compression that it is no longer taken into consideration.

As the mechanical concepts are thus abstractions on the basis of experience, so also the geometrical concepts are likely to have resulted from abstraction—be it from experience or from intuition—, and geometrical and mechanical concepts thus hardly differ from ordinary empirical concepts.

But there is more. After having found, by virtue of experience, laws which interrelate the geometrical concepts, these laws fuse in our consciousness with the concepts, and from this results the will to retain these laws given any future experiences; this however will also affect the particular application of the concepts. In mechanics, this is indeed the case. It has already often been remarked and has often been objected against an empirical origin of geometry that we usually correct experience by geometry, not the other way round. It is true that we often do so. If we put a ruler on the table and if the ruler does not fit the table in all directions, we will conclude from this that the table is not perfectly plane, but not that the propositions applying to straight lines and planes are incorrect. I nevertheless think that this point does not refute the empirical origin of geometry. In mechanics we act precisely in this way; if, e. g., a comet shows an inexplicable deceleration of its motion, we assume for the sake of the law of inertia that the space is filled with a resisting medium.

We can understand this point as follows: the primitive concepts of geometry and mechanics as well as the simplest concepts of this science are abstracted according to experience, in the case of geometry probably according to quite raw experiences. Only after the laws, obtained by experience, proved to be true on the whole and to be useful in application, one attempted to postulate them as exactly valid for points and straight lines, for masses that are perfect points, for perfectly constant forces and so forth. With these postulates we thus ascend to unlimited precision (cf. [Volkmann 1894, 17]).

The interrelated concepts of the geometric elements—point, straight line and so forth—together with the relations which can hold between these elements (e.g., a point can lie on a straight line) and the laws (axioms) postulated by us and permitting us to conclude the existence of certain relations from the existence of other relations, amount to a completely elaborated conceptual image of the real and apparent order of things which is called "space" (this concept of space is a concept of higher order which shows a logical structure and permits us to draw conclusions from, cf. [Volkelt 1886, 384]). But since we are able, as we have seen, to develop from different assumptions different coherent geometric systems, the concept of space can also be formulated in different ways. I think that the earlier philosophers who concerned themselves with the essence of space in geometry all had in mind the Euclidean concept of space, and their theories differ only in how they thought about the relation between this concept and reality. It is comprehensible that some wanted to credit such a consequential concept with a higher truth in regards to reality, whereas Kant considered it as subjective, assuming this concept to be identical with intuition and this "pure" intuition to be the form that things have to appear in because of our mental organization.

If one takes into consideration that we have a similarly consequential concept of the motions of a body and of the play of forces governing this motion, which probably rests on experience, one will be apt to believe that also the concept of space has been formed with the help of experience. It will no longer appear contradictory that, though we use this concept in some cases in order to interpret experience, we nevertheless consider it possible to check this concept—whose adequacy is hypothetical—for correspondence with experience in order to reshape the concept, if necessary, as we do with physical concepts (in the domain of physical knowledge we admittedly form concepts by the way of trial and error, reshaping them in the case of non-confirmation, cf. [Sigwart 1893, 242]). The lack of exactness which, in regard to geometry, inheres in empirical objects, does not necessarily obstruct verification. Euclid's geometry allows us to draw conclusions for extended bodies, i.e., also for points that are not exactly points, straight lines that are not exactly straight lines, and so forth, for which reason we can verify whether experiences obtained with such imprecise objects agree, within accuracy limits, with conclusions from the axioms. The same holds for the mechanical postulates which we check only in a mediate way.

Until now there was accordance between Euclidean geometry and experience; this geometry has been verified in innumerable cases, since practically any empirical application of this geometry is such a verification. At least as regards application, there is no reason to deviate from Euclid's geometry. Riemann presumes that possibly in the future new facts will be found which will determine such a deviation, and already Lobachevsky held that astronomical measurements might give reasons for this.

Such assertions have provoked rejection in philosophy (cf. [Lotze 1884, 248]; [Sigwart 1893, 82]). There is no doubt that a contradiction in astronomical measurements and calculations might be interpreted in quite different ways. We could assume that the light ray which falls in the observer's telescope is not completely straight; for even if we originally abstracted the concept of the straight line from the line of vision, it is still possible to subsequently relate the concept of the exactly straight line to the Euclidean axioms and then to distinguish between the straight line and the line of vision. But, since the astronomical measurement in question must be made at different moments and since the celestial bodies meanwhile have moved, the error might also be corrected by assuming different laws for the motion of the celestial bodies. In order to take into consideration in our hypothetical case all possibilities, the whole system of physico-geometrical concepts, insofar as it is relevant for astronomy, must be reorganized (the work of Lipschitz, who developed a mechanical theory for non-Euclidean geometry, can be regarded from this point of view, cf. [Lipschitz 1872, 116]). It is surely extremely improbable that geometry would finally be the object of this reorganization—but it is nonetheless not impossible.

[65] The sensation which makes us declare something as evident seems in my opinion to rest on this.

[66] The fact that it is impossible to define properly, i.e., constructively, the primitive concepts of mechanics has also resulted in the complete abandoning of any definition of these concepts, e.g., in [Kirchhoff 1877]. At the same time Kirchhoff declared the problem of mechanics to consist in a complete and most simple description of motions in nature. The descriptions he has in mind are effectuated in terms of equations; he considered the magnitudes of the forces and masses to be nothing else than numerical coefficients figuring in these equations [Kirchhoff 1877, 5, 12]. This conception does not do justice to the physical content of the concepts and within this conception it seems inexplicable why those numerical coefficients, which are called masses, are independent from time. Kirchhoff's claim that mechanics merely describes has been repeated too often. In a certain sense explaining of course amounts to describing, but nonetheless it fundamentally differs from a purely external description, and we should thus retain the distinguishing term. Paul Du Bois-Reymond says : "The derivation of a manifold phenomenic domain from the simplest elements of appearance is no description. I think it is better called the synthesis, the construction or the build-up of the phenomenic domain from simplest mechanisms" [du Bois-Reymond 1890, 14].

[67] As has already been remarked, the similarity between geometrical and mechanical concepts as well as between geometrical and mechanical deduction provides evidence for the empirical origin of geometry, unless certain concepts of mechanics also are considered as "a prioric", i.e., as independent from experience. This seems to be the case in Kant where he speaks of "pure" science. For example, the persistence of substance he counts among "the pure and completely *a priori* laws of nature" ([Kant 1781-1787, Pt. II, Div. I, Bk. II, Ch. II, 3., A, 301]). What should be understood, according to Kant, by persistence of substance becomes evident from the example given by him in the passage in question: "A philosopher was asked: How much does the smoke weigh? He replied: If you take away from the weight of the wood that was burnt the weight of the ashes that are left over, you will have the weight of the smoke. He thus assumed as incontrovertible that even in fire the matter (substance) never disappears but rather only suffers an alteration in its form" [Kant 1781-1787, 302]. Kant hence wants the precise law of the conservation of matter to be included in the principle of persistence. He however had a different opinion about different mechanical concepts, e.g., when he considers motion to be an empirical concept (*Doctrine of Elements*, Pt. I, Div. I, Bk. I, Ch. III, 3., [Kant 1781-1787]).

In addition to this similarity, an immediate interrelation between geometrical and mechanical concepts can be found. Helmholtz indeed assumes (annotation 14) that the geometrical concept of congruence presupposes a physical concept, e.g., the concept of the rigid body; and Benno Erdmann considers this in his *Axiome der Geometrie* ([Erdmann 1877, 91, 147, 148]) as a proof of the empirical view.

68 On "experience as a verification of correctness of knowledge", cf. [Volkelt 1886, 256].

69 In both sciences it becomes evident that a collection and examination of empirical facts has to come first in order to inductively find the laws which make possible deduction in the particular parts of the science. This becomes evident because still today empirical facts are—*intentionally*—collected by means of measurement and experiment.

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