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Foundation of Mathematics between Theory and Practice

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Résumé : Je me propose dans cet article de traiter de la théorie des ensembles, non seulement comme fondement des mathématiques au sens traditionnel, mais aussi comme fondement de la pratique mathématique. De ce point de vue, je marque une distinction entre un fondement ensembliste standard, d'une nature ontologique, grâce auquel tout objet mathématique peut trouver un succédané ensembliste, et un fondement pratique, qui vise à expliquer les phénomènes mathématiques, en donnant des conditions nécessaires et suffisantes pour prouver les propositions mathématiques. Je présente quelques exemples de cette utilisation des méthodes ensemblistes, dans le contexte des principales théories mathématiques, en termes de preuves d'indépendance et de résultats d'équiconsistance, et je discute quelques résultats récents qui montrent comment il est possible de « compléter » les structures $H(\aleph_1)$ et $H(\aleph_2)$. Ensuite, je montre que les fondements ensemblistes de mathématiques peuvent être utiles aussi pour la philosophie de la pratique mathématique, car certains axiomes de la théorie des ensembles peuvent être considérés comme des explications de phénomènes mathématiques. Dans la dernière partie de mon article, je propose une distinction plus générale entre deux différentes espèces de fondement : pratique et théorique, en tirant quelques exemples de l'histoire des fondements des mathématiques.

Abstract: In this article I propose to look at set theory not only as a foundation of mathematics in a traditional sense, but as a foundation for mathematical practice. For this purpose I distinguish between a standard, ontological, set theoretical foundation that aims to find a set theoretical surrogate to every mathematical object, and a practical one that tries to explain mathematical phenomena, giving necessary and sufficient conditions for the proof of mathematical propositions. I will present some example of this use of set theoretical methods, in the context of mainstream mathematics, in terms of independence

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proofs, equiconsistency results and discussing some recent results that show how it is possible to "complete" the structures $H(\aleph_1)$ and $H(\aleph_2)$. Then I will argue that a set theoretical foundation of mathematics can be relevant also for the philosophy of mathematical practice, as long as some axioms of set theory can be seen as explanations of mathematical phenomena. In the end I will propose a more general distinction between two different kinds of foundation: a practical one and a theoretical one, drawing some examples from the history of the foundation of mathematics.

A wonderful aspect of mathematical work is the possibility to create useful interactions between apparently different areas. This aspect, that we may call the *unity of mathematics*, is a distinctive aspect of modern mathematics. The tools and the ideas that come to light thanks to this global point of view are so powerful that they allow to overcome the Aristotelian *caveat* about the different *genus*, for example, between geometry and arithmetic. Moreover, the birth of modern mathematical logic and the need to keep together a very vast and disparate development of mathematics were among the reasons that allowed and pushed toward the foundational programs of the beginning of the last century. Nevertheless history frustrated these foundational efforts. Not only contradictions were discovered, but also a deep and unsolved tension between syntax and semantics: two very new branches of mathematical enquire. We can say that all foundational programs did not succeed in the sense they were conceived.

Nevertheless foundational enquires are still open and there are mathematical problems that have a foundational flavor. This situation calls for an explanation of what a foundation is and how it is possible to propose one nowadays. We think that among the many reasons that push for a foundation of mathematics, there is a goal that is common to every foundation, that is to shape the mathematical field. By this we mean that any kind of foundation, if it does not define, at least it distinguishes between mathematical and non-mathematical work and, in some way, characterizes mathematical practice, as being of a certain kind and obeying some specific rules. It is in this sense that we can find concerns for the unity of mathematics also in the foundational context and we believe that this is a common aspect of all different foundations of mathematics.

In this article we propose to look at set theory not only as a foundation of mathematics in a traditional sense, but as a foundation for mathematical practice. For this purpose, we distinguish between a standard, ontological, set theoretical foundation that aims to find a set theoretical surrogate to every mathematical object, and a practical one that tries to explain mathematical phenomena, giving necessary and sufficient conditions for the proof of mathematical propositions. We will present some examples of this use of set theoretical methods, in the context of mainstream mathematics, in terms of independence proofs, equiconsistency results. We will also discuss some recent results that show how it is possible to complete the structures $H(\aleph_1)$ and $H(\aleph_2)$. Moreover, in the central part of this article we will claim that a practical foundation of mathematics can be considered relevant not only for the practice of doing mathematics, but also from a truly philosophical perspective, showing its importance in the context of the philosophy of mathematical practice. This latter task will be done considering the explanatory role of some set theoretical axioms and discussing Kitcher's account on the matter of scientific explanation. In the end, we will propose a more general distinction between two different kinds of foundation: a practical one and a theoretical one, drawing some examples from the history of the foundations of mathematics.

1 Set theoretical foundation as unity

In order to explain this concept of unity we can see how it is realized in the context of the most common foundation of mathematics: set theory. This character of set theory has always been stressed by many people. We offer just one quotation for many, by Penelope Maddy:

> For all that, set theoretic foundations still play a strong unifying role: vague structures are made more precise, old theorems are given new proofs and unified with other theorems that previously seemed quite distinct, similar hypotheses are traced at the basis of disparate mathematical fields, existence questions are given explicit meaning, unprovable conjectures can be identified, new hypotheses can settle old open questions, and so on. That set theory plays this role is central to modern mathematics, that it is able to play this role is perhaps the most remarkable outcome of the search for foundations. [Maddy 1997, 34–35]

However, instead of describing the almost 'standard' set theoretical foundation following which every mathematical entity is intended to be a set, we propose to look at set theory as a means to give a foundation to mathematical practice.[1] Indeed, the universality character of set theoretical language—i.e.,

1. On this ground we will be inspired by Resnik's idea that mathematics is a science of patterns and that a set theoretical foundation can be seen as a macro-pattern: "There is another phenomenon which has greatly changed mathematics and which could be called a reduction. This is the set theorizing of mathematics. I have in mind the use of the language of set theory as the background language of working mathematics and the attendant objectification (or, in my terms, positionalization) of mathematical structures" [Resnik 1981, 540]. However, contrary to the realist structural position of [Resnik 1997] and [Resnik 1981] we will try to show how to make sense of the notion of pattern not only in an ontological, but also epistemological way. Our aim indeed will be to support a set theoretical foundation of mathematics, deprived of its standard realist proposal.

the possibility to formalize any piece of mathematics inside set theory and to find a set theoretic surrogate for any mathematical object—has not *a priori* any ontological meaning. Set theory is not to be intended here as only ZFC, as it is often the case when set theory is called upon arguing for a standard foundation, but as a general method that makes use of set theoretical principles to analyze mathematical practice. As part of this method we include also reverse mathematics and all the useful set theoretical assumptions, sometimes called axioms, that extend ZFC.[2] Clearly the term "theory" here is an abuse of language from a logical point of view, because, neither we think of a consistent set of sentences, nor of an intuitive theory with its intended interpretation. What we have in mind is a general method that is widely and sometimes tacitly used in mathematical practice.

1.1 Hilbertian origins

There are two aspects of this set theoretical point of view that we propose that model the corresponding foundation of mathematics. They explain in which sense set theory meets the requirements of unity of mathematics. These ideas can be traced back to Hilbert's foundational works. They follow the evolution of Hilbert's thought on foundational issues, belonging to two different periods, also chronologically distant. Hence they do not characterize his point of view on this subject.

The first one pertains to Hilbert's period of foundations of Geometry. In his mind the question of why a theorem is true was equivalent to the problem of elucidating the main possibility of a proof.

> I understand under the axiomatical exploration of a mathematical truth [or theorem] an investigation which does not aim at finding new or more general theorems being connected with this truth, but to determine the position of this theorem within the system of known truths in such a way that it can be clearly said which conditions are *necessary and sufficient* for giving a foundation of this truth. [Hilbert 1902-1903, 50, my italics]

Of course, despite Hilbert's ideas, history then showed that metamathematics can give rise to new truly mathematical results and it is a powerful method not only to determine general properties of the axiomatic setting of a formal theory. What is important to stress here is that this attitude is an attempt to give an answer to possible 'why questions' that can rise in the mathematical discourse. Indeed this is exactly what Hilbert was hoping to do in his foundation of geometry. In a letter to Frege, dated December 29th, 1899 Hilbert wrote:

2. Among them a special status is hold by large cardinals, but we will discuss it later.

> I wanted to make possible to *understand* and answer such questions as *why* the sum of the angles in a triangle is equal to two right angles and how this fact is connected with the parallel axiom. [Frege 1980, 38–39, italics mine]

The second idea that we would like to recover from Hilbert is the conviction that a good axiomatization of mathematics should be a catalog of the principles that we use in our mathematical practice.

> The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds. [Hilbert 1927, 475], in [Van Heijenoort 1967]

In Hilbert's program, this belief was related to the expectation that few arithmetical and logical axioms were able to characterize every piece of mathematics. Since this has been shown to be impossible, we accept this suggestion to be compatible with an open-ended list. Indeed this idea of a catalog of principles could *a priori* involve also incompatible principles. We are not looking for a categoricity theorem that permits to define what a theory is about, but a theory that can explain our mathematical work, showing its uniformity of methods and arguments, in order to account for its unity.

We think that these two ideas are also able to account for the explanation of a mathematical fact, outlining the main conditions of its proof and pointing at the reasons for accepting its truth.[3] Since we are dealing with a demonstrative context, what is often essential for overcoming the difficulty of an argument is a combinatorial aspect of the proof, that reveals the key ingredient for the solution of a problem. This is the reason why many set theoretical principles have a combinatorial character, but this does not prevent us from listing them in the catalog, as long as they contribute to account for the unity of mathematics—i.e., they are not *ad hoc* and they have many and different applications. What is relevant in showing that some principles are necessary and/or sufficient is their role in the argumentative structure of a theorem. Sometimes these principles go hand in hand with a more general understanding of a whole field.

The general idea behind this conception of set theory is that it is a method that can be applied to all other branches of mathematics. Indeed this is how it was conceived, at least by Zermelo, in the thirties.

> Our axiom system is non-categorical after all, which, in this case, is not a disadvantage, but an advantage. For the enormous significance and unlimited applicability of set theory rests precisely on this fact. [Zermelo 2010, 427]

3. Later we will argue more on this point.

A good conceptual reason for arguing in favor of set theory as an open-ended foundation lays in the fact that—like mathematics, as we will argue in the next section—the subject matter of set theory is not sufficiently clear to immediately characterize a model and isolate its axioms. This is one of the main concern in contemporary research in set theory, but what is important to stress is the distinction between a foundational role of set theory and the research that tries to single out one true model for ZFC, out of the many we can conceive. This is not an easy task, because if we have a good intuition, for example in the case of the real line, of what are the pathological aspects we would like to avoid, like the Banach-Tarski paradox and the consequent non-measurability of some sets of reals, this is much more difficult as soon as we proceed in the hierarchy of the transfinite. Far from being a weakness, this hazy boundary is what allows set theory to account for the unity of mathematics. However, this aspect of vagueness, common to both set theory and mathematics, has been criticized and has always been subject of discussions, in the foundational context, where, exactly, to draw the dividing line between mathematics and non mathematics. There are people like Feferman and Weaver who would like to put the crossbar much lower than the level of ZFC.[4] However if we are trying to explain the unity of mathematics, and therefore we are working at a foundational level, we cannot drop so easily set theory. Indeed either we discredit modern research in set theory as being mathematics, or we have to propose a sufficiently wide framework, where it is possible to place it. In a slogan: there are more things in mathematical research and mathematical practice than are dreamt of in ZFC. What we propose here is to consider the methods offered by set theory as a framework for mathematics, part of which is of course set theory.

1.2 Practical reasons

We now plan to show why set theory can offer a foundation for the practice of doing mathematics. Before we start we need a definition that is fundamental in what follows.

Definition 1.1. *We say that a theory T, that extends ZFC, has consistency strength stronger than a theory S if in first order Peano arithmetic it is possible to prove $Con(T) \rightarrow Con(S)$, where $Con(T)$ is the sentence expressing the consistency of T. Moreover, for a sentence A written in the language of set theory, we refer to $Con(A)$ as an abbreviation for $Con(ZFC + A)$.*

There are three reasons that support the idea of a set theoretical foundation of mathematical practice.

1. *Independence proofs.* This is the main subject of modern research in set theory. Since the invention of forcing,[5] in the sixties, many prob-

4. See, for example, [Weaver 2009] and [Feferman 1999].
5. See [Kunen 1980] for a very good introduction to this subject.

lems were shown to be independent from ZFC, like for example the Continuum Hypothesis (CH) and Souslin's Hypothesis. This kind of proofs is used, as Hilbert did, to prove that a set of axioms is not sufficient for a mathematical result.

2. *Combinatorial principles.* The discovery of the independence of a proposition does not conclude its mathematical analysis. Indeed the examination of an independent problem often brings together the identification of a technical *impasse* and the corresponding combinatorial principles that are sufficient for its solution. For a safe use of these principles, the method of forcing is used to show that they are consistent relative to some theory like, for example ZFC. But sometimes ZFC is not sufficient for this task. It is here that large cardinals come into play.
3. *Large cardinals.* These are hypotheses on the existence of cardinals large enough[6] to prove $Con(ZFC)$. They are used to determine the power of sentences stronger than ZFC, in terms of their consistency strength. Indeed many natural sentences stronger than ZFC can be proved to be equiconsistent—in the context of ZFC—with the existence of suitable large cardinals. Then large cardinals can be viewed, modulo equiconsistency, as necessary and sufficient conditions for the proof of sentences stronger than ZFC.

There is an important reason for using large cardinals as the backbone for the analysis of the propositions that transcend the deductive power of ZFC.

> **Empirical fact**: the order induced by the consistency strength of large cardinal hypothesis is, except in few cases, linear and well founded.

The use of large cardinals in set theory is twofold: on the one hand they serve to compare different principles, using equiconsistency results and the linear order given by their consistency strength; on the other hand they supply the means to give relative consistency proofs, being the key ingredient of theorems of the form: given an independent statement A, written in the language of set theory, if the hypothesis I stating the existence of a large cardinal κ holds, then there is a model where A holds; in other words I implies $Con(A)$.

There exists an epistemological tension between logical deduction and consistency strength and this aspect is responsible for the richness of the analysis that set theory can offer to necessary and sufficient conditions. Indeed the epistemological value of the search for necessary and sufficient conditions for the proof of a theorem consists in the discovery of its place in the logical structure of a theory. This process can work in two directions: starting from an axiomatic system and asking which of its axioms are needed for the proof of a theorem, or starting with a proposition and looking for the axioms that are

6. If κ is large enough, so that V_κ is a model of ZFC, then we say that κ is a large cardinal. However this is not a definition of what is a large cardinal.

needed for its (non-trivial) proof, without specifying the axiomatic context. In the first case this analysis is informative on the content of a theorem, like for example Hilbert's work on Desargue's theorem—where the aim is to clear its spatial content. But in the second case, when the goal is a context-free analysis that looks for the principles that are needed for the proof of a proposition—for example a proposition independent from ZFC—the discovery of necessary and sufficient conditions consists just in finding logically equivalent formulation of the proposition. In this latter case the progress in our knowledge may be given by a combinatorial character of an equivalent formulation, or its relevance in a different field, but it is not informative for what concern the possibility of its proof, nor for its content—i.e., we cannot give an answer to the question "Why this proposition is a theorem of set theory?" On the contrary, a result of equiconsistency is well more informative on the epistemological status of a proposition. Indeed, such a proof outlines the fact that we have to believe not only in the truth of a sentence, but also in the existence of a particular class of models of ZFC: the ones whose existence is guaranteed by the equiconsistency proof. Moreover, logical equivalence and equiconsistency cannot be assimilated, without collapsing truth and existence. While the former is a syntactical notion, the latter is semantical and expresses the fact that we need to believe in something, possibly, stronger than ZFC in order to believe the truth of a particular sentence. This is the reason why the use of large cardinals in the consistency proofs fills the gap between logical deduction and consistency strength, because not only we can have theorem of the form $I \to A$, but also $I \to Con(A)$, i.e., we can show that the sentence expressing the existence of a large cardinal, logically implies not only another sentence, but also the fact that there is a model where this is true.

Hence, large cardinals provide a precise answer to "why" questions. In this sense, large cardinals can be seen as more fundamental principles that give more and more powerful means, not only to prove new propositions, but also to analyze our believes in their truth.[7] After the discovery of the difference between, truth, provability and existence, we have to accept the original Sin of Gödel's theorem, but what large cardinals—together with the method of forcing—offer is a way to analyze and stratify the degree of incompleteness that we find in our mathematical practice.

7. If we accept this point of view we also have to accept the consequence that the more fundamental principle is a contradiction. As a matter of fact, many large cardinals hypothesis can be seen as stating the existence of a non trivial elementary embedding of two universes of set theory; the more similar are these classes, the higher is the consistency strength of the corresponding large cardinal. Pushing this process at the limit, we get a statement that postulates the existence of a non trivial elementary embedding of the universal class V in itself. This statement has been shown to be inconsistent with ZFC, by Kenneth Kunen.

1.3 Example of sufficient conditions

We would like here to present some examples that show how set theory is used to analyze mathematical problems. It is important here to stress not only the fine and deep explanation that is given by a set theoretical investigation, but also the fact that the problems discussed come from some of the characteristic fields of classical mathematics: group theory and functional analysis. This aspect is important since it acknowledges the importance of set theoretical method not only in logical or pathological context, but in classical domains of mathematical practice.

We shall describe the solutions given to Whitehead's problem and a recent result by Farah on operators algebras of an Hilbert space.

Definition 1.2. ***(Whitehead's problem (WP))*** *Is every Whitehead group (i.e., an abelian group A such that, whenever B is an abelian group and $f : B \to A$ is a surjective group homomorphism, whose kernel is isomorphic to the group of integers $\mathbb{Z}$, then there exists a group homomorphism $g : A \to B$ with $fg = id_A$) a free group (i.e., a group A that has a subset X, called the set of generators, such that every element of A can be written uniquely*[8] *as a finite combination of elements in X and their inverses)?*

In the seventies Shelah proved the following theorems.

Theorem 1.3. *([Shelah 1980]) If $V = L$*[9]*, then the answer to WP is yes.*

Theorem 1.4. *([Shelah 1977]) If Martin's Axiom (MA)*[10] *and the negation of the Continuum Hypothesis ($\neg CH$) both hold, then the answer to WP is no.*

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8. Modulo equivalence of the form $ab = axx^{-1}b$.

9. The set-theoretical hypothesis $V = L$ states that all sets are constructible. Indeed V is the standard notation that refers to the universal class of all sets that can be defined by stages iterating the power set operation α-many time, for every ordinal α:

- $V_0 = \varnothing$,
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
- $V_\lambda = \bigcup_{\alpha\in\lambda} V_\alpha$, for λ limit ordinal,
- $V = \bigcup_{\alpha\in Ord} V_\alpha$.

On the other hand L is the class of all constructible sets, that can also be presented as a cumulative hierarchy. For a set X let $Def(X)$ be the collection of all sets definable with parameters in X. Then L is defined as follows:

- $L_0 = \varnothing$,
- $L_{\alpha+1} = Def(L_\alpha)$,
- $L_\lambda = \bigcup_{\alpha\in\lambda} L_\alpha$, for λ limit ordinal,
- $L = \bigcup_{\alpha\in Ord} L_\alpha$.

10. We will not give the definition of MA here. What is important to know is that it is one of the weakest Forcing Axioms. We refer, for the interested reader, to [Jech 2003].

We then have another proof of the fact that V=L and ¬CH are incompatible. Moreover, since $Con(\text{ZFC} + \text{V=L}) \iff Con(\text{ZFC}) \iff Con(\text{ZFC} + \text{MA} + \neg\text{CH})$ we have sufficient conditions for both answers to WP, without exceeding the consistency strength of ZFC; that is, without an overshooting that would confuse the problem.

Another example is the following result in the context of functional analysis.

Definition 1.5. *The Calkin algebra $\mathcal{C}(H)$ is the quotient of $B(H)$, the ring of bounded linear operators on a separable infinite-dimensional Hilbert space H, by the ideal $K(H)$ of compact operators.*

It is a natural question to ask if every automorphism is inner; i.e., if it is induced by the operation of conjugation. The answer to this question is again sensible to the background set theoretical hypothesis.

Theorem 1.6. *([Phillips & Weaver 2007]) If CH holds there is an automorphism of $\mathcal{C}(H)$ that is not inner.*

Theorem 1.7. *([Farah 2010]) If the Open Coloring Axiom (OCA)*[11] *holds all automorphism of $\mathcal{C}(H)$ are inner.*

It is interesting to note that, in this case, the first version of Farah's theorem used the Proper Forcing Axiom (PFA), whose consistency strength is much higher than $Con(ZFC)$. Then, the analysis of why PFA was used in the proof led to the discovery that just OCA, that is a combinatorial consequence of PFA, was needed. Then since $Con(\text{ZFC} + \text{OCA}) \iff Con(\text{ZFC}) \iff Con(\text{ZFC} + \text{CH})$ we have found again sufficient conditions for the solution of a natural mathematical problem; the best possible solution with respect to consistency strength.

1.4 Examples of equivalence and equiconsistent results

Of course there are also examples of necessary and sufficient conditions for sufficiently natural mathematical problems. They indeed show equivalences between different principles that can be epistemically informative for their combinatorial content, or just useful for finding new and unexpected links between different areas of mathematics.

Theorem 1.8. *([Freiling 1986]) The following are equivalent, over ZFC:*

1. *Continuum Hypothesis: $2^{\aleph_0} = \aleph_1$,*

11. As in the case of MA, we refer to [Jech 2003] for the definition of OCA.

2. *Axiom of Symmetry: for any function f that associates countable sets of real numbers to real numbers, i.e., $f : \mathbb{R} \to [\mathbb{R}]^{\aleph_0}$, there are $x_0, x_1 \in \mathbb{R}$ such that $x_0 \notin f(x_1)$ and $x_1 \notin f(x_0)$.*

Nevertheless, the strength of the set theoretic method can be mostly appreciated in combination with large cardinals and so when necessary and sufficient conditions are such, up to equiconsistency. The best example is Solovay's model for the following very natural property for sets of reals, that started the study of descriptive set theory.

Definition 1.9. ***(LM, BP and PSP)*** *Given $X \subseteq \mathbb{R}$, we say that X is Lebesgue measurable (LM) if it belongs to the σ-algebra generated by the Lebesgue measure on $\mathbb{R}$. We say that X had the Property of Baire (BP), if there is an open set U such that $U \Delta X$ (the symmetric difference) is a meager set (i.e., small). We say that X has the Property of the perfect set (PSP), if it is either countable or has a nonempty perfect subset: a closed set with no isolated point.*

Theorem 1.10. *([Solovay 1970]) The following are equivalent, over ZFC:*

- *Con(ZF+ all sets of reals are LM and have BP and PS),*
- *Con(There exists an inaccessible[12] cardinal κ).*

The epistemological meaning of this theorem is that it explains what we need to believe in terms of consistency to accept that all subset of the reals behave very nicely with respect to some natural properties.

1.5 Necessary and sufficient conditions

We would like here to present a new point of view on the application of the forcing method to the general phenomenon of independence.[13] They are part of a more general program that helps in making more precise the methodology suggested in Gödel's program and that is now called *Woodin's program.*[14] This program aims at finding a satisfactory description of the universe of set theory step by step; that is, giving a sufficiently complete description of initial segments of the class $V = \{x : x = x\}$.

We need a definition in order to make precise the "step by step" methodology of this program.

12. This is the weakest notion of large cardinal. We say that κ is inaccessible if it is regular and such that for every $\lambda < \kappa$, we have $2^\lambda < \kappa$.
13. Some of the results we quote are old, but what is new is their presentation, the context in which they are placed and consequently the meaning they assume in this new context; see [Viale 2011].
14. See [Woodin 2001] for a presentation of this program.

Definition 1.11. *If x is a set we define $tc(x)$ as the transitive closure of x: the minimal set under inclusions that contains x and that is transitive, i.e. if $y \in tc(x)$ then $y \subseteq tc(x)$.*

Definition 1.12. ***(A cumulative hierarchy)*** *We can build V stage by stage in the following way: for every $\lambda \in Card$ the structure $H(\lambda)$ consists of the sets of cardinality hereditarily less than λ.*

$$H(\lambda) = \{x : |x| < \lambda \text{ and } \forall y\,(y \in tc(x) \Rightarrow |y| < \lambda)\}.$$

Then Woodin's way to phrase his program is the following:

> One attempts to understand in turn the structures $H(\aleph_0)$, $H(\aleph_1)$ and then $H(\aleph_2)$. A little more precisely, one seeks to find the relevant axioms for these structures. Since the Continumm Hypothesis concerns the structure of $H(\aleph_2)$,[15] any reasonably complete collection of axioms for $H(\aleph_2)$ will resolve the Continumm Hypothesis. [Woodin 2001, 569]

Notice that one of the main motivation is the solution of CH. Now we need a definition in order to make precise the sense in which the program attempts to complete the initial segment of the universe of set theory, ruling out the trivial incompleteness phenomena given by Gödel's sentences and the sentences expressing the consistency of a theory.

Definition 1.13. *ψ is called a solution of a structure M, that models enough of ZFC, iff for every sentence $\phi \in Th(M)$,*

$$ZFC + \psi \vdash \ulcorner M \vDash \phi \urcorner \text{ or } ZFC + \psi \vdash \ulcorner M \vDash \neg\phi \urcorner.$$

We now present some initial results of Woodin's program and some other by Viale, that show how the forcing axioms fit in this program. The importance of these results is to be found in the possibility of using forcing not only to give sufficient conditions, but also necessary. Indeed the slogan that motivates them is the following.

> **Key idea**: The method of forcing is a tool that allow to prove theorems over certain natural[16] theories T which extend ZFC.

The first result of this kind is a reformulation of Cohen's forcing theorem, that shows how any transitive model of ZFC overlaps with the Σ_1 theory of $H(\aleph_1)$.

15. Indeed $\mathbb{R} \subseteq H(\aleph_2)$, since $\mathcal{P}(\omega) \subseteq H(\aleph_1)$ and so every subset of $\mathbb{R}$ belongs to $H(\aleph_2)$.

16. Of course there is a an important philosophical problem behind the concept of natural, but we will came back on this later. Luckily the main concept of naturalness is sufficiently natural to be easily understood, but nevertheless it deserves a philosophical analysis.

Theorem 1.14. *([Cohen 1963]) Assume T extends ZFC. Then for every Σ_0-formula $\varphi(x,p)$ and every parameter p such that $T \vdash p \subset \omega$ the following are equivalent:*

- $T \vdash \ulcorner H(\aleph_1) \models \exists x \varphi(x,p) \urcorner$
- $T \vdash$ *There is a partial order P such that $\Vdash_P \exists x \varphi(x,p)$.*

Thanks to large cardinals the above theorem can be extended to all formulas, with parameters in $H(\aleph_1)$. Next theorem says that it is possible to find a solution of the theory of $L(\mathbb{R})$ (i.e., the class of all set that are constructible with real parameters), but we have that $H(\aleph_1) \subseteq L(\mathbb{R})$ and $H(\aleph_1)^{L(\mathbb{R})} = H(\aleph_1)$. So next result is really an extension of the previous one.

Theorem 1.15. *(Woodin [Larson 2004]) Assume T extends ZFC + There are class many Woodin cardinals.*[17] *Then for every formula $\varphi(p)$ and every parameter p such that $T \vdash p \subseteq \omega$ the following are equivalent:*

- $T \vdash \ulcorner L(\mathbb{R}) \models \varphi(p) \urcorner$
- $T \vdash$ *There is a partial order P such that $\Vdash_P \varphi^{L(\mathbb{R})}(p)$.*

So, modulo the method of forcing, large cardinals are a solution for the structure $H(\aleph_1)$; i.e., they decide the theory of $H(\aleph_1)$ with parameters in $H(\aleph_1)$. Indeed the fact of interpreting every result that we present in this section as a good solution for the corresponding theory depends heavily on the assumption that the forcing is the only way to obtain new models for set theory. This is true as far as other known model-theoretic methods are concerned, but there is no proof—and I doubt that there will ever be—of this fact. However, this is a theoretical innocuous assumption, that goes hand in hand with the general pragmatism of the methodology that is used in every mathematical research.

Moreover, it is possible to extend this result to a full solution of $H(\aleph_2)$, thanks to the axiom: MM^{+++}: the strongest version of a specific type of axioms called Forcing Axioms.

Theorem 1.16. *([Viale 2013]) Assume T extends $ZFC + MM^{+++} +$*[18] *There are class many super huge cardinals. Then for every formula $\varphi(x)$ in the free variable x and every parameter p such that $T \vdash p \in H(\aleph_2)$ the following are equivalent:*

- $T \vdash \ulcorner H(\aleph_2) \models \varphi(p) \urcorner$
- $T \vdash$ *There is a stationary set preserving partial order P such that $\Vdash_P \varphi^{H(\aleph_2)}(p)$ and P preserves MM^{+++}.*

17. This notion is again too technical to be defined here; see [Jech 2003]. What is important to know is just that Woodin cardinals are large cardinals.
18. See [Viale 2013] for the definition of this principle.

Hence, thanks to the Forcing Axioms, we have a good description of the structure $H(\aleph_2)$ and, in keeping with the general idea that lies behind and foundation of mathematics—as we saw in the example of the Whitehead Problem and of the Calkin Algebra—they are capable of unifying different branches of mathematics. Indeed both MA and OCA are consequences of MM^{+++}.

It is now time to come back to the main aspect of a set theoretical foundation of mathematics: unity. Indeed, we want to give more philosophical arguments to sustain the claim that set theory should be able to unify mathematics, as far as it is a foundational theory. In doing so we will also elaborate on our claim that set theory should be considered as a foundation for mathematical practice.

2 Axioms as explanations

So far we showed the relevance and usefulness of set theory for what concerns the practice of doing mathematics, but our claim is stronger and refers to the philosophy of mathematical practice: a recent tradition in the philosophy of mathematics, as a quick look at contemporary bibliography clearly shows.[19] Our main thesis of this section is that many of the principles that are used in contemporary set theory, many of which are called axioms, manifest specific characteristics that can be assimilated to, at least, one important account of mathematical explanation—one of the more studied and developed area of the philosophy of mathematical practice. Hence, our derived claim is that some set theoretical axioms can be seen as an explanation of the mathematical phenomena.

The aim of our analysis will be to link unification and explanation in a foundational context. Many authors have proposed a philosophical inquire on the notion of explanation in science in terms of unification, but few of them have proposed it as a way to understand the notion of mathematical explanation, i.e., the role of mathematics in the scientific explanation. Among them Kitcher's account, presented in terms of the unification power of scientific theories, can be compared with the unifying features that some axioms in set theory have. We will not present here his position in details, but we refer to the primary bibliography ([Kitcher 1981], [Kitcher 1989] and [Kitcher 1984]) and to Molinini's Phd thesis ([Molinini 2011]), for a good presentation of the subject, able to clear many of the obscure passages that can be found in Kitcher's work.

In trying to find some explanatory aspects in the concept of axioms in set theory, we connect our arguments with a long tradition in the philosophy of mathematics and in the foundational studies, that has its roots in the

19. See, among others [Mancosu 2008].

empiricist positions of John Stuart Mill. Paolo Mancosu, in [Mancosu 2001], has named this tradition h-inductivism: the position that sees in the success of an hypothesis of an axiom, and in its ability to give a systematization of a discipline, the main justification for its acceptance. On the same par, we can find also Russell and Gödel. In particular, the latter believed in a direct connection between the role of the axioms of set theory and their explicative power, in analogy with the process of explanation of a physical phenomenon.[20]

However, we want to be clear that we do not endorse the thesis that axioms can be seen as explanation to argue in favor of a realist conception of mathematical object. As a matter of fact we argue for a foundation of mathematics free from any ontological commitment. Instead, in what follows we will try to give arguments in favor of the thesis that, in the context of the axiomatic setting, explanation of a proof and justification of an axiom are two sides of the same coin: unification. Once this point is achieved, then, there will be no reasons for justifying the axioms in terms of an existing mathematical reality.

2.1 Applying Kitcker's account?

What interests us here is the role of mathematical explanation, if there is any, inside mathematics. Indeed mathematical explanation can mean both the use of mathematics in explaining physical phenomena and the use of explanatory considerations in the context of pure mathematics. From now on by mathematical explanation we will mean the latter: mathematical explanation of mathematical phenomena.

For the sake of precision, there is no precise account of mathematical explanation in any of the writings of Kitcher, but, instead, of scientific explanation of physical phenomena. Nevertheless the possibility to export this model from

20. See Mehlberg, in this respect: "The limited effect of the failure of Hilbert's program upon the dependability of the impressive cluster of mathematical theories which he tried to place on a common 'foundation' can be clarified by reference to certain relevant views of Gödel which he informally conveyed to me, some years ago, during a discussion we had at Princeton, N.J. According to Gödel, an axiomatization of classical mathematics on a logical basis or in terms of set theory is not literally a foundation of the relevant mathematics, i.e., a procedure aiming at establishing the truth of the relevant mathematical statements and at clarifying the meaning of the mathematical concepts involved in these theories. In Gödel's view, the role of these alleged 'foundations' is rather comparable to the function discharged, in physical theory, by explanatory hypotheses. [...] Professor Gödel suggests that so-called logical or set theoretical 'foundations' for number-theory, or any other well established mathematical theory, is explanatory, rather than really foundational, exactly as in physics" [Mehlberg 1960, 86–87].

physic to—pure—mathematics is proposed by Kitcher himself,[21] in the light of his holistic point of view on scientific knowledge.

> [G]iven my own views on the nature of mathematics, mathematical knowledge is similar to other parts of scientific knowledge, and there is no basis for a methodological division between mathematics and natural sciences. [Kitcher 1989, 423]

We will see below how weak this thesis is, but what we want to save from Kitcher's way to set the problem is the global point of view on the problem of mathematical explanation—i.e., to consider how a general theory can explain some of the phenomena it is able to formalize or deduce—as opposed to a local point of view that tries to look for the explanatory characters of a proof. Indeed, this latter task is much more complicated and there are hints that it is not possible to give a detailed and objective account of why a proof counts for more explanatory than others. A seminal, but isolated, case of such a work can be found in Paseau's study of the different proofs of the compactness theorem [Paseau 2010]. The main thesis of Paseau, with whom we agree, is that the explanatory virtue of a proof always depends on the context and so it is hard, if not impossible, to give an objective account of it. As we will see, one of the points we will make in this section is that also a global explanation depends on the background theory in a substantial way, contrary to a long tradition that dates back to Aristotle's times and that numbers, among his members, also Bolzano.[22] Leaving aside these difficulties, we want to stress how a global account of explanation can in principle fit with a global mathematical point of view, as it is the case when dealing with a foundation of mathematics.

Once we are assured that the general setting of the two problems is compatible, let us see more in details Kitcher's account and his affinities with our proposed set theoretical practical foundation for mathematics. The more important conceptual similarity is, of course, the individuation of unity as the main virtue of both a practical foundation and a global account of mathematical explanation. Indeed, when discussing Hempel's model of explanation, Kitcher describes his position in these terms:

> This unofficial view, that regards explanation as unification, is, I think, more promising than the official view. My aim in this paper is to develop the view and present its virtues. [Kitcher 1981, 508]

21. And sustained by his readers, as it is done in [Tappenden 2005, 158–159]—where Tappenden says: "However, mindful of the fact that some explanations in physics and mathematics do seem to be governed by the same principles, I'll count it as an advantage of an account that it supports a uniform treatment of some mathematical and some physical explanations. A promising candidate to support a uniform treatment of some pure mathematical cases and some non-mathematical ones is the treatment of explanation as unification as proposed in the seventies by Michael Friedman and Philip Kitcher"—and in [Mancosu & Hafner 2008].

22. See [Mancosu 1999] for a detailed historical presentation of Bolzano's theory of mathematical explanation.

This is done at least at two levels: making clear the aim of an explanation and clearing the nature of an explanation. For what concerns the former, Kitcher is explicit in saying that an explanation is an answer to a why question. Exactly how, echoing Hilbert's foundation of geometry, our practical foundation aims to do.

> I shall restrict my attention to explanation-seeking why-questions, and I shall attempt to determine the conditions under which an argument whose conclusion is S can be used to answer the question "Why is it the case that S"?[23] [Kitcher 1981, 510]

Notice that this declaration of intention is not as narrow as Kitcher seems to argue. Indeed, when we get to mathematical explanation, the question "Why is it the case that S"? can be interpreted in different ways according to the context in which the question is asked. For example, we can take S to be Fermat's Last Theorem (FLT), but if we ask: "why is it the case that FLT" in a context where it is possible to understand and state the question, like that of number theory, we cannot even formulate a possible answer because we do not have an elementary proof of it. Contrariwise, this is a reasonable question in the context of a theory sufficiently strong to incorporate scheme theory and algebraic geometry. This example is meant to show that in the context of pure mathematics the methods of proof do have a role in the determination of an answer to a why-question. Moreover, what we suggest is that, in the particular case of mathematical explanation, an answer to a why-question can hide many different problems, like for examples considerations on the purity of methods, that bring together some controversial and less objective positions towards the nature of mathematical discourse.

For what concerns the second aspect Kitcher is explicit in saying that an argument is a derivation[24] and—this is the main thesis contained in [Kitcher 1981] and [Kitcher 1989]—that, given a set of sentences K, there is an *explanatory store* $E(K)$, that consists in the best *systematization*—read formalization in the context of the axiomatic method—of K. What makes $E(K)$ the best systematization is the possibility to associate to it a set A of arguments—called a *basis*—that instantiate general arguments patterns and that, better then other systematizations, unify K, in the following sense:

> So the criterion of unification I shall try to articulate will be based on the idea that $E(K)$ is a set of derivations that makes the best tradeoff between minimizing the number of patterns of derivation

23. Notice that this quotation manifests Kitcher's global point of view. Indeed, his aim is not to show which are the properties that an argument should have to be considered as explanatory, but which are the general—global—conditions under which an argument should be considered as explanatory.

24. In Kitcher words: "a sequence of statements whose status (as a premise or as following from a previous members in accordance with some specified rule) is clearly specified" [Kitcher 1989, 431].

> employed and maximizing the number of conclusions generated. [Kitcher 1989, 432]

It is not clear, in Kitcher's work, how this minimizing-maximizing effect should act in the process of choosing an argument instead of another.[25] We maintain that Kitcher is appealing here to an intuitive principle of success that our arguments in $E(K)$ should surely fulfill.

What is important to stress here is that, for Kitcher, what is fundamental in the analysis of explanatory unification is the notion of argument pattern. Kitcher offers a description of these arguments as detailed as vague. What is important to keep from Kitcher's idea of argument pattern is that it is a general structure of an argument—not only a logical structure—sufficiently general to be applied in many different contexts and in many different forms: it is what permits to recognize that different proofs are essentially the same. We will not give a detailed presentation of the notion of argument pattern as it can be found in [Kitcher 1981] mostly because of the lack in Kitcher's work of a clear analysis of the notion of similarity.

> This suggests that our conditions on unifying power should be modified, so that, instead of merely counting the number of different patterns in a basis,[26] we pay attention to similarities among them. All the patterns in a basis may contain a common core pattern, that is, each of them may contain some pattern as a subpattern. [Kitcher 1981, 521]

Although Kitcher tries to clear what is an argument pattern, it is better to keep this notion as intuitive, although vague, as possible.[27] Then at this level of generality we could ask: which are the similarities between some axioms of set theory—especially those exceeding ZF, that we discussed in the last section—with respect to the argument patterns, as far as both are responsible for the unity of the theory? To answer this question we can recall Zermelo's idea about the "unlimited applicability of set theory". In order to make the argument more concrete, recall the problem whether all the automorphisms of the Calkin algebra are inner. The way in which the proof works is by finding enough similarities between this algebra of operators and the structure $\mathcal{P}(\omega)/fin$. Then, it is possible to use the axiom OCA to perform the same argument on

25. The problem, with this counting arguments, is that it can work just in the case of finitely many consequences. And not with a mathematical theory with countably many consequences.

26. Just to recall it, a basis is the set of arguments that instantiate in the more unifying way all the relevant argument patterns of a given systematization.

27. In passing let us just notice how this vagueness on the concept of argument pattern can be related to a more qualitative analysis of the notion of explanation. Indeed any account that aims to ascribe the explanatory power of an axiomatic setting that minimizes the *relevant* arguments patterns, seems to be promising. But without a corresponding analysis of the concept of relevance this account would be useless.

both sides. It is a general methodology when facing a mathematical problem and looking for its solution. Indeed, this is one of the main advantages in relaying on set theory as a foundation of mathematics: the concept of set and the methods used in set theory are so general and abstract that they can be applied—possibly—to any field of mathematical inquiry. Hence, the use of the axioms for set theory permits to show the similarity, in the arguments, of many different mathematical reasonings. When attacking a problem, the first attempt of a mathematician is to bring the difficulties to a more clean and comprehensible level, where a solution is easier to find. This operation, of cleaning a problem from the irrelevant aspect, amounts in recognizing similar patterns or making more evident the core of the problem that often has—as the methodology of set theory shows, in abstracting from any content—a combinatorial aspect. Here by combinatorial aspects of a proof we do not only mean the part of an argument that is performed by pure calculation without a broader overview of the structure of a proof, but also the steps of an argument that manifest a necessity character similar, for strength, to calculation, and that act like the fundamental ingredient of a theorem. In a set theoretical context, the combinatorial aspects of a proof are often found when abstracting from the particular properties of the subject matter of a theorem and when outlining the general set theoretical properties that make possible to perform an argument. Indeed the combinatorial character of some axioms, or some principles that flow from them, is capable of showing in a pure form what is needed for the proof of a sentence: they show how to overcome the main difficulty one finds in a problem, acting as the key ingredient for its solution. Indeed, this method works also in the opposite direction, from solutions to axioms and sometimes brings together the discovery of new, tacitly used, principles as it is the case of the Axiom of Choice. Even one of the more influential proponent of the more outstanding alternative foundation of mathematics: category theory, acknowledges the ability of ZFC to reduce many arguments to few ones.

> The rich multiplicity of mathematical objects and the proofs of theorems about them can be set out formally with absolute precision on a remarkably parsimonious base. [Mac Lane 1986, 358]

Moreover, whenever these principles are proved to be independent from ZFC, even if we do not have logical necessity, we have a deductively dependency of a proposition on the principles used in its proof that shows the insufficiency of other methods to solve a particular problem. Hence, as for the argument patterns, the axioms of set theory can be seen as the reasons for an argument to work. Then we can say that the axioms that extend ZFC can be considered as argument patterns.

The underlying notion of argument pattern is of course stretched to its limit and it could be argued that, in the context of the axioms of set theory, it is hardly recognizable. But our claim is not that argument patterns are set theoretical axioms, because we acknowledge that there are of course different methods of proof and argument patterns in different areas of mathematics that have nothing to do with a set theoretical methodology. What we argue is that, when it comes to the foundation of mathematics, some axioms of set theory explain why a given proposition is a mathematical theorem, providing its proof; not why it is a theorem of a particular theory, say geometry or analysis. Moreover, we are not claiming that any set theoretical axiom, singularly, can be seen as an instantiation of argument patterns, but that set theory as a whole can be seen as an explanation of why it is possible to prove a theorem—showing the core argument that allows a proof to work—once it has been cleared that the sense of explanation we use is related to a form of unification.

2.2 Kitcher's problem for mathematical explanation

We are aware of the fact that the arguing for the relevance of the role of the axioms in the context of argument patterns is a subtle and far from easy task[28]—even if Kitcher seems to support this view, as we will see in the only passage were he discusses the possibility to use his account for analyzing the notion of mathematical explanation—but there is a preliminary problem that needs to be cleared. Even if we give for granted that axioms act as—or instantiate—a form of argument patterns:[29] what is the epistemological argument in favor of

28. Notice that, at the ontological level, also Resnik does not exclude the fundamental relevance that axioms can have with respect to patterns: "But I must elaborate a bit on this answer, since one might remark that I am saying that, in effect, the premisses to which we appeal in proving the theorem 'implicitly define' the pattern or class of patterns to which the theorem pertains. Now I have no problem *per se* with calling such premisses (or a more condensed set of axioms from which they might be derived) an implicit definition, so long as this is not taken to imply that the premisses are known a priori in some absolute sense. Of course the axioms constituting the clauses of an implicit definition are trivial consequences of this definition. Thus it is a matter of definition that they characterize the pattern they help specify" [Resnik 1997, 237–238]. In general, the aims and the context of the notion of pattern, in Resnik's work, is very far from Kitcher's and ours, but when we get to the more fundamental elements of doing mathematics, even in a platonic context as Resnik's, the notions of axiom and of pattern tend to collide. What is really different here is the background idea of mathematics: a science of existing structures for Resnink, while the domain of rigorous arguments for us.

29. Notice that Kitcher rejected the arguments proposed by Friedman in [Friedman 1974]—who also proposed to identify explanation and unification—saying that the major difference between his account and Friedman's consisted in what was to be assumed as the basic notion in the process of explanation: for Friedmann it was the physical laws, while for Kitcher it was the argument patterns. In Kitcher words: "Finally, I think that it is not hard to see why Friedman's theory goes wrong. Although he rightly insists on the connection between explanation and unification, Friedman is incorrect in counting phenomena according to the number of independent laws. [...] What is much more striking than the relation between these numbers is the fact that Newton's laws of motion are used again and again and that they are always

the coincidence between unification and explanation? In other words, even if it seems at first sight a convincing matching, what are the arguments in favor of this identification? The main argument that Kitcher advances, to hold the epistemological link between the act of unifying a theory and that of explaining why some of phenomena described by the theory hold, is that the limit goal of all science is to unveil the causal structure of the world.

> The growth of science is driven in part by the desire for explanation, and to explain is to fit the phenomena into a unified picture insofar as we can. What emerges in the limit of this process is nothing less than the causal structure of the world. [Kitcher 1989, 500]

This position of course has profound consequences, one of which is the dependency of the concept of causality from that of explanation. Kitcher is well aware of this fact.

> Indeed, I have been emphasizing the idea (favored by Mill, Hempel, and many other empiricists) that causal notions are derived from explanatory notions. Thus I am committed to
>
> > (2) If F is causally relevant to P, then F is explanatory relevant to P. [Kitcher 1989, 495]

Without entering in the discussion of the robustness of this philosophical position, we want to outline the main difficulty that this position suffers in the context of an analysis of explanation internal to mathematics: mathematics is not a causal world. There is a general agreement on this point and even a truly platonistic-minded thinker, like Gödel, has always advanced only an analogy between the physical world and the mathematical realm.

As we said before, Kitcher has never fully addressed the matter of the applicability of his model to the mathematical explanation. However, in [Kitcher 1989], there is a, although short, attempt to discuss the problem.

> For even in areas of investigation where causal concepts do not apply—such as mathematics—we can make sense of the view that

supplemented by laws of the same types, to wit, laws specifying force distributions, mass distributions, initial velocity distributions, etc. Hence the unification achieved by Newtonian theory seems to consist not in the replacement of a large number of independent laws by a smaller number, but in the repeated use of a small number of types of law which relate a large class of apparently diverse phenomena to a few fundamental magnitudes and properties. Each explanation embodies a similar pattern: from the laws governing the fundamental magnitudes and properties together with laws that specify those magnitudes and properties for a class of systems, we derive the laws that apply to systems of that class" [Kitcher 1976, 212]. However Kitcher's criticism is directed to some technical points raised by M. Friedman's proposal, hence nothing prevents, in principle, to argue in favor of the possibility that axioms—or laws—can capture some essential feature of an argument pattern.

> there are patterns of derivation that can be applied again and again to generate a variety of conclusions. Moreover, the unification criterion seems to fit very well with the examples in which explanatory asymmetries occur in mathematics. Derivations of theorems in real analysis that start from premises about the properties of the real numbers instantiate patterns of derivation that can be used to yield theorems that are unobtainable if we employ patterns that appeal to geometrical properties. Similarly the standard set of axioms for group theory covers both the finite and the infinite groups, so that we can provide derivations of the major theorems that have a common pattern, while the alternative set of axioms for the theory of finite groups would give rise to a less unified treatment in which different patterns would be implied in the finite and in the infinite case. Lastly, what Lagrange seems to have aimed for is the incorporation of the scattered methods for solving equations within a general pattern, and this was achieved first in his pioneering memoir and later, with greater generality, in the work of Galois.
>
> The fact that the unification approach provides an account of explanation, and explanatory asymmetries, in mathematics stands to its credit. [Kitcher 1989, 437]

As it is clear, Kitcker argues that the bare possibility of applying the same pattern again and again is responsible for the unificatory virtue of a systematization without demanding a causal connection. This thesis is compatible with the claim that "If F is causally relevant to P, then F is explanatory relevant to P", but it is not with the idea that the unification process shows, in the limit, the causal relations between phenomena, because this would imply an even stronger thesis:

> (2^*) F is causally relevant to P if and only if F is explanatory relevant to P.

But this is of course false when dealing with mathematical explanation in pure mathematics. Then if we want to argue that axioms act as explanations we should look for a framework with different motivations than Kitcker's, in order to justify the link between unification and explanation in a non causal context.

As a matter of fact, there is a theoretical, and not just methodological, difference between argument patterns in science and axioms in pure mathematics. What differs in the two contexts is the nature of the why-questions for which the explanatory unification looks for an answer. Indeed, if not partial, an answer to a why-question is such that it is not possible to ask any further why-question. When we get to pure mathematics and we discuss explanation in an axiomatic context we need to distinguish between the explanation of "why is it the case that S", for a sentence S and "why is it the case that A", for an axiom A. Kitcher chooses to explain "why is it the case that S" appealing to the best possible unification of all sentences of the theory to which S belongs. This strategy can work as long as we remain in the context of physical phenomena, where we do not need to ask why-questions on the physical laws. Indeed the example on Newton's theory of gravitation, proposed in *Explanatory Unification*, is acceptable, since nobody would ever ask why it is the case for Newton's law of Universal Gravitation. The reason is that reality serves as a bedrock in the search for the causes of a phenomenon. As a matter of fact, when Kitcher deals properly with mathematical explanation he proposes to explain group theory by means of its axioms, but what explains the axioms—as far as they are mathematical propositions—if we cannot make reference to a physical world where groups exist, i.e., if there is no causal connection between the groups and the theorems of group theory?

Before trying to find a solution to the problem of "why is it the case that A", for an axiom A, let us look at the question of "why is it the case that S", for a mathematical sentence S, without appealing to the causal structure of the world, i.e., without appealing to the bedrock of reality that can stop the rise of new why-questions. If we are not dealing with a self-evident proposition, nor we are referring to some metaphysical property of mathematical objects,[30] an answer to a why question, in terms of argument patterns, can be considered satisfactory only when we are not anymore in the position to ask for the reasons that could explain why some proposition hold. Only in this case it is possible to give objective reasons for a proposition S. Then we cannot make reference to any extra-mathematical—informal—property, but we have to ground our answer on something as objective and indubitable as the logical structure of mathematics. In other words, the explanation needs to be internal to the mathematical discourse. So, if we accept that arguments are derivations—as it is the case in Kitcher's account—and the fact that axioms act as argument patterns, then, following Hilbert's suggestion, the best answer to "why is it the case that S" amounts in showing the necessary and sufficient conditions for the proof of S.

However, such an answer seems to be clearly unsatisfactory when we restrict it to a single sentence S, because it is often the case that logical equiv-

30. As for example Steiner seems to do in [Steiner 1978]. It seems that for him the question "why is it the case that S" needs to be answered referring to some essential properties of the mathematical objects to which S refers. For example, we can take S to be Fermat's Last Theorem (FLT) and ask: "why is it the case that FLT"? Steiner's answer is that it is the case that FLT if there is a property of the natural numbers such that for every $n \in \mathbb{N}$ there is no positive integers a, b and c such that $a^n + b^n = c^n$. This type of answer recalls closely a Tarskian definition of truth: "for every $n \in \mathbb{N}$ there is no positive integers a, b and c such that $a^n + b^n = c^n$" iff for every $n \in \mathbb{N}$ there is no positive integers a, b and c such that $a^n + b^n = c^n$. However such a move, on one side, hides a strong realist position toward the existence of mathematical objects that needs to be argued and, on the other side, is tautological and hence non explanatory.

alences are not explanatory at all.[31] But this objection misses an important aspect of doing mathematics, because the why-questions for which we are normally seeking answers are not "why is it the case that S" independently from the mathematical context, but, once the background theory T is made explicit: i.e., "why is it the case that $S \in T$". Indeed, granting that the act of explanation is a global matter, given by unification, necessary and sufficient conditions need to be given to explain the claim that S is a theorem of T. Hence when we restrict to countable languages our thesis is that a set of axioms $\mathcal{A} = \{A_i : i \in \omega\}$ can act as a unifying explanation of a theory T only if it is possible to show that, for any of its sentence S

$$S \in T \iff \exists n \in \omega \exists A_{i_0}, \ldots, A_{i_n} \in \mathcal{A} \text{ such that } A_{i_0} \wedge \ldots \wedge A_{i_n} \vdash S.$$

Nevertheless, the possibility to ask a general, context-independent, why-question has value and deserves to be considered. Then the question "why is it the case that S" becomes a question about the mathematical pedigree of S. If such a context-free question can ever find an answer, this will be in a sufficiently broad framework where it is possible to ask why we can consider S as a mathematical theorem: exactly the context given by a foundational theory as it is the case for set theory. For this reason we can say that set theoretical axioms as large cardinals and MM^{+++} can be seen as explanations of a mathematical proposition S, for what concerns the question "why is it the case that S", i.e., why S is a mathematical theorem.

Then we are finally in the position to come back to our initial problem: to give a philosophical justification of the claim that set theory can be seen as a foundation for mathematics as long as it is capable of unifying mathematical practice. The answer then is to be found in its possibility to explain mathematical phenomena, given necessary and sufficient condition, at least when it is possible to make clear reference to a theory T, that we can easily describe and recognize; as it is the case for an initial segment of the cumulative hierarchy, in terms of an $H(\theta)$, as it has been seen for Woodin's and Viale's results.

However, we are left with the problem "why is it the case that A", for an axiom A, that is, the search for justifications of the axioms. And remember that our goal, at the beginning of this section was to give a philosophical analysis of the criteria of unification. The outcome of this inquire is that explanation and justification are tied together by a sort of completeness theorem that links axioms and propositions, in the attempt to unify a theory. One side of the if-and-only-if-condition, from right to left, shows how it is possible to explain that a given proposition S belongs to some theory T—this is done by showing that S is a consequence of the set of axioms $\mathcal{A}$. Then, starting from the axioms, we have an implicit definition of T, as it was indeed the case for Hilbert's

31. This is not always the case as some equivalence theorems show. Consider for example the equivalence between the Axiom of Choice and the Well-ordering Theorem.

Axiom of Completeness. On the other hand, the implication from left to right presupposes an intuitive description of T and then asks for the axioms that can prove the whole of its theorems and, thus, unify the theory—in the sense described by Kitcher, as argument patterns. If this second implication holds then it is possible to match the intuitive theory and its axioms, and so we are able to justify the axioms in terms of their unification power. These judgments are indeed related to an informal description of a theory T and so presuppose its intuitive description. To come back to the argument we proposed at the end of our historical examples of the axiom-as-explanation position, we think that justification and explanation are two sides of the same coin: a complete unification. Indeed unification allows the proof of completeness theorem of the form we have just described, where a link is established between syntax and semantics. The correctness direction, from left to right, amounts to the justification of the axioms of a theory T, while the completeness direction, from right to left, amounts to the explanation of why the sentences can be seen as proposition of T. Hence we maintain that the answer to the question "why is it the case that A", for an axiom A, consists in its justification—whenever it is possible to give a sufficiently clear description of a theory T, for which A acts as an axiom—thanks to a completeness theorem of the form we have just outlined.

In the end, we want to be clear that our thesis is neither that explanation always comes, in mathematics, through axioms, nor that the explanatory unification of set theoretical principles is always granted by their acting as argument patterns. Indeed, as we argued, the epistemological importance of an equiconsistency proof gives different reasons for the explanatory role of large cardinal axioms. However, the possibility to apply Kitcher's model to some axioms of set theory is intended to show that explanation is part of the role of these axioms, but neither to make a general theory of the nature of the axioms in mathematics, nor to make a theory of mathematical explanation. On this latter aspect, we acknowledge that explanation does not always come in the context of an axiomatic setting, nor all the axioms are capable of explanation. As a matter of fact, for what concerns explanation we favor a more pluralist conception, capable of taking into account all the different nuances that can have a mathematical explanation, internal to the mathematical work.

Moreover, although Woodin's and Viale's results point in the direction of a complete axiomatization of set theory and we maintained that an intuitive description of a theory is needed in order to give a complete axiomatization of a theory T, we do not want to argue neither that an analysis of the concept of set, nor an intuition to this concept—*à la* Gödel—is needed in order to give a foundation of mathematics. Quite the contrary, as we hinted before, the vagueness of this concept is the main reason to argue for the set theoretical foundation of mathematics we proposed. As a matter of fact, we think that a foundation of set theory—with the word 'foundation' intended in the sense explained by Gödel: "a procedure aiming at establishing the truth of the relevant mathematical statements and at clarifying the meaning of the mathematical concepts involved in these theories" [Mehlberg 1960, 86]—is really a different task and we will try to make this distinction clearer in the last part of this section.

2.3 Towards a more general distinction

To come back to the first part of this work, we hope we have been able to show that set theory, in the extended sense considered, is a good tool in the analysis of necessary and sufficient conditions for the proof of all mathematical problems and in this sense it is to be intended as a foundation for mathematics. The instruments it provides go much beyond the possibilities that are given by the use of solely logical tools—that encouraged the vast application of the axiomatic method in the last century. As a matter of fact, thanks to equiconsistency results, it is possible to find equivalence results that are not only logical, but epistemological in character; and this analysis is a good form of explanation, in terms of the main possibility of proof—able to unveil deep combinatorial aspects. Moreover it is important to stress the difference between the set theoretical foundation we described and the standard view that sees a big ontological importance in the possibility to reduce every piece of mathematics to set theory. As a matter of fact the foundation of mathematics we argued for is ontologically and theoretically neutral: it does not even take a stand about the attempt to single out the true universe of set theory, in the context of the multiple alternatives offered by the method of forcing. This line of research is an interesting and fruitful subject, but it has important theoretical implications that cannot be compatible with a foundation that aims to explain practice and then follows the free and unforeseeable development of mathematics.

Indeed there is an important debate on the main possibility to find such a complete description of V, where platonic-minded mathematicians, like for example Woodin, are opposed to researchers that hold a multiverse point of view, like for example Hamkins [Hamkins 2012]. There are also positions in-between like Magidor's who maintains that "some set theories are more equal than others"[32] or, similarly, like Sy Friedman and Sharon Shelah, who argue in favor of the possibility to find rational arguments for choosing one model instead of another.[33] As a matter of fact, we think that a multiverse view on the nature of the set theoretic universe just confuses the foundational role of set theory with its nature of mathematical theory in itself for which the search for a good description of the intended model is a fundamental and natural demand. On the contrary, even if this would be found, it would not disqualify all the theorems that do not hold in that model. As a matter of fact the

32. This is the title of the draft of a talk that Magidor gave in Harvard in 2012.

33. On this topic see the work of Friedman [Friedman & Arrigoni 2013] and of Shelah [Shelah 2003].

distinction between foundational aspects and infra-theoretical ones is meant to legitimate both analyses.

In conclusion, we want to stress the importance of not confusing the foundational role of set theory with its nature of mathematical theory in itself, for which the search for a good description of the intended model is a fundamental and natural demand.

3 Two different foundational ideas

We believe that the difference between the two set theoretical foundations of mathematics we discussed before is the appearance of a more general phenomenon: two distinct attitudes in the foundation of mathematics. Of course, we do not pretend to give an exhaustive classification, but at least to indicate that there are two areas that deal with foundational problems with distinctive perspectives: philosophy and mathematics. These two attitudes are of course well interlaced in the foundational works of the last century, but they are, in principle, autonomous. As a matter of fact, these two dispositions act in response to different needs. The choice of the terms to indicate them could be *theoretical* and *practical*. We could have called them philosophical and mathematical but this choice is somehow misleading, because on the one hand there is no sharp distinction between the two subjects at a foundational level and, on the other hand, we do not want to suggest an opposition between philosophy and mathematics, but, on the contrary, a distinction that can produce useful interactions. The antinomy that I would like to propose with this categorization is the one existing between essence and method.

We will use the expressions "theoretical foundation" and "practical foundation" to indicate the corresponding attitude in the foundational enterprise. We will quote same examples of these approaches, but we would like to be clear that we are not proposing a classification of philosophers and mathematicians in two separate categories. On the contrary, we just delineate a distinction for what concerns goals, approaches and, sometimes, admittedly, true predilections. Indeed, it will always be difficult to draw a clear line to distinguish the two kinds of foundations in the work of an author, since the reflection on mathematics is always a difficult and broad enterprise. As we outlined in the first part of this article the main concern of a foundation is unity. For this reason a theoretical foundation and a practical foundation are both stimulated by this idea and we will describe how they achieve this purpose.

3.1 Theoretical foundation

By a theoretical foundation we mean the attitude that sees in the foundation of mathematics the possibility of a *reduction*. This stance tries to answer a question on what there is in the mathematical world and how we can give a mathematical definition of our mathematical concepts. A reduction of this kind deals mostly with ontological or semantical problems; as, for example, in the case of the reduction of mathematical objects to sets (with all the problems related to the fact of assuming that everything is a set (see [Benacerraf 1965]). See for example the very beginning of [Kunen 1980], one of the most used textbook in set theory:

> Set theory is the foundation of mathematics. All mathematical concepts are defined in terms of the primitive notions of set and membership. In axiomatic set theory we formulate a few simple axioms about these primitive notions in an attempt to capture the basic "obviously true" set theoretic principles. From such axioms, all known mathematics may be derived. [Kunen 1980, xi]

Another example of this approach can be found in Russell's logicist program, for which the reduction is even more conceptual.

> In constructing a deductive system such as that contained in the present work ... we have to analyse existing mathematics, with a view to discovering what premisses are employed, whether these premisses are mutually consistent, and whether they are capable of reduction to more fundamental premisses. ... [T]he chief reason in favor of any theory on the principles of mathematics must always lie in the fact that the theory in question enables us to deduce ordinary mathematics. [Whitehead & Russell 1910, Preface, v]

In an opposite way, also, any attempt of nominalization of the mathematical discourse can be seen as a form of reduction; a reduction of the truth value of a mathematical sentence to a syntactic game that can be played uniformly within any mathematical theory.[34] As a matter of fact, the answer to the question "on what there is" can be answered in many and incompatible ways, like, for example, everything or nothing. What is peculiar to this attitude is that it tries to give a comprehensive reduction of the whole of mathematical discourse, or sentences, or truths, to some objects or principles that are able to subsume or vanish any peculiar aspect of a particular mathematical field. Not only this kind of foundation tries to unify but also to disappear the differences, explaining that the various things we encounter in our mathematical experience are just diverse manifestations of the same phenomenon. What is common to the foundations that share this goal is an holistic and static view of mathematics, that sees mathematical practice as the field where to test if the reduction proposed is sufficiently comprehensive.

Of course there are problematic aspects of a theoretical foundation. These problems arise in trying to give a general account of mathematics and not only of its unity. First of all there is the matter of fact that mathematics is an

34. See, for example, the work of Hartry Field.

always evolving enterprise. This makes very difficult to single out, once and for all, the very characteristic marks of mathematics and moreover to confine its existence within rigid boundaries. The horizon of sense and application of mathematics is always moving and follows freely the heavy burden of its history. Secondly there are problems of reference, or aboutness, as in the case of numbers and sets, as outlined in [Benacerraf 1965]. Indeed, once a reduction is proposed, there should be arguments in favor of that particular reduction instead of another, maybe, of the same kind. For example, following Benacerraf, once we admit that numbers are sets we should be able to explain which sets are the numbers. Finally, and related to this latter point, there is always a metaphysical obscurity that surrounds any reduction: how does this reduction work? what is the relationship between what is reduced and the the tools of reduction? We will not try to give an answer to these questions, because this is not a necessary task for a theoretical foundation, even if, of course, we have to admit that these questions deserve an answer, in the context of a philosophical account of mathematics. We would like here just to outline this view, clear its weaknesses and not try to defend it.

3.2 Practical foundation

The second attitude we would like to describe is the practical foundation: it aims to *explain* the unity of mathematics without proposing a reduction and it is epistemological in character. The main question that it tries to answer is: Why can we prove a theorem? Why a proposition can be seen as a theorem of a theory? The main reason for calling it practical, in contrast with theoretical, is the attention that is devoted to mathematical practice. As a matter of fact the motivation for such a foundation is the observation that doing mathematics consists essentially in trying to prove theorems. Moreover this attitude grants that one of the most important task of a serious reflection on mathematics is to explain the nature and the possibility of mathematical knowledge. In contrast with a theoretical foundation, a practical foundation of mathematics is not confined to a fixed set of axioms or to a given set of primitive principles, as in the case of the *Principia Mathematica*, but it makes use of the axiomatic method, trying to give a detailed description of the mathematical work. In this context, the unity of mathematics is suggested as a methodological uniformity. The main goal of a practical foundation is to explain in what consists the procedure that allows to recognize an argument as a proof. To qualify something as a proof has the consequence of characterizing the proposition that is proved as a piece of mathematical work. The roots of this attitude can be found in Hilbert's foundational work on geometry. Remember the letter to Frege, dated December 29th, 1899, when Hilbert said that he wanted to understand *why* "the sum of the angles in a triangle is equal to two right angles" [Frege 1980, 38–39].

In a different way, with respect to the role of the axioms in defining the basic ideas of a theory, an attitude of this kind can be found also in Frege's foundation of arithmetic.

> By insisting that the chains of inferences do not have any gaps we succeed in bringing to light every axiom, assumption, hypothesis or whatever else you want to call it on which a proof rests; in this way we obtain a basis for judging the epistemological nature of the theorem. [Frege 1893, Introduction]

We can see here what explanation means in the context of a practical foundation. The explanation that is given is internal to the theory for which the foundation is proposed. Indeed the explanation of a theorem is given in terms of its place in the logical structure of a theory, as for Hilbert, or in terms of the elucidation, step by step, of a proof, in Frege's proposal. Then the reasons that explain are to be found in the axioms that characterize a domain of knowledge or in the tools that we use to get from the premises of an argument to its conclusion. It is important to note that in these cases nothing depends on some metaphysical property of the subject matter nor to the recognition of a sort of similarity in the nature of the things involved in the foundational analysis. With a practical foundation mathematical practice is investigated in details and the quest for the reasons terminates only when we stop asking why-questions.

We have to pause here for a moment, because a digression is needed for the role that logic plays with respect to both types of foundations; and meanwhile to illustrate why we can find aspects of the work of the two main champions of logicism, Frege and Russell, in different horns of the dichotomy we are proposing. The reason for it is the twofold nature of logic. There is an old and venerable tradition, that can be traced back to Leibniz, that acknowledges logic, on the one hand, as a *characteristica universalis* and, on the other hand, as a *calculus ratiocinator*. The former aspect stresses the fact that logic is a universal language that can express any mathematical concept, while the latter indicates the circumstance that logic can be used to perform formal deductions. This bivalent character of logic can be found also in Frege's work but the passage quoted above shows one of the relevant feature of his *Begriffsschrift*: the possibility to explain why a conclusion follows from its premises, in terms of a rigorous deduction. While Frege maintained that these two aspects of logic cannot be disentangled, he emphasized that the *characteristica universalis* aspect was the more important one.[35] However Russell goes much further in the direction of the *characteristica universalis* and he says not only that logic is the language in which mathematical concepts can be expressed, but that every piece of mathematics can be defined in terms of logic. Then, while Frege has a universal view about logic but he thought that the computational aspect

35. Contrary to Schröder's critiques, according to which Frege's foundational work tended mostly towards a *calculus ratiocinator*. See [Peckhaus 2004] for this debate.

was really necessary for any meaningful notion of logic, on the contrary Russell sees in its work a more fundamental reductive stance.

3.2.1 Single theory vs. mathematics

By looking at the quotations above, it could be thought that a practical foundation is context-depending and it works only when a single theory needs a foundation, and not the whole of mathematics. Indeed Hilbert's work was on geometry, whereas Frege's was on arithmetics and, thanks to their work, it is possible not only to explain what is needed for the proof of a proposition but also why we can recognize it as a proposition of geometry or arithmetic. However, in this case: when a foundation of a particular theory is proposed, it raises the problem of the adequacy of an axiomatization to the theory that is axiomatized. We will not tackle this problem here in its generality, because this involves issues such as clarifying what a mathematical concept is, how it is possible to formalize it and how we manage to know what we formalize. Even if the attempt to give an answer to these questions is among the central tasks of the philosophy of mathematics, it is not in the scope of this work. Both Frege and Hilbert had their personal solutions to the problem of the adequacy: the former believed in the existence of a realm of concepts, while the latter discarded the problem using implicit definitions. What is important to stress here is that there is an insolvable tension between intuition and formalization, that, in the context of a theory for which we feel to have strong intuitions about its subject matter, can rise deep philosophical questions. In the case of geometry or arithmetic there is a tentative solution that comes directly from mathematics: a categoricity proof like Hilbert's for analytic geometry, or the one that is possible to give for natural numbers, using second order Peano axioms. Leaving aside the discussion on the significance of a categoricity proof, we just acknowledge that there are also situations where there are not even such results, as it is indeed the case for the formalization of set theory proposed by Zermelo and Fraenkel, for which it is not even sufficient to use second order ZFC, as it is shown by Zermelo's theorem on the quasi-categoricity of the universes of set theory [Zermelo 1930].

Despite all the difficulties that emerge in the case of a practical foundation of a single theory, we would like to argue that this is not the case for a practical foundation of mathematics, as a whole—as the set theoretical one we discussed in the beginning. In this case, we do not have neither the problem of reference, as for a theoretical foundation of mathematics, nor the problem of adequacy, as for a practical foundation of a single mathematical theory. Indeed, it is not necessary to know the subject matter of mathematics before we can propose a practical foundation for it. Or, to put it in a different way, knowing why we can prove a theorem does not entail knowledge of what the theorem is about.

In the case of set theory, the same difficulty to develop a reliable intuition of the general concept of set was sufficient to show the independence of a practical foundation from a complete knowledge of the matter for which a foundation is sought. I would like, in conclusion, to discuss another example where, even if we feel to have strong mathematical intuition, we can still mark a conceptual distinction between a practical and a theoretical foundation. Let us consider the case of arithmetic. In general, the fact that we know which principles allow us to solve a problem in number theory does not depend on our knowledge of what natural numbers are. Indeed, there are many cases in which tools that transcend arithmetic are used to solve a problem in number theory, as in the case of Fermat's Last Theorem, while, on the contrary, just second order Peano axioms are able to fix the structure of the natural numbers. This situation could be seen—and it is often seen—as an historical accident. Indeed it is common opinion among mathematicians that for any relevant number-theoretic statement it can be found a proof in elementary number theory. This belief would involve an extensive coincidence of the set of principles that allow to give an explanation of the "epistemological nature of a theorem" in number theory and the set of axioms that are able, in second order logic, to characterize the structure of natural numbers. Then, this could be seen as a cause of ambiguity between the two different foundations that we are proposing. Nevertheless, even granted this quantitative coincidence, there is a qualitative difference in looking at the axioms as characterizing natural numbers and as tools that characterize the work in number theory. In the former case we are tempted to say that the truth of a proposition in number theory depends on the fact that Peano Arithmetic is the right formalization the "natural numbers", while in the latter that it depends on the knowledge of which principles—or axioms—we are using in its proof. This is the reason why it would be a mistake to confuse the level of explanation of why we can prove a theorem and the level of metaphysical justification of the truth of a theorem in terms of the nature of the terms involved.[36]

In the case of a practical foundation of the entire mathematics this point is even more evident, because we do not have a clear idea of what the subject matter of mathematics is. Quite the contrary, we have a vague and ambiguous intuition of it, whence spring our feeling that mathematics is completely free in its paths and development. To make clear the borders of mathematics is exactly the purpose of a foundation, hence shaping mathematics. This is the reason why we cannot know what mathematics is before proposing a foundations for it. Then it is clear that being able to explain the facts we encounter in our mathematical practice does not presuppose a precise knowledge of their subject matter.

In conclusion, we hope we have given a clear picture of these two different aims in the foundations of mathematics. However we do not want to argue in favor of a separation of a more philosophical attitude from a more mathemati-

36. This distinction echoes the disagreement on the role of the axioms between Frege and Hilbert and the point that we are trying to make is on the same line as the one of Hilbert.

cal one. Of course a useful interaction between these points of view is not only the best way to find a deep understanding of our mathematical experience, but also a good guide for our mathematical work. The recognition of a conceptual distinction between two different attitudes in the foundational studies does not involve a separation of them in practice as working tools in the attempt to account for the mathematical phenomena and to widen our mathematical knowledge.

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