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Local Analysis of Whittaker New Vectors and Global Applications

LOCAL ANALYSIS OF WHITTAKER NEW VECTORS AND GLOBAL APPLICATIONS

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ABSTRACT

The theme of this thesis is to apply ultrametric analysis to classical problems in analytic number theory. This allows one to handle situations featuring high ramification at finite places. While this strategy works in many cases, the main focus of this work is the sup-norm problem for automorphic forms on GL_2. Our treatment of the problem is spread over two main parts.

First, we have to develop the necessary local theory which splits into archimedean and p-adic cases. The results needed in the archimedean cases are mostly classical, but recalling them in some detail will provide some guidance and intuition for the non-archimedean cases. The ultrametric situation is far less developed. Here we compute explicit expressions for the p-adic Whittaker function associated to a newform. These expressions are new in most cases and lead to tight bounds for the Whittaker function in question.

Second, we use the adelic framework and the theory of automorphic representations to put the local pieces together and treat the sup-norm of automorphic forms over number fields. We establish lower bounds far up in the cusp coming from the transition region, archimedean and non-archimedean, of the global Whittaker new vector. Furthermore, we prove hybrid upper bounds, in other words estimates that are explicit in all major aspects of the automorphic form under investigation. We allow a wide variety of representations at the archimedean places and make no restrictions at the finite one. In that sense we go beyond the existing work.

To Livia.

No man should escape our universities without knowing how little he knows.

— J. Robert Oppenheimer

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DECLARATION

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's 'Regulations and Code of Practice for Research Degree Programmes' and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is my own work. Work done in collaboration with, or with the assistance of, others is indicated as such. Any views expressed in this dissertation are those of the author.

10th May 2019

Edgar Assing

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Part I

PART ONE

This part provides a general introduction to the entire thesis. We will outline the philosophy behind this work, provide some motivation for the results in the following parts, and finally we will combine everything by briefly stating the main results.

Furthermore, we will setup some notation and provide the necessary prerequisites for the rest of the text.

1

INTRODUCTION AND BACKGROUND

The language of automorphic representations is a powerful framework for the study of many problems in number theory. The obvious advantages are the uniform treatment of different types of automorphic forms as well as the efficient notation handling number fields. More importantly, it is the right scheme to attack problems featuring ramification. Indeed, it not only offers the right tools to do so, it also clears up several phenomena by analogy with the archimedean theory, which are usually better understood.

We will put this language to good use and obtain some new results towards the sup-norm problem for automorphic forms on GL_2. While doing so we will exploit all the advantages mentioned above. Indeed, the theory of automorphic representations enables us to treat a wide variety of objects such as modular forms, Maaß forms, Hilbert modular forms, and combinations of these at once. Furthermore, the number field setting will come across very natural and is far less tedious than in the classical language. We will also see that treating square full level, requires a strategy which is borrowed from the archimedean places.

It is the nature of the adelic approach that it essentially consists of two parts. The local computations or preliminaries and the global argument. Locally, the analytic machinery, to establish important estimates, is available. Globally, automorphy is exploited to piece the local results together and prove the final result. This essentially dictates the structure of this thesis. It consists of a local part and a global part. Both parts heavily rely on existing theory and ideas. But in each part we also establish new results which are of independent interest. This thesis relies heavily on the manuscripts [2, 4] by the author. We will use the next sections for a more careful introduction of the two main parts of this work.

1.1 LOCAL ANALYSIS OF THE WHITTAKER MODEL

Uniform bounds for special functions, say the K-Bessel function, are by now a standard tool in the spectral theory of automorphic forms. From the viewpoint of automorphic representations many special functions appear in the Whittaker model of admissible, irreducible, infinite dimensional representations. To illustrate this we begin the local part by computing a suitable basis for the Whittaker model of an admissible, infinite dimensional representation of GL_2 over an archimedean field. We also recall the method of stationary phase and derive useful bounds for the K-Bessel function.

Turning to the non-archimedean situation we observe that the representation theoretic point of view provides a large source of special functions over $\mathbb{Q}_p$ and other local fields. The importance of these functions is underlined through their appearance in several trace/period formulae. Unfortunately their properties are not as well understood as in the archimedean situation. We are particularly interested in certain examples of p-adic Whittaker functions.

The functions under consideration are elements of the Whittaker space

$$\mathcal{W}(\mathrm{GL}_2(\mathbb{Q}_p),\psi) = \Bigg\{ W\colon \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{C} \text{ smooth } : \\ W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)W(g) \text{ for all } x\in\mathbb{Q}_p, g\in \mathrm{GL}_2(\mathbb{Q}_p)\Bigg\},$$

where ψ is a non-trivial additive character on $\mathbb{Q}_p$. The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on this space by right translation. Thus we can look at subspaces of $\mathcal{W}(\mathrm{GL}_2(\mathbb{Q}_p),\psi)$ on which this action is irreducible. These subspaces contain a special element that we call a Whittaker new vector. The defining property is that it is right invariant by the compact open subgroup

$$K_1(n) = \left\{ \begin{pmatrix} 1+p^n a & b \\ p^n c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p)\colon a,b,c,d \in \mathbb{Z}_p \right\}$$

for minimal n. The new vector in an irreducible subspace is unique up to scaling and we normalise it by $W(1) = 1$.

It turns out that the behaviour of a new vector is dictated by the underlying representation. Thus, given an irreducible, admissible representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ we denote the corresponding subspace of the Whittaker space by $\mathcal{W}(\pi,\psi)$ and the new vector contained within will be called W_π. The first property of W_π that comes to mind is its absolute size. This leads to the invariant

$$h(\pi) = 1 + \frac{\|W_\pi\|_\infty}{\|W_\pi\|_2}$$

first defined in [82]. This was studied in [69, 82], motivated by its connection to the sup-norm of modular forms.

The methods in both papers [69, 82] rely heavily on π having a highly ramified central characters. Thus, the values of $h(\pi)$ for mildly ramified central character remained somewhat mysterious. This produced some interest in finding out the truth. The conjecture [69, Conjecture 2], based on assuming square root cancellation in certain sums of epsilon factors, predicts that

$$h(\pi) \ll_\epsilon p^{n\epsilon}$$

as long as the exponent conductor of the central character is less than $\frac{n}{2}$. However, it quickly turned out that this conjecture is not accurate. Indeed, counter examples are constructed by A. Saha and Y. Hu in an unpublished manuscript.

In this thesis we settle the question for the size of $h(\pi)$ once and for all. Indeed we prove that

$$h(\pi) \ll_F \max(q^{\frac{n}{12}}, q^{\frac{m}{3}-\frac{n}{12}})$$

where m is the exponent conductor of the central character and F is any non-archimedean local field of characteristic 0, odd residual characteristic q, and uniformiser ϖ. This upper bound is sharp in the sense that for fixed central character and fixed even n there are representations such that the upper bound is attained up to constant. Note that the transition between the two exponents happens exactly at $m = \frac{n}{2}$. If $n = m$, we recover the exponent $\frac{n}{4}$ which already appeared in [82]. Furthermore, our results show that the lower bounds obtained in [69, Theorem 2.8] are not best possible.

The upper bounds are derived using the method of stationary phase and the exceptional large values appear due to the existence of degenerate critical points. This is very similar to the archimedean situation. It is the nature of ultrametric analysis that the method of stationary phase yields a precise formula instead of an asymptotic expansion. Thus, as a by-product, we obtain several explicit expressions for W_π which we believe to be of independent interest.

Note that for odd n there are no degenerate critical points and we obtain the stronger bounds

$$h(\pi) \ll_F \max(1, q^{\frac{m}{2}-\frac{n}{4}}).$$

The starting point of the method of stationary phase is to use integral representations for the Whittaker new vector on certain special matrices $g_{t,l,v}$. Roughly we will prove that

$$W_\pi(g_{t,l,v}) = C(t,\pi)\int_{\mathfrak{O}^\times} \xi(z)\psi(\mathrm{Tr}(A(t)z) + v\varpi^{-l}\mathrm{Nr}_{E/F}(z))d\mu_E$$

for a two dimensional etalé algebra E with character ξ associated to π and explicit constants $C(t,\pi) \in \mathbb{C}$ and $A(t) \in E^\times$. The choices for E and ξ can be naturally explained for each π. Our proof of these integral representations is based on finite Fourier analysis as well as the local functional equation. Note that similar formulae have been independently obtained by N. Templier in 2011 (unpublished) and Y. Hu in 2016 (unpublished except for the case of principal series which appeared in [46]).

The work described so far is built on the results published by the author in [4]. However, in this thesis we go beyond them. More specifically, we remove the restriction that the local field is $\mathbb{Q}_p$. Furthermore, we perform the stationary phase arguments in much more detail which leads to many explicit expressions for W_π which did not appear earlier.

We will apply these results to the sup-norm problem of automorphic forms. See below for a more detailed description of the latter. However we believe that the formulae we prove for W_π are of interest beyond the applications given here.

1.2 A BRIEF INTRODUCTION TO THE SUP-NORM PROBLEM

Let M be a compact Riemannian surface with Laplace-Beltrami operator Δ. The eigenfunctions

$$\Delta\phi = \lambda_\phi \phi$$

are central objects in mathematical physics. In view of the correspondence principle one expects a close connection between the geodesic flow on M and the mass distribution properties of ϕ. The sup-norm

$$\|\phi\|_\infty = \sup_{x\in M} |\phi(x)|$$

is one measure of the latter. The sup-norm problem asks for the true size of $\|\phi\|_\infty$. The local bound, which we like to call the Hörmander bound, reads

$$\|\phi\|_\infty \ll_M \lambda_\phi^{\frac{1}{4}} \|\phi\|_2.$$

It is known as a local bound since its proof does not use any global information of M. If M has negative curvature, the geodesic flow is ergodic and the Hörmander bound is expected to be far from the truth. Indeed, in the specific case of compact surfaces with negative curvature a bound of the form $\|\phi\|_\infty \ll_\epsilon \lambda_\phi^\epsilon \|\phi\|_2$, for all $\epsilon > 0$, should hold. In the breakthrough paper [52], the authors use connections to number theory to exploit the global structure of M and prove a sub-local bound

$$\|\phi\|_\infty \ll_M \lambda_\phi^{\frac{5}{24}+\epsilon} \|\phi\|_2$$

for arithmetic surfaces M and Hecke-Maaß eigenforms ϕ. Since then there has been much work extending their method. We will give a more comprehensive survey later on in this thesis. For now let us only mention two recent results. In [70] hybrid bounds for Hecke-Maaß newforms on congruence quotients of arbitrary level and central character are proven. Furthermore, in [20] Hecke-Maaß newforms with square-free level and trivial central character on congruence quotients over number fields are considered. In this thesis we combine these two results non-trivially producing a general hybrid bound over number fields.

Let us describe our result in some detail. Let F be a number field with archimedean places $\nu \in S_\infty$. Further let $\mathfrak{n} = \mathfrak{n}_2\mathfrak{n}_0^2$ be an ideal factorised in square-free and square-full part, and ω be a Hecke character of conductor $\mathfrak{m}$. Let ϕ be a newform of level $\mathfrak{n}$ and central character ω. We assume that ϕ is holomorphic of weight k_ν at real places $\nu \in S_{hol} \subset S_\infty$, Maaß of weight 0 or 1 and spectral parameter T_ν at the remaining real places, and spherical with spectral parameter T_ν at all complex places. To measure the size of the archimedean parameters simultaneously we define

$$|T|_{sph} = \prod_{\nu \in S_\infty \setminus S_{hol}} |T_\nu|^{[F_\nu : \mathbb{R}]},\ |T|_{\mathbb{C}} = \prod_{\nu \text{ complex}} |T_\nu|^2 \text{ and } |k|_{hol} = \prod_{\nu \in S_{hol}} |k_v|.$$

The size of the ideals is determined by their absolute norms $\mathcal{N}(\mathfrak{n})$, $\mathcal{N}(\mathfrak{m})$ etc. We have the following theorem.

Theorem 1.2.1. *In the setting described above we have*

$$\frac{\|\phi\|_\infty}{\|\phi\|_2} \ll_{F,\epsilon} (|T|_{sph}|k|_{hol}\mathcal{N}(\mathfrak{n}))^\epsilon \mathcal{N}(\mathfrak{n}_0)^{\frac{1}{2}}\mathcal{N}\left(\frac{\mathfrak{m}}{(\mathfrak{m},\mathfrak{n}_0\mathfrak{n}_2)}\right)^{\frac{1}{2}} \left(|T|_{sph}^{\frac{5}{12}}|k|_{hol}^{\frac{7}{16}}\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{3}} + |T|_{sph}^{\frac{1}{4}}|T|_{\mathbb{C}}^{\frac{1}{4}}|k|_{hol}^{\frac{1}{4}}\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{4}}\right).$$

Furthermore, if $S_{hol} = \emptyset$ and $[F^{\mathbb{R}} \colon F] \geq 2$ is the maximal totally real subfield of F, then

$$\frac{\|\phi\|_\infty}{\|\phi\|_2} \ll_{F,\epsilon} (|T|_\infty \mathcal{N}(\mathfrak{n}))^\epsilon |T|_\infty^{\frac{1}{2}-\frac{1}{8[F^{\mathbb{R}}\colon F]-4}} \mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}-\frac{1}{8[F^{\mathbb{R}}\colon F]-4}}\mathcal{N}(\mathfrak{n}_0)^{\frac{1}{2}}\mathcal{N}\left(\frac{\mathfrak{m}}{(\mathfrak{m},\mathfrak{n}_0\mathfrak{n}_2)}\right)^{\frac{1}{2}}.$$

The proof consists of two main parts. First, we estimate the Whittaker expansion to gain good control high in the cusps. This can be compared to a maximum principle as used in the classical theory of PDE's. It is here where the understanding of the local Whittaker new vectors, studied in the first part, comes in handy. Indeed, it is essential to our estimate that we understand their support and their L^2-size. Second, we apply the so called amplification method. Here we exploit the global structure in form of an amplifier to bound ϕ in the bulk. This step requires some local preliminaries. At places with high ramifications we use a local test function to produce the dependence on $\mathfrak{n}_0$ and $\mathfrak{m}$ which can be seen as a p-adic version of the Hörmander bound.

We can also produce large values of Hilbert-Maaß newforms just as in [69, 82]. In this case we find examples of forms which are large in every aspect simultaneously. These are large values arising from the transition region of the Whittaker function. Thus, they appear high in the cusp and may be interpreted as resonance phenomena before the cusp form starts its decay. Large values in the bulk are of a very different origin and are much harder to construct. We do not address them in this thesis.

The theorem stated here is based on the work [2]. However, we go slightly further by including the possibility of Hilbert-Maaß forms. Over $\mathbb{Q}$ hybrid bounds for holomorphic modular forms are folklore. However, to the best of our knowledge, such bounds in this explicit form have not yet appeared in the literature. Furthermore, we keep the argument quite general so that some extensions become easy to implement.

1.3 NOTATION AND PREREQUISITES

In this section we introduce the necessary notation and provide some background. Everything should be quite standard. However, since conventions differ from source to source, we want to introduce our notation in some detail. All the background provided in this section should be well known.

Our notation is taken from [2, 4] and we draw inspiration from the papers [20, 69, 70] as well as the thesis [30]. Additionally we use classical conventions from analytic number theory. Indeed, for a positive function g we write $f \ll g$ and $g \gg f$ to mean $f = O(g)$. We may add parameters as subscript to indicate dependencies of the implied constant. Further we write $f \asymp g$ if $f \ll g$ and $g \ll f$. Note that the implied constants may differ, if they agree we write $f \sim g$. Finally we use $e(x) = e^{2\pi i x}$.

1.3.1 *The archimedean fields* $\mathbb{R}$ *and* $\mathbb{C}$

Let F be either $\mathbb{R}$ or $\mathbb{C}$. We equip F with the modulus

$$|x| = \begin{cases} \operatorname{sgn}(x)x & \text{if } F = \mathbb{R}, \\ \Im(x)^2 + \Re(x)^2 & \text{if } F = \mathbb{C}. \end{cases}$$

We equip $\mathbb{R}$ with the standard Lebesgue measure $\mu_{\mathbb{R}}$ (normalised by $\mu_{\mathbb{R}}([0,1]) = (2\pi)^{-\frac{1}{2}}$) and $\mathbb{C}$ with $\mu_{\mathbb{C}} = \mu_{\mathbb{R}} \otimes \mu_{\mathbb{R}}$. Note that these are Haar measures for the additive group. On the multiplicative group $(F^\times, \times)$ we define the Haar measure $\mu_F^\times = |\cdot|^{-1} \mu_F$.

Additive characters of F are very well understood. Indeed, we define

$$\psi(x) = \begin{cases} e(x) & \text{if } F = \mathbb{R}, \\ e(\Re(x)) & \text{if } F = \mathbb{C}. \end{cases}$$

Every character of the (locally compact topological) group $(F, +)$ is of the form $\psi_a = \psi(a\cdot)$ for some $a \in F$. Thus we can identify the character group of $(F, +)$ with $(F, +)$ itself. With this identification the Fourier transform is given by

$$[\mathcal{F}f](y) = \hat{f}(y) = \int_F f(x)\psi_y(x)^{-1} d\mu_F(x).$$

Note that our measures are normalised such that $\hat{\hat{f}}(x) = f(-x)$ for Schwartz functions $f \in \mathcal{S}(F)$.

Similarly easy are the multiplicative characters. Indeed, a generic multiplicative (quasi)-character will be of the form

$$\chi(x) = \left(\frac{x}{|x|^{\frac{1}{[F:\mathbb{R}]}}} \right)^k |x|^s,$$

for $s \in \mathbb{C}$ and $k \in \mathbb{Z}$. With this in mind we define the Mellin transform

$$[\mathfrak{M}f](\chi, s) = \int_{F^\times} f(x)\chi(x)\,|x|^s\,d\mu_F^\times(x).$$

1.3.2 *Non-archimedean local fields*

We are now turning to the non-archimedean or p-adic world. We will restrict our attention to those fields which arise as localisations of number fields. Thus, F will be a finite extension of $\mathbb{Q}_p$ for some prime p.

Basic objects

Let F be a local field of characteristic 0. More precisely, F is a topological field of characteristic 0 which is complete with respect to a discrete valuation v and has finite residue field. Due to the classification of such fields, [63, Chapter II, Proposition 5.2], F is a finite extension of $\mathbb{Q}_p$ for some prime p and thus has residue field $\mathbb{F}_q$ for $q = p^f$. We normalise the valuation v to be surjective onto $\mathbb{Z}$ and fix a uniformiser $\varpi \in F$ such that $v(\varpi) = 1$. This choice is fixed once and for all and we ignore any dependence on it that may arise. Equip F with the complete absolute value $|\cdot| = q^{-v(\cdot)}$ and let $\mathfrak{o} = \{x \in F\colon\ |x| \leq 1\}$ denote the ring of integers in F. Note that $\mathfrak{o}$ is a discrete valuation ring with unique maximal ideal $\mathfrak{p} = \varpi\mathfrak{o}$. We define the local zeta factor by

$$\zeta_F(s) = (1 - q^{-s})^{-1}.$$

Let $e = e(F/\mathbb{Q}_p)$ be the maximal ramification index of F over $\mathbb{Q}_p$ and put $\kappa_F = \lceil \frac{e}{p-1} \rceil$. Then we can define the p-adic logarithm by the convergent power series

$$\log_F(1 + \varpi^{\kappa_F} x) = \sum_{n \geq 1} (-1)^{n+1} \frac{x^n \varpi^{n\kappa_F}}{n}.$$

Measures and Volumes

Since $(F, +)$ is a locally compact group there is an up to scaling unique Haar measure μ_F, which we normalise by $\mu(\mathfrak{o}) = 1$. On the other hand we have the unique Haar measure $\mu_F^\times$ on $(F^\times, \times)$ normalised by $\mu_F^\times(\mathfrak{o}^\times) = 1$. Note that these two measures are connected by $\mu_F^\times = \zeta_F(1)\frac{\mu_F}{|\cdot|}$. The volumes of some important sets are

$$\mathrm{Vol}(\mathfrak{p}^n, \mu_F) = q^{-n} \text{ and } \mathrm{Vol}(1 + \mathfrak{p}^n, \mu_F^\times) = \zeta_F(1)q^{-n}, \text{ for } n \geq 1.$$

By abuse of notation we will write $1+\mathfrak{p}^0 = \mathfrak{o}^\times$.

Characters and integral transforms

The additive character theory is very similar to the archimedean case. We fix an additive character ψ which is trivial on $\mathfrak{o}$ and non-trivial on $\varpi^{-1}\mathfrak{o}$. Any additive character is of the form $\psi_a = \psi(a\cdot)$. Let $n(\psi_a) = -v(a)$. This is the smallest integer k such that $\psi_a|_{\mathfrak{p}^k} = 1$. With this at hand we define the Fourier transform

$$[\mathcal{F}f](y) = \hat{f}(y) = \int_F f(x)\psi_y(x)^{-1} d\mu_F(x).$$

Naturally this transform is defined for Schwartz-Bruhat functions $f \in \mathcal{S}(F)$. These are locally constant, compactly supported functions. Our measures are normalised such that $\hat{\hat{f}}(x) = f(-x)$.

The multiplicative theory is more involved. We define the set

$$\mathfrak{X} = \{\chi\colon F^\times \to S^1\colon \text{ continuous character satisfying } \chi(\varpi) = 1\}.$$

Then every (quasi)-character is of the form $\chi\,|\cdot|^s$ for $s \in \mathbb{C}$ and $\chi \in \mathfrak{X}$. Note that this decomposition depends on the choice of ϖ which we consider as fixed. An important special function in this context is the Gauß sum

$$G(y,\chi) = \int_{\mathfrak{o}^\times} \psi(xy)\chi(x) d\mu_F^\times = [\mathcal{F}(\chi\cdot \mathbb{1}_{\mathfrak{o}^\times})](-y).$$

which is essentially the Fourier transform of a multiplicative character.

To each (quasi)-character χ we associate the exponent conductor $a(\chi)$. This is the smallest integer $k \in \mathbb{N}_0$ such that $\chi|_{1+\mathfrak{p}^n} = 1$. Note that if $a(\chi) = 0$ then $\chi|_{\mathfrak{o}^\times} = 1$ and $\chi = |\cdot|^s$ for some $s \in \mathbb{C}$. If this is the case, we call χ unramified. We also define $\mathfrak{X}_n = \{\chi \in \mathfrak{X}\colon a(\chi) \leq n\}$ and $\mathfrak{X}_n' = \{\chi \in \mathfrak{X}\colon a(\chi) = n\}$. We have

$$\sharp\mathfrak{X}_n = \zeta_F(1)^{-1}q^n \text{ and } \sharp\mathfrak{X}_n' = \begin{cases} q-2 & \text{if } n=1, \\ \zeta_F(1)^{-2}q^n & \text{else.}\end{cases}$$

The Mellin transform of a Schwartz-Bruhat function $f \in \mathcal{S}(F^\times)$ is defined by

$$[\mathfrak{M}f](\chi,s) = \int_{F^\times} f(x)\chi(x)\,|x|^s\,d\mu_F^\times.$$

Furthermore, we can associate to each character χ a L-factor by

$$L(s,\chi) = \begin{cases} (1-\chi(\varpi)q^{-s})^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{else.}\end{cases}$$

Another crucial invariant is the so called ϵ-factor $\epsilon(s,\chi)$. Note that this factor also depends on the fixed additive character ψ. However, we hide this from the notation. The exact shape and behaviour of these complex numbers are quite mysterious and we will not describe them in more detail. They are connected to Gauß sum as follows.

$$G(x,\chi) = \begin{cases} 1 & \text{if } a(\chi)=0 \text{ and } v(x)\geq 0, \\ -\zeta_F(1)q^{-1} & \text{if } a(\chi)=0 \text{ and } v(x)=-1, \\ \zeta_F(1)q^{-\frac{a(\chi)}{2}}\epsilon(\frac{1}{2},\chi^{-1})\chi^{-1}(x) & \text{if } a(\chi)\geq 1 \text{ and } v(x)=-a(\mu) \\ 0 & \text{else.} \end{cases} \tag{1.3.1}$$

This evaluation appeared for example in [31, Lemma 2.3] and will be used frequently in what follows. Other important properties of ϵ factors are

$$\epsilon(s,\chi|\cdot|^c) = q^{-ca(\chi)}\epsilon(s,\chi), \quad \epsilon(s,\chi)=\epsilon(\frac{1}{2},\chi|\cdot|^{s-\frac{1}{2}}) \text{ and } \epsilon(s,\chi)\epsilon(1-s,\chi^{-1})=\chi(-1).$$

If q is odd, then $\mathfrak{X}$ contains a unique quadratic character which we will denote by χ_F. Note that χ_F lives in $\mathfrak{X}_1$. Furthermore, if $F=\mathbb{Q}_p$ it reduces to the Legendre symbol via the identification $\mathfrak{o}^\times/(1+\mathfrak{p})=\mathbb{F}_p^\times$.

Two dimensional étale algebras over F

The following notions have been introduced in [4] under the name quadratic space, but two dimensional étale algebra is the appropriate name. In the special case of two dimensions an étale algebra E over F is either a quadratic extension of F or it is simply the algebra $E=F\times F$. If we are dealing with a quadratic extension, we let $e=e(E/F)$ be the ramification index and $f=f(E/F)$ be the degree of the residual extension. In particular, we have $ef=2$. By $d=d(E/F)$ we denote the valuation of the discriminant of E/F, it satisfies $d=e-1$. The Galois group is $\mathrm{Gal}_{E/F}=\{1,\sigma\}$. The norm and the trace are defined as usual by

$$\mathrm{Tr}(z)=z+\sigma z \text{ and } \mathrm{Nr}_{E/F}(z)=z\cdot\sigma z.$$

The Haar measure on E will be normalised as follows:

$$\mathrm{Vol}(\mathfrak{O},\mu_E)=q^{-\frac{d}{2}},$$

where $\mathfrak{O}$ is the ring of integers in E. The unique maximal ideal in $\mathfrak{O}$ is denoted by $\mathfrak{P}$ it will be generated by a uniformiser Ω of E. We will usually choose uniformisers such that $\mathrm{Nr}_{E/F}(\Omega)=\varpi^f$. Note that this determines a canonical valuation v_E on E.

Further, let $\chi_{E/F}$ be the quadratic character on $F^\times$ which is trivial on $\mathrm{Nr}_{E/F}(E^\times)$ and set

$$\psi_E = \psi \circ \mathrm{Tr}.$$

By [73, Lemma 2.3.1] we have

$$n(\psi_E) = -\frac{d}{f}.$$

Which again implies that the Haar measure μ_E is normalised to be self dual with respect to ψ_E. Multiplicative characters on E are usually denoted by ξ and one can attach the same objects as we did over F.

If $E = F \times F$, we define the ring of integers to be $\mathfrak{O} = \mathfrak{o} \times \mathfrak{o}$ and the ideal $\mathfrak{P} = \mathfrak{p} \times \mathfrak{p}$. The Haar measure is simply the product measure $\mu \times \mu$ and all multiplicative characters factor into two multiplicative characters on $F^\times$. To keep notation consistent we define

$$\mathrm{Tr}(x_1, x_2) = x_1 + x_2 \text{ and } \mathrm{Nr}_{E/F}((x_1, x_2)) = x_1 x_2.$$

1.3.3 *Number Fields*

Let F be a number field of degree $n = r_1 + 2r_2$, where r_1 is the number of real embeddings and $2r_2$ is the number of complex embeddings. We write $\mathcal{O}_F$ for the ring of integers in F. Prime ideals in $\mathcal{O}_F$ are typically denoted by $\mathfrak{p}$. Each prime ideal gives rise to a non-archimedean place of F which we also denote by $\mathfrak{p}$. The corresponding local field will be called $F_\mathfrak{p}$ and it is equipped with the local structure described above. In the global setting we add a subscript $\mathfrak{p}$ everywhere to indicate that these are local objects at that particular place. Thus, we have $|\cdot|_\mathfrak{p}, v_\mathfrak{p}, \cdots$. We define $\mathcal{N}(\mathfrak{p}) = q_\mathfrak{p} = \sharp(\mathcal{O}_F/\mathfrak{p})$. By extending this multiplicatively we define the absolute norm $\mathcal{N}(\mathfrak{a})$ of a fractional ideal $\mathfrak{a}$. In a similar spirit we use ν for an archimedean place and at the same time for the corresponding embedding $\nu\colon F \to F_\nu$. We put $|\cdot|_\nu = |\cdot|^{[F_\nu:\mathbb{R}]}$. In the number field context $|\cdot|$ without a subscript always denotes the standard absolute value on $F, \mathbb{R} \subset \mathbb{C}$, while $|\cdot|_\nu$ corresponds to the modulus defined on the archimedean local field F_ν.

We define $F_\infty = \prod_\nu F_\nu$ and equip it with the modulus $|\cdot|_\infty = \prod_v |\cdot|_\nu$. Sometimes we use $|\cdot|_\mathbb{R}$ (respectively $|\cdot|_\mathbb{C}$) to denote the part of $|\cdot|_\infty$ coming from the real (respectively complex) embeddings only. Let $\mathbb{A}_{\text{fin}}$ denote the finite adéles equipped with the absolute value $|\cdot|_{\text{fin}}$ being the product of all the local absolute values. The usual adéle ring is then defined by $\mathbb{A}_F = F_\infty \times \mathbb{A}_{\text{fin}}$ and equipped with $|\cdot|_\mathbb{A}$ and $\mu_\mathbb{A}$ in the usual manner. We also define the set of totally positive field elements F^+ to contain all $x \in F$ such that $x_\nu > 0$ for all real ν. Furthermore, put $F^0(\mathbb{A}_F) = \{a \in \mathbb{A}_F \colon |a|_{\mathbb{A}_F} = 1\}$ and embed $\mathbb{R}^+ \subset F_\infty$ diagonally. Finally, we also define $F_{\infty,+} \subset F_\infty$ to be the set of vectors having positive entries over the real places. Note that the classical Minkowski space can be identified with F_∞ (as an euclidean vector space). This is the content of [63, Chapter I, Proposition 5.1]. However, it is important to keep in mind that the canonical measure on Minkowski space is $2^{r_2}(2\pi)^{\frac{n}{2}}\mu_\infty$.

Further, let us choose ideal representatives $\theta_1, \dots, \theta_{h_F} \in \hat{\mathcal{O}}_F$, where h_F denotes the narrow class number of F. By abuse of notation we will use θ_j to denote the element in $\hat{\mathcal{O}}_F$ and also for the associated ideal. We write d_F for the discriminant of F and $\mathfrak{d}$ for the different ideal of F. Then by [63, Theorem 2.9] we have $\mathcal{N}(\mathfrak{d}) = |d_F|$. For any ideal $\mathfrak{m}$ we use $[\mathfrak{m}]_\mathfrak{n} = \frac{\mathfrak{m}}{(\mathfrak{m},\mathfrak{n}^\infty)}$ for the coprime-to-$\mathfrak{n}$ part of $\mathfrak{m}$.

Note that, as explained in [20, Remark 1], we can find fundamental domains for $F_\infty^\times/\mathcal{O}_F^\times$, $F_\infty^\times/(\mathcal{O}_F^\times)^2$, $F_\infty^\times/\mathcal{O}_{F,+}^\times$ and $F_{\infty,+}^\times/\mathcal{O}_{F,+}^\times$ contained in the set

$$\{y \in F_\infty^\times \colon |y_\nu| \asymp |y|_\infty^{\frac{1}{n}} \text{ for all } \nu\}.$$

Elements in the latter set will be referred to as balanced. In particular, every (fractional) ideal is of the form $\theta_j(\alpha)$ for some j and a balanced $\alpha \in F_{\infty,+}^\times$.

Fix an additive character $\psi_\mathbb{Q} = \psi_\infty \prod_p \psi_p$ on $\mathbb{A}_\mathbb{Q}$, trivially on $\mathbb{Q}$ with $\psi_\infty = e(\cdot)$. This extends to a global character on $\mathbb{A}_F$ via

$$\psi(x) = \psi_\mathbb{Q}(\mathrm{Tr}_{A_F \backslash A_\mathbb{Q}}(x)). \tag{1.3.2}$$

Note that we can choose our local unramified characters in such a way that

$$\psi(x) = \prod_{\nu \text{ real}} \psi_\nu(x_\nu) \prod_{\nu \text{ complex}} \psi_\nu(2x_\nu) \prod_\mathfrak{p} \psi_\mathfrak{p}(\varpi_\mathfrak{p}^{-v_\mathfrak{p}(\mathfrak{d})} x_\mathfrak{p}). \tag{1.3.3}$$

The interesting multiplicative characters are the so called Hecke characters $\chi \colon F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$. The easiest such character is given by $|\cdot|_\mathbb{A}^s$ which defines an everywhere unramified Hecke character. Note that due to the tensor product theorem we have the decomposition $\chi = (\otimes_\nu \chi_\nu) \otimes (\otimes_\mathfrak{p} \chi_\mathfrak{p})$, where each χ_ν and $\chi_\mathfrak{p}$ is a character of the underlying local field. Furthermore, almost all $\chi_\mathfrak{p}$ are unramified. There is a unique integral ideal $\mathfrak{m} = \prod_\mathfrak{p} \mathfrak{p}^{a(\chi_\mathfrak{p})}$ such that χ factors through

$$F^\times \backslash \mathbb{A}_F^\times \to F^\times \prod_\mathfrak{p} (1 + \mathfrak{p}^{v_\mathfrak{p}(\mathfrak{m})}) \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times.$$

Another important structural result is (a version of) the strong approximation theorem for $F^\times$ given in [63, Chapter VI, Proposition 1.9]. We have

$$F^\times \cdot \left(F_{\infty,+} \times \prod_{\mathfrak{p}} (1+\mathfrak{p}^{v_\mathfrak{p}(m)}) \right) \setminus \mathbb{A}_F^\times \cong \mathrm{Cl}_F^\mathfrak{m}.$$

Where $\mathrm{Cl}_F^\mathfrak{m} = J_F^\mathfrak{m}/P_F^\mathfrak{m}$ is the (narrow) ray class group, for

$$J_F^\mathfrak{m} = \{\mathfrak{a} \text{ fractional ideal such that } (\mathfrak{a},\mathfrak{m}) = 1\} \text{ and } P_F^\mathfrak{m} = \{(a) \subset J_F^\mathfrak{m} \colon a \text{ totally positive}\}.$$

Note that after dealing with slight subtleties at the archimedean places the latter helps to establish a correspondence between Hecke characters and Größencharakteren and thus yields another structural description of Hecke characters. This is the content of [63, Chapter VI, Corollary 6.14].

If $\chi\colon F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}$ is a Hecke character, we associated the corresponding L-function

$$\Lambda(s,\chi) = \underbrace{\prod_\nu L_\nu(s,\chi_\nu)}_{=\gamma_\infty(s,\chi)} \underbrace{\prod_\mathfrak{p} L_\mathfrak{p}(s,\chi_\mathfrak{p})}_{=L(s,\chi)}.$$

Here we use the classical (analytic) notation which separates the archimedean and non-archimedean parts. If χ is the trivial character, this leads to the Dedekind zeta function. In this case the local factors reduce to $\zeta_\mathfrak{p}(s)$ at the finite places and we write $\zeta_\mathfrak{n}(s) = \prod_{\mathfrak{p}|\mathfrak{n}} \zeta_\mathfrak{p}(s)$. At the archimedean places we have $L_\nu(1,s) = \Gamma_\mathbb{R}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ if ν is real and $L_\nu(1,s) = \Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$ otherwise.

1.3.4 *The Group GL_2*

Let R be a commutative ring with 1. Typically this will be one of the objects introduced above. We set $G(R) = GL_2(R)$. We will also need the subgroups

$$Z(R) = \left\{ z(r) = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} : r \in R^\times \right\}, \qquad A(R) = \left\{ a(r) = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} : r \in R^\times \right\},$$

$$N(R) = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in R \right\} \qquad \text{and} \qquad B(R) = Z(R)A(R)N(R).$$

Further, let

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be the long Weyl element. If the ring R is equipped with Haar measures μ and $\mu^{\times}$, we use the identifications $N(R) = (R, +)$, $A(R) = R^{\times}$, and $Z(R) = R^{\times}$ to transport these measures to the corresponding groups. Via the same identifications we can also transport characters.

We will now describe the structure of G in more detail for some special choices of R.

GL_2 over Archimedean fields

In this case the maximal subgroup is given by

$$K = \begin{cases} U_2(\mathbb{C}) & \text{if } F \text{ is complex,} \\ O_2(\mathbb{R}) & \text{if } F \text{ is real.} \end{cases}$$

A typical element in $k(\theta) \in \mathrm{SO}_2$ is of the form

$$k(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

On the other hand, elements $k[\alpha, \beta] \in SU_2(\mathbb{C})$ is given by

$$k[\alpha, \beta] = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix},$$

for $\alpha, \beta \in \mathbb{C}$ such that $|\alpha| + |\beta| = 1$. Finally, we equip K with the unique Haar probability measure μ_K.

The representation theory of GL_2 over archimedean fields is well known and we will give a very brief summary later on.

GL_2 over Non-Archimedean fields

In this case the maximal compact subgroup is given by $K = G(\mathfrak{o})$. We will also need the following compact, open subgroups

$$K^0(n) = K \cap \begin{bmatrix} \mathfrak{o} & \varpi^n\mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} \end{bmatrix}, K_0(n) = K \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \varpi^n\mathfrak{o} & \mathfrak{o} \end{bmatrix} \text{ and}$$
$$K_1(n) = K \cap \begin{bmatrix} 1+\varpi^n\mathfrak{o} & \mathfrak{o} \\ \varpi^n\mathfrak{o} & \mathfrak{o} \end{bmatrix}, K_2(n) = K \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \varpi^n\mathfrak{o} & 1+\varpi^n\mathfrak{o} \end{bmatrix}.$$

We equip K with the Haar probability measure μ_K. We have the following decomposition of $G(F)$

$$G(F) = \bigsqcup_{t\in\mathbb{Z}} \bigsqcup_{0\leq l\leq n} \bigsqcup_{v\in\mathfrak{o}^\times/(1+\varpi^{min(l,n-l)}\mathfrak{o}^\times)} Z(F)N(F)\underbrace{a(\varpi^t)\omega n(\varpi^{-l}v)}_{=g_{t,l,v}}K_1(n). \qquad (1.3.4)$$

This suggests to define the invariants $t(g)$, $l(g)$ and $n_0(g)$ in the obvious way by writing

$$g \in Z(F)N(F)g_{t(g),l(g),v}K_1(n)$$

with $v \in \mathfrak{o}^\times/(1+\varpi^{n_0(g)}\mathfrak{o}^\times)$. We further define

$$\begin{aligned} n_1 &= \left\lceil \frac{n}{2} \right\rceil, \\ n_0 &= n - n_1, \\ n_1(g) &= \begin{cases} n_0 \text{ if } l(g) \leq n_0, \\ n_1 \text{ if } l(g) \geq n_1, \end{cases} \end{aligned} \qquad (1.3.5)$$

Let π be a infinite dimensional, admissible, irreducible representation of $GL_2(F)$. Such a representation comes with several invariants. Namely, the log-conductor $n = a(\pi)$, this is the smallest $n \in \mathbb{N}_0$ such that $\pi|_{K_1(n)}$ contains the trivial representation. Furthermore, π has central character ω_π. We write $m = a(\omega_\pi)$ for the log-conductor of the central character and define

$$m_1(g) = \max(0, n_0(g) - n + m) \leq \max(0, m - n_1) = m_1.$$

The contragredient representation will be denoted by $\tilde{\pi}$. We attach the L-factor $L(s,\pi)$ and the ϵ-factor $\epsilon(\frac{1}{2},\pi)$ to π. Without loss of generality we may twist π by an unramified character to ensure that $\omega_\pi \in \mathfrak{X}'_m$. Such representations are completely classified and we follow [4] as well as [2] to describe this classification and its consequences in more detail. More precisely we know each unitary, tempered, irreducible π belongs to one of the following families.

1. **Twists of Steinberg:** $\pi = \chi\text{St}$, for some unitary character χ satisfying $\chi(\varpi) = 1$. In this case we have $\omega_\pi = \chi^2$ and $a(\pi) = \max(1, 2a(\chi))$. Furthermore, the L-factor as well as the ϵ-factor are given by

$$L(s,\pi) = \begin{cases} L(s, |\cdot|^{\frac{1}{2}}) & \text{if } \chi = 1, \\ 1 & \text{if } \chi \neq 1, \end{cases} \text{ and } \epsilon(\frac{1}{2},\pi) = \begin{cases} -1 & \text{if } \chi = 1, \\ \epsilon(\frac{1}{2},\chi)^2 & \text{if } \chi \neq 1. \end{cases}$$

2. **Principal series:** $\pi = \chi_1 \boxplus \chi_2$, for unitary characters χ_1 and χ_2. In particular, $a(\pi) = a(\chi_1) + a(\chi_2)$ and $\omega_\pi = \chi_1\chi_1$. Concerning the L-factor we know
$$L(s,\pi) = L(s,\chi_1)L(s,\chi_2) \text{ and } \epsilon(\frac{1}{2},\pi) = \epsilon(\frac{1}{2},\chi_1)\epsilon(\frac{1}{2},\chi_2).$$

3. **Supercuspidal representations:** If π is supercuspidal, then $L(s,\pi) = 1$ and all the other invariants are more difficult to describe. However, if q is odd, we know that every supercuspidal representation is dihedral. Thus $\pi = \omega_\xi$, where ω_ξ is the Weil representation constructed from a quadratic extension E/F and an unitary multiplicative character ξ of $E^\times$. Details on the construction of ω_ξ can be found in [73]. In this scenario we call π the dihedral supercuspidal representation associated to (E,ξ). We find that $a(\pi) = fa(\xi) + d$ and
$$\epsilon(\frac{1}{2},\pi) = \gamma\epsilon(\frac{1}{2},\xi), \tag{1.3.6}$$
for some $\gamma \in S^1$, given in [53, Section 2], depending only on E. The behaviour of π under GL_1-twists is described by $\chi\pi = \omega_{\xi\cdot(\chi\circ\mathrm{Nr}_{E/F})}$ and the central character is $\omega_\pi = \chi_{E/F}\cdot\xi|_{F^\times}$.

This list can be extracted from [35] and [73]. We will sometimes also allow principal series associated to non-unitary characters χ_1 and χ_2 since these appear as local components of Eisenstein series.

It is well known, that for G each admissible, irreducible, infinite dimensional representation is generic. In other words it admits a unique ψ-Whittaker model $\mathcal{W}(\pi)$. This Whittaker model contains an up to scaling unique new vector W_π which we normalise by $W_\pi(1) = 1$. This vector is characterised by $W_\pi(gk) = W(g)$ for all $g \in G$ and all $k \in K_1(n)$. On the subgroup $A(F)$ this function is given by
$$W_\pi\left(\begin{pmatrix} v\varpi^t & 0 \\ 0 & 1\end{pmatrix}\right) = \begin{cases} q^{-t(s+1)} & \text{if } t\geq 0 \text{ and } \pi = |\cdot|^s St\,, \\ \chi_1(v\varpi^t)q^{-\frac{t}{2}} & \text{if } t\geq 0 \text{ and } \pi = \chi_1\boxplus\chi_2 \\ & \quad\text{with } a(\chi_1) > a(\chi_2) = 0, \\ \omega_\pi(v) & \text{if } t = 0 \text{ and } L(s,\pi) = 1, \\ 0 & \text{else.}\end{cases} \tag{1.3.7}$$
for $t \in \mathbb{Z}$ and $v \in \mathfrak{o}^\times$. This is stated in [69, Lemma 2.5] and proven in [31, Lemma 2.10] by reducing it to results from [73].

An important tool to understand the L- and ϵ-factors of a representation π is the local functional equation

$$\frac{Z(W,s,\mu)}{L(s,\mu\tilde{\pi})}\epsilon(s,\mu\tilde{\pi}) = \frac{Z(\tilde{\pi}(w)W,1-s,\mu^{-1}\omega_{\tilde{\pi}}^{-1})}{L(1-s,\mu^{-1}\pi)},$$

for

$$Z(W,s,\mu) = \int_{F^\times} W(a(y))\mu(y)\,|y|^{s-\frac{1}{2}}\,d\mu_F^\times y,$$

a multiplicative character $\mu \in \mathfrak{X}$, a Schwartz-Bruhat function W, and some complex number s with sufficiently large real part. The action of $\tilde{\pi}$ on a Schwartz-Bruhat function is understood by inclusion in the Kirillov model.

GL_2 over Global fields

Over a global field we consider the group $G(\mathbb{A}_F)$. In this setting we will add subscripts ν and $\mathfrak{p}$ to all the local objects to indicate the corresponding local place ν or $\mathfrak{p}$. We define the compact subgroups

$$K_1(\mathfrak{n}) = K_\infty \prod_{\mathfrak{p}} K_{1,\mathfrak{p}}(v_{\mathfrak{p}}(\mathfrak{n})) \text{ and } K = K_\infty \prod_{\mathfrak{p}} K_{\mathfrak{p}},$$

where $K_\infty = \prod_\nu K_\nu$. We view $G(F_\infty)$ as a real Lie group and associate the Lie algebra $\mathfrak{g}_\infty$ with universal enveloping algebra $U(\mathfrak{g}_\infty)$ and center of the latter $Z(\mathfrak{g}_\infty)$. The global Hecke algebra of $G(\mathbb{A}_F)$ will be denoted by $\mathscr{H}$.

We choose the product measure on K and $A(\mathbb{A}_F)$ coming from the previously defined local measures. On the group $N(\mathbb{A}_F) = \mathbb{A}_F$ we put the measure

$$\mu_{N(\mathbb{A}_F)} = \frac{2^{r_2}(2\pi)^{\frac{n}{2}}}{\sqrt{|d_F|}} \prod_\nu \mu_\nu \prod_{\mathfrak{p}} \mu_{\mathfrak{p}}.$$

This corresponds to the normalisation $\mathrm{Vol}(N(F)\setminus N(\mathbb{A}_F)) = 1$, as can be seen from strong approximation together with [63, Chapter I, Proposition 5.2].

Finally, we define

$$\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(g)d\mu(g) = \int_K \int_{\mathbb{A}_F^\times} \int_{N(\mathbb{A}_F)} f(na(y)k)d\mu_{N(\mathbb{A}_F)}(n)\frac{d\mu^\times_{\mathbb{A}_F^\times}(y)}{|y|}d\mu_K(k) \tag{1.3.8}$$

as in [35].

The global applications will be concerned with the study of cuspidal automorphic forms. We will quickly summarise the definition of such forms and their relation to representation theory following [22]. A function ϕ is called an automorphic form on $G(\mathbb{A}_F)$ if it satisfied the following properties[1]

1. $\phi(\gamma g) = \phi(g)$ for all $g \in G(\mathbb{A}_F)$ and all $\gamma \in G(F)$,
2. There is a simple element $\xi \in \mathscr{H}$ such that $f * \xi = f$,
3. There is an ideal $J \subset \mathcal{Z}(\mathfrak{g}_\infty)$ of finite co-dimension which annihilates ϕ,
4. For each $g \in G(\mathbb{A}_F)$ the function $\phi(\cdot g)|_{G(F_\infty)}$ is slowly increasing.

We say ϕ is cuspidal if

$$\int_{N(F)\backslash N(\mathbb{A}_F)} \phi(ng)dn = 0 \text{ for almost all } g \in G(\mathbb{A}_F).$$

This definition may seem technical at first. However, it encompasses all possible classical notions. Furthermore, in this form the definition works for more general reductive groups.

Of particular interest to us will be

$$\phi \in L^2_0(G(F) \setminus G(\mathbb{A}_F), \omega) \subset L^2(G(F) \setminus G(\mathbb{A}_F), \omega)$$

which are right $K_1(\mathfrak{n})$-invariant, and eigenfunctions of the Casimir element $(C_\nu)_\nu \in \mathcal{U}(\mathfrak{q}_\infty)$ with eigenvalues $(\lambda_\nu)_\nu$. These are automorphic forms in the sense described above. Thus, it is standard procedure to associate an cuspidal automorphic representation[2] π_ϕ to ϕ. As explained in [22, p. 4.6] each cuspidal automorphic representation with central character ω can be (uniquely) realised as a closed invariant subspace of $L^2_0(G(F) \setminus G(\mathbb{A}_F), \omega)$.

Let us describe the structure of the cuspidal automorphic representation π. We write V_π for the representation space of π. First note that since (π, V_π) is a cuspidal automorphic representation it is in particular unitary and admissible. For convenience we assume throughout the text that the central character ω_π of π satisfies $\omega_\pi|_{\mathbb{R}^+} = 1$. In other words, the archimedean part of ω_π is trivial on the diagonally embeded positive reals. This can be achieved without loss of generality by twisting by a character of the form $|\cdot|^{i\alpha}_{\mathbb{A}_F}$, for $\alpha \in \mathbb{R}$.

1 The term used in [22] is K_∞-automorphic form. However, since we fixed our maximal compact subgroups once and for all we dropped this from the notation. The K_∞ dependence enters because the Hecke algebra at archimedean places ν depends on K_ν.

2 We use the definition of an automorphic representation given in [22, p. 4.6]. In particular irreducibility is included in the definition.

By the tensor product theorem [34, Theorem 4] we may assume that

$$\pi \simeq \bigotimes_{\nu} \pi_{\nu} \otimes \bigotimes_{\mathfrak{p}} \pi_{\mathfrak{p}}.$$

Where $(\pi_{\mathfrak{p}}, V_{\pi,\mathfrak{p}})$ (respectively $(\pi_{\nu}, V_{\pi,\nu})$) is an irreducible representation of $G(F_{\mathfrak{p}})$ (reps $G(F_{\nu})$) with central character $\omega_{\pi,\mathfrak{p}}$ (respectively $\omega_{\pi,\nu}$). Note that this decomposition also preserves the subspaces of K-finite vectors.

As to Hecke characters, we can associate a L-function to an automorphic representation π. Indeed we set

$$\Lambda(s,\pi) = \prod_{\nu} L(s,\pi_{\nu}) \prod_{\mathfrak{p}} L(s,\pi_{\mathfrak{p}}).$$

This completed L-function has a meromorphic continuation and satisfies a functional equation. Furthermore, it encodes the arithmetic information of π and is a central object in modern analytic number theory. Sometimes it is helpful to consider a slightly more general object. We define the zeta integral

$$Z(s,\phi,\chi) = \int_{F^{\times}\backslash \mathbb{A}_F^{\times}} \phi(a(y)) \, |y|_{\mathbb{A}_F}^{s-\frac{1}{2}} \chi(y) d^{\times}y,$$

for a automorphic form $\phi \in V_{\pi}$, a Hecke character χ and $s \in \mathbb{C}$. Again this posses a functional equation

$$Z(s,\phi,\chi) = Z(1-s,\pi(w)\phi,\chi^{-1}\omega_{\pi}).$$

See [25, Chapter 3, (5.42)]. To gather more information about the structure of Z let us assume that ϕ corresponds to a pure tensor and is cuspidal, so that it has the Whittaker expansion

$$\phi(g) = \sum_{\alpha \in F^{\times}} W_{\phi}(a(\alpha)g),$$

for

$$W_{\phi}(g) = \int_{A\backslash F} \phi(n(x)g)\psi(-x)dx = \prod_{\nu} W_{\phi,\nu}(g_{\nu}) \prod_{\mathfrak{p}} W_{\phi,\mathfrak{p}}(g_{\mathfrak{p}}).$$

Unfolding reveals

$$Z(s,\phi,\chi) = \prod_{\nu} Z(s, W_{\phi,\nu}, \chi_{\nu}) \prod_{\mathfrak{p}} Z(s, W_{\phi,\mathfrak{p}}, \chi_{\mathfrak{p}}).$$

for s with sufficiently large real part. Note that these are exactly the local zeta integral for which we have a local functional equation.

Finally, let us define some more geometric objects. Indeed, it is well known that if $F = \mathbb{Q}$ then classical automorphic forms are thought to be functions on the upper half plane $\mathcal{H}^{(2)} = \{z \in \mathbb{C} \colon \Im(z) > 0\}$. Similarly, one defines upper half space by

$$\mathcal{H}^{(3)} = \{x + yj \colon x \in \mathbb{C}, y \in \mathbb{R}_+\} \subset \mathbb{H}, \tag{1.3.9}$$

here $\mathbb{H}$ are the usual (Hamilton) quaternions over $\mathbb{R}$. We write $\Im(x + yj) = y$. Further we equip $\mathcal{H}^{(2)}$ and $\mathcal{H}^{(3)}$ with the usual (euclidean) norm $\|\cdot\|$. Automorphic forms over F can then be thought (by strong approximation) as living on copies of

$$\mathcal{H} = \prod_{\nu \text{ real}} \mathcal{H}^{(2)} \times \prod_{\nu \text{ complex}} \mathcal{H}^{(3)}. \tag{1.3.10}$$

Points in $\mathcal{H}$ are usually denoted by $P = (P_\nu)_\nu$ and we define the distances

$$u_\nu(P_\nu, Q_\nu) = \frac{\|P_\nu - Q_\nu\|^2}{2\Im(P_\nu)\Im(Q_\nu)}. \tag{1.3.11}$$

We conclude this section by recalling the construction of so called Eisenstein series. These describe the continuous spectrum by intertwining from unitary principal series representations. We closely follow the exposition in [35].

We define the function $H \colon G(\mathbb{A}_F) \to \mathbb{R}_+$ via the Iwasawa decomposition as follows

$$H\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} k\right) = \left|\frac{a}{b}\right|_{\mathbb{A}_F} \text{ for all } k \in K.$$

H factors in the obvious way. We have $H = \prod_\nu H_\nu \prod_\mathfrak{p} H_\mathfrak{p}$.

We define the space

$$\begin{aligned}\tilde{\mathbf{H}}(s) = \Bigg\{\Psi \colon G(\mathbb{A}_F) \to \mathbb{C} \quad : \quad & \Psi\left[\begin{pmatrix} \alpha a u & x \\ 0 & \beta a v \end{pmatrix} g\right] = \omega_\pi(a) \left|\frac{u}{v}\right|_\infty^{s+\frac{1}{2}} \Psi(g) \\ & \text{for } \alpha, \beta \in F^\times,\ a \in \mathbb{A}_F^\times,\ x \in \mathbb{A}_F,\ u, v \in \mathbb{R}_+, \\ & \int_K \int_{F^\times \backslash F^0(\mathbb{A}_F)} |\Psi(a(y)k)|^2 \, d\mu^\times_{\mathbb{A}_F^\times}(y) d\mu_K(k) < \infty \Bigg\}.\end{aligned}$$

This defines a representation $(\pi_s, \tilde{\mathbf{H}}(s))$ of $G(\mathbb{A}_F)$ where $G(\mathbb{A}_F)$ acts by right translation. For $s \in i\mathbb{R}$ an inner product is given by

$$\langle \Psi_1, \Psi_2 \rangle_{\tilde{\mathbf{H}}(s)} = \int_K \int_{F^\times \backslash F^0(\mathbb{A}_F)} \Psi_1(a(y)k)\overline{\Psi_2(a(y)k)} d\mu^\times_{\mathbb{A}_F^\times}(y) d\mu_K(k).$$

We can also view $\tilde{\mathbf{H}}(s)$ as a trivial holomorphic fibre bundle over $\tilde{\mathbf{H}} = \tilde{\mathbf{H}}(0)$. For $\phi \in \tilde{\mathbf{H}}$ we define $\Psi(s) = \Psi \cdot H(\cdot)^s \in \tilde{\mathbf{H}}(s)$.

Further, to $\Psi \in \tilde{\mathbf{H}}$ we associate the Eisenstein series

$$E_{\Psi}(s,g) = \sum_{\gamma \in B(F)\backslash G(F)} [\Psi(s)](\gamma g).$$

Note that the space $\tilde{\mathbf{H}}$ is not irreducible. Indeed it can be decomposed in global principal series representations $\chi_1 \boxplus \chi_2$ satisfying $\chi_1\chi_2 = \omega_\pi$. This is useful when giving a more explicit description of an orthonormal basis $\mathcal{B}_{\tilde{\mathbf{H}}}$ for $\tilde{\mathbf{H}}$.

Part II

THE LOCAL THEORY

This part is dedicated to the analysis of Whittaker vectors over local fields. The uniqueness of the Whittaker model implies very strong factorisation results which allow us to understand many essential properties of global Whittaker functions by studying the corresponding local objects instead. In this section we develop the necessary local theory. We will start in the archimedean setting and then move into the p-adic world.

Of main interest to us are integral representations, explicit formula, support properties, and asymptotic expansions for certain elements in the Whittaker space of a local representation. In the case of archimedean fields these features are mostly standard. However, it is still hard work to gather all the results and present them in a unified matter. Therefore we include this case here. In the non-archimedean case many of these results appear to be new.

In this section we fix a local field F. In particular all the corresponding objects, such as ψ, ϖ, $\mathfrak{p}$, $\mathfrak{o}$ etc., are those attached to this particular field and appear without subscript.

2

THE CASE OF ARCHIMEDEAN FIELDS

In this chapter F is either real or complex. We first recall some facts about the method of stationary phase. Then we will consider Whittaker models of representations of $G(\mathbb{R})$ and $G(\mathbb{C})$ by computing some important elements in terms of classical Whittaker functions $W_{p,q}$, K-Bessel functions and even some hypergeometric functions. We conclude by recalling some useful asymptotic expansions.

Besides introducing the archimedean method of stationary phase we obtain the following key results. Corollary 2.1.7 provides general bounds for the K-Bessel function allowing complex parameter with fixed real part. Further we compute and L^2-normalise certain Whittaker vectors which will be important later on. See Lemma 2.2.2 and Lemma 2.3.4.

Note that the proper way to treat the archimedean places would be by considering representations of the Hecke algebra or by using the language of $(\mathfrak{g}, K)$-modules. However, for our rudimentary purpose it suffices to stick to simple representations.

2.1 THE METHOD OF STATIONARY PHASE

Oscillatory integrals of the form

$$I(t;y) = \int_{\mathbb{R}^m} \alpha(x) e^{it\varphi(x,y)} dx, \tag{2.1.1}$$

where y varies in some parameter space N, appear frequently in mathematics. A helpful tool for dealing with such integrals is the method of stationary phase. In this section we will summarise the tools used later on. There are many good references concerning the method of stationary phase. A very abstract version can be found in [45]. Here we follow the more explicit approach taken in [13]. We were also inspired by the account on the method of stationary phase given in [24]. Our exposition has the slight caveat that it only deals with one dimensional integrals. However, this will be enough for our applications.

2.1.1 *Basic estimates*

The function α appearing in (2.1.1) is usually referred to as *amplitude*, while φ is called the *phase*. The method of stationary phase roughly states that the main contributions of the integral (2.1.1) come from critical points of the phase. A *critical point* $(x_0, y_0) \in \mathbb{R}^m \times N$ of φ is a point which satisfies

$$\nabla \varphi_{y_0}(x_0) = 0.$$

Here $\varphi_y = \varphi(\cdot, y)$. Further, we call a critical point *non-degenerate* if the corresponding Hessian quadratic form, $Q(x_0, y_0)$, is non-degenerate.

The set

$$\mathcal{S} = \{y \in N \colon \nabla \varphi_y(x) \neq 0, \ \forall x \in \mathbb{R}^m\}$$

is called the *shadow zone*. The *light zone* $\mathcal{L}$ consists of those $y \in N$ which exhibit only non-degenerate critical points. The remaining y are contained in the so called *caustic locus* $\mathcal{C}$, which therefore features the degenerate critical points. The reason for this distinction is that the behaviour of $I(t; y)$ depends on the degeneracy of y.

For $y \in \mathcal{S}$ we can derive very good upper bounds for $I(t; y)$ simply by integration by parts. This is made rigorous in [13, Lemma 8.1], which we recall now.

Lemma 2.1.1. *Let $y \geq 1$, $X, Q, U, R > 0$ and let $K \subset [\alpha, \beta] \times \mathcal{S}$ be a compact set. Suppose $\varphi(x, y)$ is a smooth function such that*

$$\frac{d}{dx}\varphi(x, y) \geq R \text{ for all } (x, y) \in K$$

and

$$\frac{d^j}{dx^j}\varphi(x, y) \ll_j YQ^{-j} \text{ for all } (x, y) \in K, j \geq 2.$$

Then

$$\int_{\mathbb{R}} \alpha_y(x) e^{it\varphi(x,y)} dx \ll_A \begin{cases} (\beta - \alpha) X t^{-\frac{A}{2}} \left[\left(\frac{QR}{\sqrt{Y}}\right)^{-A} + (\sqrt{t} RU)^{-A} \right] \\ (\beta - \alpha) X t^{-A} \left[\left(\frac{QR}{\sqrt{Y}}\right)^{-A} + (RU)^{-A} \right], \end{cases}$$

for all $A \in \mathbb{N}_0$, $y \in pr_2(K)$ and all $\alpha \in \mathcal{C}^\infty(\mathbb{R})$ such that $supp(\alpha) \subset pr_1(K)$ and

$$\alpha_y^{(j)}(u) \ll_j XU^{-j}.$$

Proof. The first bound follows directly from [13, Lemma 8.1] with $h(u) = t\varphi(u, y)$. The second bound follows by observing that

$$\int_{-\infty}^{\infty} \alpha_y(u) e^{it\varphi(u,y)} du = t^{-n} \int_{-\infty}^{\infty} [\mathcal{D}^n \alpha_y](u) e^{it\varphi(u,y)} du$$

for $\mathcal{D} f = -\frac{d}{du}[(i\frac{d\varphi}{du})^{-1} f]$ and $n \in \mathbb{N}$. The claim follows after estimating trivially using [13, (8.5), (8.6)] to deal with $\mathcal{D}^n \alpha_y$. □

Next we will recall an asymptotic formula valid for y in the light zone. The following lemma is a slight simplification of [13, Proposition 8.2].

Lemma 2.1.2. *Let $0 < \delta < \frac{1}{10}$, $t, X, Y_1, Y_2, V, Q > 0$ be parameters and fix an integral J of length $V_1 \geq V$ and let $K \subset J \times N$ be a compact set. Define $Z = Q + X + tY_2 + V_1 + 1$ and assume that*

$$\frac{t^{\frac{1}{3}} Y_1}{Y_2^{\frac{2}{3}}} \geq Z^{\delta}, \quad Y_1 \leq 1 \leq Y_2 \leq t^{\frac{1}{3}}, \text{ and } V \geq \frac{QZ^{\frac{\delta}{2}}}{\sqrt{tY_1}}.$$

Suppose that for each $y \in pr_2(K)$ there is a unique $x_0(y) \in J$ such that $\varphi'(x_0(y); y) = 0$. Further, assume that

$$\varphi''(x, y) \gg Y_1 Q^{-2} \text{ and } \varphi^{(j)}(x, y) \ll_j Y_2 Q^{-j},$$

for all $j \geq 2$ and all $(x, y) \in K$. Then

$$I(t; y) = \int_{\mathbb{R}} \alpha_y(x) e^{it\varphi(x;y)} dx = e^{it\varphi(x_0(y);y)} \frac{e^{\frac{\pi i}{4}} \sqrt{2\pi}}{\sqrt{t\varphi''(x_0(y); y)}} \left(\alpha_y(x_0(y)) + O_\delta \left(XZ^{-\frac{\delta}{2}} \right) \right),$$

for all $\alpha_y \in \mathcal{C}^\infty(\mathbb{R})$ satisfying $supp(\alpha_y) \subset pr_1(K)$ and

$$\alpha_y^{(j)}(x) \ll_j XV^{-j}$$

for all $(x, y) \in K$.

For the sake of completeness we present the proof here. As it suffices for our purposes we only compute the leading order term. However, the proof can be extended to provide more terms of the asymptotic expansion just as in [13, Proposition 8.2].

Proof. Choose a parameter $U = \frac{QZ^{\frac{\delta}{2}}}{\sqrt{tY_1}} \leq V$. By assumption we have

$$\frac{tY_1 U^2}{Q^2} \geq Z^{\delta} \text{ and } \frac{tY_2 U^3}{Q^3} \leq 1. \tag{2.1.2}$$

Further, we fix $\kappa_0 \in \mathcal{C}^\infty(\mathbb{R})$ supported in $[-1,1]$ such that $\kappa_0(x) = 1$ for all $|x| \le \frac{1}{2}$. We write

$$I_1(t;y) = \int_{\mathbb{R}} \underbrace{\alpha_y(x)\left(1-\kappa_0\left(\frac{x-x_0(y)}{U}\right)\right)}_{f_y(x)} e^{it\varphi(x;y)}dx. \tag{2.1.3}$$

It is easy to check that $f_y^{(j)} \ll_j XU^{-j}$ on its support. Furthermore, the mean value theorem implies that

$$\left|\varphi'(x;y)\right| \gg |x-x_0(y)|\frac{Y_1}{Q^2} \gg \frac{UY_1}{Q^2}.$$

Thus we apply Lemma 2.1.1 together with (2.1.2) and get

$$I_1(t;y) \ll_A V_1 X\left(Z^{-\frac{A\delta}{2}}\sqrt{\frac{tY_1}{Y_2}} + Z^{-A\delta}\right) \ll Z^{2-A\delta}\left(1+\left(\frac{t^{\frac{1}{3}}}{Y_2^{\frac{1}{6}}}\right)^{-A}\right) \ll Z^{-B}.$$

So far we have shown that

$$I(t;y) = \int_{\mathbb{R}} \alpha_y(x)\kappa_0\left(\frac{x-x_0(y)}{U}\right)e^{it\varphi(x;y)}dx + O_{B,\delta}(Z^{-B}).$$

Let $H_y(x)$ be defined by requiring that

$$\varphi(x;y) = \varphi(x_0(y);y) + \frac{1}{2!}\varphi''(x_0(y);y)(x-x_0(y))^2 + H_y(x).$$

One checks that $H_y' \ll U^2Y_2Q^{-3}$ and $H_y'' \ll UY_2Q^{-3}$. In particular, U is chosen such that

$$tH_y^{(j)} \ll U^{-j} \text{ for } j = 1,2.$$

For $j \ge 3$ we observe

$$tH_y^{(j)}(x) = t\varphi(x;y)^{(j)} \ll tY_2Q^{-j} \ll (tY_2)^{1-\frac{j}{3}}U^{-j} \ll U^{-j}.$$

With this at hand we define

$$g_y(x) = \alpha_y(x)\kappa_0\left(\frac{x_0(y)-x}{U}\right)e^{itH_y(x)}$$

and observe that $g_y^{(j)} \ll_j XU^{-j}$. We rewrite the integral $I(t;y)$ as

$$I(g;y) = e^{it\varphi(x_0(y);y)}\int_{\mathbb{R}} g_y(x)e^{\frac{it\varphi''(x_0(y);y)}{2}(x-x_0(y))^2}dx + O_{B,\delta}(Z^{-B}). \tag{2.1.4}$$

The decay of $\hat{g}$ allows us to write

$$g_y(x) = \int_{\mathbb{R}} \hat{g}_y(z)e(zx)dz = \int_{[-\frac{Z^{\frac{\delta}{4}}}{U},\frac{Z^{\frac{\delta}{4}}}{U}]} \hat{g}_y(z)e(zx)dz + O_{\delta,B}(Z^{-B}). \tag{2.1.5}$$

Inserting this expression in (2.1.4), interchanging order of integration, and evaluating the inner integral yields

$$I(g;y) = e^{\frac{i\pi}{4}+it\varphi(x_0(y);y)}\sqrt{\frac{2\pi}{t\varphi''(x_0(y);y)}}\int_{[-\frac{Z^{\frac{\delta}{4}}}{U},\frac{Z^{\frac{\delta}{4}}}{U}]}\hat{g}_y(z)e\left(x_0(y)z-\frac{\pi z^2}{t\varphi''(x_0(y);y)}\right)dz$$
$$+O_{B,\delta}(Z^{-B}).$$

By Taylor's theorem we have

$$e\left(-\frac{\pi z^2}{t\varphi''(x_0(y);y)}\right)=\sum_{n=0}^{K}\frac{z^{2n}}{n!}\left(\frac{-2\pi^2 i}{t\varphi''(x_0(y);y)}\right)^n+O\left(XZ^{-\frac{(2K+1)\delta}{4}}\right)$$

on the domain of integration. Next, we chose $K = K(B,\delta)$ big enough, extend the truncated integral to $\mathbb{R}$, and get

$$I(g;y) = e^{\frac{i\pi}{4}+it\varphi(x_0(y);y)}\sqrt{\frac{2\pi}{t\varphi''(x_0(y);y)}}\left[\sum_{n\le K}\frac{1}{n!}\left(\frac{-2\pi^2 i}{t\varphi''(x_0(y);y)}\right)^n\right.$$
$$\left.\cdot\int_{\mathbb{R}}z^{2n}\hat{g}_y(z)e\left(x_0(y)z\right)dz\right]+O_{B,\delta}(Z^{-B}).$$

By Fourier inversion we obtain

$$I(g;y) = e^{\frac{i\pi}{4}+it\varphi(x_0(y);y)}\sqrt{\frac{2\pi}{t\varphi''(x_0(y);y)}}\sum_{n\le K}\frac{g^{(2n)}(x_0(y))}{n!}\left(\frac{i}{2t\varphi''(x_0(y);y)}\right)^n$$
$$+O_{B,\delta}(Z^{-B}).$$

In order to complete the proof we need to estimate all the higher order terms. This is done as follows. First, we recall that $\kappa_0((x_0(y)-x)/U)$ is constant in a neighbourhood of $x_0(y)$. Thus we obtain

$$g^{(2n)}(x_0(y)) \ll_n \sum_{j=0}^{2n} XV^{-j}\left|\frac{d^{2n-j}}{dx^{2n-j}}e^{itH_y(x_0(y))}\right|.$$

Further, note that $H_y^{(j)}(x_0(y)) = 0$ for $j = 0,1,2$ and $H_y^{(j)}(x_0(y)) = \varphi^{(j)}(x_0(y);y)$ for $j \ge 3$. We arrive at

$$g^{(2n)}(x_0(y)) \ll_n X\left(V^{-2n}+\left(\frac{(tY_2)^{\frac{1}{3}}}{Q}\right)^{2n}\right).$$

Using the lower bound on φ'' together with the assumptions on the parameters reveals that the n-th term in the sum is bounded by $XZ^{-\frac{n\delta}{2}}$. The result follows by estimating all but the 0-th term in the sum. □

Unfortunately this result only allows a single critical point. In practice one often encounters several critical points which might approach each other as the parameter y varies. However, due to the great flexibility in the parameters, one can handle these situations by using suitable partitions of unity. As we will see this is very cumbersome.

To complete the picture one needs satisfying expansions in some neighbourhood of the caustic locus $\mathcal{C}$. This will concern us for the rest of this section.

In the case of degenerate critical points many different outcomes are possible. The general behaviour of (2.1.1) is governed by the 'singularity type' of φ at (x_0, y_0). There is a wide range of classification results for singularities and the corresponding normal forms. For our purpose we only need one type of singularity.

We say a critical point (x_0, y_0) is a singularity of type A_2 (sometimes called fold singularity) if $Q(x_0, y_0)$ has corank 1 and there is $Z \in \mathbb{R}^m$ such that for all $X \in \text{kern}(Q(x_0, y_0))$ we have

$$\partial_v^{2+1}\varphi(x_0, y_0) \neq 0$$

where ∂_v is the partial derivative in the direction $v = \langle X, Z \rangle$.

These are exactly the singularities that lead to the appearance of Airy functions in the asymptotic expansion of $I(t; y)$. Let us note that in the case $m = 1$ an A_2 singularity boils down to a point (x_0, y_0) such that

$$\frac{d\varphi}{dx}(x_0, y_0) = \frac{d^2\varphi}{dx^2}(x_0, y_0) = 0 \neq \frac{d^3\varphi}{dx^3}(x_0, y_0).$$

For more details and further references on the theory of singularities we refer to [1].

Before we continue let us recall some basic properties of the Airy function and its derivative. The Airy function is defined by

$$\text{Ai}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(t^3/3+xt)}dt$$

for $x \in \mathbb{R}$ and can be continued analytically to the complex plane. As we can see from the definition, the Airy function is the prototype of a function with the simplest possible degenerate critical point. As such $\text{Ai}(x)$ interpolates between a slowly decaying wavefront to the left of 0 and exponential decay to the right. This is captured in the asymptotic expansions

$$\mathrm{Ai}(x) = \begin{cases} \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}x^{\frac{1}{4}}}(1+o(1)) & \text{if } x > 0, \\ \frac{\cos(\frac{2}{3}|x|^{\frac{3}{2}} - \frac{\pi}{4})}{\sqrt{\pi}|x|^{\frac{1}{4}}}(1+o(1)) & \text{if } x < 0, \\ \frac{1}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})} & \text{if } x = 0, \end{cases}$$

which can be extracted from [64, Chapter 4, Section 4.1]. Thus it is not surprising that the Airy function and its derivative play a key role in asymptotic expansions of oscillatory integrals featuring degenerate critical points of type A_2. Recall that Ai' has the integral representation

$$\mathrm{Ai}'(x) = \frac{i}{2\pi}\int_{\mathbb{R}} te^{i(t^3/3+xt)}dt$$

and the asymptotic behaviour

$$\mathrm{Ai}'(x) = \begin{cases} -\frac{z}{2\sqrt{\pi}}e^{-\frac{2}{3}z^{\frac{3}{2}}}(1+o(1)) & \text{if } x > 0, \\ |x|^{\frac{1}{4}}\,\frac{\sin(\frac{2}{3}|x|^{\frac{3}{2}} - \frac{\pi}{4})}{\sqrt{\pi}}(1+o(1)) & \text{if } x < 0, \\ \frac{-1}{3^{\frac{1}{3}}\Gamma(\frac{1}{3})} & \text{if } x = 0. \end{cases}$$

The latter can be extracted from the connection formula

$$\mathrm{Ai}'(z) = -\frac{z}{\sqrt{3}\pi}K_{\frac{2}{3}}(\frac{2}{3}z^{\frac{3}{2}}) \text{ and } \mathrm{Ai}'(-z) = \frac{z}{3}\left(J_{\frac{2}{3}}(\frac{2}{3}z^{\frac{3}{2}}) - J_{-\frac{2}{3}}(\frac{2}{3}z^{\frac{3}{2}})\right)$$

and the corresponding asymptotic expansions [64, Chapter 12 Equation 1.03 and Chapter 4 Equation 9.09].

Before we come to oscillatory integrals with degenerate critical points we will need one more preparation. Usually we are concerned with integrals involving compactly supported test functions satisfying certain decay properties. However, we will encounter situations where the test function is not compactly supported. To deal with this issue we fix a smooth partition of unity $\{\chi_n\}_{n\geq 0}$ of $\mathbb{R}$ such that

$$\mathrm{supp}(\chi_n) \subset \begin{cases} [2^{n-1}, 2^{n+1}] \cup [-2^{n+1}, -2^{n-1}] & \text{if } n \geq 1, \\ [-2, 2] & \text{if } n = 0. \end{cases}$$

Furthermore, let us assume that

$$\left|\frac{d^l}{dx^l}\chi_n(x)\right| \ll_l 2^{-ln} \text{ for all } n, l \in \mathbb{N}_0 \text{ and all } x \in \mathbb{R}.$$

It is an easy exercise to see that such a partition of unity exists.[1]

Concerning oscillatory integrals with degenerate critical points we have the following lemma, which is extracted from [45, Theorem 7.7.18]. See also [24, Lemma 16.4].

Lemma 2.1.3. *Let $(x_0, y_0) \in K \subset J \times N$ be a critical point of singularity type A2. In particular y_0 is contained in the caustic locus $\mathcal{C}$. Then, there exists a neighbourhood*

$$(x_0, y_0) \in V \times N' \subset K$$

and $a, r_0, r_1 \in \mathcal{C}^\infty(N')$ such that

$$I(t; y) = 2\pi e^{itb(y)} r_0(y) Ai(a(y)t^{\frac{2}{3}})t^{-\frac{1}{3}} - 2\pi i e^{itb(y)} r_1(y) Ai'(a(y)t^{\frac{2}{3}})t^{-\frac{2}{3}} \\ + O_{K,\varphi}(X \min(1, U)^{-2} t^{-1}),$$

for all $y \in N'$ and all $\alpha_y \in \mathcal{C}^\infty(J)$ such that $supp(\alpha_y) \subset V$ and

$$\alpha_y^{(j)} \ll_j XU^{-j}.$$

Furthermore, we have

$$r_0(y_0) = \frac{2\alpha_{y_0}(x_0)}{\varphi^{(3)}(x_0; y_0)^{\frac{1}{3}}}, \quad b(y_0) = \varphi(x_0; y_0) \text{ and } a(y_0) = 0.$$

In particular, we have

$$I(t; y_0) = \frac{2\Gamma(\frac{1}{3})}{3^{\frac{1}{6}}} \frac{e^{it\varphi(x_0;y_0)}\alpha_{y_0}(x_0)}{\varphi'''(x_0; y_0)^{\frac{1}{3}}} t^{-\frac{1}{3}} + O(t^{-\frac{2}{3}}) \text{ and}$$
$$I(t; y) = 2\pi e^{itb(y)} r_0(y) Ai(a(y)t^{\frac{2}{3}})t^{-\frac{1}{3}} + O(t^{-\frac{1}{2}}). \tag{2.1.6}$$

Note that the latter is only an asymptotic formula when y is sufficiently close to y_0. More precisely, there is a fixed $\delta > 0$ such that (2.1.6) is an asymptotic formula as long as $|y - y_0| \le \delta t^{-\frac{2}{3}}$.

Proof. By the Malgrange preparation theorem, [45, Theorem 7.5.13], there is a neighbourhood $(x_0, y_0) \in U_1 \times N_1 \subset K$, smooth functions $a, b \in \mathcal{C}^\infty(N_1)$ and $T \in \mathcal{C}^\infty(U_1 \times N_1)$ with $T(x_0, y_0) = 0 \neq \frac{\partial T}{\partial x_0}(x_0, y_0)$ such that

$$\varphi(x, y) = \frac{T(x, y)^3}{3} + a(y)T(x, y) + b(y) \text{ for all } (x, y) \in U_1 \times N_1. \tag{2.1.7}$$

In particular, we have $a(y_0) = 0$, $b(y_0) = \varphi(x_0, y_0)$ and $T'(x_0, y_0) = \frac{\varphi'''(x_0,y_0)^{\frac{1}{3}}}{2}$. Furthermore, there is a neighbourhood $(x_0, y_0) \in (U_2, N_2) \subset (U_1, N_1)$ such that

$$T_y \colon U_2 \to T_y(U_2) \subset V_2, \quad x \mapsto T(x, y)$$

1 The construction is outlined in a footnote in [24].

is invertible and satisfies $T_y'(x) \asymp 1$ for all $(x,y) \in (U_2, N_2)$. We define $V = U_2$ and

$$\tilde{\alpha}_y(z) = \frac{\alpha(T_y^{-1}(z))}{T_y'(T_y^{-1}(z))}.$$

Since $\mathrm{supp}(\alpha_y) \subset V$, we have $\mathrm{supp}(\tilde{\alpha}_y) \subset T_y(U_2) \subset V_2$. Even more, $\alpha_y^{(j)} \ll XV^{-j}$ implies $\tilde{\alpha}_y^{(j)} \ll_j X \min(1,V)^{-j}$.

By [45, Theorem 7.5.6] we find $r_0, r_1 \in \mathcal{C}^\infty(N_3)$ and $q \in V_3 \times N_3$ such that

$$\tilde{\alpha}_y(z) = q(z,y)(z^2 + a(y)) + r_1(y)z + r_0(y) \text{ for all } (z,y) \in V_3 \times N_3.$$

In particular, $r_0(y) = \tilde{\alpha}_y(0) \ll X$.

Next, fix a neighbourhood $y_0 \in N' \subset N_3$ such that $\left|z^2 + a(y)\right| \gg 1$ on $(V_3 \setminus V_4) \times N'$, for some open set $0 \in V_4 \subset V_3$. By construction $a(y) \ll 1$ for $y \in N'$.

Finally, we choose a compactly supported test function ξ_0 such that $\xi_0 | V_4 \equiv 1$ and $\mathrm{supp}(\xi_0) \subset V_3$.

With this at hand we can perform the change of variables $x \to z = T_y(x)$ and compute that

$$\begin{aligned}
I(t;y) &= e^{itb(y)} \int_R \tilde{\alpha}_y(z) e^{it[\frac{z^3}{3} + a(y)z]} dz \\
&= e^{itb(y)} \biggl(r_0(y) \int_{\mathbb{R}} e^{it[\frac{z^3}{3} + a(y)z]} dz + r_1(y) \int_{\mathbb{R}} z e^{it[\frac{z^3}{3} + a(y)z]} dz \\
&\qquad + \int_{\mathbb{R}} \xi_0(z) q(z,y)(z^2 + a(y)) e^{it[\frac{z^3}{3} + a(y)z]} dz \\
&\qquad + \int_{\mathbb{R}} (1 - \xi_0(z))(\tilde{\alpha}_y(z) - r_0(y) - r_1(y)) e^{it[\frac{z^3}{3} + a(y)z]} dz \biggr) \\
&= e^{itb(y)} \left(I_1 + I_2 + I_3 + I_4 \right).
\end{aligned}$$

for $y \in N'$. The first two integrals are identified with

$$I_1 = 2\pi \frac{r_0(y)}{t^{\frac{1}{3}}} \mathrm{Ai}(a(y)t^{\frac{2}{3}}) \text{ and } I_2 = -i2\pi \frac{r_1(y)}{t^{\frac{2}{3}}} \mathrm{Ai}'(a(y)t^{\frac{2}{3}}).$$

The third integral can be estimated trivially after performing partial integration. Indeed,

$$I_3 = \frac{-i}{t} \int_{\mathbb{R}} \frac{d}{dz} (\xi_0(z) q(z,y)) e^{it[\frac{z^3}{3} + a(y)z]} dz \ll t^{-1} \sup_{z \in V_3} \left(|q(z,y)| + \left|q'(z,y)\right| \right).$$

Finally, we have to estimate the integral I_4. Note that in this case the test function does not have compact support. However, for $z \in \mathrm{supp}(1 - \xi_0)$ the phase is well behaved. Using the partition of unity $(\chi_n)_{n \in \mathbb{N}_0}$ together with Lemma 2.1.1 shows that

$$I_4 \ll_A X \min(1,V)^{-A} t^{-A}.$$

This is absorbed in the contribution of I_3, which completes the proof of the main formula.

The special cases follow after inserting the suitable bounds for Ai$'$. □

2.1.2 *Estimating the K-Bessel function*

The K-Bessel function often occurs in the theory of automorphic forms through its appearance in the archimedean Kirillov model of certain representations of GL_2. Evaluating this function asymptotically is also a good example to see the method of stationary phase in action. As starting point we use the so called Basset integral. For all $|\arg z| < \frac{\pi}{2}$, $s > 0$ and $k \geq -1$, we have

$$K_{k/2+is}(sz) = \frac{\Gamma(k/2+is+\frac{1}{2})2^{k/2+is}}{2\sqrt{\pi}(sz)^{k/2+is}} I_k(s;z). \tag{2.1.8}$$

Where $I_k(s;z)$ is the oscillatory integral

$$I_k(s;z) = \int_{-\infty}^{\infty} \alpha_k(u) e^{-is\varphi(u;z)} du,$$

for

$$\varphi(u;z) = \log(u^2+1) + zu, \text{ and } \alpha_k(u) = (u^2+1)^{-(k+1)/2}.$$

For simplicity we assume $s, k, z \in \mathbb{R}_+$ in all the following estimates.

The phase φ has the following critical point structure:

$$\mathcal{S} = (1,\infty), \quad \mathcal{L} = (0,1) \text{ and } \mathcal{C} = 1.$$

This follows directly from

$$\frac{d}{du}\varphi(u;z) = \frac{2u}{u^2+1} + z \text{ and } \frac{d^2}{du^2}\varphi(u;z) = 2\frac{u^2-1}{(u^2+1)^2}.$$

The asymptotic expansion in the case $k = 0$ is well known. Therefore, we will go a bit further and try to give precise upper bounds which are uniform in $k \geq 0$. The method of stationary phase, as we use it, works only for a limited range of k. Thus, later on we will introduce some restrictions on k.

Estimate in the shadow zone

We use the partition of unity constructed above to write

$$I_k(s;z) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \alpha_k(u)\chi_n(u) e^{-is\phi(u;z)} du = \sum_{n=0}^{\infty} I_k^{(n)}(s;z).$$

Furthermore, we define the compact sets $K_n = \overline{\operatorname{supp}(\chi_n)} \times [1+r, C]$ for some parameters $r > 0$ and $C \gg 1$.

For $(x_0, y_0) \in K_n$ we have the lower bound

$$\left|\frac{d}{dx}\varphi(x_0, y_0)\right| \geq \inf_{(x,y)\in K_n} \left|\frac{2u}{u^2+1} + z\right| \geq r. \tag{2.1.9}$$

In the last step we used that

$$\left|\frac{2u}{u^2+1} + z\right| \geq \frac{zu^2+2u+z}{u^2+1} \geq \frac{u^2+2u+1}{u^2+1} + r \geq r$$

for every $u \in \mathbb{R}$.

By elementary means one shows that

$$\frac{d^j}{dx^j}\varphi(x, y) \ll_j 2^{-nj} \tag{2.1.10}$$

for $n \in \mathbb{N}_0$, $j \geq 2$ and $x \in \operatorname{pr}_1(K_n)$.

Because

$$\sup_{u\in\operatorname{supp}(\chi_n)} \left|\frac{d^j}{du^j}\alpha_k(u)\right| \ll_j (k+1)^j 2^{-n(j+1)},$$

we can use the product rule to bound

$$\sup_{u\in\operatorname{supp}(\chi_n)} \left|\frac{d^j}{dx^j}\alpha_k\chi_n(x)\right| \ll_l (k+1)^j 2^{-n(j+1)}. \tag{2.1.11}$$

These estimates enable us to prove the following lemma.

Lemma 2.1.4. *For any $A \in \mathbb{N}_0$ we have*

$$I_k^{(n)}(s; z) \ll_A \begin{cases} (\sqrt{s}r2^n)^{-A}\left(1 + \left(\frac{\sqrt{s}}{k+1}\right)^{-A}\right) \\ \left(\frac{sr2^n}{k+1}\right)^{-A}, \end{cases}$$

where $z \in [1+r, C]$ and $n \in \mathbb{N}_0$. Furthermore,

$$I_k(s; z) \ll_A \begin{cases} (\sqrt{s}(z-1))^{-A}\left(1 + \left(\frac{\sqrt{s}}{k+1}\right)^{-A}\right) \\ \left(\frac{s(z-1)}{k+1}\right)^{-A} \end{cases}$$

whenever $z > 1$.

Proof. We apply Lemma 2.1.1 with

$$\alpha = -2^{n+1}, \beta = 2^{n+1}, Y = 1, Q = 2^n, X = 2^{-n}, U = \frac{2^n}{k+1} \text{ and } R = r$$

to obtain the bounds for $I^{(n)}$. The estimates (2.1.9), (2.1.10) and (2.1.11) ensure that the necessary conditions are met. The bounds for I follow after summing over $n \in \mathbb{N}_0$. □

Estimate in the light zone

Lemma 2.1.5. *For $cs^{-\frac{1}{9}} \leq z < 1 - Cs^{-\frac{1}{3}+\delta}$ with suitable constants $c \leq 1 \leq C$, and $k \leq s^{\frac{2}{9}}$ we have*

$$I_k(s;z) = \frac{e^{\frac{i\pi}{4}}\sqrt{\pi}}{\sqrt{s}2^{\frac{k}{2}+1+is}(1-z^2)^{\frac{1}{4}}}\Big[e^{is(1+\sqrt{1-z^2})}(1-\sqrt{1-z^2})^{\frac{k}{2}+is} \\ - e^{is(1-\sqrt{1-z^2})}(1+\sqrt{1-z^2})^{\frac{k}{2}+is}\Big]\Big(1+O_{c,\delta}(2^{k+1}s^{-\frac{\delta}{2}})\Big).$$

Proof. Each $z \in \mathcal{L}$ comes with two non-degenerate critical points

$$u_\pm = u_\pm(z) = -\frac{1}{z}(1 \mp \sqrt{1-z^2}).$$

In order to treat these two points independently we need to modify our partition of unity $\{\chi_n\}_{n\in\mathbb{N}_0}$. We start by choosing suitable test function in order to separate the critical points. Let $\chi_\pm$ be two smooth functions satisfying $\operatorname{supp}(\chi_-) \subset [-\frac{1}{z}, 0]$ and $\operatorname{supp}(\chi_+) \subset [-\frac{2}{z}, -\frac{1}{z}]$. Furthermore, we assume that

$$\chi_+(u_+) = 1, \quad \chi_-(u_-) = 1 \text{ and } \frac{d^j}{dx^j}\chi_\pm \ll_j (1-z^2)^{-\frac{j}{2}}z^{-2j}.$$

With this at our disposal we can compute the contribution of the critical points.

We start with u_- by asymptotically evaluating

$$I_k^{(-)}(s;z) = \int_{-\infty}^{\infty} \chi_-(u)\alpha_k(u)e^{-is\varphi(u;z)}du.$$

The parameters in Lemma 2.1.2 are as follows. First, we have $V_1 \asymp z^{-1}$ and $K = \frac{1}{z}[-1,0] \times [cs^{-1}, 1-Cs^{-\frac{1}{3}+\delta}]$. Further, recall that $\alpha_k^{(j)} \ll_j (k+1)^j(\frac{1}{z^2}+1)^{-\frac{k+1}{2}}z^j$ on the support of χ_-. Thus

$$\frac{d^j}{dx^j}\alpha_k\chi_- \ll_j (1+z^2)^{-\frac{k+1}{2}}z^{k+1}\sum_{i=0}^{j}\binom{j}{i}(k+1)^i\sqrt{1-z^2}^{j-i}z^{3i-2j} \\ \ll_j \frac{z^{k+1}}{(1+z^2)^{\frac{k+1}{2}}}\left(z(k+1)+\frac{\sqrt{1-z^2}}{z^2}\right)^j.$$

and we have $X = \frac{z^{k+1}}{(z^2+1)^{\frac{k+1}{2}}}$ and $V = (zk + \sqrt{1-z^2}/z^2)^{-1}$. Naturally, $V_1 \geq V$ as $k > 0$. At last we observe that

$$\varphi''(u,z) \gg (1-z)z^2 \text{ and } \varphi^{(j)}(u;z) \ll_j z^j,$$

so that $Y_1 = 1-z$, $Y_2 = 1$ and $Q = z^{-1}$. Plugging these values into Lemma 2.1.2 reveals

$$I_k^{(-)}(s;z) = \frac{z^{k+2is}\sqrt{\pi}e^{\frac{i\pi}{4}}e^{is(1+\sqrt{1-z^2})}}{2^{\frac{k}{2}+1+is}\sqrt{s}(1+\sqrt{1-z^2})^{\frac{k-1}{2}+is}\sqrt{1-z^2+\sqrt{1-z^2}}}\left(1+O_\delta\left(2^{k+1}s^{-\frac{\delta}{2}}\right)\right).$$

Note that the necessary conditions are satisfied by assumption. Indeed

$$s^{\frac{1}{3}}(1-z) \geq Cs^{\delta} \geq \left(\frac{5}{c}\right)^{\delta} s^{\delta} \geq Z^{\delta}$$

implies $\frac{s^{\frac{1}{3}}Y_1}{Y_2^{\frac{2}{3}}} \geq Z^{\delta}$. The condition $V \geq \frac{QZ^{\frac{\delta}{2}}}{\sqrt{sY_1}}$ follows from

$$Z^{\frac{\delta}{3}}\left(zk + \frac{\sqrt{1-z^2}}{z^2}\right) \leq (1+c^{-2})\left(\frac{5}{c}\right)^{\frac{\delta}{2}} s^{\frac{\delta}{2}+\frac{2}{9}} \leq cCs^{\frac{\delta}{2}+\frac{2}{9}} \leq z\sqrt{s(1-z)}.$$

The argument for χ_+ in place of χ_- is very similar. One obtains

$$\begin{aligned} I_k^{(+)}(s;z) &= \int_{-\infty}^{\infty} \chi_+(u)\alpha_k(u)e^{-is\varphi(u;z)}du \\ &= \frac{z^{k+2is}\sqrt{\pi}e^{\frac{i\pi}{4}}e^{is(1-\sqrt{1-z^2})}}{2^{\frac{k}{2}+1+is}\sqrt{s}(1-\sqrt{1-z^2})^{\frac{k-1}{2}+is}\sqrt{1-z^2-\sqrt{1-z^2}}}\left(1+O_{\delta}\left(2^{k+1}s^{-\frac{\delta}{2}}\right)\right). \end{aligned}$$

The final step is to complete $(\chi_{\pm})_{\pm}$ to a partition of unity and show, using Lemma 2.1.1, that the remaining contribution is absorbed in the error.

The result stated above follows by adding the contributions of the two critical points together with elementary manipulations. □

At this point we note that the ranges for z are quite complicated and far from optimal. Furthermore, for most k we obtain a rather crude upper bound instead an asymptotic formula. Nevertheless, it is handy to have some results for K-Bessel functions featuring mixed order.

The transition region

Lemma 2.1.6. *There is a constant c as well as functions $b(z)$, $r_0(z)$, $r_1(z)$ and $a(y)$ such that*

$$I_k(s;z) = e^{isb(z)}\left(r_0(z)Ai(a(z)s^{\frac{2}{3}})s^{-\frac{1}{3}} - r_1(z)Ai'(a(z)s^{\frac{2}{3}}s^{-\frac{2}{3}}\right) + O_c(s^{-\frac{2}{3}-\epsilon})$$

for $k \ll s^{\frac{1}{6}-\frac{\epsilon}{2}}$ and $z \in (1-c, 1+c)$. In particular,

$$I_k(s;z) \ll_k s^{-\frac{1}{3}}$$

for $z \in (1-c, 1+c)$.

Proof. This is a consequence of Lemma 2.1.3 after localising around the degenerate critical point $(-1, 1)$ using a suitable modification of the partition of unity $\{\chi_n\}_{n\in\mathbb{N}_0}$. □

All together we recover the following standard bounds for the K-Bessel function.

Corollary 2.1.7.

$$\frac{K_{\frac{k}{2}+is}(y)}{\Gamma(\frac{k+1}{2}+is)} \ll_{k,A} \begin{cases} y^{-\frac{k}{2}} \min(s^{-\frac{1}{3}}, s^{-\frac{1}{4}} |s-y|^{-\frac{1}{4}}) & \text{if } 0 < y < s + cs^{\frac{1}{3}}, \\ y^{-\frac{k}{2}} |y-s|^{-A} & \text{if } y \geq s + cs^{\frac{1}{3}}. \end{cases}$$

Proof. According to (2.1.8) we have

$$\frac{K_{\frac{k}{2}+is}(y)}{\Gamma(\frac{k+1}{2}+is)} \ll_k y^{-\frac{k}{2}} \left| I_k(s, \frac{y}{s}) \right|.$$

If $y > s$, then Lemma 2.1.4 yields

$$\frac{K_{\frac{k}{2}+is}(y)}{\Gamma(\frac{k+1}{2}+is)} \ll_k y^{-\frac{k}{2}} (y-s)^{-A}.$$

In the range $s - cs^{\frac{2}{3}+\delta} \leq y \leq s + cs^{\frac{1}{3}+\delta}$ we refer to Lemma 2.1.6 and obtain the upper bound $y^{-\frac{k}{2}} s^{-\frac{1}{3}}$. Lemma 2.1.5 produces the desired bounds in the range $c' s^{\frac{8}{9}} \leq y \leq s - cs^{\frac{2}{3}+\delta}$. It remains to show that

$$\frac{K_{\frac{k}{2}+is}(y)}{\Gamma(\frac{k+1}{2}+is)} \ll_k s^{-\frac{1}{2}} \text{ for } 0 < y \leq c' s^{\frac{8}{9}}.$$

This follows from [38, Proposition 7.2] and Stirling's approximation for the Γ-function. □

Recently similar bounds have been established using the method of steepest descent in [84]. Furthermore, if $k = 0$, these bounds are well known and commonly used in analytic number theory. See [83, (3.1)] as an example.

2.2 THE WHITTAKER MODEL FOR REPRESENTATIONS OF $\mathrm{GL}_2(\mathbb{R})$

In this section we will give precise descriptions of special functions that appear in the Whittaker models of irreducible representations of $GL_2(\mathbb{R})$.

Let χ_1 and χ_2 be two characters given by

$$\chi_i \colon \mathbb{R}^\times \to \mathbb{C}, \quad x \mapsto \chi_i(x) = |x|_{\mathbb{R}}^{s_i} \operatorname{sgn}(x)^{m_i} \quad \text{for } i = 1, 2.$$

We construct the space

$$\mathcal{B}(\chi_1, \chi_2) = \Bigg\{ f \colon G(\mathbb{R}) \to \mathbb{C} \text{ right } \mathrm{SO}_2 \text{ finite} \colon \\ f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g \right) = \chi_1(a)\chi_2(d) \left| \frac{a}{d} \right|^{\frac{1}{2}} f(g) \Bigg\}.$$

The group $G(\mathbb{R})$ acts on this function space and its completion by right translation. The $G(\mathbb{R})$-representation arising in this way will be denoted by $\rho(\chi_1, \chi_2)$. Note that usually one considers $(\mathfrak{g}, K)$-modules or representation of the Hecke algebra in place of $G(\mathbb{R})$-representations. By abuse of notation we will not distinguish between these here, this is because for our purpose the naive perspective suffices.

By [53, Theorem 5.11] we know that $\rho(\chi_1, \chi_2)$ is irreducible unless $\chi_1\chi_2^{-1} = |\cdot|^{it}\,\mathrm{sgn}^m$ for $it \in \mathbb{Z}$ and $m \equiv 1(2)$. If $\rho(\chi_1, \rho_2)$ is irreducible, we write $\chi_1 \boxplus \chi_2$ for any representation equivalent to it. On the other hand, if $\chi_1\chi_2^{-1} = |\cdot|^{it}\,\mathrm{sgn}^m$ for $it \in \mathbb{N}_0$ and $m \equiv 1(2)$, then $\mathcal{B}(\chi_1, \chi_2)$ has a unique invariant subspace $\mathcal{B}_s(\chi_1, \chi_2)$, which is infinite dimensional. The restriction of $\rho(\chi_1, \chi_2)$ to $\mathcal{B}_s(\chi_1, \chi_2)$ and all equivalent representations will be denoted by $\sigma(\chi_1, \chi_2)$. Note that, if $\chi_1\chi_2^{-1} = |\cdot|^{-it}\,\mathrm{sgn}^m$ for $it \in \mathbb{N}_0$ and $m \equiv 1(2)$, then the unique invariant subspace is finite dimensional. In this case $\sigma(\chi_1, \chi_2)$ denotes the representation on the resulting (infinite dimensional) quotient space.

The following classification is contained in [53, Theorem 5.11].

Theorem 2.2.1. *Every admissible infinite dimensional representation of $G(\mathbb{R})$ is of the form $\chi_1 \boxplus \chi_2$ or $\sigma(\chi_1, \chi_2)$ for some quasi characters χ_1, χ_2. The only equivalences in these families are*

$$\chi_1 \boxplus \chi_2 = \chi_2 \boxplus \chi_1$$

and

$$\sigma(\chi_1, \chi_2) = \sigma(\chi_2, \chi_1) = \sigma(sgn \cdot \chi_1, sgn \cdot \chi_2) = \sigma(sgn \cdot \chi_2, sgn \cdot \chi_1).$$

We write $it = s_1 - s_2$, $m = m_1 - m_2$ and $is = s_1 + s_2$. Note that the principal series representation $\chi_1 \boxplus \chi_2$ is unitary if and only if $t, s \in \mathbb{R}$ or $s \in \mathbb{R}, it \in (-1, 1)$. Similarly, the (limits of) discrete series representations $\sigma(\chi_1, \chi_2)$ are unitarisable if and only if $s \in \mathbb{R}$. Further, the discrete series representations are square integrable and the principal series with $t, s \in \mathbb{R}$ are tempered.

We start by looking at special elements in the space $\mathcal{B}(\chi_1, \chi_2)$. For $k \equiv m \mod 2$ let $f_k \in \mathcal{B}(\chi_1, \chi_2)$ be defined by $f_k|_K(k(\theta)) = \left(e^{i\theta}\right)^k$. The Peter-Weyl theorem together with the Iwasawa decomposition implies

$$\mathcal{B}(\chi_1, \chi_2) = \bigoplus_{k \equiv m(2)} \mathbb{C} f_k.$$

Identifying K with S^1 in the obvious way leads to

$$f_k|_K(z) = z^k, \text{ for } z \in S^1.$$

By the definition of the induced representation one has

$$f(z(\pm 1)n(x)a(y)k) = (-1)^m y^{\frac{1}{2}+s_1} f(k),$$

for $x \in \mathbb{R}$, $y \in \mathbb{R}_+$ and $k \in K$. We associate a Whittaker function to f via the Jacquet integral

$$W_k(\underbrace{n(x_0)a(y)k(\theta)}_{=g}) = \int_N f_k(\omega n g)\overline{\psi}(n)dn.$$

Note that for irreducible principal series representations $(W_k)_{k\in m+2\mathbb{Z}}$ forms a natural basis of the space $\mathcal{W}(\chi_1 \boxplus \chi_2; \psi)$.

In order to compute W_k we observe that $\omega n(x)a(y) = z(y)a(y)^{-1}\omega n(\frac{x}{y})$. A simple change of variables gives

$$W_k(n(x_0)a(y)k(\theta)) = e^{2\pi i x_0}\mathrm{sgn}(y)^m |y|^{1+is} \int_{\mathbb{R}} e^{-2\pi i x y} f_k(a(y)^{-1}\omega n(x)k(\theta))\frac{dx}{\sqrt{2\pi}}.$$

Next we note that

$$\omega n(x) = \underbrace{\begin{pmatrix} 1 & \frac{-x}{x^2+1} \\ 0 & 1 \end{pmatrix}}_{=n(\cdot)} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{x^2+1}} & 0 \\ 0 & \sqrt{x^2+1} \end{pmatrix}}_{=z(\sqrt{x^2+1})a\left(\frac{1}{x^2+1}\right)} \underbrace{\begin{pmatrix} \frac{-x}{\sqrt{x^2+1}} & \frac{1}{\sqrt{x^2+1}} \\ \frac{-1}{\sqrt{x^2+1}} & \frac{-x}{\sqrt{x^2+1}} \end{pmatrix}}_{=k\left(\arg\left(\frac{-x+i}{\sqrt{x^2+1}}\right)\right)}.$$

Projecting on the K-component produces the map

$$\begin{aligned} \kappa(\omega n(\cdot))\colon \mathbb{R} &\to S^1 \\ x &\mapsto \frac{-x+i}{\sqrt{x^2+1}} = e^{i2\tan^{-1}(\sqrt{x^2+1}+x)}. \end{aligned}$$

Therefore

$$f_k|_K(\kappa(\omega n(x))) = \left(\frac{-x+i}{\sqrt{x^2+1}}\right)^k = e^{i2k\tan^{-1}(\sqrt{x^2+1}+x)} \in S^1.$$

We chose this particular inverse trigonometric function so that our phase will be smooth on $\mathbb{R}$ and corresponds to our parametrisation of S^1 with respect to angles between 0 and π. Thus

$$\begin{aligned} &W_k(n(x_0)a(y)k(\theta)) \\ &\quad= e^{2\pi i x_0}e^{ik\theta}\mathrm{sgn}(y)^{m_2}|y|^{\frac{1}{2}+s_2}\int_{\mathbb{R}}(x^2+1)^{-\frac{1}{2}-\frac{it}{2}}e^{-2\pi i x y}e^{i2k\tan^{-1}(\sqrt{x^2+1}+x)}\frac{dx}{\sqrt{2\pi}} \\ &\quad= e^{2\pi i x_0}e^{ik\theta}\mathrm{sgn}(y)^{m_2}|y|^{\frac{1}{2}+s_2}\int_{\mathbb{R}}\frac{(-x+i)^k}{(x^2+1)^{\frac{1}{2}+\frac{it}{2}+\frac{k}{2}}}e^{-2\pi i x y}\frac{dx}{\sqrt{2\pi}} \qquad (2.2.1) \\ &\quad= (2\pi)^{-\frac{1}{2}}e^{2\pi i x_0}e^{ik\theta}\mathrm{sgn}(y)^{m_2}|y|^{\frac{1}{2}+s_2} I(y;k,t). \end{aligned}$$

We further analyse this integral, as it is useful to have expressions connecting W_k to classical special functions. We observe that

$$\frac{(-x+i)^k}{(x^2+1)^{\frac{1}{2}+\frac{it}{2}+\frac{k}{2}}} = i^k(1+ix)^{\frac{k}{2}-\frac{1}{2}-\frac{it}{2}}(1-ix)^{-\frac{k}{2}-\frac{1}{2}-\frac{it}{2}}.$$

Thus, for $\Re(it) \geq 0$, we can apply[2] [36, 3.384.(9)] and get

$$W_k(a(y)) = \begin{cases} y^{\frac{is}{2}} \frac{i^k \pi^{\frac{it}{2}}}{\sqrt{2}\Gamma(\frac{1+k+it}{2})} W_{\frac{k}{2},\frac{it}{2}}(4\pi y) & \text{if } y > 0, \\ \mathrm{sgn}(y)^{m_2} |y|^{\frac{is}{2}} \frac{i^k \pi^{\frac{it}{2}}}{\sqrt{2}\Gamma(\frac{1-k+it}{2})} W_{-\frac{k}{2},\frac{it}{2}}(4\pi |y|) & \text{if } y < 0. \end{cases}$$

Up to constant this agrees with [66, (76)]. Further, if $t \in \mathbb{R}$, we observe that

$$W_0(a(y)) = \mathrm{sgn}(y)^{m_2} \frac{\sqrt{2}\pi^{\frac{it}{2}}}{\Gamma(\frac{1+it}{2})} |y|^{\frac{1}{2}+\frac{is}{2}} K_{\frac{it}{2}}(2\pi |y|).$$

On the other hand, if $it = k-1 \in \mathbb{N}$, then

$$W_k(a(y)) = \begin{cases} \frac{(2\pi)^{k-\frac{1}{2}} i^k}{\Gamma(k)} y^{\frac{k}{2}+\frac{is}{2}} e^{-2\pi y} & \text{if } y > 0, \\ 0 & \text{if } y < 0. \end{cases}.$$

Note that up to normalisation the last two examples are exactly the functions that appear in the classical Fourier expansion of Maaß forms and holomorphic modular forms respectively. Due to the classification of irreducible representations, Theorem 2.2.1, we have computed a basis for the Whittaker space for every infinite dimensional irreducible admissible representation of $G(\mathbb{R})$.

Our next goal is to check that, for irreducible, tempered, principal series $\chi_1 \boxplus \chi_2$, W_k is essentially L^2-normalised. To do so we use the classical Plancherel theorem, exploiting square integrability of $W_k(\cdot)$ for unitary principal series. Indeed we define

$$g_{k,it}(x) = \frac{(-x+i)^k}{(x^2+1)^{\frac{1}{2}+\frac{it+k}{2}}}.$$

Then (2.2.1) implies that $W_k(a(y)) = [\mathcal{F}(g_{k,it})](y)$. Thus we compute

$$\begin{aligned} \|W_k\|_2^2 &= \int_{\mathbb{R}^\times} W_k(a(y))\overline{W_k(a(y))} d^\times y = \int_{\mathbb{R}} [\mathcal{F}(g_{k,it})](y)\overline{[\mathcal{F}(g_{k,it})](y)} \frac{dy}{\sqrt{2\pi}} \\ &= \int_{\mathbb{R}} g_{k,it}(x)\overline{g_{k,it}(x)} \frac{dx}{\sqrt{2\pi}} = \int_{\mathbb{R}} (x^2+1)^{-1} \frac{dx}{\sqrt{2\pi}} = \sqrt{\frac{\pi}{2}}. \end{aligned}$$

We have shown the following lemma.

2 After analytic continuation of the integral representation to $\Re(\mu+\nu) = \frac{1}{2}$.

Lemma 2.2.2. *For any $k \in \mathbb{N}_0$ and any $s, t \in \mathbb{R}_{\geq 0}$ we have*

$$\|W_k\|_2 = \left(\frac{\pi}{2}\right)^{\frac{1}{4}}.$$

Furthermore, if $it = k-1$ one has

$$\|W_k\|_2^2 = \frac{\pi^{k-1}}{\sqrt{2}\Gamma(k)}.$$

The second part is a direct computation which we omit.

It is possible to obtain a complete asymptotic description of the size of W_k using the method of stationary phase and (2.2.1). Because this is not directly relevant to the rest of this thesis we will not pursue this here.

2.3 THE WHITTAKER MODEL FOR REPRESENTATIONS OF $\mathrm{GL}_2(\mathbb{C})$

Given two characters

$$\chi_i(z) = |z|^{2s_i}\left(\frac{z}{|z|}\right)^{m_i}, \quad i = 1, 2 \tag{2.3.1}$$

we associated the induced representation $\rho(\chi_1, \chi_2)$ of $G(\mathbb{C})$ acting on the completion of

$$\mathcal{B}(\chi_1, \chi_2) = \left\{ f\colon G(\mathbb{C}) \to \mathbb{C} \text{ right } \mathrm{SU}_2(\mathbb{R}) \text{ finite}: \right.$$
$$\left. f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi_1(a)\chi_2(d)\left|\frac{a}{d}\right| f(g)\right\}$$

by right translation. Note that $\rho(\chi_1, \chi_2)$ is irreducible unless $\chi_1\chi_2^{-1} = z^p\overline{z}^q$ for $p, q \in \mathbb{Z}$ and $pq > 0$. If $\rho(\chi_1, \chi_2)$ is irreducible, we denote it by $\chi_1 \boxplus \chi_2$. Again, by abuse of notation, this stands for the representation of $G(\mathbb{C})$, all equivalent representations, the representation of the Hecke algebra, and for the associated $(\mathfrak{g}, K)$-module.

Theorem 2.3.1. *Every infinite dimensional admissible irreducible representation of $G(\mathbb{C})$ is of the form $\chi_1 \boxplus \chi_2$ for two quasi characters χ_1, χ_2 of $\mathbb{C}^\times$.*

This is [53, Theorem 6.2]. As in the real case we define the numbers

$$it = s_1 - s_2, \quad is = s_1 + s_2 \text{ and } m = m_1 - m_2.$$

In order to find a convenient basis for the Whittaker space we recall some basic representation theory of $\mathrm{SU}(2)$. All irreducible representations of $\mathrm{SU}(2)$ are uniquely determined (up to equivalence) by dimension. We can model the irreducible representation of dimension $n+1$ on,

$$V_n = \bigoplus_{\substack{|q|\leq \frac{n}{2},\\ q\equiv \frac{n}{2}(1)}} \mathbb{C}P_q \text{ for } P_q = X^{\frac{n}{2}-q}Y^{\frac{n}{2}+q},$$

the space of homogeneous polynomials of degree n. The action σ_n is given by left translation. The invariant inner product is determined by

$$\langle P_q, P_p\rangle_{V_n} = \delta_{p=q}(\frac{n}{2}-q)!(\frac{n}{2}+q)!.$$

We define the matrix coefficients

$$\Phi^n_{p,q}(k) = \langle \sigma_n(k)P_p, P_q\rangle_{V_n}.$$

An element $f \in \mathcal{B}(\chi_1,\chi_2)$ is uniquely determined by its restriction to SU(2) and $f|_{\mathrm{SU}(2)} \in L^2(\mathrm{SU}(2))$. According to the Peter-Weyl theorem the set

$$\left\{f_{n,p} \in \mathcal{B}(\chi_1,\chi_2)\colon f_{n,p}|_{SU(2)} = \Phi^n_{p,-\frac{m}{2}}, n\equiv m(2), n\geq m, |p|\leq \frac{n}{2} \text{ and } 2p\equiv n(2)\right\}$$

spans $\mathcal{B}(\chi_1,\chi_2)$. We will denote the Whittaker function associated to $f_{n,p}$ via the Jacquet-integral by $W_{n,p}$.

Lemma 2.3.2. *We have*

$$\begin{aligned} W_{n,p}(a(y)) \;=\; & y^{1+2s_2}(\frac{n-m}{2})!(\frac{n+m}{2})! \sum_{l=\max(\frac{m}{2}-p,0)}^{\min(\frac{n+m}{2},\frac{n}{2}-p)} (-1)^{\frac{n}{2}-p-l}\binom{\frac{n}{2}-p}{l}\binom{\frac{n}{2}+p}{\frac{n+m}{2}-l} \\ & \cdot\Bigg[(2\pi y)^{-p+\frac{m}{2}}\frac{\Gamma(1+l)\Gamma(it+\frac{n}{2}-l)}{2\Gamma(1+\frac{n}{2}+it)\Gamma(1-p+\frac{m}{2})} \\ & \quad \cdot {}_1F_2(1+l;1+l-\frac{n}{2}-it;1-p+\frac{m}{2};4\pi^2y^2) \\ & +(2\pi y)^{2it+n-p-2l+\frac{m}{2}}\frac{\Gamma(l-\frac{n}{2}-it)}{2\Gamma(1+\frac{n+m}{2}-l-p+it)} \\ & \quad \cdot {}_1F_2(1+\frac{n}{2}+it;1+\frac{n+m}{2}-l-p+it,1+\frac{n}{2}-l+it;4\pi^2y^2)\Bigg], \end{aligned}$$

for $y>0$.

Note that this determines $W_{n,p}$ on all of $G(\mathbb{C})$ due to its transformation properties. However, due to the complexity of SU(2) this is not as clean as in the real case. In the spherical case, $n=p=0$, a similar computation appeared in [3].

Proof. We have

$$\begin{aligned} W_{n,p}(a(y)) &= \int_{\mathbb{C}} f_{n,p}(\omega n(z)a(y))e(-2\Re(z))\frac{dz}{2\pi} \\ &= y^{1+2s_2}\int_{\mathbb{C}} f_{n,p}(\omega n(z))e(-2y\Re(z))\frac{dz}{2\pi}. \end{aligned}$$

Recall that

$$\omega n(z) = n(\star)\begin{pmatrix} \frac{1}{\sqrt{|z|^2+1}} & 0 \\ 0 & \sqrt{|z|^2+1} \end{pmatrix} k\left[\frac{-\overline{z}}{\sqrt{|z|^2+1}}, \frac{1}{\sqrt{|z|^2+1}}\right].$$

Furthermore, writing z in polar coordinates reveals

$$k\left[\frac{-\overline{z}}{\sqrt{|z|^2+1}}, \frac{1}{\sqrt{|z|^2+1}}\right] = k[e^{-i\frac{\theta+\pi}{2}}, 0]k\left[\frac{r}{\sqrt{r^2+1}}, \frac{1}{\sqrt{r^2+1}}\right]k[e^{-i\frac{\theta+\pi}{2}}, 0].$$

Exploit that

$$\Phi^n_{p,-\frac{m}{2}}(k[e^{-i\frac{\theta+\pi}{2}}, 0]hk[e^{-i\frac{\theta+\pi}{2}}, 0]) = e^{-i(\theta+\pi)(-p+\frac{m}{2})}\Phi^n_{p,-\frac{m}{2}}(h)$$

leads to

$$W_{n,p}(a(y)) = y^{1+2s_2}\int_0^\infty \frac{rf_{n,p}(k[\frac{r}{\sqrt{r^2+1}}, \frac{1}{\sqrt{r^2+1}}])}{(r^2+1)^{1+2s_1-2s_2}}\int_0^{2\pi} e^{-i(\theta+\pi)(-p+\frac{m}{2})-4\pi yri\cos(\theta)}\frac{d\theta dr}{2\pi}.$$

The θ-integral can be computed as follows. First, we write

$$I = \int_0^{2\pi} e^{-i(\theta+\pi)(-p+\frac{m}{2})-4\pi yri\cos(\theta)}\frac{d\theta}{2\pi} = 2\int_0^\pi \cos(\theta(-p+\frac{m}{2}))e^{-4\pi yri\cos(\theta)}\frac{d\theta}{2\pi} \quad (2.3.2)$$

Applying [36, 3.715.(13)] gives

$$\Re(I) = -2\int_0^\pi \cos(\theta(-p+\frac{m}{2}))\sin(4\pi yri\cos(\theta))\frac{d\theta}{2\pi} = \sin(\frac{\pi}{2}(p-\frac{m}{2}))J_{-p+\frac{m}{2}}(4\pi yr).$$

Similarly, by using [36, 3.715.(18)], we get

$$\Im(I) = 2\int_0^\pi \cos(\theta(-p+\frac{m}{2}))\cos(4\pi yri\cos(\theta))\frac{d\theta}{2\pi} = \cos(\frac{\pi}{2}(p-\frac{m}{2}))J_{-p+\frac{m}{2}}(4\pi yr).$$

Combing real and imaginary part shows

$$I = i^{p-\frac{m}{2}}J_{-p+\frac{m}{2}}(4\pi yr).$$

The matrix coefficient can be evaluated using the binomial expansion as follows.

$$\begin{aligned} &f_{n,p}\left(k[\frac{r}{\sqrt{r^2+1}}, \frac{1}{\sqrt{r^2+1}}]\right) \\ &\quad = (\frac{n-m}{2})!(\frac{n+m}{2})!\frac{r^{p-\frac{m}{2}}}{(r^2+1)^{\frac{n}{2}}}\sum_{l=\max(\frac{m}{2}-p,0)}^{\min(\frac{n+m}{2},\frac{n}{2}-p)}(-1)^{\frac{n}{2}-p-l}\binom{\frac{n}{2}-p}{l}\binom{\frac{n}{2}+p}{\frac{n+m}{2}-l}r^{2l}. \end{aligned}$$

Finally, we compute the r-integral using [36, p. 6.565.8] with $\rho = 2 + p + 2l - \frac{m}{2}$, $\nu = p - \frac{m}{2}$, $a = 4\pi y$, $k = 1$, and $\mu = it + \frac{n}{2}$. One checks that all conditions are satisfied as long as $\Re(it) > -\frac{3}{4}$. We obtain

$$\begin{aligned}
&\int_0^\infty \frac{r^{1+p-\frac{m}{2}+2l}}{(r^2+1)^{1+it+\frac{n}{2}}} J_{-p+\frac{m}{2}}(4\pi y r)dr \qquad (2.3.3)\\
&= (2\pi y)^{-p+\frac{m}{2}} \frac{\Gamma(1+l)\Gamma(it+\frac{n}{2}-l)}{2\Gamma(1+\frac{n}{2}+it)\Gamma(1-p+\frac{m}{2})} \cdot {}_1F_2(1+l; 1+l-\frac{n}{2}-it; 1-p+\frac{m}{2}; 4\pi^2y^2)\\
&\quad + (2\pi y)^{2it+n-p-2l+\frac{m}{2}} \frac{\Gamma(l-\frac{n}{2}-it)}{2\Gamma(1+\frac{n+m}{2}-l-p+it)}\\
&\qquad \cdot {}_1F_2(1+\frac{n}{2}+it; 1+\frac{n+m}{2}-l-p+it, 1+\frac{n}{2}-l+it; 4\pi^2y^2).
\end{aligned}$$

Inserting this in the l-sum gives the result. □

In several cases this complex formula simplifies considerably.

Lemma 2.3.3. *We have*

$$W_{m,p}(a(y)) = \frac{m!(2\pi)^{it+\frac{m}{2}}}{\Gamma(1+\frac{m}{2}+it)} y^{1+\frac{m}{2}+is} K_{p+it}(4\pi y)$$

for all $|p| \leq \frac{m}{2}$, $2p \equiv m(2)$. *Furthermore,*

$$W_{n,\pm\frac{n}{2}}(a(y)) = (-1)^{\frac{n+m}{2}} \frac{n!(2\pi)^{it+\frac{n}{2}}}{\Gamma(1+\frac{n}{2}+it)} y^{1+\frac{n}{2}+is} K_{\mp\frac{m}{2}+it}(4\pi y)$$

for all $n > m$, $n \equiv m(2)$.

This agrees with the results given in [21, Section 4.2].

Proof. The additional assumptions imply that the l-sum contains exactly one element. Evaluating the r-integral in (2.3.3) using [36, 6.565.(4)] leads to the desired formula. □

We record the following lemma for later reference.

Lemma 2.3.4. *If the representation* π *is spherical, we must have* $m = 0$. *The spherical element is given by*

$$|W_{0,0}(a(z))| = \frac{|y|\, K_{it}(4\pi y)}{|\Gamma(1+it)|}.$$

It satisfies

$$\|W_{0,0}\|_2 = \pi^{-2} 2^{-\frac{5}{2}}.$$

3

THE CASE OF NON-ARCHIMEDEAN FIELDS

We turn to the ultrametric situation. Thus we assume that $(F, |\cdot|)$ is local field satisfying the strong triangle inequality

$$|x + y| \leq \max(|x|, |y|).$$

We further restrict ourselves to characteristic 0. Thus, F is a finite extension of $\mathbb{Q}_p$ for some p.

After recalling the p-adic method of stationary phase we will start a detailed analysis of the Whittaker model. In contrast to the archimedean situation we use newform theory to provide a suitable basis for the Whittaker space. As a consequence we can exploit the connection to the local L-factors instead of the representation theory of K.

Philosophically the formula for the new vector obtained by inverting the local zeta integral can be thought of as a p-adic Mellin-Barnes type representation. However, in order to apply the method of stationary phase we need a suitable Fourier type integral. The transition between the two representations requires some additional work.

The final outcome of this highly technical section are precise upper bounds for the Whittaker new vector. We cover all of $\mathrm{GL}_2(F)$ in suitable coordinates and all possible tempered, unitary, irreducible representations π. The results are summarised in Section 3.4.4.

This part is heavily based on the paper [4] by the author. However, we give significatly more background and go into greater detail here. In the stationary phase estimates later on we go beyond the original source [4] by treating arbitrary local fields of odd residual characteristic and giving more explicit evaluations.

3.1 THE p-ADIC METHOD OF STATIONARY PHASE

In this section we introduce the p-adic method of stationary phase. This is a well established formalism used to evaluate or estimate complete exponential sums. However, it is our point of view that such sums should be written as p-adic oscillatory integrals. In that language the analogy between the archimedean method of stationary phase and the p-adic one becomes very striking. There are numerous references devoted to establishing the p-adic method of stationary phase as a versatile tool for the working analytic number theorist. Let us mention a few. An easy to use formula is given in [51]. A more technical but very convenient formulation, working with Taylor expansions of the phase, is given in [62]. Finally, a more geometrically version can be found in [32]. All the references mentioned so far deal essentially with sums of the form

$$\sum_{m\in\mathbb{Z}/p^n\mathbb{Z}} \Phi(m)e\left(\frac{f(m)}{p^n}\right),$$

for a weight $\Phi\colon \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{C}$ and a phase f. In the context of this work we encounter such sums in a different form. For a Schwartz-Bruhat function $\Phi \in \mathcal{S}(F^n)$ with $\operatorname{supp}(\Phi) \subset \mathfrak{o}^d$ and a phase $f\colon \operatorname{supp}(\Phi) \to \mathfrak{o}$ we write

$$S_f(\Phi;\lambda) = \int_{\mathfrak{o}^d} \Phi(\mathbf{x})\psi(\lambda f(\mathbf{x}))d\mathbf{x}.$$

This is an oscillatory integral completely analogous to the ones studied in archimedean situation. If $F = \mathbb{Q}_p$; ψ is the standard additive character; $\lambda = p^{-n}$; $d = 1$; Φ is constant on $a + p^n\mathbb{Z}_p$; and $f(a + p^n\mathbb{Z}_p) \subset f(a) + p^n\mathbb{Z}_p$, then

$$S_f(\Phi;\lambda) = p^{-n} \sum_{m\in\mathbb{Z}/p^n\mathbb{Z}} \Phi(m)e\left(-\frac{f(m)}{p^n}\right)$$

and we find ourselves in the classical setting.

As in the archimedean case the phase f and the weight Φ may depend on parameters. But the p-adic setting introduces certain subtleties when defining regularity properties of f. Nonetheless, definitions can be made precise and one can produce stationary phase estimates similar to the archimedean ones. In the interest of space we do not go into this here.

Instead we will mostly deal with $f \in \mathfrak{o}[x]$ or $f \in \mathfrak{o}[[x]]$. Anyway we will encounter situations featuring shadow- and light zone as well as the caustic locus. It is an interesting feature, originating from character orthogonality, that if there are no critical points we achieve asymptotic vanishing. On the other hand, if there are non-degenerate critical points, the integral can be evaluate in terms of the certain multidimensional Gauß sums. These are defined by

$$G(A\varpi^{-s}, B) = \int_{\mathfrak{o}^n} \psi({}^t x A x \varpi^{-s} + B\cdot x)dx \text{ for } A \in GL_n(\mathfrak{o}) \text{ and } B \in F^n$$

and play essentially the role of the Fresnel integral.

3.1.1 *Evaluation of Gauß sums*

In this section we will evaluate the multidimensional quadratic Gauß sum $G(A,B)$. This is an essential tool for the evaluation of exponential integrals. Throughout this section we assume that q is odd. This makes several computations slightly more convenient. For later applications we only need one and two dimensional Gauß sums. Therefore we will focus on these in some detail. We start by treating the one dimensional situation and then move on to the two dimensional case.

Lemma 3.1.1 ([18], Lemma 6)**.** *Let $\rho \in \mathbb{Z}$, $A \in \mathfrak{o}^\times$, and $B \in F$. Then, for q odd, we have*

$$G(A\varpi^{-\rho}, B) = \begin{cases} \min(q^{-\frac{\rho}{2}}, 1)\gamma_F(A,\rho)\psi(-\frac{\varpi^\rho B^2}{4A}) & \text{if } B \in \mathfrak{p}^{\min(-\rho,0)}, \\ 0 & \text{else,} \end{cases}$$

where

$$\gamma_F(A,\rho) = \begin{cases} \chi_F(A)\epsilon(\frac{1}{2},\chi_F) & \text{if } \rho > 0 \text{ is odd}, \\ 1 & \text{if } \rho \leq 0 \text{ or } \rho \text{ is even}. \end{cases}$$

Proof. Observe that, since the conductor of ψ is $\mathfrak{o}$, the case $\rho \leq 0$ reduces to a complete linear sum. For $\rho \geq 1$ we calculate

$$G(A\varpi^{-\rho}, B) \quad = \quad \sum_{x \in \mathfrak{o}/\mathfrak{p}^\rho} q^{-\rho}\psi(A\varpi^{-\rho}x^2 + Bx) \int_{\mathfrak{o}} \psi(B\varpi^\rho t)dt.$$

The last integral is 1 if $B \in \mathfrak{p}^{-\rho}$, otherwise it is 0. Thus we assume $B \in \mathfrak{p}^{-\rho}$. Completing the square yields

$$G(A\varpi^{-\rho}, B) \quad = \quad q^{-\rho}\psi\left(-\frac{\varpi^\rho B^2}{4A}\right) \sum_{x \in \mathfrak{o}/\mathfrak{p}^\rho} \psi\left(A\varpi^{-\rho}\left(x + \frac{\varpi^\rho B}{2A}\right)^2\right).$$

Note that due to our current assumption on B we have $\frac{\varpi^\rho B}{2A} \in \mathfrak{o}$. We shift the summation and obtain

$$G(A,B) = q^{-\rho}\psi\left(-\frac{\varpi^\rho B^2}{4A}\right) \sum_{x \in \mathfrak{o}/\mathfrak{p}^\rho} \psi(A\varpi^{-\rho}x^2).$$

We start the evaluation of the remaining sum with the special case $\rho = 1$:

$$\begin{aligned} G(A\varpi^{-1}, B) &= q^{-1}\psi\left(-\frac{\varpi B^2}{4A}\right)\left(\sum_{x \in (\mathfrak{o}/\mathfrak{p})^\times} \sum_{\substack{\chi \in \mathfrak{X}_1, \\ \chi^2 = 1}} \chi(x)\psi(A\varpi^{-1}x) + 1\right) \\ &= \psi\left(-\frac{\varpi B^2}{4A}\right)\zeta_F(1)^{-1} \sum_{\substack{\chi \in \mathfrak{X}_1, \\ \chi^2 = 1}} \int_{\mathfrak{o}^\times} \chi(x)\psi(A\varpi^{-1}x)d^\times x + q^{-1}\psi\left(-\frac{\varpi B^2}{4A}\right). \end{aligned}$$

By using the fact that $\{\chi \in \mathfrak{X}_1 \colon \chi^2 = 1\} = \{1, \chi_F\}$ we find

$$\begin{aligned} G(A\varpi^{-1}, B) &= \zeta_F(1)^{-1} G(A\varpi^{-1}, \chi_F)\psi\left(-\frac{\varpi B^2}{4A}\right) \\ &= q^{-\frac{1}{2}}\chi_F(A)\epsilon(\frac{1}{2}, \chi_F)\psi\left(-\frac{\varpi^{1-2n}B^2}{4A}\right). \end{aligned}$$

If $\rho > 1$, we see that

$$\begin{aligned} &G(A\varpi^{-\rho}, B) \\ &= q^{-\rho}\psi\left(-\frac{\varpi^\rho B^2}{4A}\right)\left(\sum_{x\in(\mathfrak{o}/\mathfrak{p}^\rho)^\times}\sum_{\substack{\chi\in\mathfrak{X}_\rho,\\ \chi^2=1}}\chi(x)\psi(A\varpi^{-\rho}x) + \sum_{x\in\mathfrak{o}/\mathfrak{p}^{\rho-1}}\psi(A\rho^{2-\rho}x^2)\right) \\ &= \psi\left(-\frac{\varpi^\rho B^2}{4A}\right)\zeta_F(1)^{-1}\sum_{\substack{\chi\in\mathfrak{X}_\rho,\\ \chi^2=1}}\int_{\mathfrak{o}^\times}\chi(x)\psi(A\varpi^{-\rho}x)d^\times x \\ &\qquad + q^{-\rho}\psi\left(-\frac{\varpi^\rho B^2}{4A}\right)\sum_{x\in\mathfrak{o}/\mathfrak{p}^{\rho-1}}\psi(A\rho^{2-\rho}x^2). \end{aligned}$$

Since q is odd, the only quadratic characters are still 1 and χ_F. Thus $\rho > 1$ implies

$$\int_{\mathfrak{o}^\times}\chi(x)\psi(A\varpi^{-\rho}x)d^\times x = G(A\varpi^{-\rho}, \chi) = 0.$$

The stated equality follows by a simple inductive argument. □

The evaluation of multi-dimensional Gauß sums can be reduced to the previous case by a suitable diagonalisation argument. This is made precise in [32, Proposition 1.3]. We explicate this procedure in the two dimensional case in the following lemma.

Lemma 3.1.2 ([4], Lemma 4.3)**.** *Let $A \in Mat_{2\times 2}(\mathfrak{o})$ be a symmetric matrix, $B \in F^2$, $\rho \in \{0, 1\}$, and let q be odd. Then*

$$\left|G\left(\frac{\varpi^{-\rho}}{2}A, B\right)\right| \leq \begin{cases} q^{-\frac{rk(A_\mathfrak{p})}{2}} & \text{if } \rho = 1,\ rk(A_\mathfrak{p}) \in \{1,2\} \text{ and } \varpi B \in \mathfrak{o}^2, \\ 1 & \text{if } \rho = 0 \text{ or } A_\mathfrak{p} = 0 \text{ and } B \in \mathfrak{o}^2, \\ 0 & \text{else} \end{cases}$$

where $A_\mathfrak{p}$ is the image of A in $A \in Mat_{2\times 2}(\mathfrak{o}/\mathfrak{p})$.

Proof. Since $\rho \in \{0, 1\}$, the quadratic Gauß sum depends only on $A_\mathfrak{p}$. If $\rho = 0$ or $A_\mathfrak{p} = 0$, then we are simply dealing with a linear sum and the statement is obvious. Therefore we assume $\rho = 1$ and $A_\mathfrak{p} \neq 0$ for the rest of the proof. Write

$$A_\mathfrak{p} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{for } a, b, c \in \mathfrak{o}/\mathfrak{p}.$$

If $a \neq 0$, we have

$$A_\mathfrak{p} = \begin{pmatrix} 1 & 0 \\ ba^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1}\det(A_\mathfrak{p}) \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}.$$

The obvious linear change of variables yields

$$\begin{aligned} G\left(\frac{\varpi^{-1}}{2}A, B\right) &= G\left(\frac{\varpi^{-1}}{2}\begin{pmatrix} a & 0 \\ 0 & a^{-1}\det(A_\mathfrak{p}) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -a^{-1}b & 1 \end{pmatrix} B\right) \\ &= G\left(\frac{\varpi^{-1}a}{2}, B_1\right) G\left(\frac{\varpi^{-1}\det(A_\mathfrak{p})}{2a}, B_2 - \frac{b}{a}B_1\right). \end{aligned}$$

Applying Lemma 3.1.1 to the remaining one dimensional Gauß sums gives

$$G\left(\frac{\varpi^{-1}}{2}A, B\right) = \begin{cases} \gamma(A_\mathfrak{p})\psi\left(-\frac{\omega B_1^2}{2a}\right) q^{-\frac{1}{2}} \\ \qquad \text{if } \det(A_\mathfrak{p}) = 0,\ B_1, B_2 \in \mathfrak{p}^{-1} \text{ and } B_2 - \frac{b}{a}B_1 \in \mathfrak{o}, \\ \gamma(A_\mathfrak{p})\psi\left(\frac{-\varpi}{2\det(A_\mathfrak{p})}\left(aB_1^2 - 2bB_1B_2 + cB_2^2\right)\right) q^{-1} \\ \qquad \text{if } \det(A_\mathfrak{p}) \in (\mathfrak{o}/\mathfrak{p})^\times, \text{ and } B_1, B_2 \in \mathfrak{p}^{-1}, \\ 0 \\ \qquad \text{else} \end{cases}$$

with

$$\gamma(A_\mathfrak{p}) = \begin{cases} \chi_F(\frac{a}{2})\epsilon(\frac{1}{2}, \chi_F) & \text{if } \mathrm{rk}(A_\mathfrak{p}) = 1, \\ \chi_F(\det(A_\mathfrak{p}))\epsilon(\frac{1}{2}, \chi_F)^2 & \text{if } \mathrm{rk}(A_\mathfrak{p}) = 2. \end{cases}$$

If $a = 0$ but $c \neq 0$, then the argument is essentially the same, one simply exchanges the roles of a and c as well as B_1 and B_2.

If $a = c = 0$, then we must have $b \neq 0$. Observing

$$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{b}{2} & 0 \\ 0 & \frac{b}{2} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

and making a linear change of variables yields

$$\begin{aligned} G\left(\frac{\varpi^{-1}}{2}A, B\right) &= G(\varpi^{-1}b, -B_1 + B_2)G(\varpi^{-1}b, B_1 + B_2) \\ &= \gamma(A_\mathfrak{p})\psi(\frac{-\varpi}{2b}(B_1^2 + B_2^2))q^{-1}. \end{aligned}$$

The bounds stated above are special cases of these explicit evaluations. □

3.1.2 *An explicit p-adic method of stationary phase and some tricks*

In this section we will finally introduce a version ot the p-adic method of stationary phase which will be used later on to evaluate the ramified Whittaker new vectors.

A useful tool to turn multiplicative oscillations in additive ones is the p-adic logarithm. The following result is well known and will be used multiple times in the upcoming computations.

Lemma 3.1.3 ([4], Lemma 4.1). *Let $e = e(F/\mathbb{Q}_p)$ be the absolute ramification index. We define $\kappa_F = \lceil \frac{e}{p-1} \rceil$. For a multiplicative character χ with $a(\chi) \geq \kappa_F$ there is $b_\chi \in \mathfrak{o}^\times$, uniquely determined modulo $\mathfrak{p}^{a(\chi)-\kappa_F}$, such that*

$$\chi(1 + z\varpi^{\kappa_F}) = \psi\left(\frac{b_\chi}{\varpi^{a(\chi)}} \log_F(1 + z\varpi^{\kappa_F})\right) \text{ for all } z \in \mathfrak{o}.$$

Furthermore, if $\frac{a(\chi)}{3} \leq \alpha \in \mathbb{N}$, then there is $b_\chi \in \mathfrak{o}^\times$ such that

$$\chi(1 + z\varpi^{\alpha}) = \psi\left(\frac{b_\chi}{\varpi^{a(\chi)}} \left(z\varpi^\alpha - \frac{z^2}{2}\varpi^{2\alpha}\right)\right) \text{ for all } z \in \mathfrak{o}.$$

In particular, if $\frac{a(\chi)}{2} \leq \alpha \in \mathbb{N}$, then

$$\chi(1 + z\varpi^{\alpha}) = \psi(zb_\chi\varpi^{\alpha - a(\chi)}) \text{ for all } z \in \mathfrak{o}.$$

Note that in the last two cases we do not make any assumption on κ_F.

The proof of this result can be found in [18, Section 2.3]. Next, we will recall some results on quadratic congruences.

Lemma 3.1.4 ([56], Lemma 9.6). *Let $a, b, c \in \mathfrak{o}$. We set*

$$\begin{aligned} S &= \{x \in \mathfrak{o}/\mathfrak{p}^n \colon ax^2 + bx + c \in \mathfrak{p}^n\}, \\ \Delta &= b^2 - 4ac = \Delta'\varpi^{\delta_0} \text{ for } \Delta' \in \mathfrak{o}^\times. \end{aligned}$$

If $v(a) = 0$, we have

$$S = \left\{-\frac{b}{2a} \pm \frac{Y}{2a}\varpi^\delta + \alpha\varpi^{n-\delta} \colon \alpha \in \mathfrak{o}/\mathfrak{p}^\delta\right\} \tag{3.1.1}$$

with

$$Y = \begin{cases} 0 & \text{if } \delta_0 \geq n, \\ Y_0 & \text{if } Y_0^2 = \Delta' \text{ and } \delta_0 < n \text{ is even,} \end{cases} \quad \text{and } \delta = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } \delta_0 \geq n, \\ \delta' & \text{if } \delta_0 = 2\delta' < n. \end{cases}$$

In particular,

$$\sharp S \leq 2q^{\delta}.$$

If $v(a) > 0$ and $v(b) = 0$, we have $\sharp S = 1$. Furthermore, the solution $x_0 \in S$ has valuation $v(c)$.

For $|\lambda| = q$ and $\mathfrak{p}$ periodic f the basic estimates for $S_f(\mathbb{1}_{\mathfrak{o}}; \varpi^{-1})$ rely on algebraic methods and are highly non-trivial. As pointed out in [4, Section 4], this case reduces to a complete exponential sum over the finite field $\mathfrak{o}/\mathfrak{p}$. In the one dimensional situation we have the following very strong bound due to Weil, [86]. Let $g(x) = \prod_{i=1}^{l}(x-\xi_i)^{a_i}$ be a rational function with coefficients in $\mathfrak{o}/\mathfrak{p}$. Furthermore, let χ be a multiplicative character of $(\mathfrak{o}/\mathfrak{p})^{\times}$ such that no a_i is a multiple of the order of χ. We have

$$\left| \sum_{\substack{x \in \mathfrak{o}/\mathfrak{p}, \\ x \neq \xi_i \text{ for } 1 \leq i \leq l}} \chi(g(x))\psi(\varpi^{-1}f(x)) \right| \leq (N+l-1)\sqrt{q}, \tag{3.1.2}$$

for each polynomial $f \in (\mathfrak{o}/\mathfrak{p})[x]$ of degree N satisfying $f(0) = 0$. For $d > 1$ we have to involve some heavy machinery and we need some additional assumptions. Let us assume that $f \in \mathfrak{o}[X_1, \ldots, X_d]$ has degree N_f co-prime to p. We write $N_{f,\mathfrak{p}}$ for the degree of the reduced polynomial $\tilde{f} \in (\mathfrak{o}/\mathfrak{p})[X_1, \ldots, X_d]$ and assume that the homogeneous part of degree $N_{f,\mathfrak{p}}$ defines a smooth projective hypersurface. Then [57, Example 19.(5)] yields

$$\left| S_f(\mathbb{1}_{\mathfrak{o}^d}; \varpi^{-1}) \right| \leq (N_{f,\mathfrak{p}} - 1)^d q^{-\frac{d}{2}}. \tag{3.1.3}$$

If $|\lambda| > q$, the situation is completely different because there are several elementary methods that can be used for the evaluation. These are parallel to the classical method of stationary phase. More precisely, one will split the integral in suitable pieces each of which can be expressed in terms of Gauß sums.

In the one dimensional situation over $F = \mathbb{Q}_p$ an estimate for $S_f(\mathbb{1}_{\mathbb{Z}_p}, \lambda)$ allowing very general phase functions f is given in [28, (5.3)]. We now translate this result in our setting.

Lemma 3.1.5 ([28], (5.3); [4], Lemma 4.4). *Let $F = \mathbb{Q}_p$ for $p > 2$. Furthermore, let f be a polynomial, with degree $d_{\mathfrak{p}} > 0$ modulo $\mathfrak{p}$. If $\tau = v(f')$ and every α solving the critical point congruence*

$$\varpi^{-\tau} f'(\alpha) \in \mathfrak{p} \tag{3.1.4}$$

has multiplicity less then M, then we have

$$\left|S_f(\mathbb{1}_{\mathbb{Z}_p};\varpi^{-m})\right| \leq (d_{\mathfrak{p}}-1)q^{-\frac{1}{M+1}(m-\tau)}$$

for all $m \geq \tau + 2$.

Points satisfying the congruence (3.1.4) are called *critical points*. As already mentioned in [4], there are different types of critical points. If they have multiplicity one, they are referred to as *non-degenerate critical points*. The contribution of non-degenerate critical points is well behaved. Indeed, if there are only such critical points, one can evaluate the corresponding oscillatory integral explicitly. On the other hand, if there are critical points with multiplicity bigger than one, the situation becomes more complicated. Such critical points are called *degenerate critical points*, their existence usually destroys square root cancellation in $S_f(\Phi;\lambda)$. Analogously to the archimedean case degenerate critical points are responsible for the appearance of new special functions. We define

$$\mathrm{Ai}_{\psi}(a;b) = q^{-\frac{v(a)}{3}}\int_{\mathfrak{o}}\psi(ax^3+bx)dx,$$

which is a p-adic version of the Airy function. Note that $\mathrm{Ai}_{\psi}(a;b) = 0$ if $v(b) < \min(0, v(a))$. In general we have the bound

$$|\mathrm{Ai}_{\psi}(a,b)| \leq 2.$$

The following lemma provides a very general device to treat p-adic oscillatory integrals. It is a good example for the usual approach taken to evaluate highly ramified (complete) exponential sums.

Lemma 3.1.6. *Suppose that for $m > \kappa \geq \frac{m}{3}$ we have*

$$f(\mathbf{x}+\varpi^{\kappa}\mathbf{t}) \in f(\mathbf{x}) + \varpi^{\kappa}\langle \mathbf{g}(\mathbf{x}),\mathbf{t}\rangle + \frac{\varpi^{2\kappa}}{2}{}^t\mathbf{t}A_{\mathbf{x}}\mathbf{t} + \mathfrak{p}^m \text{ for all } t \in \mathfrak{o}^d.$$

For some function $\mathbf{g}\colon \mathfrak{o}^d \to \mathfrak{o}^d$ and a matrix A. Further, assume that Φ is $\mathfrak{p}^{\kappa}$ periodic. Define $\Phi_{\chi_1\cdots\chi_d} = \Phi\cdot\prod_{i=1}^d \chi_i\circ pr_i$. We have

$$S_f(\Phi_{\chi_1,\cdots,\chi_d},\varpi^{-m}) = \zeta_F(1)^d q^{-d\kappa}\sum_{\substack{\mathbf{x}\in(\mathfrak{o}/\mathfrak{p}^{\kappa})^d,\\ \mathbf{g}(\mathbf{x})+\mathbf{h}(\mathbf{x})\in\mathfrak{p}}}\Phi(\mathbf{x})\chi_1(x_1)\ldots\chi_d(x_d)\psi(\varpi^{-m}f(\mathbf{x}))$$
$$\cdot G\left(\frac{\varpi^{2\kappa-m}}{2}(B_{\mathbf{x}}-A_{\mathbf{x}}),\varpi^{\kappa-m}(\mathbf{g}(\mathbf{x})+\mathbf{h}(\mathbf{x}))\right)$$

for

$$\mathbf{g}(\mathbf{x}) = \left(\frac{b_{\chi_i}}{x_i}\varpi^{m-a(\chi_i)}\right)_{1\leq i\leq d} \text{ and } B_{\mathbf{x}} = \mathrm{diag}\left(\frac{b_{\chi_1}}{x_1^2}\varpi^{m-a(\chi_1)},\ldots,\frac{b_{\chi_d}}{x_d^2}\varpi^{m-a(\chi_d)}\right).$$

Note that one can refine the congruence condition in the x-sum by evaluating the Gauß sum. In one dimension this is similar to [18, Lemma 7]. The proof is straight forward and left to the reader.

An important two dimensional example that lies at the heart of the upcoming sections is the integral

$$K(\chi_1\otimes\chi_2,(\varpi^{-l_1},\varpi^{-l_2}),v\varpi^{-l}) = \int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times}\chi_1(x_1)\chi_2(x_2)\psi(x_1\varpi^{-l_1}+x_2\varpi^{-l_2}+vx_1x_2\varpi^{-l})d^\times x_1 d^\times x_2$$

attached to the algebra $E=F\times F$.

Lemma 3.1.7 ([4], Lemma 4.6). *Suppose χ_1 and χ_2 are characters on $F^\times$ such that $a(\chi_1)\geq a(\chi_2)\geq 1$. Put $k=\max(a(\chi_1),l)=2r+\rho$ for some $r\in\mathbb{N}_0$ and $\rho\in\{0,1\}$. For $0<l_1,l_2\leq k$ and $r>0$ we have*

$$\begin{aligned} K(\chi_1\otimes\chi_2,&(\varpi^{-l_1},\varpi^{-l_2}),v\varpi^{-l}) \\ &= \zeta_F(1)^2q^{-2r}\sum_{(x_1,x_2)\in S}\chi_1(x_1)\chi_2(x_2)\psi(x_1\varpi^{-l_1}+x_2\varpi^{-l_2}+vx_1x_2\varpi^{-l}) \\ &\qquad\cdot G\left(\frac{\varpi^{-\rho}}{2}A_{x_1,x_2},\varpi^{-r-\rho}B_{x_1,x_2}\right)\end{aligned}$$

for

$$\begin{aligned} A_{x_1,x_2} &= \begin{pmatrix} -b_1\varpi^{k-a(\chi_1)} & vx_1x_2\varpi^{k-l} \\ vx_1x_2\varpi^{k-l} & -b_2\varpi^{k-a(\chi_2)}\end{pmatrix}, \\ B_{x_1,x_2} &= \begin{pmatrix} b_1\varpi^{k-a(\chi_1)}+x_1\varpi^{k-l_1}+vx_1x_2\varpi^{k-l} \\ b_2\varpi^{k-a(\chi_2)}+x_2\varpi^{k-l_2}+vx_1x_2\varpi^{k-l}\end{pmatrix}, \\ S &= \{x_1,x_2\in(\mathfrak{o}/\mathfrak{p}^r)^\times : B_{x_1,x_2}\in(\mathfrak{p}^r)^2\}\end{aligned}$$

where b_1 and b_2 are the constants associated to the characters χ_1 and χ_2 using Lemma 3.1.3. In particular we have

$$\left|K(\chi_1\otimes\chi_2,(\varpi^{-l_1},\varpi^{-l_2}),v\varpi^{-l})\right| \leq \zeta_F(1)^2q^{-2r}\sharp S\sup_{x_1,x_2\in S}\left|G\left(\frac{\varpi^{-\rho}}{2}A_{x_1,x_2},\varpi^{-r-\rho}B_{x_1,x_2}\right)\right|.$$

Proof. We use Lemma 3.1.3 to rewrite the integral as

$$\begin{aligned}K(\chi_1\otimes\chi_2,(\varpi^{-l_1},\varpi^{-l_2}),v\varpi^{-l}) \\ = \zeta_F(1)^2q^{-2r}\sum_{x_1,x_2\in(\mathfrak{o}/\mathfrak{p}^r)^\times}\chi_1(x_1)\chi_2(x_2)\psi(x_1\varpi^{-l_1}+x_2\varpi^{-l_2}+vx_1x_2\varpi^{-l}) \\ \cdot G\left(\frac{\varpi^{-\rho}}{2}A_{x_1,x_2},\varpi^{-r-\rho}B_{x_1,x_2}\right).\end{aligned}$$

Using the support properties of the Gauß sum contained in Lemma 3.1.2 it is clear that we can restrict the summation to S. □

3.1.3 *Evaluating twisted Kloosterman sums*

We will end this section by applying the theory developed so far to twisted Kloosterman sums (generalised Salié sums). These are oscillatory integrals of the form

$$S_\chi(A,B,m)=\int_{\mathfrak{o}^\times}\chi(x)\psi\left(\frac{Ax+Bx^{-1}}{\varpi^m}\right)d^\times x.$$

Because $\mathrm{Vol}(\mathfrak{o}^\times,d^\times)=1$, the trivial bound is 1. If $A,B\in\mathfrak{o}^\times$, $m=1$, and $a(\chi)\leq 1$ we have the stronger bound

$$|S_\chi(A,B,1)|\leq 2\zeta_F(1)q^{-\frac{1}{2}}. \tag{3.1.5}$$

This is essentially due to Weil. For a reference see [51, Chapter 11, Excercise 1]. We will now apply the method of stationary phase to the situations $m>1$. This has been studied very well. However, we could not locate a reference dealing with general sums $S_\chi(A,B,m)$ over arbitrary local fields F of characteristic 0. Classical references are [18, 27, 51, 56]. Furthermore, as the estimate of the K-Bessel function before, this is a good exercise to get used to the p-adic method of stationary phase.

Lemma 3.1.8. *Let q be odd, $m\geq 2$, $l\in\mathbb{N}_0$, $\chi\in\mathfrak{X}$ a multiplicative character, and $a\in\mathfrak{o}^\times$. If $m>a(\chi)$, then*

$$S_\chi(1,a\varpi^l,m)=\begin{cases}\zeta_F(1)q^{-\frac{m}{2}}\sum_\pm\gamma_F(\pm\sqrt{a},m)\chi(y_\pm)\psi\left(\frac{2a}{y_\pm}\varpi^{-m}-b_\chi\varpi^{-a(\chi)}\right) & \\ \qquad \text{if } l=0 \text{ and } a\in\mathfrak{o}^{2\times}, \\ 0 \qquad \text{else.}\end{cases}$$

Here $y_\pm$ are the two solutions of $y^2+yb_\chi\varpi^{m-a(\chi)}-a$ in $\mathfrak{o}^\times$. In the opposite situation, $m<a(\chi)$, we have

$$S_\chi(1,a\varpi^l,m)=0.$$

If $m = a(\chi)$ and $l > 0$, then

$$S_\chi(1, a\varpi^l, m) = \zeta_F(1)\gamma_F(-2b_\chi, m)\chi(y_0)\psi((2y_0 + b_\chi)\varpi^{-m})q^{-\frac{m}{2}},$$

where y_0 is the unique solution to $y(y + b_\chi) = a\varpi^l$ in $\mathfrak{o}^\times$.

Finally, assume $m = a(\chi)$ and $l = 0$. Let $\Delta = b_\chi^2 + 4a$ and $y_\pm = -\frac{b_\chi}{2} \pm \frac{\sqrt{\Delta}}{2}$. If $\Delta \in \mathfrak{p}$ and $\lceil \frac{m}{4} \rceil \geq \kappa_F$, then

$$S_\chi(1, a\varpi^l, m) = \begin{cases} \zeta_F(1)q^{-\frac{m}{2}} \sum_\pm \gamma_F(\pm 2\sqrt{\Delta}, m)\chi(y_\pm)\psi\left(\left(\frac{2a}{y_\pm} - b_\chi\right)\varpi^{-m}\right) \\ \qquad \text{if } \Delta \in \mathfrak{o}^{2\times}, \\ \zeta_F(1)q^{-\frac{m}{2}+\frac{v(\Delta)}{4}} \sum_\pm \gamma_F(\pm(\Delta)_0^{\frac{1}{2}}, \frac{v(\Delta)}{2} + m)\chi(y_\pm)\psi\left(\left(\frac{2a}{y_\pm} - b_\chi\right)\varpi^{-m}\right) \\ \qquad \text{if } 0 < v(\Delta) < \lceil \frac{m}{2} \rceil \text{ even}, \\ \zeta_F(1)q^{-\frac{m}{3}}\chi(-\frac{b_\chi}{2})\psi(-\frac{\Delta}{2b_\chi}\varpi^{-m}) \\ \quad \cdot Ai_\psi\left(-\frac{4b_\chi^4}{3}\varpi^{3\lceil\frac{r+\rho}{2}\rceil - m}; (8b_\chi^2 - \Delta\varpi^{-2\lceil\frac{r+\rho}{2}\rceil})\varpi^{3\lceil\frac{r+\rho}{2}\rceil - m}\right) \\ \qquad \text{if } \lceil \frac{m}{2} \rceil \geq v(\Delta), \\ 0 \qquad \text{else.} \end{cases}$$

Proof. Let us start with the exceptional cases. Assume $m > a(\chi)$. Let us write $m = 2r + \rho$ for $r \geq 1$ and $\rho \in \{0, 1\}$. We will apply the method of stationary phase. Before we do so let us recall the geometric series

$$\frac{a\varpi^l}{y + t\varpi^r} = \frac{a\varpi^l}{y}\sum_{j\geq 0}\left(-\frac{t}{y}\varpi^r\right)^j,$$

which converges because $r \geq 1$. Thus, since $r \geq \frac{a(\chi)}{2}$, we use Lemma 3.1.3 to see that

$$\begin{aligned} S_\chi(1, a\varpi^l, m) &= \zeta_F(1)q^{-r} \sum_{y \in (\mathfrak{o}/\mathfrak{p}^r)^\times} \chi(y)\psi\left(\left(y + \frac{a\varpi^l}{y}\right)\varpi^{-m}\right) \\ &\qquad G\left(\frac{a}{y^3}\varpi^{l-\rho}, [y^2 + yb_\chi\varpi^{m-a(\chi)} - a\varpi^l]y^{-2}\varpi^{-r-\rho}\right). \end{aligned}$$

Before we can evaluate the quadratic Gauß sum using Lemma 3.1.1 we have to solve the congruence

$$y^2 + yb_\chi\varpi^{m-a(\chi)} - a\varpi^l \in \mathfrak{p}^r \text{ for } y \in (\mathfrak{o}/\mathfrak{p}^r)^\times.$$

If $l > 0$, then this has obviously no solution $y \in \mathfrak{o}^\times$. Otherwise the discriminant $\Delta = b_\chi^2\varpi^{2m-2a(\chi)} + 4a$ is a unit and according to Lemma 3.1.4 we have

$$y_\pm = -\frac{b_\chi}{2}\varpi^{m-a(\chi)} \pm (\Delta)_{\frac{1}{2}}.$$

Evaluating the Gauß sum gives the desired result.

If $m \leq a(\chi)$, we use a similar trick. Write $a(\chi) = 2r + \rho$ and apply Lemma 3.1.3. We arrive at

$$S_\chi(1, a\varpi^l, m) = \zeta_F(1) q^{-r} \sum_{y \in (\mathfrak{o}/\mathfrak{p}^r)^\times} \chi(y)\psi\left(\left(y + \frac{a\varpi^l}{y}\right)\varpi^{-m}\right)$$
$$\cdot G\left(\frac{a}{y^3}\varpi^{l+2r-m} - \frac{b_\chi}{2y^2}\varpi^{-\rho}, [y^2\varpi^{a(\chi)-m} + yb_\chi - a\varpi^{l+a(\chi)-m}]y^{-2}\varpi^{-r-\rho}\right).$$

Thus, as before, we have to evaluate the Gauß sum. Note that, if $m < a(\chi)$, the latter vanishes for all $y \in \mathfrak{o}^\times$. Therefore we assume $m = a(\chi)$ from now on. If $l > 0$, we look at the congruence

$$y(y + b_\chi) \in a\varpi^l + \mathfrak{p}^r,$$

which has one solution $y_0 \in \mathfrak{o}^\times$. Further, the quadratic term of the Gauß sum reduces to $-\frac{b_\chi}{2y_0^2}\varpi^{-\rho}$. We conclude this case by appealing to Lemma 3.1.1.

Finally, we have to deal with the possibly degenerate situation $a(\chi) = m$ and $l = 0$. We treat several cases according to the p-adic size of $\Delta = b_\chi^2 + 4a$. This is the discriminant of the quadratic equation $y^2 + yb_\chi - a = 0$.

Case I: $\Delta \in \mathfrak{o}^\times$**.** This is the non-degenerate situation. By Lemma 3.1.4 we have two solutions $y_\pm$ to the equation in question and we conclude using Lemma 3.1.1.

Case II: $0 < v(\Delta) < \lceil \frac{m}{2} \rceil$**.** In this case we note that $2a - y_\pm b_\chi \in \mathfrak{p}$, so that

$$S_\chi(1, a\varpi^l, m) = \zeta_F(1) q^{-r} \sum_{y \in S} \chi(y)\psi\left(\left(y + \frac{a\varpi^l}{y}\right)\varpi^{-m}\right),$$

where $S = \{y_\pm + \alpha\varpi^{r+\rho-\frac{v(\Delta)}{2}} : \alpha \in \mathfrak{o}/\mathfrak{p}^{\frac{v(\Delta)}{2}-\rho}\}$ is given by Lemma 3.1.4. Here we used the fact that S is empty if $v(\Delta)$ is odd. Observe that $r + \rho - \frac{v(\Delta)}{2} \geq \lceil \frac{m}{4} \rceil \geq \kappa_F$ and apply Lemma 3.1.3. This leads to

$$S_\chi(1, a\varpi^l, m) = \zeta_F(1) q^{-r} \sum_\pm \chi(y_\pm)\psi\left(\left(y_\pm + \frac{a\varpi^l}{y_\pm}\right)\varpi^{-m}\right)$$
$$\cdot \sum_{x \in \mathfrak{o}/\mathfrak{p}^{\frac{v(\Delta)}{2}-\rho}} \psi\left(\mp \frac{b_\chi\sqrt{(\Delta)_0}}{2y_\pm^3}x^2\varpi^{\rho-\frac{v(\Delta)}{2}} + \frac{a}{3y_\pm^4}x^3\varpi^{r+2\rho-\frac{3}{2}v(\Delta)}\right).$$

If $v(\Delta) = 2\rho > 0$, our current assumption implies $r > 1$ in which case the terms of the sum are trivial. For $v(\Delta) > 2\rho$ we can run another stationary phase argument. In both cases the remaining sums are easily evaluated and one obtains the result stated above.

Case III: $v(\Delta) \geq \lceil \frac{m}{2} \rceil$**.** In this case we use Lemma 3.1.4, 3.1.3, and 3.1.1 to see that

$$S_\chi(1, a\varpi^l, m) = \zeta_F(1) q^{-r} \chi(-\frac{b_\chi}{2}) \psi(-\frac{\Delta}{2b_\chi}\varpi^{-m})$$
$$\cdot \sum_{x \in \mathfrak{o}/\mathfrak{p}^{t-\lceil \frac{r+\rho}{2} \rceil}} \psi\left(-\frac{\Delta}{b_\chi^2} x\varpi^{\lceil \frac{r+\rho}{2} \lceil -m} - \frac{2\Delta}{b_\chi^3} x\varpi^{2\lceil \frac{r+\rho}{2} \lceil -m} - \frac{16ax^3}{3b_\chi^4}\varpi^{3\lceil \frac{r+\rho}{2} \rceil -m}\right).$$

By completing the cube we can express the remaining sum in terms of Ai_ψ. We get

$$S_\chi(1, a\varpi^l, m) = \zeta_F(1) q^{-r+r-\lceil \frac{r+\rho}{2} \rceil + \lceil \frac{r+\rho}{2} \rceil - \frac{m}{3}} \chi(-\frac{b_\chi}{2}) \psi(-\frac{\Delta}{2b_\chi}\varpi^{-m})$$
$$\cdot \mathrm{Ai}_\psi\left(-\frac{4b_\chi^4}{3}\varpi^{3\lceil \frac{r+\rho}{2} \rceil - m}; (8b_\chi^2 - \Delta\varpi^{-2\lceil \frac{r+\rho}{2} \rceil})\varpi^{3\lceil \frac{r+\rho}{2} \rceil - m}\right).$$

This completes the proof. □

Throughout the proof we have seen non-degenerate and degenerate critical points. In particular, the twisted Kloosterman sum features a transition region just as the K-Bessel function. If we view l, m and a as parameters, we can also classify their degeneracy. The caustic locus is given by

$$\mathcal{C} = \left\{ m = a(\chi),\ l = 0,\ a = -\frac{b_\chi^2}{4} \right\}.$$

The light zone is

$$\mathcal{L} = \{m > a(\chi),\ a \in \mathfrak{o}^{2\times}\} \cup \{m = a(\chi),\ l > 0,\ a \in \mathfrak{o}^\times\}$$
$$\cup \left\{ m = a(\chi),\ l = 0,\ a \notin -\frac{b_\chi^2}{4} + p \right\}.$$

The rest is made up by the shadow zone, where S_χ vanishes.

Remark 3.1.9. *As there are different types of Bessel functions there are also other types of Kloosterman sums. Let E be a quadratic extension of F and let d_H denote the Haar probability measure on the hypersurface $\mathrm{Nr}_{E/F}^{-1}(1)$. We define*

$$S_\xi(A|E) = \int_{\mathrm{Nr}_{E/F}^{-1}(1)} \xi(x)\psi_E(Ax) d_H x,$$

for a multiplicative character ξ and $A \in E$.

This function does not come out of thin air as we will encounter it in our analysis of the Whittaker new vector. However, this generalisation of the classical Kloosterman sum seem not to be standard in the literature. A complete evaluation of these sums would be interesting but is beyond the scope of this thesis. We will derive the estimates needed later on in an ad-hoc manner.

3.2 COMPUTING FINITE FOURIER COEFFICIENTS

This section lays the foundation for most of the computations in this chapter. We will explicitly compute the finite Fourier expansion of local Whittaker new vectors. In doing so we built on the circle of ideas introduced in [69]. Let π be an irreducible, admissible representation of $GL_2(F)$ and let W_π be a normalised ψ-Whittaker new vector.

The new vector is an element of fundamental importance. Indeed, the set of vectors

$$W_\pi(\cdot), \quad W_\pi(\cdot a(\varpi^{-1})), \quad W_\pi(\cdot a(\varpi^{-2})), \quad \ldots$$

spans the complete Whittaker space. This can be extracted from [26]. Furthermore, we have

$$W_\pi(g_{t,l,v}a(\varpi^{-i})) = W_\pi(g_{t-i,\min(0,i-l),v}).$$

We conclude that by understanding the new vector on the matrices $g_{t,l,v}$ we understand a very convenient basis of the whole space.[1]

One notes that the function

$$v \mapsto W_\pi(g_{t,l,v})$$

is well defined for $v \in \mathfrak{o}^\times/(1+\mathfrak{p}^l)$. This follows from the identity

$$g_{t,l,v(1+x\varpi^l)} = g_{t,l,v}n(vx).$$

Thus we can expand it in its finite Fourier expansion as follows.

$$W_\pi(g_{t,l,v}) = \sum_{\mu \in \mathfrak{X}_l} c_{t,l}(\mu)\mu(v). \tag{3.2.1}$$

The Fourier coefficients in question are the constants $c_{t,l}(\mu)$. They can be computed using the basic identity:

$$\begin{aligned} &\sum_{t=-\infty}^{\infty} q^{(t+a(\mu\pi))(\frac{1}{2}-s)} c_{t,l}(\mu) \\ &\quad = \omega_\pi(-1)\epsilon(\frac{1}{2},\mu\pi)^{-1}\frac{L(s,\mu\pi)}{L(1-s,\mu^{-1}\omega_\pi^{-1}\pi)}\sum_{a=0}^{\infty} W_\pi(a(\varpi^a))q^{-a(\frac{1}{2}-s)}G(\varpi^{a-l},\mu^{-1}). \end{aligned} \tag{3.2.2}$$

The proof of this formula, given in [69, Propositon 2.23], is valid for any $l \geq 0$ as long as $\omega_\pi(\varpi) = \mu(\varpi) = 1$.

1 In general it is still hard to understand arbitrary vectors as they can be complex linear combinations of our basis. In some cases there are more direct ways to compute some special vectors directly. One example of such are the minimal vectors studied in [48].

With the help of this identity we will give explicit expressions for the constants $c_{t,l}(\mu)$ defined above. The resulting representation of $W_\pi(g_{t,l,v})$ can be thought of as a p-adic Mellin-Barnes representation.

The upcoming calculations work in great generality. Indeed, we can handle any non-archimedean local field F, and any (not necessarily unitary) irreducible, admissible representation π with unitary central character $\omega_\pi \in \mathfrak{X}_n$. This section is organised in subsections, each of which deals with a particular type of $\mathrm{GL}_2(F)$-representation on its own. We closely follow the exposition in [4, Section 2]. Further we end this section with a summary adapted from [5].

3.2.1 *Supercuspidal representations*

Let π be a supercuspidal representation. Because $L(s, \mu\pi) = 1$, for all μ, the basic identity takes the simple form

$$\sum_{t=-\infty}^{\infty} q^{(t+a(\mu\pi))(\frac{1}{2}-s)} c_{t,l}(\mu) = \omega_\pi(-1)\epsilon(\frac{1}{2}, \mu\pi)^{-1} G(\varpi^{-l}, \mu^{-1}). \tag{3.2.3}$$

By comparing coefficients we arrive at

$$c_{t,l}(\mu) = \begin{cases} \omega_\pi(-1)\frac{G(\varpi^{-l},\mu^{-1})}{\epsilon(\frac{1}{2},\mu\pi)} & \text{if } t = -a(\mu\pi), \\ 0 & \text{else.} \end{cases} \tag{3.2.4}$$

Evaluating the Gauß sum yields

$$c_{t,l}(\mu) = \begin{cases} \epsilon(\frac{1}{2}, \omega_\pi^{-1}\pi) & \text{if } l = 0,\ t = -n, \text{ and } \mu = 1, \\ -\zeta_F(1)q^{-1}\epsilon(\frac{1}{2}, \omega_\pi^{-1}\pi) & \text{if } l = 1,\ t = -n, \text{ and } \mu = 1, \\ \zeta_F(1)q^{-\frac{l}{2}}\epsilon(\frac{1}{2}, \mu)\epsilon(\frac{1}{2}, \mu^{-1}\omega_\pi^{-1}\pi) & \text{if } \mu \in \mathfrak{X}_l',\ t = -a(\mu\pi), \text{ and } l > 0, \\ 0 & \text{else.} \end{cases}$$

These expressions essentially appeared in [69, Section 2.7].

3.2.2 *Twists of Steinberg*

If $\pi = \chi\mathrm{St}$, then the situation is slightly more complicated. This is because $L(s, \mathrm{St})$ is non-trivial.

Lemma 3.2.1 ([4], Lemma 2.1). *Let $l \in \mathbb{N}_0$ and $\mu \in \mathfrak{X}_l$. If $\pi = \chi St$ for $\chi \neq 1$, then the constants $c_{t,l}(\mu)$ are given by*

$$c_{t,l}(\mu) = \begin{cases} \epsilon(\frac{1}{2}, \mu^{-1}\omega_\pi^{-1}\pi)G(\varpi^{-l}, \mu^{-1}) & \text{if } \mu \neq \chi^{-1} \text{ and } t = -2a(\mu\chi), \\ q^{-1}G(\varpi^{-l}, \mu^{-1}) & \text{if } \mu = \chi^{-1} \text{ and } t = -2, \\ -\zeta_F(2)^{-1}q^{-1-t}G(\varpi^{-l}, \mu^{-1}) & \text{if } \mu = \chi^{-1} \text{ and } t > -2, \\ 0 & \text{else.} \end{cases}$$

If $\pi = St$, then we have

$$c_{t,l}(\mu) = \begin{cases} q^{a(\mu)-l}G(-\varpi^{-a(\mu)}, \mu) & \text{if } \mu \neq 1 \text{ and } t = -l - a(\mu), \\ -q^{-t-1} & \text{if } \mu = 1,\ l = 0, \text{ and } t \geq -1, \\ q^{-t-2l} & \text{if } \mu = 1,\ l \geq 1, \text{ and } t \geq -l, \\ -\zeta_F(1)q^{-l} & \text{if } \mu = 1,\ l \geq 1, \text{ and } t = -l-1, \\ 0 & \text{else.} \end{cases}$$

Proof. If $\chi \neq 1$ and $\mu \neq \chi^{-1}$, then the basic identity is as in (3.2.3). It is easy to compare coefficients.

We continue by considering $\chi \neq 1$ and $\mu = \chi^{-1}$. In this case we have

$$\sum_{t=-\infty}^{\infty} q^{(t+1)(\frac{1}{2}-s)}c_{t,l}(\chi^{-1}) = -\omega_\pi(-1)\frac{L(s, |\cdot|^{\frac{1}{2}})}{L(1-s, |\cdot|^{\frac{1}{2}})}G(\varpi^{-l}, \chi).$$

For suitable s one can expand

$$\frac{L(s, |\cdot|^{\frac{1}{2}})}{L(1-s, |\cdot|^{\frac{1}{2}})} = -q^{-\frac{3}{2}+s} + \zeta_F(2)^{-1}\sum_{a=0}^{\infty} q^{-\frac{a}{2}-as}.$$

Inserting this expression together with the explicit evaluation of the Gauß sum and comparing coefficients completes this case.

Next we look at $\chi = 1$ and $\mu \neq 1$. Using the support of the Gauß sum, (1.3.1), and evaluating the Whittaker function, (1.3.7), yields a basic identity of the form

$$\sum_{t=-\infty}^{\infty} q^{(t+a(\mu\pi))(\frac{1}{2}-s)}c_{t,l}(\mu) = \epsilon(\frac{1}{2}, \mu^{-1}\pi)G(\varpi^{-a(\mu)}, \mu^{-1})q^{-(l-a(\mu))(\frac{3}{2}-s)}.$$

Note that $a(\mu\pi) = 2a(\mu)$ and $a(\pi) = n = 1$. Since we are assuming $\mu \neq 1$, we must have $l \geq 1$. We complete this case by observing that, because $\pi = \mathrm{St}$,

$$\epsilon(\frac{1}{2}, \mu^{-1}\pi)G(\varpi^{-a(\mu)}, \mu^{-1}) = G(-\varpi^{-a(\mu)}, \mu).$$

Further, we consider $\chi = \mu = 1$ and $l = 0$. In this case the basic identity simplifies to

$$\sum_{t=-\infty}^{\infty} q^{(t+1)(\frac{1}{2}-s)} c_{t,0}(1) = -L(s, |\cdot|^{\frac{1}{2}}) = -\sum_{a=0}^{\infty} q^{-a(\frac{1}{2}+s)}.$$

Again we can compare coefficients.

It remains to check the case $\chi = \mu = 1$ and $l \geq 1$. The basic identity becomes

$$\begin{aligned}\sum_{t=-\infty}^{\infty} q^{(t+1)(\frac{1}{2}-s)} c_{t,l}(1) &= -q^{-(l-1)(\frac{3}{2}-s)} \frac{L(s, |\cdot|^{\frac{1}{2}})}{L(1-s, |\cdot|^{\frac{1}{2}})} \left(q^{-\frac{3}{2}+s} L(1-s, |\cdot|^{\frac{1}{2}}) - \zeta_F(1) q^{-1}\right) \\ &= -q^{-l(\frac{3}{2}-s)} \zeta_F(1) L(s, |\cdot|^{\frac{1}{2}})(1 - q^{\frac{1}{2}-s}) \\ &= -\zeta_F(1) q^{-l(\frac{3}{2}-s)} + \sum_{a \geq 1} q^{-(a-l)s - \frac{3}{2}l - \frac{1}{2}a + 1}.\end{aligned}$$

In the last step we expanded $L(s, |\cdot|^{\frac{1}{2}})$ as a geometric series. The result is derived by comparing coefficients. □

3.2.3 *Irreducible principal series*

We turn to the situation $\pi = \chi_1 \boxplus \chi_2$. The invariants attached to π are given explicitly in terms of χ_1 and χ_2. Further, the assumption $\omega_\pi \in \mathfrak{X}_n$ implies that $\chi_1\chi_2(\varpi) = 1$. Some values for $c_{t,l}(\mu)$ have been computed in [69, Proposition 2.39, 2.40]. We refine and complete these computations in order to list precise expressions for all possible t, l and μ.

Lemma 3.2.2 ([2], Lemma 2.2)**.** *Let $\pi = \chi_1 \boxplus \chi_2$ with $a(\chi_i) > 0$ for $i = 1, 2$. If $\chi_1|_{\mathfrak{o}^\times} \neq \chi_2|_{\mathfrak{o}^\times}$, then*

$$c_{t,l}(\mu) = \begin{cases} \epsilon(\frac{1}{2}, \mu^{-1}\omega_\pi^{-1}\pi) G(\varpi^{-l}, \mu^{-1}) \\ \qquad \text{if } a(\mu\chi_1), a(\mu\chi_2) \neq 0 \text{ and } t = -a(\mu\chi_1) - a(\mu\chi_2), \\ -q^{-\frac{1}{2}} \chi_i(\varpi^{-1}) \epsilon(\frac{1}{2}, \mu^{-1}\omega_\pi^{-1}\pi) G(\varpi^{-l}, \mu^{-1}) \\ \qquad \text{if } a(\mu\chi_j) \neq a(\mu\chi_i) = 0 \text{ for } \{j, i\} = \{1, 2\}, \text{ and } t = -a(\mu\chi_j) - 1, \\ \zeta_F(1)^{-1} q^{-\frac{t+a(\mu\pi)}{2}} \chi_i(\varpi^{t+a(\mu\pi)}) \epsilon(\frac{1}{2}, \mu^{-1}\omega_\pi^{-1}\pi) G(\varpi^{-l}, \mu^{-1}) \\ \qquad \text{if } a(\mu\chi_j) \neq a(\mu\chi_i) = 0 \text{ for } \{j, i\} = \{1, 2\}, \text{ and } t \geq -a(\mu\chi_j), \\ 0 \\ \qquad \text{else.} \end{cases}$$

If $\chi_1|_{\mathfrak{o}^\times} = \chi_2|_{\mathfrak{o}^\times}$, *then*

$$c_{t,l}(\mu) = \begin{cases} \epsilon(\frac{1}{2}, \mu^{-1}\omega_\pi^{-1}\pi)G(\varpi^{-l}, \mu^{-1}) \\ \qquad\qquad \text{if } a(\mu\chi_1), a(\mu\chi_2) \neq 0 \text{ and } t = -a(\mu\chi_1) - a(\mu\chi_2), \\ q^{-1}G(\varpi^{-l}, \mu^{-1}) \\ \qquad\qquad \text{if } a(\mu\chi_1) = a(\mu\chi_2) = 0, \text{ and } t = -2, \\ -q^{-\frac{1}{2}}\zeta_F(1)^{-1}G(\varpi^{-l}, \mu^{-1})(\chi_1(\varpi) + \chi_2(\varpi)) \\ \qquad\qquad \text{if } a(\mu\chi_1) = a(\mu\chi_2) = 0, \text{ and } t = -1, \\ q^{-\frac{t}{2}}G(\varpi^{-l}, \mu^{-1})\Big(- q^{-1}\zeta_F(1)^{-1}(\chi_1(\varpi^{t+2}) + \chi_2(\varpi^{t+2})) \\ \qquad\qquad\qquad + \zeta_F(1)^{-2}\sum_{k=0}^{t}\chi_1(\varpi^k)\chi_2(\varpi^{t-k})\Big) \\ \qquad\qquad \text{if } a(\mu\chi_1) = a(\mu\chi_2) = 0, \text{ and } t \geq 0, \\ 0 \\ \qquad\qquad \text{else.} \end{cases}$$

Proof. The case $\mu \neq \chi_1, \chi_2$ is straight forward. We start by considering $\chi_1 = \mu^{-1}|\cdot|^c \neq \chi_2|\cdot|^{2c}$. The same calculation will work with the roles of χ_1 and χ_2 interchanged. The basic identity reads

$$\sum_{t=-\infty}^{\infty} q^{(t+a(\mu\pi))(\frac{1}{2}-s)}c_{t,l}(\mu) = G(\varpi^{-l}, \mu^{-1})\epsilon(\frac{1}{2}, \mu^{-1}\omega_\pi^{-1}\pi)\frac{L(s, |\cdot|^c)}{L(1-s, |\cdot|^{-c})}.$$

Expanding the quotient of L-factors into a power series and recalling $\chi_1(\varpi) = q^{-c}$ yields

$$\frac{L(s, |\cdot|^c)}{L(1-s, |\cdot|^{-c})} = -q^{-1}\chi_1(\varpi^{-1})q^s + \zeta_F(1)^{-1}\sum_{a=0}^{\infty}\chi_1(\varpi^a)q^{-as}. \tag{3.2.5}$$

Inserting this in the basic identity enables us to compare coefficients, which concludes this case.

In the end we consider the situation where both, χ_1 and χ_2, are unramified twists of μ. Since the central character is trivial on the uniformiser we have $\chi_1(\varpi) = \chi_2^{-1}(\varpi) = |\varpi|_F^c$ for some $c \in \mathbb{C}$. The basic identity becomes

$$\sum_{t=-\infty}^{\infty} q^{t(\frac{1}{2}-s)}c_{t,l}(\mu) = \omega_\pi(-1)G(\varpi^{-l}, \mu^{-1})\frac{L(s, |\cdot|^c)L(s, |\cdot|^{-c})}{L(1-s, |\cdot|^c)L(1-s, |\cdot|^{-c})}.$$

We use (3.2.5) twice to obtain

$$\frac{L(s,|\cdot|^c)L(s,|\cdot|^{-c})}{L(1-s,|\cdot|^c)L(1-s,|\cdot|^{-c})} = q^{-2}q^{2s} - q^{-1}\zeta_F(1)^{-1}(\chi_1(\varpi)+\chi_2(\varpi))q^s$$
$$+\sum_{a=0}^{\infty}\Big(-q^{-1}\zeta_F(1)^{-1}(\chi_1(\varpi^{a+2})+\chi_2(\varpi^{a+2}))+\zeta_F(1)^{-2}\sum_{l=0}^{a}\chi_1(\varpi^l)\chi_2(\varpi^{a-l})\Big)q^{-as}.$$

We may compare coefficients to conclude the proof. □

Lemma 3.2.3 ([4], Lemma 2.3). *Let $\pi = \chi_1 \boxplus \chi_2$ with $n = a(\chi_1) > a(\chi_2) = 0$. We have*

$$c_{t,l}(\mu) = \begin{cases} \mu(-1)\epsilon(\frac{1}{2},\mu^{-1}\omega_\pi^{-1})\zeta_F(1)q^{-\frac{l}{2}}\chi_2(\varpi^{a(\mu\omega_\pi)-l}) \\ \qquad \text{if } \mu \neq \omega_\pi^{-1},\ l>0, \text{ and } t=-a(\mu\omega_\pi)-l, \\ \epsilon(\frac{1}{2},\omega_\pi^{-1})q^{-\frac{1}{2}(t+n)}\chi_2(\varpi^{t+2n}) \\ \qquad \text{if } l=0, \text{ and } t\geq -n, \\ -\omega_\pi(-1)\zeta_F(1)q^{-\frac{1}{2}(l+1)}\chi_2(\varpi^{1-l}) \\ \qquad \text{if } \mu=\omega_\pi^{-1} \text{ and } t=-l-1, \\ \omega_\pi(-1)q^{-\frac{t}{2}-l}\chi_2(\varpi^{-t-2l}) \\ \qquad \text{if } \mu=\omega_\pi^{-1} \text{ and } t\geq -l, \\ 0 \qquad \text{else.} \end{cases}$$

Note that by isomorphy this covers the case $a(\chi_2) > a(\chi_1) = 0$ as well.

Proof. From (1.3.7) we infer that

$$W_\pi(a(\varpi^a)) = \chi_1(\varpi^a)q^{-\frac{a}{2}} \text{ for } a \geq 0.$$

First, we consider $\mu \neq 1$ and assume that μ is not an unramified twist of χ_1^{-1}. Using the support of the Gauß sum we write the basic identity in the form

$$\sum_{t=-\infty}^{\infty} q^{(t+a(\mu\pi))(\frac{1}{2}-s)}c_{t,l}(\mu)$$
$$= \epsilon(\frac{1}{2},\mu^{-1}\omega_\pi^{-1}\pi)\chi_1(\varpi^{l-a(\mu)})q^{-(l-a(\mu))(1-s)}G(\varpi^{-a(\mu)},\mu^{-1})$$
$$= \mu(-1)\epsilon(\frac{1}{2},\mu^{-1}\omega_\pi^{-1})\zeta_F(1)q^{-l+\frac{a(\mu)}{2}}\chi_1(\varpi^{l-a(\mu\omega_\pi)})q^{(l-a(\mu))s}.$$

Comparing coefficients yields the desired constants.

Next, we consider $\mu = \omega_\pi^{-1} = \chi_1^{-1}\chi_2^{-1}$. Again the basic identity reduces to

$$\begin{aligned}\sum_{t=-\infty}^{\infty} & q^{(t+a(\mu\pi))(\frac{1}{2}-s)} c_{t,l}(\mu) \\ &= \epsilon(\frac{1}{2},\chi_1)\frac{L(s,\chi_2^{-1})}{L(1-s,\chi_2)}\chi_1(\varpi^{l-a(\mu)})G(\varpi^{-a(\mu)},\mu^{-1})q^{-(l-n)(1-s)} \\ &= \omega_\pi(-1)\zeta_F(1)\chi_1(\varpi^l)q^{-l+\frac{n}{2}}\frac{L(s,\chi_2^{-1})}{L(1-s,\chi_2)}q^{(l-n)s}.\end{aligned}$$

The quotient of L-factors can be evaluated using (3.2.5). One obtains

$$\begin{aligned}\sum_{t=-\infty}^{\infty} q^{(t+a(\mu\pi))(\frac{1}{2}-s)} c_{t,l}(\mu) &= -\omega_\pi(-1)\zeta_F(1)q^{-1-l+\frac{n}{2}}\chi_2(\varpi^{-l+1})q^{(l-n+1)s} \\ &\quad +\omega_\pi(-1)\sum_{a=0}^{\infty} q^{-l+\frac{n}{2}}\chi_2(\varpi^{-a-l})q^{(-a-n+l)s}.\end{aligned}$$

With this at hand it is easy to evaluate $c_{t,l}(\mu)$.

The last case to consider is $\mu = 1$. This is a very degenerated situation which splits into two sub cases. First, we look at $l > 0$. In this situation the basic identity has the form

$$\begin{aligned}\sum_{t=-\infty}^{\infty} & q^{(t+a(\pi))(\frac{1}{2}-s)} c_{t,l}(1) \\ &= \epsilon(\frac{1}{2},\chi_1^{-1})\frac{L(s,\chi_2)}{L(1-s,\chi_2^{-1})}\left(\sum_{a=l}^{\infty}\chi_1(\varpi^a)q^{-a(1-s)} - \zeta_F(1)q^{-1}\chi_1(\varpi^{l-1})q^{-(l-1)(1-s)}\right) \\ &= \epsilon(\frac{1}{2},\chi_1^{-1})\chi_1(\varpi^l)q^{-l(1-s)}\left(L(1-s,\chi_2^{-1}) - \zeta_F(1)\chi_2(\varpi)q^{-s}\right)\frac{L(s,\chi_2)}{L(1-s,\chi_2^{-1})} \\ &= \zeta_F(1)\epsilon(\frac{1}{2},\chi_1^{-1})\chi_1(\varpi^l)q^{-l(1-s)}\end{aligned}$$

This nice formula makes comparing coefficients easy.

Second, if $l = 0$, the situation is slightly different. Indeed the basic identity reads

$$\sum_{t=-\infty}^{\infty} q^{(t+a(\pi))(\frac{1}{2}-s)} c_{t,0}(1) = \epsilon(\frac{1}{2},\chi_1^{-1})L(s,\chi_2) = \epsilon(\frac{1}{2},\chi_1^{-1})\sum_{a=0}^{\infty}\chi_1(\varpi^{-a})q^{-as}.$$

This concludes the proof. □

3.2.4 *Summary*

We will now summarise our findings focusing on non-zero situations. This is taken from [5, Appendix A]. For future applications we slightly rescale the coefficients. We set

$$c_{t,l}(\mu) = c(\pi,l,t,\mu)\zeta_F(1)q^{-\frac{l+t+a(\mu\pi)}{2}}\lambda_{\mu\pi}(\mathfrak{p}^{t+a(\mu\pi)+\delta_{\mu\pi}}),$$

for some $\delta_{\mu\pi} \in \mathbb{N}$ which will be defined case by case. In most cases it turns out to be the degree of the Euler-factor of $\mu\pi$. This new constants satisfy

$$|c(\pi, l, t, \mu)| \leq 5q^{\frac{1}{2}} t \max_{i=1,2}(|\alpha_i|^t), \tag{3.2.6}$$

for $\alpha_i = \chi_i(\varpi)$ if $\pi = \chi_1 \boxplus \chi_2$ and $\alpha_i = 1$ otherwise. Note that, since we are dealing with admissible, unitary representations π, we have $|\alpha_i| = 1$ except for χ_1 equals χ_2 up to unramified twist.

Supercuspidal representations π. Recall that in this case $\lambda_{\mu\pi}(\mathfrak{p}^m) = \delta_{m=0}$ and $\delta_{\mu\pi} = 0$ for all μ. This leads to the table below.

$c(\pi, l, t, \mu)$	$\mu = 1$	$\mu \in \mathfrak{X}_l \setminus \{1\}$
$l = 0$	$\epsilon(\frac{1}{2}, \tilde{\pi})\zeta_F(1)^{-1}$	$-$
$l = 1$	$-q^{-\frac{1}{2}}\epsilon(\frac{1}{2}, \tilde{\pi})$	$\epsilon(\frac{1}{2}, \mu)\epsilon(\frac{1}{2}, \mu^{-1}\tilde{\pi})$
$l > 1$	0	$\epsilon(\frac{1}{2}, \mu)\epsilon(\frac{1}{2}, \mu^{-1}\tilde{\pi})$

Twists of Steinberg. Here we consider $\pi = \chi St$ for some ramified character χ. We have

$$\lambda_{\chi_\mu \pi}(\mathfrak{p}^m) = \begin{cases} \delta_{m=0} & \text{if } \mu \neq \chi^{-1}, \\ q^{-\frac{m}{2}}\delta_{m\geq 0} & \text{if } \mu = \chi^{-1}. \end{cases}$$

Set $\delta_{\mu\pi} = 1$ if $\mu = \chi^{-1}$, and $\delta_{\mu\pi} = 0$ otherwise. One obtains the following evaluations.

$c(\pi, l, t, \mu)$	$\mu = 1$	$\mu = \chi^{-1}$	$\mu \in \mathfrak{X}' \setminus \{1, \chi^{-1}\}$
$l = 0$	$\epsilon(\frac{1}{2}, \tilde{\pi})\zeta_F(1)^{-1}$	$-$	$-$
$l = 1$	$-\epsilon(\frac{1}{2}, \tilde{\pi})q^{-\frac{1}{2}}$	$\epsilon(\frac{1}{2}, \mu)q^{-\frac{3}{2}}$ if $t \leq -2$ $-\epsilon(\frac{1}{2}, \mu)\frac{q^{\frac{1}{2}}}{\zeta_F(2)}$ if $t > -2$	$\epsilon(\frac{1}{2}, \mu^{-1}\tilde{\pi})\epsilon(\frac{1}{2}, \mu)$
$l > 1$	0	$\epsilon(\frac{1}{2}, \mu)q^{-\frac{3}{2}}$ if $t \leq -2$ $-\epsilon(\frac{1}{2}, \mu)\frac{q^{\frac{1}{2}}}{\zeta_F(2)}$ if $t > -2$	$\epsilon(\frac{1}{2}, \mu^{-1}\tilde{\pi})\epsilon(\frac{1}{2}, \mu)$

Irreducible principal series. In this section we treat three cases. First, we look at $\pi = \chi_1 \boxplus \chi_2$ with $\chi_1|_{\mathfrak{o}^\times} \neq \chi_2|_{\mathfrak{o}^\times}$. In this case $\delta_{\mu\pi} = 1$ if $\mu|_{\mathfrak{o}^\times} = \chi_i^{-1}|_{\mathfrak{o}^\times}$ and 0 otherwise. Furthermore,

$$\lambda_{\mu\pi}(\mathfrak{p}^m) = \begin{cases} \delta_{m=0} & \text{if } \mu|_{\mathfrak{o}^\times} \neq \chi_i^{-1}|_{\mathfrak{o}^\times}, \\ \chi_i(\varpi^m)\delta_{m\geq 0} & \text{if } \mu|_{\mathfrak{o}^\times} = \chi_i^{-1}|_{\mathfrak{o}^\times}. \end{cases}$$

We get the following table.

$c(\pi, l, t, \mu)$	$\mu = 1$	$\mu\|_{\mathfrak{o}^\times} = \chi_i^{-1}\|_{\mathfrak{o}^\times}$	$\mu \in \mathfrak{X}' \setminus \{1, \chi_i^{-1}\}$
$l = 0$	$\epsilon(\frac{1}{2}, \tilde{\pi})\zeta_F(1)^{-1}$	–	–
$l = 1$	$-\epsilon(\frac{1}{2}, \tilde{\pi})q^{-\frac{1}{2}}$	$-\epsilon(\frac{1}{2}, \mu^{-1}\tilde{\pi})\epsilon(\frac{1}{2}, \mu)\chi_i^{-1}(\varpi)q^{-1}$ if $t \leq -a(\mu\pi) - 1$; $\epsilon(\frac{1}{2}\mu^{-1}\tilde{\pi})\epsilon(\frac{1}{2}, \mu)\chi_i^{-1}(\varpi)\zeta_F(1)^{-1}$ if $t > -a(\mu\pi) - 1$	$\epsilon(\frac{1}{2}, \mu^{-1}\tilde{\pi})\epsilon(\frac{1}{2}, \mu)$
$l > 1$	0	$-\epsilon(\frac{1}{2}, \mu^{-1}\tilde{\pi})\epsilon(\frac{1}{2}, \mu)\chi_i^{-1}(\varpi)q^{-1}$ if $t \leq -a(\mu\pi) - 1$; $\epsilon(\frac{1}{2}\mu^{-1}\tilde{\pi})\epsilon(\frac{1}{2}, \mu)\chi_i^{-1}(\varpi)\zeta_F(1)^{-1}$ if $t > -a(\mu\pi) - 1$	$\epsilon(\frac{1}{2}, \mu^{-1}\tilde{\pi})\epsilon(\frac{1}{2}, \mu)$

Next we look at $\pi = \chi_1 \boxplus \chi_2$ where $\chi_1|_{\mathfrak{o}^\times} = \chi_2|_{\mathfrak{o}^\times}$. In this case $\delta_{\mu\pi} = 2$ if $\mu|_{\mathfrak{o}^\times} = \chi_1^{-1}|_{\mathfrak{o}^\times}$ and 0 otherwise. Furthermore,

$$\lambda_{\mu\pi}(\mathfrak{p}^m) = \begin{cases} \delta_{m=0} & \text{if } \mu|_{\mathfrak{o}^\times} \neq \chi_1^{-1}|_{\mathfrak{o}^\times}, \\ \frac{\chi_1(\mathfrak{p}^{m+1}) - \chi_2(\mathfrak{p}^{m+1})}{\chi_1(\varpi) - \chi_2(\varpi)}\delta_{m\geq 0} & \text{if } \mu|_{\mathfrak{o}^\times} = \chi_1^{-1}|_{\mathfrak{o}^\times} \end{cases}$$

We produce the following table.

$c(\pi,l,t,\mu)$	$\mu=1$	$\mu\vert_{\mathfrak{o}^\times}=\chi_1^{-1}\vert_{\mathfrak{o}^\times}$	$\mu\in\mathfrak{X}'\setminus\{1,\chi_1^{-1}\}$
$l=0$	$\epsilon(\frac{1}{2},\tilde{\pi})\zeta_F(1)^{-1}$	–	–
$l=1$	$-\epsilon(\frac{1}{2},\tilde{\pi})q^{-\frac{1}{2}}$	$\epsilon(\frac{1}{2},\mu)q^{-2}$ if $t\leq -2$ $-\epsilon(\frac{1}{2},\mu)\frac{q^{-1}}{\zeta_F(1)}$ if $t=-1$ $\epsilon(\frac{1}{2},\mu)(\frac{1+q^{-1}-q^{-2}}{\zeta_F(1)^2}\frac{\lambda_{\mu\pi}(\mathfrak{p}^t)}{\lambda_{\mu\pi}(\mathfrak{p}^{t+2})}-\zeta_F(1)^{-1})$ if $t\geq 0$	$\epsilon(\frac{1}{2},\mu^{-1}\tilde{\pi})\epsilon(\frac{1}{2},\mu)$
$l>1$	0	$\epsilon(\frac{1}{2},\mu)q^{-2}$ if $t\leq -2$ $-\epsilon(\frac{1}{2},\mu)\frac{q^{-1}}{\zeta_F(1)}$ if $t=-1$ $\epsilon(\frac{1}{2},\mu)(\frac{1+q^{-1}-q^{-2}}{\zeta_F(1)^2}\frac{\lambda_{\mu\pi}(\mathfrak{p}^t)}{\lambda_{\mu\pi}(\mathfrak{p}^{t+2})}-\zeta_F(1)^{-1})$ if $t\geq 0$	$\epsilon(\frac{1}{2},\mu^{-1}\tilde{\pi})\epsilon(\frac{1}{2},\mu)$

Finally, we need to look at $\pi=\chi_1\boxplus\chi_2$ with $a(\chi_1)>a(\chi_2)=0$. In this case we have

$$\lambda_{\mu\pi}(\mathfrak{p}^m)=\begin{cases}\delta_{m=0} & \text{if } \mu\neq\omega_\pi^{-1},\\ \chi_2(\varpi^m)\delta_{m\geq 0} & \text{if } \mu=\omega_\pi^{-1}.\end{cases}$$

Also, $\delta_{\mu\pi}=1$ if $\mu=\omega_\pi^{-1}$ and 0 otherwise. For technical reasons we put $\delta_\pi=l$. One has the following results.

$c(\pi,l,t,\mu)$	$\mu=1$	$\mu=\omega_\pi^{-1}$	$\mu\in\mathfrak{X}'\setminus\{1,\omega_\pi^{-1}\}$
$l=0$	$\epsilon(\frac{1}{2},\tilde{\pi})\zeta_F(1)^{-1}$	–	–
$l>1$	$\epsilon(\frac{1}{2},\tilde{\pi})\chi_1(\varpi^l)q^{-\frac{l}{2}}$	$-\omega_\pi(-1)\chi_2(\varpi^{1-l})q^{-1}$ if $t\leq -a(\mu\pi)-1$ $\omega_\pi(-1)\chi_2(\varpi^{1-l})$ if $t>-a(\mu\pi)-1$	$\epsilon(\frac{1}{2},\mu^{-1}\tilde{\pi})\epsilon(\frac{1}{2},\mu)$

3.3 INTEGRAL REPRESENTATIONS FOR WHITTAKER NEW VECTORS

In this section, following [4, Section 3], we use our description of $c_{t,l}(\mu)$ to evaluate Whittaker new vectors on the special matrices $g_{t,l,v}$. We will obtain expressions for the local Whittaker functions featuring several (p-adic) special functions. These functions are analogues of well known special functions that appear in the archimedean representation theory of GL_2 and are interesting in their own right. Probably the most prestiges function we will encounter is the Kloosterman sum and its twisted generalisation (generalised Salié sum). A more general function is

$$K(\xi, A, B) = \int_{\mathfrak{O}^\times} \xi(x)\psi(\mathrm{Tr}(Ax) + B\mathrm{Nr}_{E/F}(x))d_E x,$$

which we associated to a multiplicative character $\xi\colon E^\times \to S^1$ on some étale algebra E over F. Here $A \in E$ and $B \in F$.

However, the focus of this section is to describe the support of the Whittaker functions as precisely as possible. This will help us later on to exclude several choices for t and l for which $W_\pi(g_{t,l,v})$ vanishes.

We consider each type of representation on its own. The case of supercuspidal representations has already been considered in [69, Proposition 2.30]. However, in many cases the sums of ϵ-factors simplify considerably.

The results in this section hold for any non-archimedean field F and any irreducible, admissible, unitary representation π with central character $\omega_\pi \in \mathfrak{X}_n$.

3.3.1 *Dihedral supercuspidal representations*

Here we will derive an expression of the Whittaker new vector for dihedral supercuspidal representations which goes beyond the one given in [69, Proposition 2.30]. The following results hold for any dihedral representation even if $2 \mid q$. However, in this case not every supercuspidal representation is dihedral.

If π is dihedral supercuspidal, then so is $\tilde{\pi}$. Therefore we find a quadratic extension E/F and a multiplicative character ξ such that $\tilde{\pi} = \omega_\xi$. We now use the properties of dihedral supercuspidal representations, in particular (1.3.6), to calculate the Whittaker function.

Lemma 3.3.1 ([4], Lemma 3.1). *If π is dihedral supercuspidal, then*

$$W_\pi(g_{t,l,v}) = \begin{cases} \epsilon(\frac{1}{2},\tilde{\pi}) & \text{if } l=0 \text{ and } t=-n, \\ \gamma q^{\frac{n}{2}} K(\xi^{-1},\Omega^{-\frac{n}{f}},v\varpi^{-l}) & \text{if } 0<l<\frac{n}{2} \text{ and } t=-n, \\ \gamma q^{-\frac{t}{2}} K(\xi^{-1},\Omega^{\frac{t}{f}},v\varpi^{-l}) & \text{if } l=\frac{n}{2} \text{ and } -n\le t<0, \\ \gamma q^{l} K(\xi^{-1},\Omega^{-\frac{2l}{f}},v\varpi^{-l}) & \text{if } \frac{n}{2}<l<n \text{ and } t=-2l, \\ \omega_\pi(-v^{-1})\psi(-v^{-1}\varpi^{-l}) & \text{if } n\le l \text{ and } t=-2l, \\ 0 & \text{else.} \end{cases}$$

Proof. First, we apply [73, Lemma 1.1.1] in the setting of E and obtain

$$\begin{aligned}&\epsilon(\frac{1}{2},\xi\cdot(\mu^{-1}\circ \mathrm{Nr}_{E/F}))\\ &= q^{\frac{f}{2}(n(\psi_E)-a(\xi\cdot(\mu^{-1}\circ\mathrm{Nr}_{E/F}))}\int_{\Omega^{n(\psi_E)-a(\xi\cdot(\mu^{-1}\circ\mathrm{Nr}_{E/F}))}\mathfrak{O}^\times}\xi^{-1}(x)\mu(\mathrm{Nr}_{E/F}(x))\psi_E(x)d\mu_E(x).\end{aligned}$$

Note that, if $t=-a(\mu\pi)$, we get

$$n(\psi_E)-a(\xi\cdot(\mu^{-1}\circ\mathrm{Nr}_{E/F}))=\frac{t}{f}.$$

With this at hand we proceed computing the Whittaker new vector for $l>0$ using (3.2.4). We obtain

$$\begin{aligned}W_\pi(g_{t,l,v}) &= \sum_{t=-a(\mu\pi)}\epsilon(\frac{1}{2},\mu^{-1}\tilde{\pi})G(\varpi^{-l},\mu^{-1})\mu(v)\\ &=\gamma q^{\frac{t}{2}}\int_{\Omega^{\frac{t}{f}}\mathfrak{O}^\times}\int_{\mathfrak{o}^\times}\xi^{-1}(x)\psi(\mathrm{Tr}(x)+v\varpi^{-l}y)\sum_{\mu\in\mathfrak{X}_l}\mu(\mathrm{Nr}_{E/F}(x)y^{-1})d_Exd^\times y\\ &=\gamma q^{-\frac{t}{2}}\int_{\mathfrak{O}^\times}\int_{\mathfrak{o}^\times}\xi^{-1}(\Omega^{\frac{t}{f}}x)\psi(\mathrm{Tr}(\Omega^{\frac{t}{f}}x)+v\varpi^{-l}y)\sum_{\mu\in\mathfrak{X}_l}\mu(\mathrm{Nr}_{E/F}(x)y^{-1})d_Exd^\times y\\ &=\gamma q^{-\frac{t}{2}}\int_{\mathfrak{O}^\times}\xi^{-1}(x)\psi(\mathrm{Tr}(\Omega^{\frac{t}{f}}x)+v\varpi^{-l}\mathrm{Nr}_{E/F}(x))d_Ex.\end{aligned}$$

If $l=0$, the only term in the expansion of $W_\pi(g_{t,l,v})$ is $c_{t,0}(1)$, which makes this case easy.

Finally, if $l\ge n$, we have the matrix identity

$$g_{t,l,v}\begin{pmatrix}0 & 1\\ \varpi^n & 0\end{pmatrix} = n(-v^{-1}\varpi^{t+l})z(v\varpi^{n-l})g_{t-n+2l,0,v^2}\begin{pmatrix}1 & 1+v^{-1}\varpi^{l-n}\\ 0 & -v^{-2}\end{pmatrix}.$$

Furthermore, [69, Lemma 2.17, Corollary 2.27, Proposition 2.28] imply

$$\epsilon(\frac{1}{2},\pi)W_\pi\left(g\begin{pmatrix}0 & 1\\ \varpi^n & 0\end{pmatrix}\right) = \omega_\pi(\det(g))W_{\tilde{\pi}}(g).$$

With this at hand we compute

$$\begin{aligned}W_\pi(g_{t,l,v}) &= \epsilon(\frac{1}{2},\tilde{\pi})W_{\tilde{\pi}}\left(g_{t,l,v}\begin{pmatrix}0 & 1\\ \varpi^n & 0\end{pmatrix}\right)\\ &= \omega_{\tilde{\pi}}(v)\psi(-v^{-1}\varpi^{t+l})\epsilon(\frac{1}{2},\tilde{\pi})W_{\tilde{\pi}}(g_{t+2l-n,0,v^2}).\end{aligned}$$

The expression claimed above follows by using the $l=0$ case for $W_{\tilde{\pi}}$. □

Remark 3.3.2. *It is clear from the proof that, if there is no μ such that $-t=a(\mu\pi)$ and $a(\mu)=l$, then $W_\pi(g_{t,l,v})=0$. In particular, if $n_0(\pi)=\min_{\mu\in\mathfrak{X}}(a(\mu\pi))$, then $W_\pi(g_{t,l,v})=0$ for all $t>-n_0(\pi)$. Even more, if π comes from an unramified extension E of F, then $W_\pi(g_{t,l,v})\neq 0$ forces t to be even. On the other hand, if π comes from a ramified extension E of F, it is a theorem due to Tunnell, [85, Proposition 3.5], that π is twist-minimal if and only if $a(\pi)$ is odd. We observe that in this case, if $a(\pi)$ is even and $l=\frac{a(\pi)}{2}$, then $t=-n_0(\pi)$ is the largest and only odd value for t that can appear for non-zero $W_\pi(g_{t,l,v})$.*

3.3.2 *Twists of Steinberg*

Throughout this subsection E will denote the algebra $F\times F$. In this case any multiplicative character ξ on $E^\times$ factors in $\xi=(\chi_1\circ\mathrm{pr}_1)\cdot(\chi_2\circ\mathrm{pr}_2)$ for two characters χ_1 and χ_2 of $F^\times$. However, at the moment we will only encounter the special situation where $\chi_1=\chi_2$. In other words, ξ factors through the norm map. We will now compute the local Whittaker functions in terms of $K(\xi,A,B)$ and other well known exponential sums.

We start of with the simplest case.

Lemma 3.3.3 ([4], Lemma 3.2). *For $\pi=St$ we have*

$$W_\pi(g_{t,l,v}) = \begin{cases}-q^{-t-1} & \text{if } t\geq -1 \text{ and } l=0,\\ q^{-t-2l}\psi(-\varpi^{l+t}v^{-1}) & \text{if } l\geq 1 \text{ and } -2l\leq t,\\ 0 & \text{else.}\end{cases}$$

Proof. By the definition of $c_{t,l}(\mu)$ we have

$$W_\pi(g_{t,l,v}) = \sum_{\mu\in\mathfrak{X}_l} c_{t,l}(\mu)\mu(v).$$

We will now insert the expressions given in Lemma 3.2.1. The interesting cases are obviously $t \leq -l-1$ and $l \geq 1$. The values for $c_{t,l}(\mu)$ calculated above yield

$$\begin{aligned} W(g_{t,l,v}) &= q^{-t-2l} \sum_{\mu\in\mathfrak{X}'_{-l-t}} G(-\varpi^{l+t},\mu)\mu(v) - \delta_{t=-l-1}\zeta_F(1)q^{-l} \\ &= q^{-t-2l} \sum_{\mu\in\mathfrak{X}_{-l-t}} G(-\varpi^{-l-t}v^{-1},\mu) \\ &= q^{-t-2l}\sharp\mathfrak{X}_{-l-t} \int_{1+\varpi^{-l-t}\mathfrak{o}} \psi(-\varpi^{l+t}v^{-1}y)d^\times y \\ &= q^{-t-2l}\psi(-\varpi^{l+t}v^{-1}). \end{aligned}$$

□

We move on to the slightly more complicated situation of $\pi = \chi\mathrm{St}$ for a non trivial χ.

Lemma 3.3.4 ([4], Lemma 3.3)**.** *Let $\pi = \chi St$ where χ is a character such that $a(\chi) > 0$ and $\chi(\varpi) = 1$. If $a(\chi) \geq 1$ and $l \neq a(\chi) = \frac{n}{2}$, then we have*

$$W(g_{t,l,v}) = \begin{cases} \epsilon(\frac{1}{2},\tilde{\pi}) & \text{if } t=-n \text{ and } l=0, \\ q^{-\frac{t}{2}}\zeta_F(1)^{-2}K(\chi\otimes\chi,(\varpi^{\frac{t}{2}},\varpi^{\frac{t}{2}}),v\varpi^{-l}) & \text{if } t=-\max(n,2l) \text{ and } 0<l<n, \\ \chi^2(-v^{-1})\psi(-v^{-1}\varpi^{-l}) & \text{if } t=-2l \text{ and } l\geq n,, \\ 0 & \text{else.} \end{cases}$$

Finally, if $a(\chi) \geq 1$ and $l = a(\chi) = \frac{n}{2}$, we are in the transition region and have

$$W(g_{t,l,v}) = \begin{cases} -\zeta_F(2)^{-1}\zeta_F(1)q^{-1-\frac{a(\chi)}{2}-t}\chi(v^{-1})\epsilon(\frac{1}{2},\chi^{-1}) & \text{if } t>-2, \\ q\zeta_F(1)^{-2}K(\chi\otimes\chi,(\varpi^{-1},\varpi^{-1}),v\varpi^{-l}) & \text{if } t=-2 \text{ and } l=1, \\ \chi(v^{-1})\epsilon(\frac{1}{2},\chi^{-1})\zeta_F(1)^{-1}q^{1-\frac{a(\chi)}{2}}S(1,-b_\chi v^{-1},1) & \text{if } t=-2 \text{ and } l>1, \\ q^{-\frac{t}{2}}\zeta_F(1)^{-2}K(\chi\otimes\chi,(\varpi^{\frac{t}{2}},\varpi^{\frac{t}{2}}),v\varpi^{-l}) & \text{if } -2l\leq t<-2 \text{ even}, \\ 0 & \text{else.} \end{cases}$$

Proof. We start by expanding

$$W_\pi(g_{t,l,v}) = \sum_{\mu\in\mathfrak{X}_l} c_{t,l}(\mu)\mu(v).$$

Using Lemma 3.2.1 we first observe that, if $t > -2$, the only character $\mu \in \mathfrak{X}_l$ with non-zero $c_{t,l}(\mu)$ is $\mu = \chi^{-1}$. Similarly, if $l = 0$, the only character to consider is $\mu = 1$, which contributes only if $t = -2a(\chi) = -a(\pi)$. We move on to the more interesting cases and assume $l > 0$ and $t \leq -2$.

If $t = -2$, we obtain

$$W_\pi(g_{-2,l,v}) = q^{-1}G(\varpi^{-l},\chi)\chi^{-1}(v) + \sum_{\substack{\mu\in\mathfrak{X}_l,\\ a(\mu\chi)=1}} \epsilon(\frac{1}{2},\mu^{-1}\chi^{-1})^2 G(\varpi^{-l},\mu^{-1})\mu(v).$$

Reversing the evaluation of the Gauß sum given in (1.3.1) reveals

$$\begin{aligned} W_\pi(g_{-2,l,v}) &= q\zeta_F(1)^{-2}\sum_{\substack{\mu\in\mathfrak{X}_l,\\ a(\mu\chi)\leq 1}} G(\varpi^{-1},\mu\chi)^2 G(v\varpi^{-l},\mu^{-1}) \\ &= q\zeta_F(1)^{-2}\sum_{\mu\in\mathfrak{X}_l} G(\varpi^{-1},\mu\chi)^2 G(v\varpi^{-l},\mu^{-1}). \end{aligned} \tag{3.3.1}$$

To exploit cancellation in the μ-average we write the Gauß sums as an integral. This leads to

$$\begin{aligned} W_\pi(g_{-2,l,v}) = q\zeta_F(1)^{-2}\int_{(\mathfrak{o}^\times)^3} &\chi(y_1y_2)\psi(y_1\varpi^{-1}+y_2\varpi^{-1}+y_3v\varpi^{-l}) \\ &\cdot\sum_{\mu\in\mathfrak{X}_l}\mu(y_1y_2y_3^{-1})d^\times y_3 d^\times y_2 d^\times y_1. \end{aligned}$$

We observe

$$\sum_{\mu\in\mathfrak{X}_l}\mu(y_1y_2y_3^{-1}) = \begin{cases} \sharp\mathfrak{X}_l & \text{if } y_1y_2y_3^{-1}\in 1+\varpi^l\mathfrak{o}, \\ 0 & \text{else.}\end{cases}$$

Using this to simplify the integral we obtain

$$\begin{aligned} &W_\pi(g_{-2,l,v}) \\ &= q\zeta_F(1)^{-2}\sharp\mathfrak{X}_l\mathrm{Vol}(1+\varpi^l\mathfrak{o},d^\times)\int_{(\mathfrak{o}^\times)^2}\chi(y_1y_2)\psi(y_1\varpi^{-1}+y_2\varpi^{-1}+y_1y_2v\varpi^{-l})d^\times y_1 d^\times y_2 \\ &= \zeta_F(1)^{-2}qK(\chi\otimes\chi,(\varpi^{-1},\varpi^{-1}),v\varpi^{-l}). \end{aligned}$$

If $l = 1 = a(\chi)$, we will leave this expression as it is. However, in the other cases we write

$$W_\pi(g_{-2,l,v}) = q\zeta_F(1)^{-2}\int_{\mathfrak{o}^\times}\chi(y_1)\psi(y_1\varpi^{-1})G(\varpi^{-1}+y_1v\varpi^{-l},\chi)d^\times y_1$$

instead. Here we have to consider two different cases. First, if $l > 1$, the Gauß sum vanishes unless $l = a(\chi)$ (which would also imply $a(\chi) > 1$). Thus, if $l = a(\chi)$, we obtain

$$W_\pi(g_{-2,a(\chi),v}) = q^{1-\frac{a(\chi)}{2}}\zeta_F(1)^{-1}\epsilon(\frac{1}{2},\chi^{-1})\int_{\mathfrak{o}^\times}\chi(y_1)\chi^{-1}(y_1v+\varpi^{a(\chi)-1})\psi(y_1\varpi^{-1})d^\times y_1$$
$$= \chi(v^{-1})\epsilon(\frac{1}{2},\chi^{-1})\zeta_F(1)^{-1}q^{1-\frac{a(\chi)}{2}}\int_{\mathfrak{o}^\times}\chi^{-1}(1+v^{-1}y_1^{-1}\varpi^{a(\chi)-1})\psi(y_1\varpi^{-1})d^\times y_1$$
$$= \chi(v^{-1})\epsilon(\frac{1}{2},\chi^{-1})\zeta_F(1)^{-1}q^{1-\frac{a(\chi)}{2}}S(1,-b_\chi v^{-1},1).$$

In the last step we observed that, if $1 < l = a(\chi)$, we have $a(\chi)-1 \geq \frac{a(\chi)}{2}$ and thus Lemma 3.1.3 can be used to find the desired $b_\chi \in \mathfrak{o}^\times$.

Second, if $l = 1$, the situation is completely different. In this case we have

$$W_\pi(g_{-2,1,v}) = q^{1-\frac{a(\chi)}{2}}\zeta_F(1)^{-1}\epsilon(\frac{1}{2},\chi^{-1}) \cdot\int_{-v^{-1}+\varpi^{1-a(\chi)}\mathfrak{o}^\times}\chi(y_1)\chi^{-1}(y_1v+\varpi^{a(\chi)-1})\psi(y_1\varpi^{-1})d^\times y_1.$$

If $a(\chi) > 1$, we can rewrite this as follows.

$$\int_{-v^{-1}+\varpi^{1-a(\chi)}\mathfrak{o}^\times}\chi(y_1)\chi^{-1}(y_1v+\varpi^{a(\chi)-1})\psi(y_1\varpi^{-1})d^\times y_1$$
$$= \psi(\frac{-1}{v\varpi})\int_{\varpi^{1-a(\chi)}\mathfrak{o}^\times}\chi(y_1-v^{-1})\chi^{-1}(y_1v-1+\varpi^{a(\chi)-1})\psi(y_1\varpi^{-a(\chi)})\frac{\zeta_F(1)dy_1}{|y_1-v^{-1}|}$$
$$= \psi(\frac{-1}{v\varpi})\int_{\mathfrak{o}^\times}\underbrace{\chi(y_1\varpi^{1-a(\chi)}-v^{-1})\chi^{-1}(y_1\varpi^{1-a(\chi)}v-1+\varpi^{a(\chi)-1})}_{=1}\psi(y_1\varpi^{-a(\chi)})d^\times y_1$$
$$= 0.$$

This implies that, if $a(\chi) > 1$ we have

$$W_\pi(g_{-2,1,v}) = 0.$$

Similarly, if $t < -2$, we get

$$\begin{aligned} W_\pi(g_{t,l,v}) &= \sum_{\substack{\mu\in\mathfrak{X}_l,\\ t=-2a(\mu\chi)}}\epsilon(\frac{1}{2},\mu^{-1}\chi^{-1})^2G(\varpi^{-l},\mu^{-1})\mu(v) \\ &= q^{-\frac{t}{2}}\zeta_F(1)^{-2}\sum_{\mu\in\mathfrak{X}_l}G(\varpi^{\frac{t}{2}},\mu\chi)^2G(\varpi^{-l}v,\mu^{-1}). \end{aligned} \tag{3.3.2}$$

At this point we expand the Gauß sums into integrals and use cancellation between the characters $\mu \in \mathfrak{X}_l$. This yields

$$\begin{aligned} W_\pi(g_{t,l,v}) &= q^{-\frac{t}{2}}\zeta_F(1)^{-2}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times}\chi(y_1y_2)\psi(y_1\varpi^{\frac{t}{2}}+y_2\varpi^{\frac{t}{2}}+y_1y_2v\varpi^{-l})d^\times y_1d^\times y_2 \\ &= q^{-\frac{t}{2}}\zeta_F(1)^{-2}K(\chi\otimes\chi,(\varpi^{\frac{t}{2}},\varpi^{\frac{t}{2}}),v\varpi^{-l}). \end{aligned} \tag{3.3.3}$$

In several cases we can obtain further simplification by writing

$$W_\pi(g_{t,l,v}) = q^{-\frac{t}{2}}\zeta_F(1)^{-2}\int_{\mathfrak{o}^\times}\chi(y_1)\psi(y_1\varpi^{\frac{t}{2}})G(\varpi^{\frac{t}{2}}+y_1v\varpi^{-l},\chi)d^\times y_1.$$

Let us assume for the moment that $-l \neq \frac{t}{2}$. This implies $\left|y_1^{-1}\varpi^{\frac{t}{2}}+v\varpi^{-l}\right| = \max(q^l,q^{-\frac{t}{2}})$. Therefore the Gauß sum vanishes whenever $\max(-\frac{t}{2},l)\neq a(\chi)$. Thus, if the Whittaker function is non-zero, we must have $a(\chi)=l>-\frac{t}{2}$ or $a(\chi)=-\frac{t}{2}>l$.

At last we consider $l=-\frac{t}{2}$. From (3.3.2) we deduce that $a(\mu\chi)=l$. Since the support of the Gauß sum implies $\mu\in\mathfrak{X}_l'$ we can assume $l\geq a(\chi)$. Whenever $l<n$ we are happy with the expression given in (3.3.3). On the other hand, if $l\geq n$, we can evaluate the Gauß sum and calculate

$$\begin{aligned}W_\pi(g_{t,l,v}) &= q^{\frac{-t-a(\chi)}{2}}\zeta_F(1)^{-1}\epsilon(\frac{1}{2},\chi^{-1})\int_{-v^{-1}+\varpi^{l-a(\chi)}\mathfrak{o}^\times}\chi\left(\frac{y_1}{1+y_1v}\right)\psi(y_1\varpi^{\frac{t}{2}})d^\times y_1\\ &= q^{\frac{a(\chi)}{2}}\zeta_F(1)^{-1}\epsilon(\frac{1}{2},\chi^{-1})\chi(v^{-1})\psi(\frac{-1}{v\varpi^l})\int_{\mathfrak{o}^\times}\underbrace{\chi(\varpi^{l-a(\chi)}-v^{-1}y_1^{-1})}_{=\chi^{-1}(-vy_1)}\psi(y_1\varpi^{-a(\chi)})d^\times y_1.\end{aligned}$$

This reduces to another Gauß sum which can be evaluated and almost everything cancels out. □

3.3.3 *Irreducible Principal Series*

In this subsection we will treat the Whittaker functions associated to irreducible principle series representations. In this case we work with the algebra $E=F\times F$. For two characters χ_1 and χ_2 on $F^\times$ we write $\chi_1\otimes\chi_2$ for the obvious character on $E^\times$.

We start with the most degenerate case. We are talking about $\pi=\omega_\pi\left|\cdot\right|^s\boxplus\left|\cdot\right|^{-s}$. For notational simplicity we will sometimes write $\chi_1=\omega_\pi\left|\cdot\right|^s$ and $\chi_2=\left|\cdot\right|^{-s}$. In this case we exploit that $K(\omega_\pi^{-1}\otimes 1,\cdot,\cdot)$ degenerates to completely explicit expressions in characters obeying some congruence condition.

Lemma 3.3.5. *Let $\pi = \omega_\pi |\cdot|^s \boxplus |\cdot|^{-s}$. In this situation $n = a(\pi) = a(\omega_\pi)$ and we have*

$$W_\pi(g_{t,l,v}) = \begin{cases} \chi_2(\varpi^{t+2n})q^{-\frac{t+n}{2}}\epsilon(\frac{1}{2},\omega_\pi^{-1}) & \text{if } l = 0 \text{ and } t \geq -n, \\ q^{\frac{l}{2}}\chi_2(\varpi^{-t-2l})\epsilon(\frac{1}{2},\omega_\pi^{-1}) & \text{if } 0 < l \leq \lfloor \frac{n}{2} \rfloor, t = -n-l, \\ & \quad \text{and } v \in b_{\omega_\pi}^{-1} + \mathfrak{p}^l, \\ \omega_\pi(-v^{-1})\psi(-v^{-1}\varpi^{t+l})\chi_2(\varpi^{-t-2l})q^{\frac{n-l}{2}} & \text{if } \lceil \frac{n}{2} \rceil \leq l < n,\ t = -n-l, \\ & \quad \text{and } v \in b_{\omega_\pi}^{-1} + \mathfrak{p}^{n-l}, \\ \omega_\pi(-v^{-1})q^{-\frac{t+2l}{2}}\chi_2(\varpi^{-t-2l})\psi(-v^{-1}\varpi^{t+l}) & \text{if } l \geq n > 0 \text{ and } t \geq -2l, \\ 0 & \text{else.} \end{cases}$$

Proof. The strategy is, as before, to use Lemma 3.2.3 together with the finite Fourier expansion, (3.2.1), of W_π. One sees directly that, if $l = 0$, there is only one contribution. The same is true for $l > 0$ and $t > -l$. Therefore we assume $l > 0$ and $t < -l$. We obtain

$$\begin{aligned} W_\pi(g_{t,l,v}) &= \sum_{\substack{\mu \in \mathfrak{X}_l \setminus \{\omega_\pi^{-1}\}, \\ t+l=-a(\mu\omega_\pi)}} \zeta_F(1)q^{-\frac{l}{2}}\chi_2(\varpi^{-t-2l})\epsilon(\frac{1}{2},\mu^{-1}\omega_\pi^{-1})\mu(-v) \\ &\quad -\delta_{\substack{t=-l-1, \\ \mu=\omega_\pi^{-1}}}\omega_\pi(-v^{-1})\zeta_F(1)q^{-\frac{l+1}{2}}\chi_2(\varpi^{-l+1}). \end{aligned}$$

At this point we consider two cases. First, if $l \leq \lfloor \frac{n}{2} \rfloor$, then we use [69, Lemma 2.37] to obtain

$$\epsilon(\frac{1}{2},\mu^{-1}\omega_\pi^{-1}) = \epsilon(\frac{1}{2},\omega_\pi^{-1})\mu(-b_{\omega_\pi}),$$

where b_{ω_π} is the constant attached to ω_π via Lemma 3.1.3. From this we deduce[2]

$$\begin{aligned} W_\pi(g_{t,l,v}) &= \zeta_F(1)q^{-\frac{l}{2}}\chi_2(\varpi^{-t-2l})\epsilon(\frac{1}{2},\omega_\pi^{-1})\sum_{\mu\in\mathfrak{X}_l}\mu(b_\chi v) \\ &= \begin{cases} q^{\frac{l}{2}}\chi_2(\varpi^{-t-2l})\epsilon(\frac{1}{2},\omega_\pi^{-1}) & \text{if } v \in b_\chi^{-1} + \mathfrak{p}^l, \\ 0 & \text{else.} \end{cases} \end{aligned} \tag{3.3.4}$$

On the other hand, if $l \geq \lceil \frac{n}{2} \rceil$, we write the Gauß sum as integral and take the character sum inside the integral. This leads to

$$W_\pi(g_{t,l,v}) = \omega_\pi(-v^{-1})\chi_2(\varpi^{-t-2l})\zeta_F(1)^{-1}q^{-\frac{t}{2}}\int_{1+\varpi^l\mathfrak{o}}\omega_\pi(x)\psi(-xv^{-1}\varpi^{t+l})d^\times x.$$

2 This is basically the argument used in the proof of [69, Proposition 2.39].

If $l \geq n$, the character is constant in the range of integration. We obtain

$$\begin{aligned} W_\pi(g_{t,l,v}) &= \omega_\pi(-v^{-1})\chi_2(\varpi^{-t-2l})\zeta_F(1)^{-1}q^{-\frac{t}{2}}\int_{1+\varpi^l\mathfrak{o}}\psi(-xv^{-1}\varpi^{t+l})d^\times x \\ &= \omega_\pi(-v^{-1})\chi_2(\varpi^{-t-2l})q^{-\frac{t}{2}}\int_{1+\varpi^l\mathfrak{o}}\psi(-xv^{-1}\varpi^{t+l})dx \\ &= \omega_\pi(-v^{-1})\chi_2(\varpi^{-t-2l})q^{-\frac{t}{2}-l}\psi(-v^{-1}\varpi^{t+l})\underbrace{\int_{\mathfrak{o}}\psi(-xv^{-1}\varpi^{t+2l})dx}_{=\delta_{t\geq -2l}}. \end{aligned}$$

Finally, for $\lceil \frac{n}{2} \rceil \leq l < n$ we can use Lemma 3.1.3 to compute

$$W_\pi(g_{t,l,v}) = \omega_\pi(-v^{-1})\psi(-v^{-1}\varpi^{t+2l})\chi_2(\varpi^{-t-2l})q^{\frac{n-l}{2}}\int_{\mathfrak{o}}\psi(\varpi^{l-n}(b_\chi - v^{-1}))dx.$$

The remaining integral can be evaluated using orthogonality of characters. □

Remark 3.3.6. *Note that, if $\chi \in \mathfrak{X}_n'$ for n even, then the lemma above implies that*

$$\epsilon(\frac{1}{2},\chi) = \chi(b_\chi^{-1})\psi(b_\chi\varpi^{-n}),$$

for $b_\chi \in \mathfrak{o}^\times$ given by Lemma 3.1.3. It is not hard to see that for n odd one has

$$\epsilon(\frac{1}{2},\chi) = \gamma_F(\frac{b_\chi}{2},n)\chi(b_\chi^{-1})\psi(b_\chi\varpi^{-n}).$$

One can check that this is consistent with twist stability. Indeed, if $\mu \in \mathfrak{X}_{\frac{n}{2}}$ we have

$$\begin{aligned} \epsilon(\frac{1}{2},\mu\chi) &= \gamma_F(\frac{b_\chi}{2},n)\mu^{-1}(b_\chi + b_\mu\varpi^{n-a(\mu)})\chi^{-1}(b_\chi + b_\mu\varpi^{n-a(\mu)})\psi(b_\chi\varpi^{-n} + b_\mu\varpi^{-a(\mu)}) \\ &= \mu^{-1}(b_\chi)\epsilon(\frac{1}{2},\chi). \end{aligned}$$

Which is what we expect from [69, Lemma 2.37].

Remark 3.3.7. *We can also give a nice integral representation for $W_\pi(g_{t,l,v})$. Indeed, for $0 < l < n$, one can compute that*

$$W_\pi(g_{t,l,v}) = \chi_2(\varpi^{-t-2l})\omega_\pi(-v^{-1})\zeta_F(1)^{-1}q^{-\frac{t}{2}}G_l\left(-\frac{\varpi^{t+l}}{v},\omega_\pi\right).$$

Where

$$G_l(y,\chi) = \int_{1+\mathfrak{p}^l}\psi(yx)\omega_\pi(y)d^\times y$$

is an incomplete Gauß sum. This is the path taken in [4, Lemma 3.4]. Comparing this to the results from Lemma 3.3.5 we obtain

$$G_l(y\varpi^k,\chi) = \begin{cases} \epsilon(\frac{1}{2},\chi^{-1})\zeta_F(1)\chi^{-1}(y)q^{-\frac{k}{2}} & \text{if } l \leq \lfloor \frac{a(\chi)}{2} \rfloor,\ k=-a(\chi) \text{ and } y \in -b_\chi + \mathfrak{p}^l, \\ \zeta_F(1)\psi(y\varpi^k) & \text{if } \lceil \frac{a(\chi)}{2} \rceil \leq l,\ k=-a(\chi) \text{ and } y \in -b_\chi + \mathfrak{p}^{a(\chi)-l}, \\ 0 & \text{else,} \end{cases}$$

for $\chi \in \mathfrak{X}$, $y \in \mathfrak{o}^\times$ and $0 < l < n$. This is an extension of the upper bound given in [4, Remark 5.9].

Next we will look at another degenerate situation.

Lemma 3.3.8 ([4], Lemma 3.5)**.** *Let $\pi = \chi|\cdot|^s \boxplus \chi|\cdot|^{-s}$ for a non trivial character χ. If $a(\chi) \geq 1$ and $l \neq a(\chi) = \frac{n}{2}$, we have*

$$W(g_{t,l,v}) = \begin{cases} \epsilon(\frac{1}{2},\tilde{\pi}) & \text{if } t=-n \text{ and } l=0, \\ q^{-\frac{t}{2}}\zeta_F(1)^{-2}K(\chi\circ Nr_{E/F},(\varpi^{\frac{t}{2}},\varpi^{\frac{t}{2}}),v\varpi^{-l}) & \text{if } t=-\max(n,2l) \\ & \quad \text{and } 0<l<n, \\ \chi^2(-v^{-1})\psi(-v^{-1}\varpi^{-l}) & \text{if } t=-2l \text{ and } l \geq n, \\ 0 & \text{else.} \end{cases}$$

Finally, if $a(\chi) > 1$ and $l = a(\chi) = \frac{n}{2}$, we have the degenerate situation

$$W(g_{t,l,v}) = \begin{cases} q^{-\frac{t}{2}}G(v\varpi^{-l},\chi)\big(-q^{-1}\zeta_F(1)^{-1}(q^{s(t+2)}+q^{-s(t+2)}) & \\ \qquad +\zeta_F(1)^{-2}\sum_{k=0}^{t} q^{s(t-2k)}\big) & \text{if } t \geq 0, \\ -q^{-\frac{1}{2}}\zeta_F(1)^{-1}G(v\varpi^{-l},\chi)(q^s+q^{-s}) & \text{if } t=-1, \\ q\zeta_F(1)^{-2}K(\chi\circ Nr_{E/F},(\varpi^{-1},\varpi^{-1}),v\varpi^{-l}) & \text{if } t=-2 \text{ and } l=1, \\ \chi(v^{-1})\epsilon(\frac{1}{2},\chi^{-1})\zeta_F(1)^{-1}q^{1-\frac{a(\chi)}{2}}S(1,-b_\chi v^{-1},1) & \text{if } t=-2 \text{ and } l>1, \\ q^{-\frac{t}{2}}\zeta_F(1)^{-2}K(\chi\circ Nr_{E/F},(\varpi^{\frac{t}{2}},\varpi^{\frac{t}{2}}),v\varpi^{-l}) & \text{if } -2l \leq t < -2 \text{ even}, \\ 0 & \text{else.} \end{cases}$$

Proof. Interesting situations occur only for $t \leq -2$. For those cases we have

$$\begin{aligned} W_\pi(g_{t,l,v}) &= \sum_{\substack{\mu \in \mathfrak{X}_l, \\ t=-2a(\mu\chi)}} \epsilon(\frac{1}{2}, \mu^{-1}\chi^{-1})^2 G(v\varpi^{-l}, \mu^{-1}) + \delta_{t=-2} q^{-1} G(v\varpi^{-l}, \chi) \\ &= \zeta_F(1)^{-2} q^{-\frac{t}{2}} \sum_{\mu \in \mathfrak{X}_l} G(\varpi^{\frac{t}{2}}, \mu\chi)^2 G(v\varpi^{-l}, \mu^{-1}). \end{aligned}$$

We have seen this exact sum already in (3.3.1) and (3.3.2). The rest of the proof is left to the reader. □

Finally, we treat the general irreducible principal series.

Lemma 3.3.9 ([4], Lemma 3.6). *Let* $\pi = \chi_1 |\cdot|^s \boxplus \chi_2 |\cdot|^{-s}$ *with* $a(\chi_1) \geq a(\chi_2) > 0$. *If* $l \notin \{a(\chi_1), a(\chi_2)\}$, *then*

$$W_\pi(g_{t,l,v}) = \begin{cases} \epsilon(\frac{1}{2}, \tilde{\pi}) & \text{if } l = 0 \text{ and } t = -n, \\ \zeta_F(1)^{-2} q^{-\frac{t}{2}} q^{s(a(\chi_1)-a(\chi_2))} K(\chi_1 \otimes \chi_2, (\varpi^{-a(\chi_1)}, \varpi^{-a(\chi_2)}), v\varpi^{-l}) & \text{if } t = -n \text{ and } 0 < l < a(\chi_2), \\ \zeta_F(1)^{-2} q^{-\frac{t}{2}} q^{s(a(\chi_1)-l)} K(\chi_1 \otimes \chi_2, (\varpi^{-a(\chi_1)}, \varpi^{-l}), v\varpi^{-l}) & \text{if } t = -l - a(\chi_1) \text{ and } a(\chi_2) < l < a(\chi_1), \\ \zeta_F(1)^{-2} q^{-\frac{t}{2}} K(\chi_1 \otimes \chi_2, (\varpi^{-l}, \varpi^{-l}), v\varpi^{-l}) & \text{if } t = -2l \text{ and } a(\chi_1) < l < n, \\ \omega_\pi(-v^{-1}) \psi(-v^{-1}\varpi^{-l}) & \text{if } t = -2l \text{ and } l \geq n\ , \\ 0 & \text{else.} \end{cases}$$

If $l = a(\chi_i) \neq a(\chi_j)$ *for* $\{i,j\} = \{1,2\}$, *then*

$$W_\pi(g_{t,l,v}) = \begin{cases} \zeta_F(1)^{-2} q^{-\frac{t}{2}} q^{s(2a(\chi_1)+t)} K(\chi_1 \otimes \chi_2, (\varpi^{-a(\chi_1)}, \varpi^{a(\chi_1)+t}), v\varpi^{-l}) \\ \qquad\qquad \text{if } l = a(\chi_2) \text{ and } -n \leq t < -a(\chi_1), \\ \zeta_F(1)^{-2} q^{-\frac{t}{2}} q^{st} G(\varpi^{-a(\chi_1)}, \chi_2^{-1}\chi_1) G(v\varpi^{-a(\chi_2)}, \chi_2) \\ \qquad\qquad \text{if } l = a(\chi_2) \text{ and } t \geq -a(\chi_1), \\ \zeta_F(1)^{-2} q^{-\frac{t}{2}} q^{s(-t-2l)} K(\chi_1 \otimes \chi_2, (\varpi^{l+t}, \varpi^{-l}), v\varpi^{-l}) \\ \qquad\qquad \text{if } l = a(\chi_1) \neq a(\chi_2) \text{ and } -2l \leq t < -a(\chi_1), \\ \zeta_F(1)^{-2} q^{-\frac{t}{2}} q^{-st} G(\varpi^{-a(\chi_1)}, \chi_2\chi_1^{-1}) G(v\varpi^{-a(\chi_1)}, \chi_1) \\ \qquad\qquad \text{if } l = a(\chi_1) \text{ and } t \geq -a(\chi_1), \\ 0 \\ \qquad\qquad \text{else.} \end{cases}$$

And if $l = a(\chi_1) = a(\chi_2)$, *then*

$$W_\pi(g_{t,l,v}) = \begin{cases} \zeta_F(1)^{-2} q^{-\frac{t}{2}} \sum_{l_2=1}^{l} q^{s(-t-2l_2)} K(\chi_1 \otimes \chi_2, (\varpi^{t+l_2}, \varpi^{-l_2}), v\varpi^{-l}) \\ +\delta_{t \geq -a(\chi_1^{-1}\chi_2)} \zeta_F(1)^{-2} q^{-\frac{t}{2}} \Big[G(v\varpi^{-l}, \chi_1) G(\varpi^{-a(\chi_1^{-1}\chi_2)}, \chi_1^{-1}\chi_2) q^{-st} \\ \qquad\qquad + G(v\varpi^{-l}, \chi_2) G(\varpi^{-a(\chi_2^{-1}\chi_1)}, \chi_2^{-1}\chi_1) q^{st} \Big] \\ \qquad\qquad\qquad \text{if } -n \leq t \leq -2, \\ \sum_{\{i,j\}=\{1,2\}} \chi_i(v^{-1}\varpi^{t+l}) q^{-\frac{t+a(\chi_i^{-1}\chi_j)+l}{2}} \epsilon(\tfrac{1}{2}, \chi_i) \epsilon(\tfrac{1}{2}, \chi_i^{-1}\chi_j) \\ \qquad\qquad\qquad \text{if } t > -2, \\ 0 \\ \qquad\qquad\qquad \text{else.} \end{cases}$$

Proof. Let us consider the interesting situation $l > 0$. For $t > -2$ the only contribution comes from the characters $\mu \in \{\chi_1^{-1}, \chi_2^{-1}\}$ which is easily written down. We thus assume $t \leq -2$. Applying the usual tricks we end up with

$$\begin{aligned} W_\pi(g_{t,l,v}) =& \zeta_F(1)^{-2} q^{-\frac{t}{2}} \sum_{\substack{t=-l_1-l_2,\\ l_1,l_2>0}} q^{s(l_1-l_2)} \sum_{\mu\in\mathfrak{X}_l} G(\varpi^{-l_1}, \mu\chi_1) G(\varpi^{-l_2}, \mu\chi_2) G(v\varpi^{-l}, \mu^{-1}) \\ &+ \delta_{t\geq -a(\chi_1^{-1}\chi_2)} \zeta_F(1)^{-2} q^{-\frac{t}{2}} \Big[G(v\varpi^{-l}, \chi_1) G(\varpi^{-a(\chi_1^{-1}\chi_2)}, \chi_1^{-1}\chi_2) q^{-st} \\ &+ G(v\varpi^{-l}, \chi_2) G(\varpi^{-a(\chi_2^{-1}\chi_1)}, \chi_2^{-1}\chi_1) q^{st} \Big]. \end{aligned} \tag{3.3.5}$$

As earlier we can compress the μ-sum to $K(\chi_1 \otimes \chi_2, (\varpi^{-l_1}, \varpi^{-l_2}), v\varpi^{-l})$. This gives

$$\begin{aligned} W_\pi(g_{t,l,v}) =& \zeta_F(1)^{-2} q^{-\frac{t}{2}} \sum_{\substack{t=-l_1-l_2,\\ l_1,l_2>0}} q^{s(l_1-l_2)} K(\chi_1 \otimes \chi_2, (\varpi^{-l_1}, \varpi^{-l_2}), v\varpi^{-l}) \\ &+ \delta_{t\geq -a(\chi_1^{-1}\chi_2)} \zeta_F(1)^{-2} q^{-\frac{t}{2}} \Big[G(v\varpi^{-l}, \chi_1) G(\varpi^{-a(\chi_1^{-1}\chi_2)}, \chi_1^{-1}\chi_2) q^{-st} \\ &+ G(v\varpi^{-l}, \chi_2) G(\varpi^{-a(\chi_2^{-1}\chi_1)}, \chi_2^{-1}\chi_1) q^{st} \Big]. \end{aligned} \tag{3.3.6}$$

We will treat different ranges of l case by case.

First, consider $0 < l < a(\chi_2)$. In this case the δ-term does not contribute. Furthermore, we are only in a non-zero situation if $t = -n$. This is because only $l_1 = a(\chi_1)$ and $l_2 = a(\chi_2)$ contribute to the sum.

Next, we look at $l > a(\chi_1)$. This case is quite similar. Indeed, the only contribution comes from $l_1 = l_2 = l$, so that $t = -2l$. The δ-term can't appear. If we further assume $l \geq n$, then it reduces to a normal Gauß sum involving χ_2 and we obtain

$$\begin{aligned} W_\pi(g_{-2l,l,v}) &= \zeta_F(1)^{-1} q^{l-\frac{a(\chi_1)}{2}} \epsilon(\frac{1}{2}, \chi_1^{-1}) \int_{-v^{-1}+\varpi^{-a(\chi_1)+l}\mathfrak{o}^\times} \chi_2(y)\chi_1^{-1}(1+vy)\psi(y\varpi^{-l}) d^\times y \\ &= \omega_\pi(-v^{-1})\psi(-v^{-1}\varpi^{-l}). \end{aligned}$$

Let us investigate $a(\chi_2) < l < a(\chi_1)$. Again no δ-term occurs and the only non-zero situation is $t = -l - a(\chi_1)$ with $l_1 = a(\chi_1)$ and $l_2 = l$.

If $l = a(\chi_2) < a(\chi_1)$, we observe that for $t \geq -a(\chi_1)$ only the δ-term contributes and we obtain

$$W_\pi(g_{t,a(\chi_2),v}) = \zeta_F(1)^{-2} q^{-\frac{t}{2}} q^{st} G(\varpi^{-a(\chi_2^{-1}\chi_1)}, \chi_2^{-1}\chi_1) G(v\varpi^{-a(\chi_2)}, \chi_2).$$

On the other hand, for $-n \leq t < -a(\chi_1)$, no δ-term occurs and we have $l_1 = a(\chi_1)$ and $l_2 = -t - a(\chi_1)$.

Similarly, when $l = a(\chi_1) > a(\chi_2)$, only the δ-term contributes to $t \geq -a(\chi_1)$. This gives

$$W_\pi(g_{-a(\chi_1),a(\chi_1),v}) = \zeta_F(1)^{-2} q^{-\frac{t}{2}} q^{-st} G(\varpi^{-a(\chi_1^{-1}\chi_2)}, \chi_1^{-1}\chi_2) G(v\varpi^{-a(\chi_1)}, \chi_1).$$

Therefore the interesting case is $-2l \leq t < -a(\chi_1)$. This forces $l_2 = l$ and $l_1 = -t - l$.

Finally, we are left with the critical situation $l = a(\chi_1) = a(\chi_2)$. Without loss of generality we assume $t \geq -n = -2l$. We can rewrite (3.3.6) as

$$\begin{aligned} W_\pi(g_{t,l,v}) =& \zeta_F(1)^{-2} q^{-\frac{t+l}{2}} \sum_{l_2=1}^{l} q^{s(-t-2l_2)} K(\chi_1 \otimes \chi_2, (\varpi^{t+l_2}, \varpi^{-l_2}), v\varpi^{-l}) \\ &+ \delta_{t \geq -a(\chi_1^{-1}\chi_2)} \zeta_F(1)^{-2} q^{-\frac{t}{2}} \Big[G(v\varpi^{-l}, \chi_1) G(\varpi^{-a(\chi_1^{-1}\chi_2)}, \chi_1^{-1}\chi_2) q^{-st} \\ &+ G(v\varpi^{-l}, \chi_2) G(\varpi^{-a(\chi_2^{-1}\chi_1)}, \chi_2^{-1}\chi_1) q^{st} \Big]. \end{aligned}$$

□

Remark 3.3.10. *If $a(\chi_1) = a(\chi_2) = l$, one can see that Lemma 3.3.9 fails to provide a simple integral representation of $W_\pi(g_{t,l,v})$. Instead we end up having a sum of several integrals. We will see later, by investigating the K-integrals for different l_2, that all but one or maximally two terms in the sum are zero. However, we can also sketch a simpler argument here. We consider two cases.*

First, let $t \geq -2a(\chi_1\chi_2^{-1})$. Suppose there is μ such that $-t = l_1 + l_2$ with $l_1 = a(\mu\chi_1)$ and $l_2 = a(\mu\chi_2)$. These are exactly the values of l_1 and l_2 we need to consider. Suppose $l_1 \neq l_2$, then

$$a(\chi_1\chi_2^{-1}) = a((\mu\chi_1) \cdot (\mu\chi_2)^{-1}) = \max(l_1, l_2).$$

But this implies $-t < 2a(\chi_1\chi_2^{-1})$, which is a contradiction. Thus, for $t \geq -2a(\chi_1\chi_2^{-1})$, the only possible configuration is $l_1 = l_2 = -\frac{t}{2}$.

Second, let $t < -2a(\chi_1\chi_2^{-1})$. Suppose that $l_1 = l_2$. Then $a(\chi_1\chi_2^{-1}) \leq l_1$. But this implies $-t = 2l_1 \geq -a(\chi_1\chi_2^{-1})$, which is a contradiction. We conclude that $l_1 \neq l_2$. Using the same trick as above implies that $\max(l_1, l_2) = a(\chi_1\chi_2^{-1})$. This leaves us with exactly two possible configurations, namely $l_1 = a(\chi_1\chi_2^{-1})$ and $l_2 = -a(\chi_1\chi_2^{-1}) - t$ or $l_2 = a(\chi_1\chi_2^{-1})$ and $l_1 = -a(\chi_1\chi_2^{-1}) - t$.

3.3.4 *Summary*

For convenience we will summarise the results of this section in a condensed form. We start by recalling the generalised Atkin-Lehner relation. Indeed, for every π one has[3]

$$W_\pi(g_{t,l,v}) = \epsilon(\frac{1}{2},\tilde{\pi})\omega_{\tilde{\pi}}(v)\psi(-v^{-1}\varpi^{t+l})W_{\tilde{\pi}}(g_{t+2l-n,n-l,-v})$$

for $0 \leq l \leq n$.

We state our expressions for the Whittaker new vector focusing only on the non-zero configurations. Thus, every quadruple (π, t, l, v) not mentioned in the following implies $W_\pi(g_{t,l,v}) = 0$.

First, we look at some general formulae that are valid for every π. With

$$L(s,\pi) = \sum_{k\in\mathbb{Z}} \lambda_\pi(\mathfrak{p}^t)q^{-st}$$

we have[4]

$$W_\pi(g_{t,0,v}) = \epsilon(\frac{1}{2},\tilde{\pi})q^{-\frac{t+n}{2}}\lambda_\pi(\mathfrak{p}^{t+n})$$

and

$$W_\pi(g_{t,l,v}) = \omega_{\tilde{\pi}}(-v)\psi(-v^{-1}\varpi^{-l})q^{-\frac{t+2l}{2}}\lambda_{\tilde{\pi}}(\mathfrak{p}^{t+2l})$$

for $l \geq n$. We now treat the remaining cases of l. The best description of W_π is available for twist-minimal non-supercuspidal representations. The case $\pi = \mathrm{St}$ is covered by the formulae given above. For $\pi = \omega_\pi |\cdot|^s \boxplus |\cdot|^{-s}$ we have

$W_\pi(g_{t,l,v})$	l	t	v
$\epsilon(\frac{1}{2},\tilde{\pi})q^{\frac{l}{2}-sl}$	$0 < l \leq \lfloor\frac{n}{2}\rfloor$	$-n-l$	$v \in b_{\omega_\pi}^{-1} + \mathfrak{p}^l$
$\omega_{\tilde{\pi}}(-v)\psi(-v^{-1}\varpi^{-n})q^{\frac{n-l}{2}-s(l-n)}$	$\lceil\frac{n}{2}\rceil \leq l < n$	$-n-l$	$v \in b_{\omega_\pi}^{-1} + \mathfrak{p}^{n-l}$

Also twist-minimal supercuspidal representation have very nice properties, in particular concerning their support. However, at this point it seems more practical to treat them with the remaining cases. Here we distinguish two main cases. If $l \neq \frac{n}{2}$, we have

$$W_\pi(g_{-\max(n,2l),l,v}) = C_\pi q^{-\frac{t}{2}}K(\xi, A_{\pi,l}, v\varpi^{-l}).$$

We record the values of C_π, ξ and $A_{\pi,l}$ in the following table.

3 This is a combination of [69, Lemma 2.18, Corollary 2.26 and Proposition 2.28].

4 These can also be proven directly using Atkin-Lehner relations and the well known values of W_π on the diagonal. This is because the upper triangular matrices making up the small Bruhat cell can be written as $Ng_{*,n,*}K_1(n)$.

π	C_π	ξ	$A_{\pi,l}$
Supercuspidal	$\xi\colon E^\times \to \mathbb{C}^\times$ such that $\tilde{\pi} = \omega_{\xi^{-1}}$	γ	$\Omega^{-\frac{\max(n,2l)}{f}}$
χSt	$\chi \otimes \chi\colon F \times F \to \mathbb{C}^\times$	$\zeta_F(1)^{-2}$	$(\varpi^{-\frac{\max(n,2l)}{2}}, \varpi^{-\frac{\max(n,2l)}{2}})$
$\chi_1 \lvert\cdot\rvert^s \boxplus \chi_2 \lvert\cdot\rvert^{-s}$	$\chi_1 \otimes \chi_2\colon F \times F \to \mathbb{C}^\times$	$\zeta_F(1)^{-2} \cdot q^{s\max(a(\chi_1),l)} \cdot q^{-s\max(a(\chi_2),l)}$	$(\varpi^{-\max(a_1,l)}, \varpi^{-\max(a_2,l)})$

With the slight addition,

$$W_\pi(g_{t,a(\chi_2),v}) = \begin{cases} \zeta_F(1)^{-2} q^{-\frac{t}{2}+s(2a(\chi_1)+t)} K(\chi_1 \otimes \chi_2, (\varpi^{-a(\chi_1)}, \varpi^{a(\chi_1)+t}), v\varpi^{-l}) \\ \qquad \text{if } -n < t < -a(\chi_1), \\ \epsilon(\frac{1}{2}, \pi_{\min})\epsilon(\frac{1}{2}, \chi_2^{-1})\chi_2^{-1}(v) q^{-\frac{t+n}{2}} \lambda_{\tilde{\pi}_{\min}}(\mathfrak{p}^{t+a(\chi_1)}) \\ \qquad \text{if } -a(\chi_1) \le t \end{cases}$$

and

$$W_\pi(g_{t,a(\chi_1),v}) = \begin{cases} \zeta_F(1)^{-2} q^{-\frac{t}{2}-s(2a(\chi_1)+t)} K(\chi_1 \otimes \chi_2, (\varpi^{a(\chi_1)+t}, \varpi^{-a(\chi_1)}), v\varpi^{-l}) \\ \qquad \text{if } -2a(\chi_1) < t < -a(\chi_1), \\ \epsilon(\frac{1}{2}, \tilde{\pi}_{\min})\epsilon(\frac{1}{2}, \chi_1^{-1}\chi_1^{-1}(v)) q^{-\frac{t+2a(\chi_1)}{2}} \lambda_{\pi_{\min}}(\mathfrak{p}^{t+a(\chi_1)}) \\ \qquad \text{if } -a(\chi_1) \le t \end{cases}$$

if $\pi = \chi_1 \lvert\cdot\rvert^s \boxplus \chi_2 \lvert\cdot\rvert^{-s}$ for $a(\chi_1) > a(\chi_2) > 0$, we cover everything outside the transition range.

If $l = \frac{n}{2}$, we are in the transition region and the results are slightly more complicated. We define $\pi_{\min} = \chi\pi$ for χ such that $n_0(\pi) = \min_\xi a(\xi\pi) = a(\chi\pi)$. Note that this is not uniquely determined. For principal series representations we require $\pi_{\min} = \lvert\cdot\rvert^s \boxplus \chi_1^{-1}\chi_2 \lvert\cdot\rvert^{-s}$. If we are assuming that $\omega_{\pi_{\min}}(\varpi) = 1$, the only other choice is $\tilde{\pi}_{\min}$. In the case of supercuspidal π we will only need the invariant $n_0(\pi) = a(\pi_{\min})$ which is well defined. With this at hand the results of this section are reflected in the following table.

π	$W_\pi(g_{t,\frac{n}{2},v})$	t
$\tilde{\pi} = \omega_\xi$	$\gamma q^{-\frac{t}{2}} K(\xi^{-1}, \Omega^{\frac{t}{f}}, v\varpi^{-l})$	$t \in [-n, -n_0(\pi)]$ and $\frac{t}{f} \in \mathbb{Z}$
χSt	$\zeta_F(1)^{-2} q^{-\frac{t}{2}}$ $\cdot K(\chi \circ \mathrm{Nr}_{E/F}, (\varpi^{\frac{t}{2}}, \varpi^{\frac{t}{2}}), v\varpi^{-l})$	$t \in [-n, -n_0(\pi))$ and t even
	$-\frac{\zeta_F(1)}{\zeta_F(2)} \epsilon(\frac{1}{2}, \chi^{-1}) \chi^{-1}(v) q^{-\frac{n}{4} - \frac{t+n_0(\pi)}{2}}$ $\cdot \lambda_{\pi_{\min}}(\mathfrak{p}^{t+n_0(\pi)})$	$t \geq -n_0(\pi)$
$\chi \lvert\cdot\rvert^s \boxplus \chi \lvert\cdot\rvert^{-s}$	$\zeta_F(1)^{-2} q^{-\frac{t}{2}}$ $\cdot K(\chi \circ \mathrm{Nr}_{E/F}, (\varpi^{\frac{t}{2}}, \varpi^{\frac{t}{2}}), v\varpi^{-l})$	$t \in [-n, -2]$ and t even
	$\epsilon(\frac{1}{2}, \chi^{-1}) \chi^{-1}(v) q^{-\frac{1}{2} - \frac{n}{4}} \lambda_{\pi_{\min}}(\mathfrak{p})$	$t = -1$
	$\epsilon(\frac{1}{2}, \chi^{-1}) \chi^{-1}(v) q^{-\frac{t}{2} - \frac{n}{4}}$ $\cdot (\lambda_{\pi_{\min}}(\mathfrak{p}^t) - q^{-1} \lambda_{\pi_{\min}}(\mathfrak{p}) \lambda_{\pi_{\min}}(\mathfrak{p}^{t+1}))$	$t \geq 0$
$\chi_1 \lvert\cdot\rvert^s \boxplus \chi_2 \lvert\cdot\rvert^{-s}$, $a(\chi_1) > a(\chi_2) > 0$	$\zeta_F(1)^{-2} q^{-\frac{t}{2} - s(a(\chi_1) - \frac{n}{2})}$ $\cdot K(\chi_1 \otimes \chi_2, (\varpi^{-a(\chi_1)}, \varpi^{-\frac{n}{2}}), v\varpi^{-l})$	$t = -\frac{n}{2} - a(\chi_1)$
$\chi_1 \lvert\cdot\rvert^s \boxplus \chi_2 \lvert\cdot\rvert^{-s}$, $a(\chi_1) = a(\chi_2)$	$q^{-\frac{t+n_0(\pi)}{2} - \frac{n}{4}}$ $\cdot \Big[\epsilon(\frac{1}{2}, \tilde{\pi}_{\min}) \epsilon(\frac{1}{2}, \chi_1^{-1}) \chi_1^{-1}(v) \lambda_{\pi_{\min}}(\mathfrak{p}^{t+n_0(\pi)})$ $+ \epsilon(\frac{1}{2}, \pi_{\min}) \epsilon(\frac{1}{2}, \chi_2^{-1}) \chi_2^{-1}(v) \lambda_{\tilde{\pi}_{\min}}(\mathfrak{p}^{t+n_0(\pi)})\Big]$	$t \geq -n_0(\pi)$
	$\zeta(1)^{-2} q^{-\frac{t}{2}} \Big[q^{s(t+2n_0(\pi))}$ $\cdot K(\chi_1 \otimes \chi_2, (\varpi^{-n_0(\pi)}, \varpi^{t+n_0(\pi)}), v\varpi^{-l})$ $+ q^{-s(t+2n_0(\pi))}$ $K(\chi_1 \otimes \chi_2, (\varpi^{t+n_0(\pi)}, \varpi^{-n_0(\pi)}), v\varpi^{-l})\Big]$	$-n_0(\pi) > t$, $t > -2n_0(\pi)$
	$\zeta(1)^{-2} q^{-\frac{t}{2}} K(\chi_1 \otimes \chi_2, (\varpi^{\frac{t}{2}}, \varpi^{\frac{t}{2}}), v\varpi^{-l})$	$-n \leq t \leq -2n_0(\pi)$

Note that in the case $a(\chi_1) = a(\chi_2)$ we also implemented the results from Lemma 3.4.15 below.

3.4 THE SIZE OF WHITTAKER NEW VECTORS

In this section we estimate the size of Whittaker new vectors, extending [4, Section 5]. To do so we will built on the integral representations for $W_\pi(g_{t,l,v})$ derived in the previous section. This reduces the problem to estimate $K(\xi, A, B)$ in several situations. Due to the generality of this sum there are many cases which seem quite different in nature and the estimation turns out to be Sisyphus work. This upcoming case study relies heavily on repeated use of the method of stationary phase as described earlier. Throughout this section we assume that F has odd residual characteristic.

3.4.1 *Dihedral supercuspidal representations*

There are two slightly different types of dihedral supercuspidal representations. We start with representations associated to unramified quadratic extensions of E/F.

Lemma 3.4.1 ([4], Lemma 5.1). *Let π be a dihedral supercuspidal representation associated to an unramified quadratic extension $E = F(\sqrt{\zeta})$ of F and a character $\xi\colon E^\times \to S^1$. Then we have*

$$W_\pi(g) \ll_F q^{\frac{n}{12}}.$$

If $\kappa_F = 1$ and $n \geq 6$, the implicit constant is less than 2.

Even more, for $n > 2$, we define $k = \max(n, 2l)$, and write $b_\xi = b_1 + \sqrt{\zeta} b_2$. We obtain the following more detailed results. If $0 < l < \frac{n}{2}$, we have

$$W_\pi(g_{-n,l,v}) = \gamma\gamma_F(\zeta Nr_{E/F}(b), \frac{n}{2})\xi^{-1}(x_0 + \sqrt{\zeta}b_2)\psi((x_1 + b_1)\varpi^{-\frac{n}{2}}),$$

where $x_0 \in \mathfrak{o}^\times$ is the unique solution to

$$v\varpi^{\frac{k}{2}-l}x^2 + x - (b_1 + v\zeta b_2^2\varpi^{k-\frac{n}{2}-l})\varpi^{\frac{k-n}{2}} = 0. \tag{3.4.1}$$

If $l = \frac{n}{2}$, $Nr_{E/F}(b) \notin \mathfrak{o}^{2\times}$ and $\Delta = 1 + 4v^2b_2^2\zeta + 4vb_1 \in \mathfrak{o}^\times$, we have

$$W_\pi(g_{-n,\frac{n}{2},v}) = \begin{cases} \gamma \sum_\pm \gamma_F(\zeta(Nr_{E/F}(b) - Nr_{E/F}(x_\pm)^2v^2), \frac{n}{2})\xi^{-1}(x_\pm + \sqrt{\zeta}b_2)\psi((x_\pm + b_1)\varpi^{-\frac{n}{2}}) \\ \qquad \text{if } \Delta \in \mathfrak{o}^{2\times}, \\ 0 \qquad \text{else} \end{cases}$$

where $x_{\pm} \in \mathfrak{o}^{\times}$ are the unique solutions to (3.4.1). If $\mathrm{Nr}_{E/F}(b) \in \mathfrak{o}^{2\times}$, degenerate critical points are possible and we end up with $\left|W_{\pi}(g_{-n,\frac{n}{2},v})\right| \leq 2\max(q^{\frac{1}{2}}, q^{\frac{n}{12}})$. For $t > -n$ we have the upper bounds

$$W_{\pi}(g_{t,\frac{n}{2},v}) \ll \begin{cases} q^{-\frac{n+t}{4}} & \text{if } t < -n_0(\pi), \\ q^{-\frac{n+t}{4}+\frac{n_0(\pi)}{12}} & \text{if } t = -n_0(\pi). \end{cases}$$

Finally, if $\frac{n}{2} < l < n$, we have

$$W_{-2l,l,v} = (-1)^l \gamma \xi^{-1}(x_0 + \sqrt{\zeta} b_2 \varpi^{l-\frac{n}{2}}) \psi(x_0 \varpi^{-l} + b_1 \varpi^{-\frac{n}{2}}),$$

where $x_0 \in \mathfrak{o}^{\times}$ is the unique solution to (3.4.1).

Proof. We start by recalling some facts concerning the extension E/F. Since it is unramified we have $e = 1$, $f = 2$, and $d = 0$. In particular, $n = 2a(\xi)$, $a(\psi_E) = 0$, and $\mathrm{Vol}(\mathfrak{O}, d_E) = 1$. Furthermore, because the extension is unramified, we have $\zeta \in \mathfrak{o}^{\times} \setminus \mathfrak{o}^{2\times}$. Note that $\mathfrak{O} = \mathfrak{o} \oplus \mathfrak{o}\sqrt{\zeta}$ and $\mathfrak{O}^{\times} = (\mathfrak{o}^{\times} \oplus \mathfrak{o}\sqrt{\zeta}) \cup (\mathfrak{o} \oplus \mathfrak{o}^{\times}\sqrt{\zeta})$. We choose uniformisers such that $\Omega = \varpi$. For $x = a + b\sqrt{\zeta}$ we compute

$$\mathrm{Nr}_{E/F}(1+x) = (1 + a + b\sqrt{\zeta})(1 + a - b\sqrt{\zeta}) = 1 + \mathrm{Tr}(x) + \mathrm{Nr}_{E/F}(x). \tag{3.4.2}$$

We put $k = \max(2l, n)$. According to Lemma 3.3.1 we need to consider $t = -k$ if $l \neq \frac{n}{2}$ and $0 > t \geq -k$ if $l = \frac{n}{2}$, since otherwise $W_{\pi}(g_{t.l.v})$ vanishes. In these cases we have

$$W_{\pi}(g_{t,l,v}) = \gamma q^{-\frac{t}{2}} \int_{\mathfrak{O}^{\times}} \xi^{-1}(x) \psi(\varpi^{\frac{t}{2}} \mathrm{Tr}(x) + \varpi^{-l} v \mathrm{Nr}_{E/F}(x)) d_E x. \tag{3.4.3}$$

We write $\frac{k}{2} = 2r + \rho$ for some $r \in \mathbb{N}_0$ and $\rho \in \{0, 1\}$. First, we note that if $r = 0$, then we must have $\rho = 1$ and thus $a(\xi) = l = -\frac{t}{2} = 1$. By [69, Corollary 2.35] we have

$$W_{\pi}(g_{t,l,v}) \leq \sqrt{2p}.$$

From now on we assume $r \geq 1$. In this case we can calculate

$$\begin{aligned}
W_\pi(g_{t,l,v}) &= \gamma q^{-\frac{t}{2}} \int_{\mathfrak{O}^\times} \xi^{-1}(x)\psi\left(\varpi^{\frac{t}{2}}\mathrm{Tr}(x) + \varpi^{-l} v \mathrm{Nr}_{E/F}(x)\right) d_E x \\
&= \gamma q^{-2r-\frac{t}{2}} \sum_{x \in (\mathfrak{O}/\mathfrak{P}^r)^\times} \xi^{-1}(x)\psi\left(\varpi^{\frac{t}{2}}\mathrm{Tr}(x)\right) \\
&\quad \cdot \int_{\mathfrak{O}} \xi^{-1}\left(1 + \frac{y}{x}\Omega^r\right)\psi_E\left(\Omega^{r+\frac{t}{2}} y + \frac{v\Omega^{-l}}{2}\mathrm{Nr}_{E/F}(x + y\Omega^r)\right) d_E y \\
&= \gamma q^{-2r-\frac{t}{2}} \sum_{x \in (\mathfrak{O}/\mathfrak{P}^r)^\times} \xi^{-1}(x)\psi\left(\varpi^{\frac{t}{2}}\mathrm{Tr}(x) + v\varpi^{-l}\mathrm{Nr}_{E/F}(x)\right) \\
&\quad \cdot \int_{\mathfrak{O}} \psi_E\Bigg(\left(-b_\xi \Omega^{\frac{k}{2}-a(\xi)} + x\Omega^{\frac{k}{2}+\frac{t}{2}} + v\mathrm{Nr}_{E/F}(x)\Omega^{\frac{k}{2}-l}\right) y \Omega^{-r-\rho} \\
&\qquad + \left(\frac{v\Omega^{\frac{k}{2}-l}}{2}\mathrm{Nr}_{E/F}(xy) + \frac{b_\xi \Omega^{\frac{k}{2}-a(\xi)}}{2} y^2\right)\varpi^{-\rho}\Bigg) d_E y.
\end{aligned}$$

Next we will transform the remaining integral in a 2-dimensional Gauß sum. To do so we recall that $\mathfrak{O} = \mathfrak{o} \oplus \mathfrak{o}\sqrt{\zeta}$ and view the integral as an two dimensional integral over $\mathfrak{o}$. The quadratic term is

$$\mathrm{Tr}\left(\frac{v\varpi^{\frac{k}{2}-l}}{2}\mathrm{Nr}_{E/F}(xy) + \frac{b_\xi \varpi^{\frac{k}{2}-a(\xi)}}{2} y^2\right) = {}^t y \left(v\mathrm{Nr}_{E/F}(x)\varpi^{\frac{k}{2}-l} A_1 + \varpi^{\frac{k}{2}-a(\xi)} A_2\right) y$$

for

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -\zeta \end{pmatrix}, A_2 = \begin{pmatrix} b_1 & \zeta b_2 \\ \zeta b_2 & \zeta b_1 \end{pmatrix},$$

and $y \in \mathfrak{o}^2$. This can be checked by a simple calculation. In particular $\det(A_2) = \zeta \mathrm{Nr}_{E/F}(b)$. Since $b \in \mathfrak{O}^\times$, we have $\Re(b) \in \mathfrak{o}^\times$ or $\Im(b) \in \mathfrak{o}^\times$, thus at least one entry of $vA_1 + A_2$ is a unit.

Similarly we can write the linear term as

$$\mathrm{Tr}\left(\left(-b_\xi \varpi^{\frac{k}{2}-a(\xi)} + x\varpi^{\frac{k}{2}+\frac{t}{2}} + v\mathrm{Nr}_{E/F}(x)\varpi^{\frac{k}{2}-l}\right) y\right) = 2^t B y$$

for

$$B = \begin{pmatrix} -b_1 \varpi^{\frac{k}{2}-a(\xi)} + x_1 \varpi^{\frac{k}{2}+\frac{t}{2}} + v(x_1^2 - \zeta x_2^2)\varpi^{\frac{k}{2}-l} \\ -b_2 \varpi^{\frac{k}{2}-a(\xi)} + x_2 \varpi^{\frac{k}{2}+\frac{t}{2}} \end{pmatrix}.$$

In this notation we obtain

$$\begin{aligned}
W_\pi(g_{t,l,v}) = \gamma q^{-2r-\frac{t}{2}} &\sum_{x \in S^\times} \xi^{-1}(x)\psi(\varpi^{\frac{t}{2}}\mathrm{Tr}(x) + v\varpi^{-l}\mathrm{Nr}_{E/F}(x)) \\
&\cdot G\left(\varpi^{-\rho}\left(v\mathrm{Nr}_{E/F}(x)\varpi^{\frac{k}{2}-l} A_1 + \varpi^{\frac{k}{2}-a(\xi)} A_2\right), 2\varpi^{-r-\rho} B\right). \quad (3.4.4)
\end{aligned}$$

Here we restricted the sum to

$$x \in S^{\times} = \left\{ x \in (\mathfrak{O}/\mathfrak{p}_E^r)^{\times} \colon -b_{\xi}\Omega^{\frac{k}{2}-a(\xi)} + x\Omega^{\frac{k}{2}+\frac{t}{2}} + v\mathrm{Nr}_{E/F}(x)\Omega^{\frac{k}{2}-l} \in \mathfrak{P}^r \right\},$$

since otherwise the Gauß sum vanishes due to Lemma 3.1.2. Writing $x = x_1 + x_2\sqrt{\zeta}$ we reformulate the congruences defining $S^{\times}$ to

$$-b_1\varpi^{\frac{k}{2}-a(\xi)} + x_1\varpi^{\frac{k}{2}+\frac{t}{2}} + v(x_1^2 - \zeta x_2^2)\varpi^{\frac{k}{2}-l} \in \mathfrak{p}^r,$$
$$-b_2\varpi^{\frac{k}{2}-a(\xi)} + x_2\varpi^{\frac{k}{2}+\frac{t}{2}} \in \mathfrak{p}^r,$$

for x_1 or x_2 in $\mathfrak{o}^{\times}$.

We will compute the set $S^{\times}$ in several cases and deduce the size of $W_{\pi}(g_{t,l,v})$ using (3.4.4).

Case I: $0 < l < \frac{n}{2}$**.** In this situation we have $t = -k = -n$ and the structure of $S^{\times}$ is very simple. Indeed, we have

$$x_2 \in b_2 + \mathfrak{p}^r.$$

This leads to the quadratic congruence

$$v\varpi^{\frac{k}{2}-l}x_1^2 + x_1 - (b_1 + \zeta v b_2^2\varpi^{\frac{k}{2}-l}) \in \mathfrak{p}^r.$$

In the notation of Lemma 3.1.4 this puts us in the situation where $v(b) = 0$ and $v(a) = \frac{k}{2} - l > 0$. Thus there is one solution. Even more, if $b_2 \notin \mathfrak{o}^{\times}$, then x_2 is not. This forces $b_1 \in \mathfrak{o}^{\times}$ and the unique solution satisfies $z_0 = x_1 + x_2\sqrt{\zeta} \in \mathfrak{O}^{\times}$. We obtain

$$W_{\pi}(g_{t,l,v}) = \gamma q^{\rho}\xi^{-1}(z_0)\psi(\varpi^{\frac{t}{2}}\mathrm{Tr}(z_0) + v\varpi^{-l}\mathrm{Nr}_{E/F}(z_0))G(\varpi^{-\rho}A_2, 2\varpi^{-r-\rho}B).$$

Furthermore, since $\det(A_2) = \zeta\mathrm{Nr}_{E/F}(b) \in \mathfrak{o}^{\times}$ and A_2 has entries in $\mathfrak{o}$, we use Lemma 3.1.2 to see that

$$W_{\pi}(g_{t,l,v}) = \gamma\gamma_F(\zeta\mathrm{Nr}_{E/F}(b), \frac{n}{2})\xi^{-1}(x_1 + \sqrt{\zeta}b_2)\psi((x_1 + b_1)\varpi^{-\frac{n}{2}}),$$

where $x_1 \in \mathfrak{o}$ is the only solution of

$$v\varpi^{\frac{k}{2}-l}x^2 + x - (b_1 + v\zeta b_2^2\varpi^{\frac{k}{2}-l}) = 0.$$

Case II: $l = \frac{n}{2}$**.** In this case $k = n = 2l$ and $-n \le t < 0$. We call x_2 admissible if it satisfies

$$x_2\varpi^{\frac{k}{2}+\frac{t}{2}} \in b_2 + \mathfrak{p}^r.$$

In order to determine the structure of $S^\times$ we have to solve the quadratic congruence

$$vx_1^2 + x_1\varpi^{\frac{k}{2}+\frac{t}{2}} - (b_1 + v\zeta x_2^2) \in \mathfrak{p}^r,$$

for each admissible x_2. To simplify notation we write $a = \frac{k}{2} + \frac{t}{2}$ and $b = v(b_2)$.

Case II.1: $t = -n$. In this situation we have exactly one admissible x_2 given by

$$x_2 = b_2 \in \mathfrak{o}/\mathfrak{p}^r.$$

Abusing notation we will identify x_2 with the fixed representative $b_2 \in \mathfrak{o}$. The quadratic equation for x_1 has discriminant $\Delta = \Delta(v) = 1 + 4v^2b_2^2\zeta + 4vb_1$. If $\Delta \in \mathfrak{o}^\times$, we have up to two possibilities for x_1, so that $\sharp S^\times \leq 2$. We now turn towards the matrix $vA_1 + A_2$. We can compute

$$\det(v\mathrm{Nr}_{E/F}(x)A_1 + A_2) = \zeta\mathrm{Nr}_{E/F}(b_\xi) - \zeta\mathrm{Nr}_{E/F}(x)^2v^2.$$

Note that for certain compositions of v, b_1, and b_2 the case $\det(v\mathrm{Nr}_{E/F}(x)A_1 + A_2) \in \mathfrak{p}$ can not be excluded. However, if $\mathrm{Nr}_{E/F}(b)$ is not a square, then the determinant in question is always a unit. Thus, if $\Delta \in \mathfrak{o}^{2\times}$, we use Lemma 3.1.2 and obtain the desired evaluation. On the other hand, if $\mathrm{Nr}_{E/F}(b) \in \mathfrak{o}^{2\times}$ we can not exclude situations where $\Delta \in \mathfrak{o}^\times$ and the determinant degenerates. In these cases we estimate trivially to get a bound of $2\sqrt{q}$.

Unfortunately, viewing Δ as an quadratic equation in v it turns out that, if $\mathrm{Nr}_{E/F}(b) \in \mathfrak{o}^{2\times}$, there are possibilities for v such that $\Delta \in \mathfrak{p}$. If this happens, we use Lemma 3.1.4 to parametrise the set $S^\times$ and define

$$A_\pm = -\frac{1}{2v} + b_2\sqrt{\zeta} \pm \frac{Y}{2v}\varpi^\delta \in \mathfrak{O}^\times.$$

Inserting the so obtained parametrisation in (3.4.4) yields

$$\begin{aligned} W_\pi(g_{t,l,v}) = {} & q^\rho \sum_\pm \gamma_\pm \psi(\varpi^{\frac{t}{2}}\mathrm{Tr}(A_\pm) + v\mathrm{Nr}_{E/F}(A_\pm)\varpi^{-l}) \\ & \cdot \sum_{x\in\mathfrak{o}/\mathfrak{p}^\delta} \xi^{-1}(A_\pm + x\Omega^{r-\delta})\psi((1\pm Y\varpi^\delta)x\varpi^{-\rho-r-\delta} + vx^2\varpi^{-\rho-2\delta}) \\ & \cdot G\left(\varpi^{-\rho}\Big(v\mathrm{Nr}_{E/F}(x)A_1 + A_2\Big), 2\varpi^{-r-\rho}B_x\right). \end{aligned} \tag{3.4.5}$$

Here we make the convention that $\gamma_\pm = \frac{\gamma}{2}$ if $\delta \geq r$ and $\gamma_\pm = \gamma$ otherwise.

Note that

$$x_2 = b_2 \text{ and } x_1 \in -\frac{1}{2v} + \mathfrak{p}.$$

Further, we have

$$b_1 + \mathrm{Nr}_{E/F}(x)v \in -2v\zeta b_2^2 + \mathfrak{p} \text{ and } b_1 - \mathrm{Nr}_{E/F}(x)v \in -\frac{1}{2v} + \mathfrak{p} \subset \mathfrak{o}^\times.$$

In particular, $b_1 - \mathrm{Nr}_{E/F}(x)v$ is a unit and $\det(v\mathrm{Nr}_{E/F}(x)A_1 + A_2) \in \mathfrak{p}$. We obtain

$$A_{\mathfrak{p}} = \begin{pmatrix} \frac{-\zeta b_2}{b_1 - \mathrm{Nr}_{E/F}(x)v} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2v} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-\zeta b_2}{b_1 - \mathrm{Nr}_{E/F}(x)v} & -1 \\ 1 & 0 \end{pmatrix}.$$

The Gauß sum can be evaluated using Lemma 3.1.2. Recall, that the degeneracy of the Gauß sum imposes a stronger congruence condition on x_1. Indeed we find

$$x_1 = -\frac{1}{2v} \pm \frac{Y}{2v}\varpi^\delta + \alpha\varpi^{r+\rho-\delta}, \text{ for } \alpha \in \mathfrak{o}/\mathfrak{p}^{\delta-\rho}.$$

Since E/F is an unramified we have $\kappa_E = \kappa_F$. Therefore we can make use of the p-adic logarithm over E without convergence issues and apply Lemma 3.1.3 to write

$$\begin{aligned} &W_\pi(g_{t,l,v}) \\ &= q^{\delta-\frac{\rho}{2}}\gamma(A_{\mathfrak{p}}) \sum_{\pm} \gamma_\pm \xi^{-1}(A_\pm)\psi(\varpi^{\frac{t}{2}}\mathrm{Tr}(A_\pm) + v\mathrm{Nr}_{E/F}(A_\pm)\varpi^{-l}) \\ &\quad \cdot \int_{\mathfrak{o}} \psi\Bigg((1 \pm Y\varpi^\delta)x\varpi^{-r-\delta} + vx^2\varpi^{\rho-2\delta} \\ &\qquad - \mathrm{Tr}\left(\frac{b_\xi}{\Omega^{2r+\rho}}\log_E\left(1 + \frac{x}{A_\pm}\Omega^{r+\rho-\delta}\right)\right)\Bigg)dx. \end{aligned}$$

Next we open the Taylor expansion of the logarithm and obtain

$$I = \int_{\mathfrak{o}} \psi(a_1 x\varpi^{-r-\delta} + a_2x^2\varpi^{\rho-2\delta} + a_3x^3\varpi^{r+2\rho-3\delta})dx,$$

for

$$\begin{aligned} a_1 &= 1 \pm Y\varpi^\delta - \mathrm{Tr}\left(\frac{b_\xi}{A_\pm}\right) = 0, \\ a_2 &= v + \frac{1}{2}\mathrm{Tr}\left(\frac{b_\xi}{A_\pm^2}\right) \text{ and} \\ a_3 &= -\frac{1}{3}\mathrm{Tr}\left(\frac{b_\xi}{A_\pm^3}\right). \end{aligned}$$

Our remaining task is to find further cancellation in I. Furthermore we can check that

$$a_2 \in \Delta\mathfrak{o}.$$

In particular, $\Delta = 0$ implies $a_1 = a_2 = 0$. We further compute

$$a_3 \in \frac{4b_1^2 - \zeta b_2^2}{12v^2\mathrm{Nr}_{E/F}(A_\pm)^3} + \mathfrak{p},$$

which implies that $a_3 \in \mathfrak{o}^\times$. Thus, in the worst case scenario, we obtain $I \ll_F q^{\frac{r}{3}+\frac{2}{3}\rho-\delta}$. We conclude

$$\left| W_\pi(g_{-n,\frac{n}{2},v}) \right| \ll_F 2q^{\frac{n}{12}}.$$

Case II.2: $\frac{t}{2} + a(\xi) < r$ and $t > -n_0(\pi)$. This and the following cases are slightly (computationally) involved. For brevity we only treat the case $\rho = 0$. As can be seen from the previous case the general situation is very similar but introduces some more details to keep track of. These appear to be purely technical and do not effect the end result.

Under the current assumptions we have $0 < a < b$. Further, we may assume that $\frac{a}{2} \geq \kappa_F$, since otherwise bounding trivially produces the desired answer.

In our cases all admissible x_2 are given by

$$x_2 = b_2\varpi^{-a} + \alpha\varpi^{r-a} \text{ for } \alpha \in \mathfrak{o}/\mathfrak{p}^a.$$

The discriminant governing the possible solutions for x_1 is

$$\Delta(x_2) = \eta_0(1 + \eta_1\alpha\varpi^{b+r-2a} + \eta_2\alpha^2\varpi^{2r-2a}),$$

for

$$\eta_0 = 4vb_1 + \varpi^{2a} + 4v^2\zeta b_2^2\varpi^{-2a}, \quad \eta_0\eta_1 = 8v^2\zeta(b_2)_0, \text{ and } \eta_0\eta_2 = 4v^2\zeta.$$

We see that $\eta_0, \eta_1, \eta_2 \in \mathfrak{o}^\times$. Furthermore, in order to have solutions for x_1 we need to assume that $vb_1 \in \mathfrak{o}^{2\times}$, since this implies $\eta_0 \in \mathfrak{o}^{2\times}$. We will do so and write $\eta_0 = \lambda^2$. We find

$$x_1 = -\frac{\varpi^a}{2v} \pm \frac{Y}{2v},$$

where $Y^2 = \Delta$. Upon noting that x_1 is well defined modulo $\mathfrak{p}^r$ we can expand

$$Y = Y(\alpha) = \lambda f(\alpha),$$

where

$$f(x) = \sqrt{1 + \eta_1 x\varpi^{b+r-2a} + \eta_2 x^2\varpi^{2r-2a}}.$$

Inserting this into (3.4.4) yields

$$W_\pi(g_{t,\frac{n}{2},v}) = \gamma q^{-a}\psi\left(\left(b_1 - \frac{\varpi^{2a}}{2v}\right)\varpi^{-2r}\right) \cdot \sum_{\pm,\alpha\in\mathfrak{o}/\mathfrak{p}^a} \xi^{-1}(x_1(\alpha) + \sqrt{\zeta}x_2(\alpha))\psi\left(\pm\frac{Y(\alpha)}{2v}\varpi^{\frac{t}{2}}\right).$$

To continue we write $a = 2s + \delta$. Let us consider the following expansion:

$$\pm \frac{Y(y+\varpi^s\beta)}{2v} = b_0(y) + \beta b_1(y)\varpi^{\min(r-a,b-a)+r-a+s} + b_2(y)\beta^2\varpi^{2r-2a+2s} + b_3(y)\beta^3\varpi^{b+3r-4a+3s} + \dots,$$

for

$$\begin{aligned} b_0(y) &= \pm\frac{Y(y)}{2v}, \\ b_1(y) &= 2v\zeta\frac{(b_2)_0\varpi^{\max(0,b-r)} + y\varpi^{\max(0,r-b)}}{Y(y)} \text{ and} \\ b_2(y) &= \pm\frac{4v^2\zeta b_1 + v\zeta\varpi^{2a}}{Y(y)^3}. \end{aligned}$$

Note that $Y(y+\varpi^s\beta)\varpi^{\frac{t}{2}}$ can be truncated after the quadratic term. Furthermore,

$$\begin{aligned} x_1(y+\beta\varpi^s) + \sqrt{\zeta}x_2(y+\beta\varpi^s) &= C_0(y) + C_1(y)\beta\varpi^{r-a+s} + C_2(y)\beta^2\varpi^{2r-2a+2s} \\ &\quad + C_3(y)\beta^3\varpi^{b+3r-4a+3s} + \dots, \end{aligned}$$

with coefficients

$$\begin{aligned} &C_0(y) = b_0(y) - \frac{\varpi^a}{2v} + \sqrt{\zeta}((b_2)_0\varpi^{b-a} + y\varpi^{r-a}) \in \mathfrak{o}^\times + \sqrt{\zeta}\mathfrak{p}, \\ &C_1(y) = \sqrt{\zeta} + b_1(y)\varpi^{\min(b-a,r-a)} \in \mathfrak{p} + \sqrt{\zeta}, \\ &C_2(y) = b_2(y), \text{ and } C_3(y) = b_3(y). \end{aligned}$$

Expanding the logarithm yields

$$\begin{aligned} \varpi^{-a(\xi)}\log_E(\dots) =& \frac{C_1(y)}{C_0(y)}\beta\varpi^{-r-a+s} + \left(-\frac{C_1(y)^2}{2C_0(y)^2} + \frac{C_2(y)}{C_0(y)}\right)\beta^2\varpi^{-2a+2s} \\ &+ \left(\frac{C_1(y)^3}{C_0(y)^3} - \frac{C_1(y)C_2(y)}{C_0(y)^2} + \frac{C_3(y)}{C_0(y)}\varpi^{b-a}\right)\beta^3\varpi^{r-3a+3s} + \dots . \end{aligned}$$

After putting things together we obtain

$$\begin{aligned} W_\pi(g_{t,\frac{n}{2},v}) =& \gamma q^{-s}\psi\left(\left(b_1 - \frac{\varpi^{2a}}{2v}\right)\varpi^{-2r}\right)\sum_{\pm,y\in\mathfrak{o}/\mathfrak{p}^s}\xi^{-1}(C_0(y))\psi\left(b_0(y)\varpi^{\frac{t}{2}}\right) \\ &\cdot\int_{\mathfrak{o}}\psi_E\Bigg(\left(\frac{b_1(y)}{2}\varpi^{\min(b,r)} - b_\xi\frac{C_1(y)}{C_0(y)}\right)\beta\varpi^{-r-a+s} \\ &\quad + \left(\frac{b_\xi C_1(y)^2}{2C_0(y)^2} - \frac{b_\xi C_2(y)}{C_0(y)} + \frac{b_2(y)}{2}\varpi^a\right)\beta^2\varpi^{-2a+2s} \\ &\quad - b_\xi\left(\frac{C_1(y)^3}{3C_0(y)^3} - \frac{C_1(y)C_2(y)}{C_0(y)^2} + \frac{C_3(y)}{C_0(y)}\varpi^{b-a}\right)\beta^3\varpi^{r-3a+3s} + \dots\Bigg)d\beta. \end{aligned}$$

Before dealing with the important terms we compute that

$$\mathrm{Nr}_{E/F}(C_0(y)) = \frac{b_1}{v} + \frac{\varpi^{2a}}{2v^2} \mp \frac{Y(y)\varpi^a}{2v^2}.$$

The linear term turns out to be $\frac{\pm 4\zeta b_1 y}{\lambda f(y)}\varpi^r$. Furthermore, one checks that the quadratic term is contained in $\varpi^a \mathfrak{o}^\times$. Combining these two facts leaves us with up to one choice for y modulo $\mathfrak{p}^s$ and a quadratic Gauß sum. We obtain the desired bound

$$W_\pi(g_{t,l,v}) \ll 2q^{-\frac{a}{2}} = 2q^{-\frac{n+t}{4}}.$$

Case II.3: $t = 2v(b_2) - n$ and $\frac{t}{2} < -r - \rho$. As before we assume $\rho = 0$ for technical convenience. All admissible x_2 are given by

$$x_2 = (b_2)_0 + \alpha\varpi^{r-a}, \quad \alpha \in \mathfrak{o}/\mathfrak{p}^a.$$

We assume that

$$\Delta = \Delta(x_2) = \varpi^{2a} + 4vb_1 + 4v^2\zeta x_2^2 = Y^2\varpi^{2\delta} + \mathfrak{p}^r$$

and deal with the case $Y = 0$ later. Note that, if $\delta = 0$, the analysis is analogous to the previous case and we will omit it here. With this at hand,

$$x_1 = -\frac{\varpi^a}{2v} \pm \frac{Y\varpi^\delta}{2v} + \beta\varpi^{r-\delta}, \quad \beta \in \mathfrak{o}/\mathfrak{p}^\delta.$$

We define

$$c_\pm(x_2) = -\frac{\varpi^a}{2v} \pm \frac{Y\varpi^\delta}{2v} + \sqrt{\zeta}x_2$$

and get

$$W_\pi(g_{t,l,v}) = \gamma q^{-a} \sum_{\substack{x_2 = (b_2)_0 + \alpha\varpi^{r-a},\\ \alpha\in\mathfrak{o}/\mathfrak{p}^a}} \sum_{\pm} \xi^{-1}(c_\pm(x_2))\psi(b_1\varpi^{-l} - \frac{\varpi^{2a-l}}{2v} \pm \frac{Y\varpi^{a+\delta-l}}{2v})$$
$$\cdot \sum_{\beta\in\mathfrak{o}/\mathfrak{p}^\delta} \xi^{-1}(1 + \frac{\beta}{c_\pm(x_2)}\varpi^{r-\delta})\psi((\varpi^a \pm Y\varpi^\delta)\beta\varpi^{-r-\delta} + v\beta^2\varpi^{-2\delta}).$$

Expanding the p-adic logarithm and straight forward computations reveal that the β-sum reads

$$\sum_{\beta\in\mathfrak{o}/\mathfrak{p}^\delta} \psi\Bigg(-\frac{2\zeta(b_2)_0}{\mathrm{Nr}_{E/F}(c_\pm(x_2))}\alpha\beta\varpi^{-\delta}$$
$$+ \left(\frac{Yb_1}{v} + \frac{Y\varpi^{2a}}{2v^2} - 2\zeta(b_2)_0\alpha(Y\varpi^{r-\delta} + \varpi^{r-\delta+2a})\right) \frac{\beta^2\varpi^{a-\delta}}{2v\mathrm{Nr}_{E/F}(c_\pm(x_2))^2} + \dots\Bigg).$$

We have to consider two cases. First, suppose $\delta \leq a$. In this case we are dealing with a linear sum, which vanishes whenever $\alpha \notin \mathfrak{p}^{\delta}$. The result is,

$$W_\pi(g_{t,l,v}) = \gamma q^{\delta-a} \sum_{\substack{x_2=(b_2)_0+\alpha\varpi^{\delta+r-a},\\ \alpha\in\mathfrak{o}/\mathfrak{p}^{a-\delta}}} \sum_{\pm} \xi^{-1}(c_\pm(x_2))\psi\left(b_1\varpi^{-l} - \frac{\varpi^{2a-l}}{2v} \pm \frac{Y\varpi^{a+\delta-l}}{2v}\right).$$

This can be handled exactly as in the previous case. Note that, if $v(\Delta) \geq r$ and $\lceil \frac{r}{2} \rceil \leq a$, the analysis is analogous.

Second, if $\delta > a$, the β-sum is a quadratic Gauß sum, which is non-zero only when $\alpha \in \mathfrak{p}^a$. Thus the α-sum collapses completely. However, in this case we have to consider the cubic term in the β-sum. We will show the computation for the worst case scenario $v(\Delta) \geq r$. Indeed, if this is the case, we compute

$$\sum_{\beta\in\mathfrak{o}/\mathfrak{p}^\delta} \psi\Bigg(\left(-\frac{8v^2\zeta(b_2)_0\alpha + \Delta\varpi^{a-r}}{\mathrm{Nr}_{E/F}(c_\pm(x_2))4v^2}\varpi^a\right)\beta\varpi^{-\delta} + \frac{\zeta x_2^2}{4v\mathrm{Nr}_{E/F}(c_\pm(x_2))^2}\left(\frac{1}{v}-1\right)\beta^2\varpi^{2a-2\delta} - \frac{b_1+\mathfrak{p}}{2v\zeta(b_2)_0^4}\beta^3\varpi^{a-3\delta+r}\Bigg).$$

We obtain the upper bound

$$|W_\pi(g_{t,l,v})| \leq 2q^{-\frac{n+t}{4}-\frac{t}{12}}.$$

Case II.4: $\frac{t}{2} \geq -r-\rho$. In particular, $v(b_2) \geq r$ and the congruence condition degenerates to

$$\mathrm{Nr}_{E/F}(x) \in \frac{b_1}{v} + \mathfrak{p}^r.$$

From (3.4.4) it follows that we have to evaluate

$$W_\pi(g_{t,\frac{n}{2},v}) = \gamma\gamma_F(b_1,\rho)q^{-2r-\frac{t}{2}-\frac{\rho}{2}} \sum_{\substack{x\in(\mathfrak{O}/\mathfrak{P}^r)^\times,\\ \mathrm{Nr}_{E/F}(x)\in\frac{b_1}{v}+\mathfrak{p}^r}} \xi^{-1}(x)\psi\left(\varpi^{\frac{t}{2}}\mathrm{Tr}(x) + v\varpi^{-l}\mathrm{Nr}_{E/F}(x)\right).$$

We make the Ansatz

$$x_2 = y + \varpi^\kappa\beta.$$

If y is chosen such that $\zeta y^2 + \frac{b_1}{v} \in \mathfrak{o}^{2\times}$, then

$$x_1 = \pm\sqrt{\zeta x_2^2 + \frac{b_1}{v}} = \pm f(x_2) = \pm f(y) \pm \frac{\zeta y}{f(y)}\beta\varpi^\kappa \pm \frac{\zeta b_1}{2vf(y)^3}\beta^2\varpi^{2\kappa} \mp \frac{\zeta^2 b_1 y}{vf(y)^5}\beta^3\varpi^{3\kappa} \pm \cdots.$$

Summing over β results in the sum

$$\sum_{\beta\in\mathfrak{o}/\mathfrak{p}^{r-\kappa}} \psi\Bigg(\frac{2\zeta}{f(y)} \left(\mp(b_2)_0\varpi^{b-a} \pm y\right) \beta\varpi^{\kappa+\frac{t}{2}} \pm \left(\frac{\zeta b_1 + \zeta^2 v(b_2)_0 y\varpi^{b-a}}{vf(y)^3} \right) \beta^2\varpi^{2\kappa+\frac{t}{2}} + \dots \Bigg).$$

Here we assume that $\kappa \geq \kappa_F$. Thus, after choosing $\kappa = \lfloor -\frac{t}{4} \rfloor$, we see that there is one such y modulo $\mathfrak{p}^\kappa$. Therefore the situation under consideration contributes $\ll_F 2q^{-\frac{a}{2}}$.

We still have to account for those x_2 which force $x_1 \in \mathfrak{p}$. This can only happen if $-\frac{b_1}{\zeta v} \in \mathfrak{o}^{2\times}$. If this is the case, we exchange the roles of x_1 and x_2. Indeed, we put

$$x_1 = \varpi y + \alpha\varpi^\kappa \text{ and } f(x) = \sqrt{-\frac{b_1}{v\zeta} + \frac{x^2}{\zeta}},$$

so that

$$x_2 = \pm f(\varpi y) \pm \frac{\varpi y}{\zeta f(\varpi y)}\alpha\varpi^\kappa \mp \frac{b_1}{2v\zeta^2 f(\varpi y)^3}\alpha^2\varpi^{2\kappa} \pm \frac{b_1\varpi y}{v\zeta^3 f(\varpi y)^5}\alpha^3\varpi^{2\kappa} \pm \cdots .$$

The α-sum reads

$$\psi(b_1\varpi^{-l} + 2y\varpi^{1+\frac{t}{2}})\xi^{-1}(\varpi y \pm \sqrt{\zeta}f(\varpi y)) \sum_{\alpha\in\mathfrak{o}/\mathfrak{p}^{r-\kappa}} \psi\Bigg(2\left[1 \mp \frac{(b_2)_0}{f(\varpi y)}\varpi^{b-a} \right] \alpha\varpi^{\kappa+\frac{t}{2}} \pm \frac{(b_2)_0\varpi y}{2\zeta f(\varpi y)^3}\beta^2\varpi^{b-a+2\kappa+\frac{t}{2}} + \dots \Bigg).$$

We see that this can not contribute unless $a = b$. We further observe that for the latter sum to be non-zero we need

$$1 \pm \frac{(b_2)_0}{f(\varpi y)} \in \mathfrak{p}.$$

However, this contradicts $\zeta y^2 + \frac{b_1}{v} \in \mathfrak{o}^{2\times}$ for $y \in (b_2)_0 + \mathfrak{p}$. We conclude that, depending on v and b_ξ, we either have contributions from $x_1 \in \mathfrak{p}$ or $x_1 \in \mathfrak{o}^\times$ but never from both cases simultaneously. To conclude this case we examine the worst case situation $\frac{b_1}{v} + (b_2)_0^2\zeta = 0$. Choosing $\kappa = \lfloor -\frac{t}{4} \rfloor$ produces a linear α-sum and we arrive at

$$W_\pi(g_{t,l,v}) = \gamma\gamma_F(b_1,\rho)\psi(b_1\varpi^{-l})q^{-\frac{n}{4}-\frac{t}{2}-\kappa} \cdot \sum_{\substack{\pm;\ y\in\mathfrak{o}/\mathfrak{p}^{\kappa-1}, \\ f(\varpi y)\mp(b_2)_0\in\mathfrak{p}^{\lceil -\frac{t}{4}\rceil}}} \xi^{-1}(\varpi y \pm \sqrt{\zeta}f(\varpi y))\psi(2y\varpi^{1+\frac{t}{2}}).$$

Writing $y = x\varpi^{\lceil\frac{\lceil -\frac{t}{4}\rceil}{2}\rceil}$ for $x \in \mathfrak{o}/\mathfrak{p}^{\kappa-1-\lceil\frac{\lceil -\frac{t}{4}\rceil}{2}\rceil}$ and computing the Taylor expansion of f at 0 produces the bound

$$W_\pi(g_{t,l,v}) = \gamma\gamma_F(b_1,\rho)\psi(b_1\varpi^{-l})\xi^{-1}(\sqrt{\zeta}(b_2)_0)q^{-\frac{n}{4}-\frac{t}{2}-1-\lceil\frac{\lceil -\frac{t}{4}\rceil}{2}\rceil} \cdot \int_{\mathfrak{o}} \psi\left(\frac{4x^3}{3\zeta(b_2)_0^2}\varpi^{\lceil\frac{\lceil -\frac{t}{4}\rceil}{2}\rceil+3+\frac{t}{2}}+\dots\right)dx \ll q^{-\frac{n+t}{4}-\frac{t}{12}}.$$

This completes our treatment of the transition region.

Case III: $\frac{n}{2} < l$**.** Here we have $k = 2l = -t$. Observe that

$$x_2 \in b_2\varpi^{l-a(\xi)} + \mathfrak{p}^r$$

is no unit. This leads to the condition

$$vx_1^2 + x_1 + (b_1\varpi^{l-a(\xi)} - v\zeta b_2^2\varpi^{2l-n}) \in \mathfrak{p}^r, \text{ for } x_1 \in (\mathfrak{o}/\mathfrak{p}^r)^\times.$$

In particular, we use Lemma 3.1.4 with $v(a) = v(b) = v(\Delta) = 0$ and find that $\sharp S^\times \leq 2$. Even more, since x_1 is only admissible if it is a unit, we find that $S^\times = \{x_0 + \sqrt{\zeta}b_2\varpi^{l-a(x)}\}$ where $x_0 \in \mathfrak{o}^\times$ is the unique solution to

$$vx^2 + x + (b_1\varpi^{l-a(\xi)} - v\zeta b_2^2\varpi^{2l-n}) = 0.$$

The $S^\times$-sum in (3.4.4) has only one term and we get

$$W_\pi(g_{-2l,l,v}) = (-1)^l\gamma\xi^{-1}(x_0 + \sqrt{\zeta}b_2\varpi^{l-\frac{n}{2}})\psi(x_0\varpi^{-\frac{t}{2}} - b_1\varpi^{-\frac{n}{2}}).$$

This was the last case to consider and the proof is complete. □

Remark 3.4.2. *Note that if $n = 2$, $l = 1$ and $t = -2$ we have*

$$W_\pi(g_{t,l,v}) = \gamma q^{-1}\sum_{x\in\mathbb{F}_{q^2}^\times}\tilde{\xi}^{-1}(x)\psi_{\mathbb{F}_{q^2}}(x + x^q + \tilde{v}x^{q+1}).$$

This is a sum over a finite field. However, due to the large exponent, Weil's bound (3.1.2) only gives the estimate

$$|W_\pi(g_{t,l,v})| \leq q+1.$$

This is worse than the local bound in this case.

Remark 3.4.3. *Let us make some remarks concerning twist-minimal supercuspidal representations. For simplicity let us assume that $\kappa_F = 1$. There is $b_\xi \in \mathfrak{O}^\times$ such that*

$$\xi(x) = \psi_E\left(\frac{b_\xi}{\Omega^{a(\xi)}}\log_E(x)\right), \tag{3.4.6}$$

for all $x \in 1+\mathfrak{P}$. *Furthermore, we can write* $b_\xi = b_1 + b_2\sqrt{\zeta}$ *as above. We make some observations.*

First, since $Tr \circ \log_E = \log_F \circ Nr_{E/F}$ *and* ξ *does not factor through the norm, we have* $b_2 \notin \mathfrak{p}_E^{a(\xi)}$.

Second, let $\chi = \xi \cdot (\mu \circ Nr_{E/F})$ *and let* $b_\chi, b_\xi \in \mathfrak{O}^\times$ *and* $b_\mu \in \mathfrak{o}^\times$ *be the numbers attached via the logarithm. We have*

$$\begin{aligned} \psi_E(\frac{b_\chi}{\Omega^{a(\chi)}}\log_E(x)) &= \chi(x) = \xi(x)\mu(Nr_{E/F}(x)) \\ &= \psi_E(\frac{b_\xi}{\Omega^{a(\xi)}}\log_E(x))\psi_F(\frac{b_\mu}{\varpi^{a(\mu)}}\log_F(Nr_{E/F}(x))) \\ &= \psi_E\left(\left[\frac{b_\xi}{\Omega^{a(\xi)}} + \frac{b_\mu}{\Omega^{a(\mu)}}\right]\log_E(x)\right) \end{aligned}$$

for all suitable x. *Suppose that* $a(\mu\pi) < a(\pi)$, *in other words* π *is not twist-minimal, then we must have* $a(\mu) = a(\xi) = \frac{a(\pi)}{2} > \frac{a(\mu\pi)}{2} = a(\chi)$. *In particular,*

$$\mathfrak{O}^\times \ni \Omega^{a(\chi)-a(\xi)} \underbrace{(b_\xi + b_\mu)}_{b_1+b_\mu+b_2\sqrt{\zeta}} \equiv b_\chi \mod \mathfrak{P}^{a(\chi)}.$$

Because b_μ *is in* F *this implies* $(b_1+b_\mu) \in \mathfrak{p}$ *and* $b_2 \in \mathfrak{p}$.

We conclude that, for $a(\pi) > 2$, π *is twist-minimal if and only if* $b_2 \in \mathfrak{o}^\times$. *Furthermore,* $n_0(\pi) = n - 2v(b_2)$. *If* $a(\pi) = 2$, *the representation is automatically twist-minimal.*

Concerning the degeneration of Whittaker new vectors we can say the following. If π *is twist-minimal, we might encounter degenerate critical points leading to large values of* W_π. *However,* W_π *features nice support properties. More precisely, we can always assume* $t = -\max(n, 2l)$ *since otherwise the new vector vanishes. Further, we can always find a twist* μ *such that* $Nr_{E/F}(b_\chi) \notin \mathfrak{o}^{2\times}$ *which ensures that there are no degenerate critical points and we get the expected size. If* π *is not twist-minimal, the support degenerates and degenerate critical points appear for* $t = -n$ *and* $t = -n_0(\pi)$.

We turn to supercuspidal representations associated to ramified extensions of F.

Lemma 3.4.4 ([4], Lemma 5.2). *Let* π *be a dihedral supercuspidal representation associated to a ramified quadratic extension* E/F *and a multiplicative character* ξ. *Then we have*

$$W_\pi(g) \ll_F q^{\frac{n}{12}}.$$

If $\kappa_F = 1$, *the implicit constant is bounded by* 2.

Furthermore, if $0 < l < \frac{n}{2}$*, we have*

$$W_\pi(g_{-n,l,v}) = \begin{cases} \gamma\gamma_E(2b_\xi,1)\xi^{-1}(x_0+\Omega b_2)\psi(x_0\varpi^{-\frac{n}{2}}+b_1\varpi^{-\frac{n}{2}}) & \text{if } n \text{ is even and } 2l > \lceil\frac{n}{2}\rceil, \\ \gamma\xi^{-1}(b_1+\Omega x_0)\psi((x_0+b_2)\varpi^{\frac{n-1}{2}}) & \text{if } n \text{ is odd and } 2l > \lceil\frac{n}{2}\rceil, \\ \epsilon(\frac{1}{2},\pi)\psi(v\varpi^{-l}Nr_{E/F}(b_\xi)) & \text{if } 2l \leq \lceil\frac{n}{2}\rceil, \end{cases}$$

where $x_0 \in \mathfrak{o}$ *is the unique solution to*

$$\begin{aligned} v\varpi^{\frac{n}{2}-l}x_1^2 + x_1 - b_1 + vb_2^2\varpi^{\frac{n}{2}-l+1} &= 0 \qquad \text{if } n \text{ is even,} \\ -v\varpi^{\frac{n+1}{2}-l}x_2^2 + x_2 - b_2 + vb_1^2\varpi^{\frac{n-1}{2}-l} &= 0 \qquad \text{if } n \text{ is odd.} \end{aligned}$$

In the transition region $l = \frac{n}{2}$ *we have the bound*

$$|W_\pi(g_{t,l,v})| \ll 2q^{-\frac{n+t}{4}-\frac{t}{12}}.$$

Even more, if $t \neq -n, -n_0(\pi)$*, we have the stronger bound* $|W_\pi(g_{t,l,v})| \ll 2q^{-\frac{n+t}{4}}$*.*

Finally, if $\frac{n}{2} < l < n$*, then we have*

$$W_\pi(g_{-2l,l,v}) = \gamma\gamma_E(2v,1)\xi^{-1}(x_0+\Omega\Im(b_\xi\Omega^{2l-n}))\psi((x_0+\Re(b_\xi\Omega^{2l-n}))\varpi^{-l}),$$

where $x_0 \in \mathfrak{o}^\times$ *is the unique solution to*

$$vx_1^2 + x_1 - v\varpi\Im(b_\xi\Omega^{2l-n})^2 - \Re(b_\xi\Omega^{2l-n}) = 0.$$

Proof. Since E/F is a ramified extension we have $f = 1$, $e = 2$ and $d = 1$. In particular, $n(\psi_E) = -1$ and the additive measure on E is normalised so that

$$\mathrm{Vol}(\mathfrak{D},\mu_E) = q^{-\frac{1}{2}}.$$

Without loss of generality we assume that $E = F(\sqrt{\varpi})$ and choose $\Omega = \sqrt{\varpi}$. The identity (3.4.2) still holds.

The log-conductor of π is given by $n = a(\pi) = a(\xi) + 1$. We observe that

$$|W_\pi(g_{-n,0,v})| = \left|\epsilon(\frac{1}{2},\tilde{\pi})\right| = 1.$$

Thus we can assume $l > 0$ and define $k = \max(a(\xi), 2l-1) = 2r + \rho$. Using Lemma 3.3.1 and Lemma 3.1.3 we compute

$$\begin{aligned} W_\pi(g_{t,l,v}) =& \gamma q^{-\frac{t}{2}} K(\xi^{-1}, \Omega^t, v\varpi^{-l}) \\ =& \gamma q^{-\frac{t}{2}-r} \sum_{x \in (\mathfrak{O}/\mathfrak{P}^r)^\times} \xi^{-1}(x)\psi(\mathrm{Tr}(x\Omega^t) + v\varpi^{-l}\mathrm{Nr}_{E/F}(x)) \\ & \cdot \int_{\mathfrak{O}} \psi_E\Big(\frac{v\mathrm{Nr}_{E/F}(xt)}{2}\Omega^{k-2l+1-\rho-1} + \frac{b_\xi}{2} t^2 \Omega^{k-a(\chi)-\rho-1} \\ & + \left(-b_\xi \Omega^{k-a(\xi)} + x\Omega^{k+1+t} + v\mathrm{Nr}_{E/F}(x)\Omega^{k-2l+1}\right) t\Omega^{-r-\rho-1}\Big) d_E t. \end{aligned} \tag{3.4.7}$$

We need to estimate this for $t \geq -k$ if $l = \frac{n}{2}$ and for $t = -k-1$ otherwise. In all these cases the remaining integral reduces to a quadratic Gauß sum over E. Thus we can restrict the x-sum to

$$S = \{x \in (\mathfrak{O}/\mathfrak{P}^r)^\times \colon -b_\xi \Omega^{k-n+1} + x\Omega^{k+t+1} + v\mathrm{Nr}_{E/F}(x)\Omega^{k-2l+1} \in \mathfrak{P}^r\}.$$

Case I: $0 < l < \frac{n}{2}$. Due to the support properties of W_π we can assume that $t = -n = -a(\xi) - 1$. Obviously $k = a(\xi)$. The set S is determined by the congruence

$$-b_\xi + x + v\mathrm{Nr}_{E/F}(x)\Omega^{n-2l} \in \mathfrak{P}^r.$$

If $n - 2l \geq r$, there is exactly one solution. Namely $x = b_\xi$ modulo $\mathfrak{P}^r$. Otherwise we write $x = x_1 + x_2\Omega$. If n is even, this leads to the two congruences

$$\begin{aligned} -b_1 + x_1 + v\mathrm{Nr}_{E/F}(x)\varpi^{\frac{n}{2}-l} &\in \mathfrak{p}^{\lceil \frac{r}{2} \rceil}, \\ -b_2 + x_2 &\in \mathfrak{p}^{\lfloor \frac{r}{2} \rfloor}. \end{aligned}$$

Using Lemma 3.1.4 we observe that there is a unique solution $(x_1, x_2) \in \mathfrak{o}/\mathfrak{p}^{\lceil \frac{r}{2} \rceil} \times \mathfrak{o}/\mathfrak{p}^{\lfloor \frac{r}{2} \rfloor}$. Furthermore, one quickly checks that x_1 is a unit. If n is odd, we have to solve

$$\begin{aligned} -b_1 + x_1 &\in \mathfrak{p}^{\lceil \frac{r}{2} \rceil}, \\ -b_2 + x_2 + v\mathrm{Nr}_{E/F}(x)\varpi^{\frac{n-1}{2}-l} &\in \mathfrak{p}^{\lfloor \frac{r}{2} \rfloor}. \end{aligned}$$

instead. We find $x = b_1 + \sqrt{\zeta}x_0$, where x_0 is the unique solution in $\mathfrak{o}$ to the corresponding quadratic equation determining x_2.

We recall $n(\psi_E) = -1$, $\mathrm{Vol}(\mathfrak{D}, \mu_E) = q^{-\frac{1}{2}}$ and $k - l - \rho > 0$. With this in mind we evaluate the quadratic Gauß sum and obtain

$$W_\pi(g_{-n,l,v}) = \begin{cases} \gamma\gamma_E(2b_\xi, 1)\xi^{-1}(x_0 + \Omega b_2)\psi(x_0\varpi^{-\frac{n}{2}} + b_1\varpi^{-\frac{n}{2}}) & \text{if } n \text{ is even and } 2l > \lceil \frac{n}{2} \rceil, \\ \gamma\xi^{-1}(b_1 + \Omega x_0)\psi((x_0 + b_2)\varpi^{\frac{n-1}{2}}) & \text{if } n \text{ is odd and } 2l > \lceil \frac{n}{2} \rceil, \\ \epsilon(\frac{1}{2}, \pi)\psi(v\varpi^{-l}\mathrm{Nr}_{E/F}(b_\xi)) & \text{if } 2l \leq \lceil \frac{n}{2} \rceil. \end{cases}$$

Case II: $l = \frac{n}{2}$**.** In this case $k = 2l - 1 = a(\xi)$ and $\rho = 1$. We write $x = x_1 + x_2\Omega$ and $b_\xi = b_1 + b_2\Omega$, to transform the congruence condition defining S into a system of congruences over F. For simplicity we consider further subcases.

Case II.1: $t \geq -\frac{n}{2} - 1$**.** In this case the congruence defining S degenerates too

$$\mathrm{Nr}_{E/F}(x) \in \frac{b_1}{v} + \mathfrak{p}^{\lceil \frac{r}{2} \rceil}.$$

This case can only occur if $v(b_2) \geq \lfloor \frac{r}{2} \rfloor$. We get

$$W_\pi(g_{t,l,v}) = \gamma\gamma_E(b_\xi, 1)\psi(b_1\varpi^{-l})q^{-\frac{t}{2}-r-1} \sum_{\substack{x \in \mathfrak{O}/\mathfrak{P}^r, \\ \mathrm{Nr}_{E/F}(x) = \frac{b_1}{v}}} \xi^{-1}(x)\psi(\mathrm{Tr}(x\Omega^t)).$$

Note that the latter sum in non-empty if and only if $b_1 v \in \mathfrak{o}^{2\times}$. We will assume so for the rest of this case. Take

$$x_2 = y + \alpha\varpi^\kappa \text{ for } \alpha \in \mathfrak{o}/\mathfrak{p}^{\lfloor \frac{r}{2} \rfloor - \kappa}.$$

Any given x_2 determines x_1 up to sign by the (convergent) Taylor expansion of the square root. Indeed,

$$x_1 = \pm f(y) \pm \frac{\varpi y}{f(y)}\alpha\varpi^\kappa + \frac{\varpi b_1}{2v f(y)^3}\alpha^2\varpi^{2\kappa} \pm \cdots,$$

for $f(y) = \sqrt{\frac{v}{b_1} + \varpi y^2}$. We compute

$$\begin{aligned} \xi(x)^{-1} =& \psi_E\Bigg(- \left(\frac{b_\xi(\pm f'(y) + \Omega)}{\pm f(y) + \Omega y} \right) \varpi^{\kappa-r-1}\alpha \\ &+ \left(\frac{b_\xi(\pm f'(y) + \Omega)^2}{(\pm f(y) + \Omega y)^2} - \frac{\pm f''(y) b_\xi}{(\pm f(y) + \Omega y)} \right) \varpi^{2\kappa-r-1}\alpha^2 + \dots \Bigg) \\ =& \psi\left(\pm \frac{2(b_2)_0}{f(y)}\alpha\varpi^{b-r-1+\kappa} \pm \frac{(b_2)_0 y}{(f(y)^3)\alpha^2\varpi^{b+1-r+\kappa}} + \dots \right). \end{aligned}$$

If t is even, then $\mathrm{Tr}(x\Omega^t) = 2x_1\varpi^{\frac{t}{2}}$ and we obtain

$$W_\pi(g_{t,l,v}) = \gamma\gamma_E(b_\xi,1)\psi(b_1\varpi^{-l})q^{-\frac{t}{2}-\lceil\frac{r}{2}\rceil-\kappa-1}\sum_{\pm,y\in\mathfrak{o}/\mathfrak{p}^\kappa}\xi^{-1}(\pm f(y)+\Omega y)\psi(\pm 2f(y)\varpi^{\frac{t}{2}})$$
$$\cdot\int_{\mathfrak{o}}\psi\left(\pm\frac{2}{f(y)}(y\pm(b_2)_0\varpi^{b-a})\alpha\varpi^{\kappa+\frac{t}{2}+1}\pm(b_1v^{-1}+(b_2)_0y\varpi^{b-a+1})\frac{\alpha^2\varpi^{2\kappa+\frac{t}{2}+1}}{f(y)^3}+\dots\right).$$

Thus, choosing $\kappa = \lfloor -\frac{t+2}{4}\rfloor$, collapses the y-sum to one term and produces the estimate

$$|W_\pi(g_{t,l,v})| \ll 2q^{-\frac{n+t}{4}}.$$

If t is odd, things turn out slightly different. Indeed, $\mathrm{Tr}(x\Omega^t) = 2x_2\varpi^{\frac{t+1}{2}}$, so that we find

$$W_\pi(g_{t,l,v}) = \gamma\gamma_E(b_\xi,1)\psi(b_1\varpi^{-l})q^{-\frac{t}{2}-\lceil\frac{r}{2}\rceil-\kappa-1}\sum_{\pm,y\in\mathfrak{o}/\mathfrak{p}^\kappa}\xi^{-1}(\pm f(y)+\Omega y)\psi(2y\varpi^{\frac{t+1}{2}})$$
$$\cdot\int_{\mathfrak{o}}\psi\left(\pm\frac{2}{f(y)}(f(y)\pm(b_2)_0\varpi^{b-r-\frac{t+1}{2}})\alpha\varpi^{\kappa+\frac{t+1}{2}}\pm\frac{(b_2)_0y\alpha^2\varpi^{1+b-r+2\kappa}}{f(y)^3}+\dots\right).$$

This forces $b = r+\frac{t+1}{2}$. Furthermore, taking $\kappa = \lfloor -\frac{t+1}{4}\rfloor$, the integral only features a linear phase. This introduces an additional congruence condition on y, which reads

$$f(y)\pm(b_2)_0 \in p^{\lceil -\frac{t+1}{4}\rceil}.$$

Note that, depending on b_2, only $+$ or $-$ can contribute. Without loss of generality we assume it to be $+$. Furthermore, we only investigate the worst case situation when $(b_2)_0^2 - \frac{b_1}{v} = 0$. We can parametrise the possible y's by

$$y = \beta\varpi^{\kappa'} \text{ for } \kappa' = \left\lceil\frac{\lceil -\frac{t+1}{4}\rceil}{2}\right\rceil.$$

Further, we find that

$$f(y) = \sqrt{\frac{b_1}{v}}+\frac{v}{4b_1}\beta^2\varpi^{2\kappa'+1}+\cdots.$$

Inserting these observations in our formula for W_π gives

$$\begin{aligned}W_\pi(g_{t,l,v}) &= \gamma\gamma_E(b_\xi,1)\psi(b_1\varpi^{-l})q^{\lceil -\frac{t+1}{4}\rceil-\lceil\frac{r}{2}\rceil-\frac{1}{2}}\\ &\quad\cdot\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}^{\lfloor -\frac{t+1}{4}\rfloor},\\ f(y)\pm(b_2)_0\in\mathfrak{p}^{\lceil -\frac{t+1}{4}\rceil}}}\xi^{-1}(\pm f(y)+\Omega y)\psi(2y\varpi^{\frac{t+1}{2}})\\ &= \gamma\gamma_E(b_\xi,1)\xi^{-1}\left(\sqrt{\frac{b_1}{v}}\right)\psi(b_1\varpi^{-l})q^{2\lceil -\frac{t+1}{4}\rceil-\kappa'-\lceil\frac{r}{2}\rceil-\frac{1}{2}}\\ &\quad\cdot\int_{\mathfrak{o}}\psi\left(\frac{4v(b_2)_0}{3b_1}\sqrt{\frac{v}{b_1}}\beta^3\varpi^{3\kappa'+1+\frac{t+1}{2}}+\dots\right)d\beta.\end{aligned}$$

We obtain the upper bound

$$|W_\pi(g_{t,l,v})| \ll 2q^{-\frac{n+t}{4}-\frac{t}{12}}.$$

Recall that the only odd value t that gives a non-zero outcome is $t = -n_0(\pi)$.

Case II.2: $n+t>0$ **even and** $t < -\frac{n}{2} - 1$**.** We set $b = v(b_2)$ and $a = l + \frac{t}{2}$. We arrive at the congruences

$$\begin{aligned} -b_1 + x_1\varpi^a + vx_1^2 - v\varpi x_2^2 &\in \mathfrak{p}^{\lceil \frac{r}{2} \rceil}, \\ -b_2 + x_2\varpi^a &\in \mathfrak{p}^{\lfloor \frac{r}{2} \rfloor}. \end{aligned}$$

The first equation is quadratic in x_1 with discriminant $\Delta(v, x_2) = \varpi^{2a} + 4v(b_1 + \varpi v x_2^2)$. Since $b_\xi \in \mathfrak{O}^\times$ we have $b_1 \in \mathfrak{o}^\times$ and therefore $\Delta \in \mathfrak{o}^\times$. There are up to two solutions modulo $\mathfrak{p}^{\lceil \frac{r}{2} \rceil}$ for x_1 for each given x_2. Furthermore, we can assume that

$$x_2 = (b_2)_0\varpi^{b-a} + \alpha\varpi^{\lfloor \frac{r}{2} \rfloor - a} \text{ for } \alpha \in \mathfrak{o}/\mathfrak{p}^a.$$

Thus the two congruences define a set S modulo $\mathfrak{P}^r$ given by

$$S = \{-\frac{\varpi^a}{2v} \pm \frac{Y(x_2)}{2v} + \Omega x_2 \colon x_2 = (b_2)_0\varpi^{b-a} + \alpha\varpi^{\lfloor \frac{r}{2} \rfloor - a} \text{ for } \alpha \in \mathfrak{o}/\mathfrak{p}^a\}$$

for $Y(x_2) = \sqrt{4vb_1 + 4\varpi v^2 x_2^2 + \varpi^{2a}}$. In particular, we are assuming that $4vb_1 \in \mathfrak{o}^{2\times}$, since otherwise the set S is empty.

We have to deal with the sum

$$\begin{aligned} W_\pi(g_{t,l,v}) = \gamma q^{-\frac{t}{2}-r-1}\psi\left(b_1\varpi^{-l} - \frac{\varpi^{a+\frac{t}{2}}}{2v}\right) \\ \cdot \sum_{\pm, x_2} \xi^{-1}\left(-\frac{\varpi^a}{2v} \pm \frac{Y(x_2)}{2v} + \Omega x_2\right)\psi\left(\pm\frac{Y(x_2)}{2v}\varpi^{\frac{t}{2}}\right). \end{aligned}$$

Write $a = 2s + \delta$ and $x_2 = y + \alpha\varpi^{\lfloor \frac{r}{2} \rfloor - s - \delta}$, for $y \in \mathfrak{o}/\mathfrak{p}^{\lfloor \frac{r}{2} \rfloor - s - \delta}$ and $y \in (b_2)_0\varpi^{b-a} + \mathfrak{p}^{\lfloor \frac{r}{2} \rfloor - a}$. The Taylor expansion of $Y(x_2)$ at y reads

$$Y(x_2) = Y(y) + \frac{4v^2 y}{Y(y)}\alpha\varpi^{1+\lfloor \frac{r}{2} \rfloor - s - \delta} + \frac{8\varpi v^3 b_1 + 2v^2\varpi^{2a}}{Y(y)^3}\alpha^2\varpi^{2\lfloor \frac{r}{2} \rfloor - a - \delta} \dots .$$

Thus, with

$$\begin{aligned} C_0(y) &= \pm\frac{Y(y)}{2v} - \frac{\varpi^a}{2v} + \Omega y, \\ C_1(y) &= \pm\frac{2vy\varpi}{Y(y)} + \Omega \text{ and} \\ C_2(y) &= \pm\frac{4v^2 b_1\varpi + v\varpi^{2a}}{Y(y)^3} \end{aligned}$$

we obtain

$$W_\pi(g_{t,l,v}) = \gamma q^{-s-\frac{1}{2}}\psi\left(b_1\varpi^{-l} - \frac{\varpi^{a+\frac{t}{2}}}{2v}\right) \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}^{\lfloor\frac{r}{2}\rfloor-s-\delta},\\ y\in(b_2)_0\varpi^{b-a}+\mathfrak{p}^{\lfloor\frac{r}{2}\rfloor-a}}} \xi^{-1}\left(C_0(y)\right)\psi\left(\pm\frac{Y(y)}{2v}\varpi^{\frac{t}{2}}\right)$$
$$\cdot\int_{\mathfrak{o}}\psi_E\Bigg(\left(-\frac{b_\xi C_1(y)}{C_0(y)} \pm \frac{vy\varpi^{a+1}}{Y(y)}\right)\alpha\varpi^{-\lceil\frac{r}{2}\rceil-1-s-\delta}$$
$$+\left(\frac{b_\xi C_1(y)^2}{2C_0(y)^2} - \frac{b_\xi C_2(y)}{C_0(y)} + \frac{C_2(y)\varpi^{a+1}}{2}\right)\alpha^2\varpi^{2\lfloor\frac{r}{2}\rfloor-r-1-a-\delta} + \cdots\Bigg)d\alpha.$$

As in the case of an unramified extension we can compute the linear and the quadric contributions to find the following. The linear term is of the form

$$\left(\frac{4b_1y}{\mathrm{Nr}_{E/F}(C_0(y))Y(y)} + \mathfrak{p}\right)\alpha\varpi^{s-\lceil\frac{r}{2}\rceil}.$$

Furthermore, the quadratic term is contained in $\mathfrak{p}^{a+1}$. This leads to the bound

$$|W_\pi(g_{t,l,v})| \ll q^{-\frac{n+t}{4}}.$$

Case II.2: $2l+t>0$ **odd and** $t<-\frac{n}{2}-1$**.** We write $2l+t=2\beta-1$ for some $\beta\in\mathbb{N}$. We have to solve the congruences

$$-b_2 + x_1\varpi^{\beta-1} \in \mathfrak{p}^{\lfloor\frac{r}{2}\rfloor}, \tag{3.4.8}$$
$$-v\varpi x_2^2 + \varpi^\beta x_2 - b_1 + vx_1^2 \in \mathfrak{p}^{\lceil\frac{r}{2}\rceil}. \tag{3.4.9}$$

Looking at the first congruence reveals that, unless $\beta-1=v(b_2)<\lfloor\frac{r}{2}\rfloor$, there are no solutions for $x_1\in\mathfrak{o}^\times$. Thus, without loss of generality, we assume the latter. There are $q^{\beta-1}$ solutions for x_1. Namely

$$x_1 = (b_2)_0 + \alpha\varpi^{\lfloor\frac{r}{2}\rfloor-b}, \text{ for } \alpha\in\mathfrak{o}/\mathfrak{p}^b.$$

Let S_{x_1} be the set of admissible x_2 given x_1. Note that S_{x_1} is empty unless $v(b_2)_0^2-b_1\in\mathfrak{p}$. In non-empty situations we have

$$S_{x_1} = \left\{\frac{\varpi^{\beta-1}}{2v} \pm \frac{Y(x_1)}{2v}\varpi^\delta + \beta\varpi^{\lceil\frac{r}{2}\rceil-1-\delta} \colon \beta\in\mathfrak{o}/\mathfrak{p}^{\delta+1}\right\}.$$

Here $\Delta(x_1)=\varpi^{2\beta-2}+4v(vx_1^2-b_1)\varpi^{-1}=Y(x_1)^2\varpi^{2\delta}$ with the usual convention that $\delta=\lfloor\frac{\lceil\frac{r}{2}\rceil-1}{2}\rfloor$ and $Y(x_1)=0$ if $\Delta(x_1)\in\mathfrak{p}^{\lceil\frac{r}{2}\rceil-1}$. Let us execute the β-sum first. To shorten notation we write $x=z_0(x_1)+\beta\varpi^{\lceil\frac{r}{2}\rceil-1-\delta}\Omega$ and $-e=2\lceil\frac{r}{2}\rceil-r-1$. We arrive at

$$W_\pi(g_{t,l,v}) = \gamma\gamma_E(b_\xi,1)q^{\frac{1}{2}-b}\sum_{x_1\in(\mathfrak{o}/\mathfrak{p}^r)^\times}\psi(\mathrm{Tr}(z_0(x_1)\Omega)\varpi^{\frac{t-1}{2}} + v\varpi^{-l}\mathrm{Nr}_{E/F}(z_0(x_1)))$$
$$\xi^{-1}(z_0(x))\sum_{\beta\in\mathfrak{o}/\mathfrak{p}^{\delta+1}}\xi^{-1}(1+\frac{\beta}{z_0(x_1)}\varpi^{\lceil\frac{r}{2}\rceil-1-\delta}\Omega)$$
$$\cdot\psi((\varpi^b \mp Y(x_1)\varpi^\delta)\beta\varpi^{\lceil\frac{r}{2}\rceil-1-\delta-r} - v\beta^2\varpi^{2\lceil\frac{r}{2}\rceil-2\delta-r-2}).$$

To investigate the phase of this integral we note that

$$\mathrm{Nr}_{E/F}(z_0(x_1)) = \frac{b_1}{v} - \frac{\varpi^{2\beta-1}}{2v^2} \mp \frac{Y(x_1)}{2v^2}\varpi^{\beta+\delta}.$$

We logarithmically expand ξ^{-1} and compute the resulting linear, quadratic and cubic term. This produces the β-sum:

$$\sum_{\beta\in\mathfrak{o}/\mathfrak{p}^{\delta+1}} \psi\Bigg(\left((b_2)_0\alpha + \alpha^2\varpi^{\lfloor\frac{r}{2}\rfloor-b}\right)\frac{2\beta}{\mathrm{Nr}_{E/F}(z_0(x_1))}\varpi^{-\delta-1} + \left(\pm 2vY(x_1)\varpi^{\delta+b}+\Delta\right)\frac{b_1\beta^2}{2v^2}\varpi^{-2\delta-e} + \left(\frac{b_1(b_2)_0^2}{3v}+\mathfrak{p}\right)\beta^3\varpi^{\lceil\frac{r}{2}\rceil+b-3\delta-1-e}+\dots\Bigg).$$

If $b>\delta$, then this reduces to a linear sum. Furthermore, we pick up the condition $\alpha \in \mathfrak{p}^{\delta+1}$. One can deal with the remaining sum as before. In the case $b\leq\delta$ and $Y(x_1)\neq 0$ we obtain the strict condition $\alpha = 0$ and bounding the remaining exponential sum gives a satisfying result. We are left with the most degenerate case $Y(x_1)=0$, in which the valuation of the cubic coefficient forces the condition $\alpha=0$. We obtain the bound

$$|W_\pi(g_{t,l,v})| \ll 2q^{-\frac{n+t}{4}-\frac{t}{12}}.$$

Case II.3: $t=-2l$**.** If this is the case, there is only one admissible x_2 given by $x_2 = b_2+\mathfrak{p}^{\lfloor\frac{r}{2}\rfloor}$. Everything boils down to the quadratic congruence

$$vx_1^2 + x_1 - \varpi v b_2^2 - b_1 \in \mathfrak{p}^{\lceil\frac{r}{2}\rceil}$$

with discriminant

$$\Delta = 1+4vb_1+4v^2b_2^2\varpi.$$

Depending on the p-adic size of Δ we have to examine different cases.

First, assume $v(\Delta)\geq 1$. Then x_1 is of the form

$$x_\pm = \begin{cases} -\frac{1}{2v} \pm \frac{Y}{2v}\varpi^\delta + \alpha\varpi^{\lceil\frac{r}{2}\rceil-\delta} & \text{if } \Delta = Y^2\varpi^{2\delta} \text{ for some } \delta<\frac{1}{2}\lceil\frac{r}{2}\rceil \text{ for } \alpha\in\mathfrak{o}/\mathfrak{p}^\delta, \\ -\frac{1}{2v} + \alpha\varpi^{\lceil\frac{r}{2}\rceil-\delta} & \text{if } v(\Delta)\geq\lceil\frac{r}{2}\rceil \text{ with } \delta=\lfloor\frac{1}{2}\lceil\frac{r}{2}\rceil\rfloor \text{ and } \alpha\in\mathfrak{o}/\mathfrak{p}^\delta. \end{cases}$$

We define $B_\pm$ to be the α independent part of $x_\pm$. This determines $x\in S$ up to $\mathfrak{P}^r$. We obtain

$$S \quad = \quad \{B_\pm + \alpha\varpi^{\lceil\frac{r}{2}\rceil-\delta} + b_2\Omega\colon \alpha\in\mathfrak{o}/\mathfrak{p}^\delta\}.$$

Next, we reinsert this parametrisation in (3.4.7). Each element of S is of the shape $A_\pm + \alpha\Omega^{2\lceil\frac{r}{2}\rceil-2\delta}$, for $A_\pm = B_\pm + b_2\Omega$. We find

$$\begin{aligned} W_\pi(g_{-n,\frac{n}{2},v}) &= \sum_\pm \gamma_\pm\gamma_E(-2v,1)\xi^{-1}(A_\pm)\psi(\mathrm{Tr}(A_\pm)+v\varpi^{-l}\mathrm{Nr}_{E/F}(A_\pm)) \\ &\quad \sum_{\substack{\alpha\in\mathfrak{o}/\mathfrak{p}^\delta,\\ \beta\in\mathfrak{o}/\mathfrak{p}}} \xi^{-1}(1+\frac{\alpha}{A_\pm}\Omega^{2\lceil\frac{r}{2}\rceil-2\delta}) \\ &\quad\quad \cdot\psi(2(1+vB_\pm)\alpha\varpi^{-\lfloor\frac{r}{2}\rfloor-\delta-1}+v\alpha^2\varpi^{2\lceil\frac{r}{2}\rceil-r-2\delta-1}). \end{aligned}$$

We use Lemma 3.1.3 to transform ξ into an additive oscillation. The Taylor expansion of $\log_E$ is given by

$$\begin{aligned} &-\mathrm{Tr}\left(\frac{b_\xi}{\Omega^{a(\xi)+1}}\log_E(1+\frac{\alpha}{A_\pm}\Omega^{2\lceil\frac{r}{2}\rceil-2\delta})\right) \\ &\qquad = \sum_{j\geq 1}\frac{(-1)^j}{j}\mathrm{Tr}\left[\frac{b_\xi}{A_\pm^j}\alpha^j\right]\varpi^{j\lceil\frac{r}{2}\rceil-r-j\delta-1}. \end{aligned}$$

Since $r-4\delta\geq 0$ we can truncate after the third term. Writing $\frac{b_\xi}{A_\pm^j}=a_j'+a_j''\Omega$ allows us to take a closer look at the coefficients. We give more explicit descriptions in the critical case $B_\pm=-\frac{1}{2v}$.

$$\begin{aligned} \mathrm{Tr}\left[\frac{b_\xi}{A_\pm}\right] &= 2a_1' = \frac{\Delta}{\mathrm{Nr}_{E/F}(A_\pm)}-2(1+vB_\pm), \\ \mathrm{Tr}\left[\frac{b_\xi}{A_\pm^2}\right] &= 2a_2' \in -v+\Delta\mathfrak{o}, \\ \mathrm{Tr}\left[\frac{b_\xi}{A_\pm^3}\right] &= 2a_3' \in \frac{2b_1B_\pm^3}{\mathrm{Nr}_{E/F}(A_\pm)^3}\alpha^3+\mathfrak{p}\subset\mathfrak{o}^\times. \end{aligned}$$

This shows

$$\begin{aligned} &W_\pi(g_{-n,\frac{n}{2},v}) \\ &= \sum_\pm \gamma_\pm\gamma_\pm\gamma_E(-2v,1)\xi^{-1}(A_\pm)\psi(\mathrm{Tr}(A_\pm)+v\varpi^{-l}\mathrm{Nr}_{E/F}(A_\pm)) \\ &\quad\cdot\sum_{\alpha\in\mathfrak{o}/\mathfrak{p}^\delta}\psi(-\frac{2}{3}a_3'\alpha^3\varpi^{3\lceil\frac{r}{2}\rceil-r-1-3\delta}+(v+a_2')\alpha^2\varpi^{2\lceil\frac{r}{2}\rceil-r-1-2\delta} \\ &\qquad\qquad +2(1+v\Re(\overline{A_\pm})-a_1')\alpha\varpi^{-\lfloor\frac{r}{2}\rfloor-1-\delta}). \end{aligned}$$

In particular, the worst case turns out to be cubic cancellation. Even more, if $\Delta=0$, we are left with a clean cubic coefficient and obtain

$$\left|W_\pi(g_{-n,\frac{n}{2},v})\right|\leq 2q^{\frac{r+1}{6}}=2q^{\frac{n}{12}}.$$

If $\Delta \in \mathfrak{o}^\times$, one obtains

$$W_\pi(g_{-n,\frac{n}{2},v}) \leq 2.$$

The details are left to the reader.

Case III: $\frac{n}{2} < l$**.** In this case we have $k = 2l-1$, $\rho = 1$ and $t = -k-1 = -2l$. In order to compute S we write $x = x_1 + x_2\Omega$ and obtain the system of equations

$$\begin{aligned} -b_2' + x_2 &\in \mathfrak{p}^{\lfloor \frac{r}{2} \rfloor}, \\ vx_1^2 + x_1 - v\varpi x_2^2 - b_1' &\in \mathfrak{p}^{\lceil \frac{r}{2} \rceil}, \end{aligned}$$

where $b_1' + b_2'\Omega = b_\xi \Omega^{2l-n}$. Because $2l > n$, we obtain a quadratic equation for x_1 with discriminant in $\mathfrak{o}^\times$. Thus, due to the special shape of this equation, we obtain exactly one $x_0 \in (\mathfrak{O}/\mathfrak{P}^r)^\times$ solving the quadratic congruence. We have

$$S = \{x_0\} \subset (\mathfrak{O}/\mathfrak{P}^r)^\times.$$

Inserting this parametrisation of S in (3.4.7) yields

$$W_\pi(g_{-2l,l,v}) = \gamma\gamma_E(2v,1)\xi^{-1}(x_0)\psi(\mathrm{Tr}(\Omega^t x_0) + v\varpi^{-l}\mathrm{Nr}_{E/F}(x_0)).$$

In particular, $|W_\pi(g_{-2l,l,v})| \leq 1$.

□

Remark 3.4.5. *We observe that, if π is a twist-minimal supercuspidal representation coming from a ramified extension E/F, then there are no degenerate critical points and we get explicit evaluations of $W_\pi(g_{t,l,v})$ in every case as well as the very satisfying bound $|W_\pi(g)| \leq 1$. However, if π is not twist-minimal, then there are always v for which there are degenerate critical points.*

3.4.2 *Twists of Steinberg*

In this section we analyse the behaviour of W_π when π is a twist of Steinberg. If $\pi = \mathrm{St}$, a complete evaluation is given in Lemma 3.3.3. If χ has large enough ramification, we can use the method of stationary phase to derive (more or less) explicit evaluations of the Whittaker new vector. This is captured in the next lemma.

Lemma 3.4.6 ([4], Lemma 5.5)**.** *Let $\pi = \chi St$ for some unitary character χ with $a(\chi) > 1$. We can evaluate the Whittaker function explicitly as follows.*

If $l \neq \frac{n}{2}, 0$, then

$$W_\pi(g_{t,l,v}) = \begin{cases} \chi^2(x_0)\psi((x_0-b)\varpi^{-\frac{k}{2}}) & \text{if } t=-k, \\ 0 & \text{else,} \end{cases}$$

where x_0 is the unique solution to $v\varpi^{\frac{k}{2}-l}x^2+x+b\varpi^{\frac{k}{2}-a(\chi)}=0$ satisfying $v(x_0)=0$.

If $l=\frac{n}{2}$ and $-2>t>-n$, then $W_\pi(g_{t,l,v})=0$ unless $-bv\in\mathfrak{o}^{2\times}$. In the latter case we observe that

$$\begin{aligned} W_\pi(g_{t,l,v}) &= q^{-\frac{n+t}{4}}\gamma_F(-\frac{b}{2},l)\psi(-b\varpi^{-l}) \\ &\cdot\sum_{\pm}\begin{cases} \gamma_F(\pm Y,-\frac{t}{2})\chi(-\frac{b}{v})\psi(\pm 2Y\varpi^{\frac{t}{2}}) \\ \text{if } Y^2=-\frac{b}{v}\in\mathfrak{o}^{2\times} \text{ and } t\geq -2\lceil\frac{l}{2}\rceil, \\ \gamma_F(2v,-\frac{t}{2})\chi\left(b\frac{A_\pm+x_0\varpi^{-\frac{t}{2}-\lfloor\frac{l}{2}\rfloor}}{vA_\mp-x_0\varpi^{-\frac{t}{2}-\lfloor\frac{l}{2}\rfloor}}\right)\psi\left(A_\pm\varpi^{\frac{t}{2}}+x_0\varpi^{\lfloor\frac{l}{2}\rfloor}\right) \\ \text{if } Y^2=-4bv+\varpi^{n+t}=\Delta\in\mathfrak{o}^{2\times} \text{ and } t<-2\lceil\frac{l}{2}\rceil, \\ \text{where } x_0\in\mathfrak{o} \text{ solves (3.4.13) below and } A_\pm=-\frac{\varpi^{l+\frac{t}{2}}}{2v}\pm\frac{Y}{2v}. \end{cases}\end{aligned}$$

If $-\frac{t}{2}=l=\frac{n}{2}$, we define $\Delta=1-4vb$. One has

$$\begin{aligned} W_\pi(g_{t,l,v}) &= q^{\frac{n}{12}}\gamma_F(-2v,l)\chi^{-1}(4v^2)\psi(\frac{3}{4v}\varpi^{-\frac{n}{2}}) \\ &\quad\cdot Ai_\psi(-16bv^3\varpi^{\lceil\frac{l}{2}\rceil+2\{\frac{l}{2}\}-3\lfloor\frac{1}{2}\lceil\frac{l}{2}\rceil\rfloor};\Delta\varpi^{-\lfloor\frac{l}{2}\rfloor-\lfloor\frac{1}{2}\lceil\frac{l}{2}\rceil\rfloor}), \end{aligned}$$

for $v(\Delta)\geq\lceil\frac{l}{2}\rceil$, and $\lceil\frac{n}{4}\rceil\geq\kappa_F$; and

$$\begin{aligned} W_\pi(g_{t,l,v}) &= \delta_{\Delta\in F^{2\times}}q^{\frac{v(\Delta)}{4}}\sum_{\pm}\gamma_F(-1\pm\sqrt{\Delta},\rho)\gamma_F(\Delta\pm\sqrt{\Delta},l-\frac{v(\Delta)}{2}) \\ &\quad\cdot\chi^2(-\frac{1}{2v}\pm\frac{\sqrt{\Delta}}{2v})\psi(\varpi^{-\frac{n}{2}}(\frac{\Delta-3}{4v})), \end{aligned}$$

for $0<v(\Delta)<\lceil\frac{l}{2}\rceil$, and $\lceil\frac{n}{4}\rceil\geq\kappa_F$ or $v(\Delta)=0$.

The cases $l=0$, $l\geq n$ or $t\geq -2$ have been treated to our satisfaction in Lemma 3.3.4 and are ignored at this point.

Proof. Define $k=\max(n,2l)$. For $0<l<n$ and $t<-2$ Lemma 3.3.4 implies

$$W_\pi(g_{t,l,v}) \quad=\quad \zeta_F(1)^{-2}q^{-\frac{t}{2}}K(\chi\circ\mathrm{Nr}_{E/F},(\varpi^{\frac{t}{2}},\varpi^{\frac{t}{2}}),v\varpi^{-l}),$$

for $E = F \times F$. We will evaluate the oscillatory integral K starting from the prototype given in Lemma 3.1.7. To do so we need to investigate the structure of the critical set S in several cases.

Case I: $0 < l < \frac{n}{2}$. In this situation we have $k = n = -t$. The matrix A_{x_1,x_2} given in Lemma 3.1.7 is diagonal modulo $\mathfrak{p}$ and independent of x_1 and x_2. Furthermore, the congruence conditions for $(x_1, x_2) \in S$ read

$$\begin{aligned} x_1 - x_2 &\in \mathfrak{p}^r, \\ v\varpi^{\frac{k}{2}-l}x_1^2 + x_1 + b &\in \mathfrak{p}^r. \end{aligned}$$

By Lemma 3.1.4 we conclude that $\sharp S = 1$. Therefore we have

$$\begin{aligned} W_\pi(g_{t,l,v}) &= q^\rho \sum_{(x_0,x_0)\in S} \chi^2(x_0)\psi(2x_0\varpi^{-\frac{k}{2}} + v\varpi^{-l}x_0^2) \\ &\qquad G(-\frac{b}{2}\varpi^{-\rho}, (b + x_0 + vx_0^2\varpi^{\frac{k}{2}-l})\varpi^{-r-\rho})^2. \end{aligned}$$

By Lemma 3.1.2 we arrive at

$$W_\pi(g_{t,l,v}) = \chi(x_0)^2\psi((x_0 - b)\varpi^{-\frac{k}{2}})$$

where x_0 is the unique solution of $v\varpi^{\frac{k}{2}-l}x_1^2 + x_1 + b = 0$ satisfying $v(x_0) = 0$.

Case II: $l = \frac{n}{2}$. This is the transition region where the Whittaker function can be non-zero for several t. Recall that the congruences defining S are

$$b + \varpi^{l+\frac{t}{2}}x_1 + vx_1x_2 \in \mathfrak{p}^r, \tag{3.4.10}$$

$$b + \varpi^{l+\frac{t}{2}}x_2 + vx_1x_2 \in \mathfrak{p}^r. \tag{3.4.11}$$

Case II.1: $-a(\chi) - \rho \leq t < -2$. The congruences simplify to $x_1x_2 \in -\frac{b}{v} + \mathfrak{p}^r$ and has a unique solution x_2 for each $x_1 \in (\mathfrak{o}/\mathfrak{p}^r)^\times$. Using the fact that the S-sum in Lemma 3.1.7 is well defined modulo $\mathfrak{p}^r$ we obtain

$$\begin{aligned} W_\pi(g_{t,a(\chi),v}) &= q^{-\frac{t}{2}-2r}\chi\left(-\frac{b}{v}\right)\psi\left(-b\varpi^{-a(\chi)}\right) \sum_{x\in(\mathfrak{o}/\mathfrak{p}^r)^\times} \psi\left(x\varpi^{\frac{t}{2}} - x^{-1}\frac{b}{v}\varpi^{\frac{t}{2}}\right) \\ &\qquad G\left(\frac{\varpi^{-\rho}}{2}\begin{pmatrix} -b & -b \\ -b & -b \end{pmatrix}, \varpi^{r+\frac{t}{2}}\begin{pmatrix} x \\ -\frac{b}{vx} \end{pmatrix}\right). \end{aligned}$$

Evaluating the Gauß sum using Lemma 3.1.2 yields

$$G\left(\frac{\varpi^{-\rho}}{2}\begin{pmatrix} -b & -b \\ -b & -b \end{pmatrix}, \varpi^{r+\frac{t}{2}}\begin{pmatrix} x \\ -\frac{b}{vx} \end{pmatrix}\right) = \gamma_F(-\frac{b}{2}, \rho)q^{-\frac{\rho}{2}}$$

if $t \geq -a(\chi)$ or $t = -a(\chi) - 1$ and $x^2 \in -\frac{b}{v} + \mathfrak{p}$. Otherwise, the Gauß sum vanishes. Thus, for $t \geq -a(\chi)$, we get

$$W_\pi(g_{t,a(\chi),v}) = \gamma_F\left(-\frac{b}{2},\rho\right)\chi\left(-\frac{b}{v}\right)\psi\left(-b\varpi^{-l}\right)\zeta_F(1)^{-1}q^{-\frac{t}{2}-r-\frac{\rho}{2}}S_1\left(1,-\frac{b}{v},-\frac{t}{2}\right).$$

Evaluating the Kloosterman sum using Lemma 3.1.8 reveals

$$W_\pi(g_{t,a(\chi),v}) = q^{-\frac{t}{4}-\frac{n}{4}}\sum_{\pm}\gamma_F\left(-\frac{b}{2},\rho\right)\gamma_F\left(\pm Y,-\frac{t}{2}\right)\chi\left(-\frac{b}{v}\right)\psi((\pm 2Y\varpi^{\frac{t}{2}}-b)\varpi^{-l}) \tag{3.4.12}$$

if $Y^2 = -\frac{b}{v} \in \mathfrak{o}^{2\times}$. Otherwise, the Whittaker function vanishes.

For $t = -a(\chi) - 1$, we observe that the critical points of the Kloosterman sum are congruent to Y modulo $\mathfrak{p}$. Thus, we also arrive at (3.4.12).

Case II.2: $-n < t < -a(\chi) - \rho$**.** The first congruence, (3.4.10), can be rewritten as

$$x_1 \in -b(\varpi^{l+\frac{t}{2}} + vx_2)^{-1} + \mathfrak{p}^r.$$

Substituting this in the second congruence, (3.4.11), yields

$$vx_2^2 + \varpi^{l+\frac{t}{2}}x_2 + b \in \mathfrak{p}^{-\frac{t}{2}-r-\rho}.$$

It is easy to see that the discriminant of this equation satisfies $v(\Delta) = 0$. We can parametrise x_2 using Lemma 3.1.4. We compute

$$\begin{aligned} W_\pi(g_{t,l,v}) &= q^{-\frac{t}{2}-l+\rho}\chi(-b)\psi(-b\varpi^{-l})\sum_{x_2}\chi(x_2)\chi^{-1}(vx_2+\varpi^{l+\frac{t}{2}})\psi(x_2\varpi^{\frac{t}{2}}) \\ &\quad \cdot G\left(\frac{v^2x_2^2}{2}\begin{pmatrix}-b & -b\\ -b & -b\end{pmatrix}\varpi^{-\rho},\varpi^{r+\frac{t}{2}}\begin{pmatrix}0\\ vx_2^2+x_2\varpi^{l+\frac{t}{2}}+b\end{pmatrix}\right).\end{aligned}$$

We use Lemma 3.1.4 to parametrise the family x_2 and set $A_\pm = -\frac{\varpi^{l+\frac{t}{2}}}{2v} \pm \frac{\sqrt{\Delta}}{2v} \in \mathfrak{o}^\times$ to shorten notation. Observe that $vA_\pm + \varpi^{\frac{t}{2}+l} = -vA_\mp$. For[5] $-t \geq 2\kappa_F + a(\chi) - \rho$ we can use Lemma 3.1.3 and get

$$\begin{aligned} W_\pi(g_{t,l,v}) &= q^{\frac{\rho}{2}}\gamma_F(-\frac{b}{2},\rho)\sum_{\pm}\chi\left(\frac{bA_\pm}{vA_\mp}\right)\psi(A_\pm\varpi^{\frac{t}{2}}-b\varpi^{-l}) \\ &\quad \cdot\int_{\mathfrak{o}}\psi\left(\sum_{j\geq 2}\frac{b}{j}((-1)^jA_\pm^j - A_\mp^j)\left(\frac{-vt}{b}\right)^j\varpi^{-(\frac{t}{2}+r)j-l}\right)dt.\end{aligned}$$

5 Note that since t is even this always holds if $\kappa_F = 1$. In case this assumption fails we can still estimate the x_2-sum trivial and obtain $|W_\pi(g_{t,l,v})| \leq 2$. However, this is not as satisfactory as an explicit evaluation.

One checks that $A_\pm^2 - A_\mp^2 = \pm\frac{\sqrt{\Delta}}{v^2}\varpi^{l+\frac{t}{2}}$. Furthermore, the binomial expansion shows that $((-1)^j A_\pm^j - A_\mp^j) \in \mathfrak{p}^{l+\frac{t}{2}}$. Evaluating the remaining oscillatory integral yields

$$W_\pi(g_{t,l,v}) = q^{-\frac{t}{4}-\frac{n}{4}}\psi(-b\varpi^{-l})\gamma_F(-\frac{b}{2},\rho) \cdot \sum_{\pm}\gamma_F\left(\frac{\sqrt{\Delta}}{2b},-\frac{t}{2}\right)\chi\left(b\frac{A_\pm + x_0\varpi^{-\frac{t}{2}-\lfloor\frac{l}{2}\rfloor}}{vA_\mp - x_0\varpi^{-\frac{t}{2}-\lfloor\frac{l}{2}\rfloor}}\right)\psi\left(A_\pm\varpi^{\frac{t}{2}} + x_0\varpi^{\lfloor\frac{l}{2}\rfloor} - b\varpi^{-l}\right),$$

where $x_0 \in \mathfrak{o}$ is the unique solution to

$$\sum_{j\geq 2} -v\left(-\frac{v}{b}x\right)^{j-1}\frac{(-1)^j A_\pm^j - A_\mp^j}{\varpi^{l+\frac{t}{2}}}\varpi^{-rj-(j-1)\frac{t}{2}} = 0. \tag{3.4.13}$$

Case II.3: $t = -n$**.** In this case we have to solve the congruences

$$\begin{aligned} x_1 - x_2 &\in \mathfrak{p}^r, \\ vx_1^2 + x_1 + b &\in \mathfrak{p}^r. \end{aligned}$$

The quadratic congruence has discriminant

$$\Delta = \Delta(v) = 1 - 4bv.$$

For some v the discriminant might be (p-adically) small, so that there are many solutions for x_1. In this case we have to argue slightly more carefully.

Using Lemma 3.1.4 we parametrise

$$S = \left\{\left(-\frac{1}{2v} \pm \frac{Y}{2v}\varpi^\delta + \alpha\varpi^{r-\delta}, -\frac{1}{2v} \pm \frac{Y}{2v}\varpi^\delta + \alpha\varpi^{r-\delta}\right) \in ((\mathfrak{o}/\mathfrak{p}^r)^\times)^2 \colon \alpha \in \mathfrak{o}/\mathfrak{p}^\delta\right\}. \tag{3.4.14}$$

We set

$$A_\pm = -\frac{1}{2v} \pm \frac{Y}{2v}\varpi^\delta$$

and $\gamma_\pm = 1$ if $v(\Delta) < r$ and $\gamma\pm = \frac{1}{2}$ otherwise. We can rewrite the Gauß sum from Lemma 3.1.7 as

$$G\left(\frac{\varpi^{-\rho}}{4}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}-b - vx_1^2 & 0\\ 0 & -b + vx_1^2\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}, \varpi^{-r-\rho}B_{x_1,x_2}\right) = G\left(\frac{\varpi^{-\rho}x_1}{4},0\right)G\left(\frac{-\varpi^{-\rho}}{4}(-2b - x_1), \varpi^{-r-\rho}(x_1 + vx_1^2 + b)\right).$$

First, we consider the degenerate case, $v(\Delta) > 0$. In particular, we have $-2b - x_1 \in \mathfrak{p}$ and $v(\Delta) \geq 2 > \rho$. Therefore, we obtain the stronger congruence $vx_1^2 + x_1 + b \in \mathfrak{p}^{r+\rho}$. Using our parametrisation of S in Lemma 3.1.7 yields

$$W_\pi(g_{t,l,v}) = q^{\frac{\rho}{2}} \sum_{\pm} \gamma_F(A_\pm, \rho)\chi^2(A_\pm)\psi(\varpi^{-\frac{n}{2}}(2A_\pm + vA_\pm^2)) \cdot \sum_{\alpha \in \mathfrak{o}/\mathfrak{p}^{\delta-\rho}} \chi^2(1 + \frac{\alpha}{A_\pm}\varpi^{r+\rho-\delta})\psi((1 \pm Y\varpi^\delta)\alpha\varpi^{-r-\delta} + v\alpha^2\varpi^{\rho-2\delta}).$$

For $\lceil \frac{n}{4} \rceil \geq \kappa_F$ we apply Lemma 3.1.3 and get

$$W_\pi(g_{t,l,v}) = q^{\delta-\frac{\rho}{2}} \sum_{\pm} \gamma_F(A_\pm, \rho)\chi^2(A_\pm)\psi(\varpi^{-\frac{n}{2}}(2A_\pm + vA_\pm^2)) \cdot \int_{\mathfrak{o}} \psi(\underbrace{(1 \pm Y\varpi^\delta + \frac{2b}{A_\pm})}_{=\frac{-\Delta}{2vA_\pm} + \frac{Y^2\varpi^{2\delta}}{2vA_\pm} \in \mathfrak{p}^{r+\rho}} t\varpi^{-r-\delta} + \underbrace{(v - \frac{b}{A_\pm^2})}_{\in \frac{\Delta \mp Y\varpi^\delta}{2vA_\pm^2} + \mathfrak{p}^{r+\rho}} t^2\varpi^{\rho-2\delta} + \frac{2b}{3A_\pm^3}t^3\varpi^{r+2\rho-3\delta})dt.$$

The sum is truncated after the third term because $2r - 4\delta \geq 0$. It is obvious that the cubic coefficient is a unit. If $v(\Delta) \geq r + \rho$, we have

$$W_\pi(g_{t,l,v}) = q^{\frac{n}{12}}\gamma_F(-2v, \rho)\chi^{-1}(4v^2)\psi(\frac{3}{4v}\varpi^{-\frac{n}{2}})\mathrm{Ai}_\psi(-16bv^3\varpi^{r+2\rho-3\delta}; \Delta\varpi^{-r-\delta}).$$

If $\rho < \Delta < r + \rho$, we arrive at

$$W_\pi(g_{t,l,v}) = q^{\frac{\delta}{2}} \sum_{\pm} \gamma_F(-1 \pm \sqrt{\Delta}, \rho)\gamma_F(\Delta \pm \sqrt{\Delta}, \delta - \rho) \cdot \chi^2\left(-\frac{1}{2v} \pm \frac{\sqrt{\Delta}}{2v}\right)\psi\left(\varpi^{-\frac{n}{2}}\left(\frac{\Delta - 3}{4v}\right)\right). \quad (3.4.15)$$

Note that in this case $W_\pi(g_{t,l,v})$ vanishes when $\Delta \notin F^{2\times}$.

Second, if $0 = v(\Delta)$, one easily checks that (3.4.15) holds as well.

Case III: $\frac{n}{2} < l < n$**.** Here we have $k = 2l$ and the Whittaker function is non-zero only for $t = -k$. The congruence conditions defining S yield the system of equations

$$\begin{aligned} x_1 - x_2 &\in \mathfrak{p}^r, \\ (x_1 + \frac{1}{2v})^2 &\in \frac{1}{4v^2} - \frac{b}{v}\varpi^{\frac{k}{2}-a(\chi)} + \mathfrak{p}^r \subset \mathfrak{o}^\times. \end{aligned}$$

The quadratic equation has a unique solution modulo $\mathfrak{p}^r$ which is in $\mathfrak{o}^\times$. Thus $\sharp S = 1$. From Lemma 3.1.7 and Lemma 3.1.2 we obtain

$$W_\pi(g_{-2l,l,v}) = \chi(x_0)^2\psi((x_0 - b\varpi^{\frac{k}{2}-a(\chi)})\varpi^{-\frac{k}{2}})$$

where x_0 is the unique solution of $vx_1^2 + x_1 + b\varpi^{\frac{k}{2}-a(\chi)} = 0$ satisfying $v(x_0) = 0$.

This was the last case to be considered and the proof is complete. □

Corollary 3.4.7. *If $\pi = \chi St$ with $a(\chi) \geq 1$, then*

$$W_\pi(g) \ll_F q^{\frac{n}{12}} \text{ and } \sup_g |W_\pi(g)| \gg_F q^{\frac{n}{12}}.$$

Even more, if $\lceil \frac{n}{4} \rceil \geq \kappa_F$, the implicit constant in the upper bound is less than 2.

Proof. If the necessary conditions on κ_F, t and n are full-filled, the explicit evaluations given above imply imply the desired bound. If $a(\chi) = 1$, $l = 1$, and $t = -2$, we have

$$\begin{aligned} W_\pi(g_{-2,1,v}) &= q\zeta_F(1)^{-2} K(\chi \circ \mathrm{Nr}_{E/F}, (\varpi^{-1}, \varpi^{-1}), v\varpi^{-1}) \\ &= q^{\frac{1}{2}} \zeta_F(1)^{-1} \epsilon(\frac{1}{2}, \chi^{-1}) \int_{\mathfrak{o}^\times \setminus (-v^{-1} + \mathfrak{p})} \chi\left(\frac{x}{1+vx}\right) \psi(\varpi^{-1}x) d^\times x. \end{aligned}$$

This is an exponential sum over a finite field and (3.1.2) implies

$$|W_\pi(g_{t,l,v})| \leq 2.$$

In all the remaining cases trivial estimates are sufficient.

The lower bounds follow from

$$\left| W_\pi(g_{-n,\frac{n}{2},\frac{1}{4b}}) \right| = q^{\frac{n}{12}} \left| \mathrm{Ai}_\psi \left(-\frac{\varpi^{r+2\rho-3\lceil \frac{r}{2} \rceil}}{4b^2}; 0 \right) \right|.$$

□

Remark 3.4.8. *If we write $\sqrt{1-4bv\varpi^{|\frac{n}{2}-l|}} = 1 + f_b(v)\varpi^{|\frac{n}{2}-l|}$ for the (in F convergent) power series*

$$f_b(v) = 2bv - 2b^2v^2\varpi^{|\frac{n}{2}-l|} + 4b^3v^3\varpi^{2|\frac{n}{2}-l|} - 10b^4v^4\varpi^{|3\frac{n}{2}-l|} + \cdots,$$

then we have

$$W_\pi(g_{t,l,v}) = \omega_\pi\left(\frac{f_b(v)}{2v}\right) \psi\left(\left(\frac{f_b(v)}{2v} - b\right)\varpi^{-\frac{n}{2}}\right)$$

as long as $0 < l < \frac{n}{2}$. The case $n > l > \frac{n}{2}$ can be treated similarly. Indeed one obtains $x_0 = -\frac{1}{v} + f_b(v)\varpi^{l-\frac{n}{2}}$.

Remark 3.4.9 ([4], Remark 5.6). *There are several other ways to evaluate the integral $K(\chi \circ Nr_{E/F}, \cdot, \cdot)$. For example one may compute that*

$$K(\chi \circ Nr_{E/F}, (\varpi^{-k}, \varpi^{-k}), v\varpi^{-l}) = \int_{\mathfrak{o}^\times} \chi(x)\psi(xv\varpi^{-l}) S_1(1, x, k) d^\times x.$$

For $k > 1$ the Kloosterman sum can be evaluated and one is left with a twisted quadratic Gauß sum. The remaining sum is amenable to the (1-dimensional) method of stationary phase. This turns out to be similar in spirit to the calculation in [18, Lemma 10].

3.4.3 *Irreducible principal series*

The last category of representations to deal with are principal series representations. It is already known from [69] that in certain degenerate situations the Whittaker function can be as large as the local bound [69, Corollary 2.35]. We consider several different situations starting with one that is similar to the twisted Steinberg representations.

Lemma 3.4.10 ([4], Lemma 5.7). *Let $\pi = \chi|\cdot|^s \boxplus \chi|\cdot|^{-s}$ for some $s \in i\mathbb{R} \cup (-\frac{1}{2}, \frac{1}{2})$. Then we have*

$$W_\pi(g) \ll_F q^{\frac{n}{12}} \text{ and } \sup_g |W_\pi(g)| \gg_F q^{\frac{n}{12}}.$$

Even more, for $t < -2$ and $a(\chi) > 1$ we can evaluate the $W_\pi(g_{t,l,v})$ explicitly and obtain the same expressions as in Lemma 3.4.6.

Note that in practice (as long as the Ramanujan conjecture is not known in general) one might encounter real parameters s as constitutes of automorphic representations. However, in this case, one can restrict $s \in (-7/64, 7/64)$ which is the currently best known bound towards the conjecture. See [8]. We ignore the unitary complementary series representations with $|s| \in (1/2, 1)$ since in this case $W_\pi(g_{t,\frac{n}{2},v})$ does not seem to be bounded for $t > \frac{n}{4}$.

Proof. As in the proof of Lemma 3.3.8 we see that for $t \leq -2$ we are in the same situation as for $\pi = \chi\mathrm{St}$. The remaining cases can be estimated trivially using Lemma 3.3.8. □

We move on to the unbalanced principal series $\pi = \chi_1 \boxplus \chi_2$ with unramified χ_2.

Lemma 3.4.11 ([4], Lemma 5.8). *Let $\pi = \chi|\cdot|^s \boxplus |\cdot|^{-s}$ for $s \in i\mathbb{R}$ and put $n = a(\chi) > 0$. Then*

$$|W_\pi(g)| \leq q^{\frac{1}{2}\lfloor\frac{n}{2}\rfloor}.$$

This follows from [69, Corollary 2.35] as well as from our explicit expressions given in Lemma 3.3.5. In [69, Proposition 2.39] it is shown that this bound is sharp. Note that this features the case of twist minimal principal series representations.

We move towards more generic situations.

Lemma 3.4.12 ([4], Lemma 5.10). *Let $\pi = \chi_1|\cdot|^s \boxplus \chi_2|\cdot|^{-s}$ be a irreducible principle series representation. Also assume $a(\chi_1) > a(\chi_2)$ and $s \in i\mathbb{R}$. Define $m = \max(2l, n)$. We have the following evaluations for the Whittaker new vector.*

If $l < \frac{a_1}{2}$,

$$W_\pi(g_{t,l,v}) = \epsilon(\frac{1}{2}, \tilde{\pi}) q^{-\frac{t+n}{2}+s(t+n)} \cdot \begin{cases} \chi_2^{-1}(1 - vb_1\varpi^{a_2-l}) & \text{if } l < a_2 \text{ and } t = -n, \\ \chi_2^{-1}(-vb_1 + \varpi^{n+t}) & \text{if } l = a_2 \text{ and } -n < t < -a_1 \\ & \text{or } t = -n \text{ and } v \notin b_1^{-1} + \mathfrak{p}, \\ \chi_2^{-1}(1 - vb_1) & \text{if } l > a_2, t = -a_1 - l \text{ and } v \in b_1^{-1} + \varpi^{l-a_2}\mathfrak{o}^\times, \\ 0 & \text{else.} \end{cases}$$

If $\frac{a_1}{2} \leq l \leq a_2$,

$$W_\pi(g_{t,l,v}) = \begin{cases} \epsilon(\frac{1}{2}, \tilde{\pi}) q^{-\frac{t+n}{2}+s(t+n)} \chi_1(b_1^{-1}(x_0\varpi^{t+2a_1} + b_{\chi_1^{-1}\chi_1})) \chi_2^{-1}(-b_2 x_0^{-1}) \\ \qquad \cdot \psi(x_0\varpi^{t+a_1} + b_2\varpi^{-a_2}) \\ \qquad \text{if } l < a_2 \text{ and } t = -n \text{ or } l = a_2 \text{ and } -n < t < -a_2 \\ \qquad \text{or } l = a_2, t = -n, \text{ and } v \notin b_1^{-1} + \mathfrak{p}, \\ 0 \qquad \text{else,} \end{cases}$$

where $x_0 \in \mathfrak{o}^\times$ *is the unique solution to*

$$vx^2\varpi^{t+2a_1} + x(\varpi^{t+n} + vb_{\chi_1^{-1}\chi_2}) + b_2 = 0.$$

If $a_2 < l < \frac{a_1+a_2}{2}$ *and* $\lfloor \frac{a_2}{2} \rfloor \geq \kappa_F$,

$$W_\pi(g_{t,l,v}) = \begin{cases} \epsilon(\frac{1}{2}, \tilde{\pi}) q^{-\frac{t+n}{2}+s(t+n)} \chi_1^{-1}(1 + x_0(\varpi^{a_1-l}) \chi_2(-\frac{x_0}{vb_2}) \psi(x_0 v^{-1}\varpi^{-l} + b_2\varpi^{-a_2}) \\ \qquad \text{if } t = -l - a_1 \text{ and } v \in b_1^{-1} + \varpi^{l-a_2}\mathfrak{o}^\times, \\ 0 \qquad \text{else,} \end{cases}$$

where $x_0 \in \mathfrak{o}^\times$ *is the unique solution to* (3.4.19) *below.*

If $l = \frac{a_1+a_2}{2}$, $\frac{a_2}{2} \geq \kappa_F$, *and* $\Delta = \beta^2 - 4b_1b_2 \in \mathfrak{o}^\times$,

$$W_\pi(g_{-l-a_1,l,v}) = q^{\frac{l-a_2}{2}} \epsilon(\frac{1}{2}, \chi_1^{-1}) \chi_2^{-1}(v) \sum_{\pm} \gamma_F(v(1 - \frac{\beta}{\sqrt{\Delta}}), a_2) \\ \cdot \chi_2(x_\pm) \chi_1^{-1}(1 + \varpi^{\frac{a_1-a_2}{2}} x_\pm) \psi(x_\pm v^{-1} \varpi^{-\frac{l}{2}}),$$

where the sum is understood to be taken over the (up to two) solutions of (3.4.22). *If* $\Delta \in \mathfrak{p}$, *we have the upper bound*

$$|W_\pi(g_{-l-a_1,l,v})| \leq 2q^{\frac{l}{2}-\frac{a_2}{3}}.$$

If $\frac{a_1+a_2}{2} < l < a_1$ *and* $\lfloor \frac{a_2}{2} \rfloor \geq \kappa_F$,

$$W_\pi(g_{t,l,v}) = \begin{cases} q^{\frac{a_1-l}{2}+s(a_1-l)}\chi_2(b_2vx_0^{-1})\chi_1(-v^{-1}-v^{-1}x_0\varpi^{l-a_2}) \\ \qquad\qquad \cdot\psi(-v^{-1}(1+x_0\varpi^{l-a_2})\varpi^{-a_1}-b_2\varpi^{-a_2}) \\ \qquad\qquad \text{if } t=-l-a_1 \text{ and } v \in b_1^{-1}+\varpi^{a_1-l}\mathfrak{o}^\times \\ 0 \qquad \text{else,} \end{cases}$$

where $x_0 \in \mathfrak{o}^\times$ *is the unique solution to* (3.4.23) *below.*

If $a_1 \leq l < n$,

$$W_\pi(g_{t,l,v}) = \begin{cases} q^{-\frac{t+2l}{2}-s(t+2l)}\chi_1(x_0)\chi_2(x_0\varpi^{t+2l}-b_{\chi_1^{-1}\chi_2})\psi(x_0\varpi^{t+l}-b_2\varpi^{-a_2}) \\ \qquad\qquad \text{if } l=a_1 \text{ and } t>-2l \text{ or } l=a_1,\ t=-2l, \text{ and } v \notin b_1^{-1}+\mathfrak{p} \\ \chi_1(x_1+b_{\chi_1^{-1}\chi_2}\varpi^{l-a_1})\chi_2(x_1)\psi(x_1\varpi^{-l}-b_2\varpi^{-a_2}), \\ \qquad\qquad \text{if } a_1<l<n \text{ and } t=-2l, \\ 0 \qquad \text{else,} \end{cases}$$

where $x_0 \in \mathfrak{o}^\times$ *(resp.* $x_1 \in \mathfrak{o}^\times$*) is the unique solution to*

$$vx^2\varpi t+2l+x(\varpi^{t+2l}-vb_{\chi_1^{-1}\chi_2})+b_1=0 \quad (\text{resp. } vx^2+x(1+vb_{\chi_1^{-1}\chi_2})+b_2\varpi^{l-a_2}=0).$$

Proof. To simplify notation we write $a_1 = a(\chi_1)$ and $a_2 = a(\chi_2)$ and put $k = \max(a_1, l) = 2r+\rho$. For $i = 1, 2$ we set $b_i = b_{\chi_i} \in \mathfrak{o}^\times$ for the constant associated to χ_i via Lemma 3.1.3. We will focus on the cases $0 < l < n$ and $t < -a_1$. We have to understand

$$K(\chi_1 \otimes \chi_2, (\varpi^{-l_1}, \varpi^{-l_2}), v\varpi^{-l})$$

for suitable $0 < l_1, l_2 \leq l$. If l is small, we find it easier to exploit the stability of ϵ-factors directly instead of using the method of stationary phase. This is similarly to the approach taken in [69, Proposition 2.40].

Case I: $l \leq \frac{a_1}{2}$ **and** $t < -a_1$**.** In this case we have $l_1 = a_1$, and $l_2 = -t - a_1$. Since we assume $t < -a_1$, the δ-term in (3.3.5) does not contribute. We have

$$W_\pi(g_{t,l,v}) = \zeta_F(1)^{-2}q^{-\frac{t}{2}}q^{s(l_1-l_2)}\sum_{\mu\in\mathfrak{X}_l} G(\varpi^{-a_1},\mu\chi_1)G(\varpi^{-l_2},\mu\chi_2)G(v\varpi^{-l},\mu^{-1}).$$

Recall from [69, Lemma 2.37] that

$$\epsilon(\frac{1}{2},\mu^{-1}\chi_1^{-1}) = \epsilon(\frac{1}{2},\chi_1^{-1})\mu(-b_1).$$

This implies

$$G(\varpi^{-a_1},\mu\chi_1) = \zeta_F(1)q^{-\frac{a_1}{2}}\mu(-b_1)\epsilon(\frac{1}{2},\chi_1^{-1}).$$

Inserting this expression above yields

$$W_\pi(g_{t,l,v}) = \zeta_F(1)^{-1}q^{-\frac{a_1+t}{2}}q^{s(l_1-l_2)}\epsilon(\frac{1}{2},\chi_1^{-1})\sum_{\mu\in\mathfrak{X}_l} G(\varpi^{-l_2},\mu\chi_2)G(-vb_1\varpi^{-l},\mu^{-1}).$$

We can evaluate the μ-sum by writing the Gauß sum as an integral, taking the μ-sum inside, and exploiting full cancellation. One arrives at

$$W_\pi(g_{t,l,v}) = \zeta_F(1)^{-1}q^{-\frac{a_1+t}{2}}q^{s(l_1-l_2)}\epsilon(\frac{1}{2},\chi_1^{-1})G(\varpi^{t+a_1}-\varpi^{-l}vb_1,\chi_2).$$

By evaluating the remaining Gauß sum we obtain

$$\begin{aligned} W_\pi(g_{t,l,v}) &= \epsilon(\frac{1}{2},\tilde{\pi})q^{-\frac{t+n}{2}+s(t+n)} \\ &\cdot\begin{cases} \chi_2^{-1}(1-vb_1\varpi^{a_2-l}) & \text{if } l<a_2 \text{ and } t=-n, \\ \chi_2^{-1}(-vb_1+\varpi^{n+t}) & \text{if } l=a_2 \text{ and } -n<t<-a_1 \\ & \quad\text{or } t=-n \text{ and } v\notin b_1^{-1}+\mathfrak{p}, \\ \chi_2^{-1}(1-vb_1) & \text{if } l>a_2, t=-a_1-l \text{ and } v\in b_1^{-1}+\varpi^{l-a_2}\mathfrak{o}^\times, \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Case II: $\frac{a_1}{2}<l<a_2$**.** In this case we have $l_1=a_1$, $l_2=a_2$ and $t=-n$. Note that $a_1-a_2\geq r$ implies $a_2\leq\frac{a_1}{2}$. This situation was covered in Case I, so that we assume $a_1-a_2<r$. Our starting point is Lemma 3.1.7 together with Lemma 3.3.9. Using Lemma 3.1.2 we compute the Gauß sum to be

$$G\left(\frac{\varpi^{-\rho}}{2}A_{x_1,x_2},\varpi^{-r-\rho}B_{x_1,x_2}\right) = q^{-\frac{\rho}{2}}\gamma_F(-\frac{b_1}{2},\rho),$$

whenever x_1 and x_2 satisfy

$$b_1+x_1+vx_1x_2\varpi^{a_1-l}\in\mathfrak{p}^r \text{ and } b_2\varpi^{a_1-a_2}+x_2\varpi^{a_1-a_2}+vx_1x_2\varpi^{a_1-l}\in\mathfrak{p}^{r+\rho}.$$

This can be reformulated to

$$\begin{aligned} &x_1\in x_2\varpi^{a_1-a_2}+b_{\chi_1^{-1}\chi_2}+\mathfrak{p}^r, \\ &x_2^2v\varpi^{a_1-l}+x_2(1+vb_{\chi_1^{-1}\chi_2}\varpi^{a_2-l})+b_2\in\mathfrak{p}^{r+\rho+a_2-a_1}. \end{aligned}$$

According to Lemma 3.1.4 $x_2 \in \mathfrak{o}^\times$ is uniquely determined modulo $\mathfrak{p}^{r+a_2-a_1}$. If x_0 is the unique solution to

$$x^2 v\varpi^{a_1-l} + x(1+vb_{\chi_1^{-1}\chi_2}\varpi^{a_2-l}) + b_2 = 0$$

in $\mathfrak{o}^\times$, then we have

$$S = \{(x_0\varpi^{a_1-a_2} - b_1 + b_2\varpi^{a_1-a_2}, x_0 + \alpha\varpi^{r+\rho+a_2-a_1})\colon \alpha \in \mathfrak{o}/\mathfrak{p}^{a_1-a_2-\rho}\}.$$

We insert this parametrisation in the S-sum from Lemma 3.1.7. Elementary rearrangements yield

$$\begin{aligned} &W_\pi(g_{t,l,v}) \\ &= q^{\frac{a_2}{2}-r+s(t+2a_1)}\gamma_F(-\frac{b_1}{2},a_1)\sum_{(x_1,x_2)\in S}\chi_1(x_1)\chi_2(x_2)\psi(x_1\varpi^{-a_1}+x_2\varpi^{-a_2}+vx_1x_2\varpi^{-l}) \\ &\quad q^{\frac{a_2}{2}+s(t+2a_1)}\gamma_F(-\frac{b_1}{2},a_1)\chi_1(x_0\varpi^{a_1-a_2}+b_{\chi_1^{-1}\chi_2})\chi_2(x_0)\psi(x_0\varpi^{-a_2}+b_{\chi_1^{-1}\chi_2}\varpi^{-a_1}) \\ &\qquad \cdot\zeta_F(1)^{-1}G_{r+\rho+a_2-a_1}(-b_2\varpi^{-a_2},\chi_2). \end{aligned}$$

After checking that $r+a_2-a_1 \leq \frac{a_2}{2}$ we apply Remark 3.3.7 and obtain

$$\begin{aligned} W_\pi(g_{t,l,v}) = q^{s(t+2a_1)}\gamma_F(-\frac{b_1}{2},a_1)\epsilon(\frac{1}{2},\chi_2^{-1})\chi_1(x_0\varpi^{a_1-a_2}+b_{\chi_1^{-1}\chi_2})\chi_2^{-1}(-b_2x_0^{-1}) \\ \cdot\psi(x_0\varpi^{-a_2}+b_{\chi_1^{-1}\chi_2}\varpi^{-a_1}). \end{aligned}$$

In view of Remark 3.3.6 we get

$$W_\pi(g_{t,l,v}) = \epsilon(\frac{1}{2},\tilde{\pi})q^{s(t+n)}\chi_1(b_1^{-1}(x_0\varpi^{a_1-a_2}+b_{\chi_1^{-1}\chi_2}))\chi_2^{-1}(-b_2x_0^{-1})\psi((x_0+b_2)\varpi^{-a_2}).$$

Case III: $\frac{a_1}{2} < l = a_2$. We find $k = 2r = a_1$, $l_1 = a_1$, and $l_2 = -t-a_1$ for $-a_1 > t \geq -n$. The situation turns out to very similar to the one in Case II. Let x_0 be the solution of

$$vx^2\varpi^{t+2a_1} + x(\varpi^{t+n} + vb_{\chi_1^{-1}\chi_2}) + b_2 = 0.$$

Note that if $t = -n$ and $v \in b_1^{-1} + \mathfrak{p}$ then this has no solution in $\mathfrak{o}$. According to Lemma 3.1.7 we obtain

$$\begin{aligned} W_\pi(g_{t,a_2,v}) = \gamma_F(-\frac{b_1}{2},a_1)q^{-\frac{t+a_1}{2}-r+s(t+2a_1)}\chi_1(x_0\varpi^{t+2a_1}+b_{\chi_1^{-1}\chi_2})\psi(x_0\varpi^{t+a_1}-b_1\varpi^{-a_1}) \\ \cdot\sum_{\alpha\in\mathfrak{o}/\mathfrak{p}^{a_1-a_2-\rho}}\chi_2(x_0+\alpha\varpi^{a_2-r})\psi((\varpi^{t+n}+vx_0\varpi^{t+2a_1}+vb_{\chi_1^{-1}\chi_2})\alpha\varpi^{-r}). \end{aligned}$$

We rewrite this as

$$W_\pi(g_{t,a_2,v}) = \gamma_F(-\frac{b_1}{2}, a_1) q^{-\frac{t+a_1}{2}+s(t+2a_1)} \zeta_F(1)^{-1} \chi_1(x_0\varpi^{t+2a_1} + b_{\chi_1^{-1}\chi_2}) \cdot \psi(x_0\varpi^{t+a_1} + b_{\chi_1^{-1}\chi_2}\varpi^{-a_1}) \chi_2(x_0) G_{a_2-r}(-b_2\varpi^{-a_2}, \chi_2).$$

Using Remark 3.3.7 and Remark 3.3.6 we arrive at

$$W_\pi(g_{t,a_2,v}) = \begin{cases} \epsilon(\frac{1}{2},\tilde{\pi}) q^{-\frac{t+n}{2}+s(t+n)} \chi_1(b_1^{-1}(x_0\varpi^{t+2a_1} + b_{\chi_1^{-1}\chi_2})) \\ \qquad \cdot \chi_2^{-1}(-b_2 x_0^{-1}) \psi(x_0\varpi^{t+a_1} + b_2\varpi^{-a_2}) \\ \qquad \text{if } -n < t < -a_1 \text{ or } t = -n \text{ and } v \notin b_1^{-1} + \mathfrak{p}, \\ 0 \qquad \text{else.} \end{cases}$$

Case IV: $l = a_1$**.** We have $l_1 = -t - l$ and $-a_1 > t \geq -2l$. The congruences defining S can be simplified to

$$x_2 \in x_1\varpi^{t+2l} + b_1 - b_2\varpi^{a_1-a_2} + \mathfrak{p}^r,$$
$$x_1^2 v\varpi^{t+2l} + x_1(vb_1 + \varpi^{t+2l} - vb_2\varpi^{a_1-a_2}) + b_1 \in \mathfrak{p}^r.$$

Depending on t the behaviour can be quite different. The matrix $A_\mathfrak{p}$ turns out to be

$$A_\mathfrak{p} = \begin{pmatrix} -b_1 & vx_1x_2 \\ vx_1x_2 & 0 \end{pmatrix}.$$

Case IV.1: $t > -2l$**.** In this case we have $(vb_1 + \varpi^{t+2l} - vb_2\varpi^{a_1-a_2}) \in \mathfrak{o}^\times$. Thus, Lemma 3.1.4 implies that $S = \{x_0\}$ for $x_0 \in \mathfrak{o}^\times$ solving

$$x^2 v\varpi^{t+2l} + x(\varpi^{t+2l} - vb_{\chi_1^{-1}\chi_2}\varpi^{a_1-a_2}) + b_1 = 0. \tag{3.4.16}$$

We obtain

$$W_\pi(g_{t,a_1,v}) = q^{-\frac{t+2l}{2}-s(t+2l)} \chi_1(x_0)\chi_2(x_0\varpi^{t+2l} - b_{\chi_1^{-1}\chi_2}) \psi(x_0\varpi^{t+l} - b_2\varpi^{-a_2}).$$

Case IV.2: $t = -2l$**.** Each x_1 determines a unique x_2 modulo $\mathfrak{p}^r$. The quadratic congruence determining x_1 reads

$$vx_1^2 + x_1(vb_1 + 1 - vb_2\varpi^{a_1-a_2}) + b_1 \in \mathfrak{p}^r$$

and has discriminant

$$\Delta = (1 + vb_1 - vb_2\varpi^{a_1-a_2})^2 - 4vb_1 = (1-vb)^2 - 4vb_2\varpi^{a_1-a_2}.$$

We can rewrite the quadratic equation as

$$(2vx_1 + 1 + vb)^2 \in \Delta + \mathfrak{p}^r,$$

for $b = b_1 - b_2\varpi^{a_1-a_2}$. Suppose that $v(\Delta) > 0$. In particular, $1 - vb \in \mathfrak{p}$. So that any admissible x_1 satisfies

$$x_1 \in -\frac{1}{2v} - \frac{b_1}{2} + \mathfrak{p}.$$

Each admissible x_1 determines a unique x_2 by

$$x_2 \in x_1 + b + \mathfrak{p}^r \subset -\frac{1}{2v} + \frac{b_1}{2} + \mathfrak{p} \subset \mathfrak{p}.$$

But we need x_2 to be a unit. We conclude that if $v(\Delta) > 0$, then $S = \emptyset$ and therefore $W_\pi(g_{-2a_1,a_1,v}) = 0$.

If $v(\Delta) = 0$, we have $\sharp S = 1$. This is because when $x_\pm$ are the two solutions of (3.4.16), then we have $x_+ \in \mathfrak{p}$ and $x_- \in \mathfrak{o}^\times$. We get

$$W_\pi(g_{-2a_1,a_1,v}) = \begin{cases} \chi_1(x_-)\chi_2(x_- - b_{\chi_1^{-1}\chi_2})\psi((x_- - b_2)\varpi^{-l}) & \text{if } v \notin b_1^{-1} + \mathfrak{p} \\ 0 & \text{else.} \end{cases}$$

Case V: $a_1 < l < n$. Here we have $t = -2l$, and $l_1 = l_2 = l$. The set S from Lemma 3.1.7 is given by the system of congruences

$$\begin{aligned} &x_1 \in x_2 - b_1\varpi^{l-a_1} + b_2\varpi^{l-a_2} + \mathfrak{p}^r, \\ &vx_2^2 + x_2(1 - vb_1\varpi^{l-a_1} + vb_2\varpi^{l-a_2}) + b_2\varpi^{l-a_2} \in \mathfrak{p}^r. \end{aligned}$$

One can check that the discriminant of the quadratic equation determining x_2 satisfies $\Delta \in 1 + \mathfrak{p}$. Therefore $S = \{x_0\}$, where $x_0 \in \mathfrak{o}^\times$ is the unique solution to

$$vx^2 + x(1 + vb_{\chi_1^{-1}\chi_2}\varpi^{l-a_1}) + b_2\varpi^{l-a_2} = 0.$$

Furthermore,

$$A_\mathfrak{p} = \begin{pmatrix} 0 & vx_1x_2 \\ vx_1x_2 & 0 \end{pmatrix}.$$

Thus we obtain

$$W_\pi(g_{-2l,l,v}) = \chi_1(x_0 - b_{\chi_1^{-1}\chi_2}\varpi^{l-a_1})\chi_2(x_0)\psi(x_0\varpi^{-l} - b_2\varpi^{-a_2}).$$

Case VI: $\max(\frac{a_1}{2}, a_2) < l < a_1$**.** This constitutes the transition region and it turns out that the approach we took before mutates into very messy calculations. We therefore choose to take a different approach. We calculate

$$\begin{aligned} K(\chi_1 \otimes \chi_2, (\varpi^{-a_1}, \varpi^{-l}), v\varpi^{-l}) \\ &= \int_{\mathfrak{o}^\times} \chi_2(x_2)\psi(\varpi^{-l}x_2)G(\varpi^{-a_1}(1+\varpi^{a_1-l}vx_2), \chi_1)d^\times x_2 \\ &= \epsilon(\frac{1}{2}, \chi_1^{-1})\zeta_F(1)q^{-\frac{a_1}{2}} \int_{\mathfrak{o}^\times} \chi_1^{-1}(1+\varpi^{a_1-l}vx)\chi_2(x)\psi(x\varpi^{-l})d^\times x. \end{aligned}$$

Using the same trick in the x_2 integral yields

$$\begin{aligned} K(\chi_1 \otimes \chi_2, (\varpi^{-a_1}, \varpi^{-l}), v\varpi^{-l}) &= \int_{\mathfrak{o}^\times} \chi_1(x_1)\psi(\varpi^{-a_1}x_1)G(\varpi^{-l}(1+vx), \chi_2)d^\times x_1 \\ &= \epsilon(\frac{1}{2}, \chi_2^{-1})\chi_1(-\frac{1}{v})\psi(-v^{-1}\varpi^{-a_1})\zeta_F(1)q^{\frac{a_2}{2}-l} \\ &\quad \cdot \int_{\mathfrak{o}^\times} \chi_1(1-v\varpi^{l-a_2}x)\chi_2^{-1}(x)\psi(x\varpi^{l-a_1-a_2})d^\times x. \end{aligned}$$

We insert these expressions in Lemma 3.3.9 and obtain

$$\begin{aligned} W_\pi(g_{-l-a_1,l,v}) = \chi_2(v^{-1})\epsilon(\frac{1}{2}, \chi_1^{-1})\zeta_F(1)^{-1}q^{\frac{l}{2}+s(a_1-l)} \\ \cdot \int_{\mathfrak{o}^\times} \chi_1^{-1}(1+\varpi^{a_1-l}x)\chi_2(x)\psi(v^{-1}x\varpi^{-l})d^\times x, \end{aligned} \tag{3.4.17}$$

as well as

$$\begin{aligned} W_\pi(g_{-l-a_1,l,v}) = \epsilon(\frac{1}{2}, \chi_2^{-1})[\chi_1\chi_2^{-1}](-\frac{1}{v})\psi(-v^{-1}\varpi^{-a_1})\zeta_F(1)^{-1}q^{\frac{n-l}{2}+s(a_1-l)} \\ \cdot \int_{\mathfrak{o}^\times} \chi_1(1+\varpi^{l-a_2}x)\chi_2^{-1}(x)\psi(-v^{-1}x\varpi^{l-a_1-a_2})d^\times x. \end{aligned} \tag{3.4.18}$$

Note that estimating the integrals trivially recovers the local bound given in [69, Corollary 2.35]. We will use the 1-dimensional method of stationary phase to find further cancellation. We consider different cases.

Case VI.1: $a_2 < l < \frac{a_1+a_2}{2}$**.** We start from the integral appearing in (3.4.17). Suppose $a_2 = 1$ then the current situation implies $l < \frac{a_1+1}{2}$ but obviously this yields $l \leq \frac{a_1}{2}$ which is excluded from Case IV. Thus we assume $a_2 > 1$ and write $a_2 = 2r + \rho$ for $\rho \in \{0, 1\}$ and $r \in \mathbb{N}$. Assuming $a_1 - l \geq \kappa_F$, for any $\max(r, \kappa_F) \leq \kappa \leq a_2$, we calculate

$$\begin{aligned} &W_\pi(g_{-l-a_1,l,v}) \\ &\quad = \chi_2(v^{-1})\epsilon(\frac{1}{2}, \chi_1^{-1})q^{\frac{l}{2}-\kappa+s(a_1-l)} \sum_{y \in (\mathfrak{o}/\mathfrak{p}^\kappa)^\times} \chi_1^{-1}(1+y\varpi^{a_1-l})\chi_2(y)\psi(v^{-1}y\varpi^{-l}) \\ &\quad \cdot \int_{\mathfrak{o}} \psi\left(-\frac{b_1}{\varpi^{a_1}}\log_F(1+\frac{t\varpi^{\kappa+a_1-l}}{1+\varpi^\kappa y\varpi^{a_1-l}}) + \frac{b_2}{\varpi^{a_2}}(\frac{t}{y}\varpi^\kappa - \frac{t^2}{2y^2}\varpi^{2\kappa}) + v^{-1}t\varpi^{\kappa-l}\right) dt. \end{aligned}$$

The Taylor expansion of the logarithm reads

$$-\frac{b_1}{\varpi^{a_1}}\log_F(1+\frac{t\varpi^{\kappa+a_1-l}}{1+\varpi^{\kappa}y\varpi^{a_1-l}}) = \sum_{j\geq 1}\frac{(-1)^j b_1 t^j}{j}(1+y\varpi^{a_1-l})^{-j}\varpi^{j\kappa+(j-1)a_1-jl}.$$

We first observe that, since $l < \frac{a_1+a_2}{2}$, we can choose $\kappa = r$ and truncate the Taylor expansion after the 1st term. We get

$$\begin{aligned} W_\pi(g_{-l-a_1,l,v}) &= \chi_2(v^{-1})\epsilon(\frac{1}{2},\chi_1^{-1})q^{\frac{l}{2}-\kappa+s(a_1-l)}\sum_{y\in(\mathfrak{o}/\mathfrak{p}^\kappa)^\times}\chi_1(1+y\varpi^{a_1-l})\chi_2(y) \\ &\cdot\psi(v^{-1}y\varpi^{-l})G\left(-\frac{b_2}{2y^2}\varpi^{-\rho},\left(\left(v^{-1}-\frac{b_1}{1+y\varpi^{a_1-l}}\right)\varpi^{a_2-l}+\frac{b_2}{y}\right)\varpi^{-r-\rho}\right). \end{aligned}$$

We see straight away that there is a unique solution $x_0 \in \mathfrak{o}^\times$ to

$$v^{-1}y^2\varpi^{a_1+a_2-2l} + y((v^{-1}-b_1)\varpi^{a_2-l}+b_2\varpi^{a_1-l}) + b_2 = 0 \tag{3.4.19}$$

if and only if

$$1 - vb_1 \in \varpi^{l-a_2}\mathfrak{o}^\times.$$

Evaluating the Gauß sum yields

$$W_\pi(g_{-l-a_1,l,v}) = \begin{cases} \epsilon(\frac{1}{2},\tilde{\pi})q^{\frac{l-a_2}{2}+s(t+n)}\chi_1^{-1}(1+x_0\varpi^{a_1-l})\chi_2(\frac{-x_0}{vb_2})\psi(\frac{x_0}{v}\varpi^{-l}+b_2\varpi^{-a_2}) \\ \qquad\qquad \text{if } v\in b_1^{-1}+\varpi^{l-a_2}\mathfrak{o}^\times, \\ 0 \qquad\qquad \text{else,} \end{cases}$$

as long as $\lfloor\frac{a_2}{2}\rfloor \geq \kappa_F$.

Case IV.2: $l = \frac{a_1+a_2}{2}$**.** Note that this implies that a_1 and a_2 have the same parity so that n must be even. Similar to Case IV.1, assuming $a_1 - a_2 \geq 2\kappa_F$, we deduce from (3.4.17) that

$$v^{-1} - b_1 = \beta\varpi^{\frac{a_1-a_2}{2}} \text{ for some } \beta = \beta(v,\chi_1) \in \mathfrak{o}^\times.$$

Assume $\lfloor\frac{a_2}{2}\rfloor \geq \kappa_F$ and split up the integral from (3.4.17) in q^r-pieces as above. Using suitable Taylor expansions we can write

$$\begin{aligned} W_\pi(g_{t,l,v}) &= \chi_2(v^{-1})\epsilon(\frac{1}{2},\chi_1^{-1})q^{\frac{l}{2}-r}\sum_{y\in(\mathfrak{o}/\mathfrak{p}^r)^\times}\chi_2(y)\chi_1^{-1}(1+\varpi^{\frac{a_1-a_2}{2}}y)\psi(\frac{y}{v}\varpi^{-\frac{a_1+a_2}{2}}) \\ &\cdot G\left(\left(\frac{b_1}{2(1+y\varpi^{a_1-l})^2}-\frac{b_2}{2y^2}\right)\varpi^{-\rho},\frac{v^{-1}y^2+y(\beta+b_2\varpi^{a_1-l})+b_2}{y(1+y\varpi^{a_1-l})}\right). \end{aligned}$$

Thus we have to investigate the equation

$$v^{-1}y^2 + y(\beta+b_2\varpi^{a_1-l}) + b_2 \in \mathfrak{p}^r \tag{3.4.20}$$

with discriminant

$$\Delta = (\beta + b_2\varpi^{a_1-l})^2 - 4\frac{b_2}{v}.$$

If y solves this congruence, then we can compute that

$$\frac{b_1}{2(1+y\varpi^{a_1-l})^2} - \frac{b_2}{2y^2} \in \frac{v}{4y^2}(\Delta \mp \beta\sqrt{\Delta}) + \mathfrak{p}. \tag{3.4.21}$$

Note that the existence of $\sqrt{\Delta}$ in F is necessary for the existence of solutions y. If $\Delta \in \mathfrak{o}^\times$, we are in a non-degenerate situation. In particular, since $v^{-1} \in b_1 + \mathfrak{p}$ we have $\Delta \in \mathfrak{o}^\times$ if and only if $\beta^2 - 4b_1b_2 \in \mathfrak{o}^\times$. Further, we remark that, if $\sqrt{\Delta} \in \pm\beta + \mathfrak{p}$, then we only have one solution to the congruence which is a unit. Evaluating the Gauß sum yields

$$W_\pi(g_{t,l,v}) = q^{\frac{l-a_2}{2}}\epsilon(\frac{1}{2},\chi_1^{-1})\chi_2^{-1}(v)\sum_{\pm}\gamma_F(v(1-\frac{\beta}{\sqrt{\Delta}}),a_2)\chi_2(x_\pm) \cdot \chi_1^{-1}(1+\varpi^{\frac{a_1-a_2}{2}}x_\pm)\psi(x_\pm v^{-1}\varpi^{-\frac{l}{2}}),$$

where $x_\pm \in \mathfrak{o}^\times$ are the only solutions to

$$v^{-1}y^2 + y(\beta + b_2\varpi^{a_1-l}) + b_2 = 0. \tag{3.4.22}$$

Depending on the number of such solutions the $\pm$-sum can have 0, 1 or 2 terms.

We still need to consider the case $\Delta \in \mathfrak{p}$. In this case the y-sum is potentially long. By using Lemma 3.1.4 we obtain

$$W_\pi(g_{t,l,v}) = \chi_2(v^{-1})\epsilon(\frac{1}{2},\chi_1^{-1})q^{\frac{l}{2}-r}\psi(A_\pm v^{-1}\varpi^{-\frac{a_1+a_2}{2}}) \cdot \sum_{y\in(\mathfrak{o}/\mathfrak{p}^{\delta-\rho})^\times}\chi_2(A_\pm + y\varpi^{r+\rho-\delta})\chi_1^{-1}(1+A_\pm\varpi^{\frac{a_1-a_2}{2}}+y\varpi^{\frac{a_1+\rho}{2}-\delta})\psi(\frac{y}{v}\varpi^{-\delta-\frac{a_1-\rho}{2}}).$$

After expanding the characters using Lemma 3.1.3 it is clear that we get an oscillatory integral with a cubic phase. Even more, only χ_2 contributes to the leading coefficient. Thus, as many times before, we get the bound

$$|W_\pi(g_{t,l,v})| \leq 2q^{l-\frac{a_2}{3}}.$$

At this point we could give an expression for W_π involving the p-adic Airy function but we will not pursue this here.

Case VI.3: $l > \frac{a_1+a_2}{2}$**.** This case is very similar to Case IV.1. However, one uses (3.4.18) instead of (3.4.17). We assume that $\frac{a_2}{2} \geq \kappa_F$. For $\kappa = r$ we have

$$\begin{aligned} W_\pi(g_{t,l,v}) =& \epsilon(\frac{1}{2}, \chi_2^{-1})[\chi_1^{-1}\chi_2](-v)\psi(-v^{-1}\varpi^{-a_1})q^{\frac{n-l}{2}-\kappa+s(a_1-l)} \\ &\cdot \sum_{y\in(\mathfrak{o}/\mathfrak{p}^\kappa)^\times} \chi_1(1+y\varpi^{l-a_2})\chi_2^{-1}(y)\psi(-v^{-1}y\varpi^{l-a_1-a_2}) \\ &\cdot G\left(-\frac{b_2}{2y^2}\varpi^{-\rho}, \left(-v^{-1}\varpi^{l-a_1-r-\rho} - \frac{b_2\varpi^{-r-\rho}}{y} + \frac{b_1\varpi^{l-a_1-r-\rho}}{1+y\varpi^{l-a_2}}\right)\right). \end{aligned}$$

We deduce that

$$v \in b_1^{-1} + \varpi^{a_1-l}\mathfrak{o}^\times$$

is necessary for the Whittaker vector to be non-zero. Evaluating the Gauß sum gives

$$\begin{aligned} W_\pi(g_{t,l,v}) =& \chi_2(b_2vx_0^{-1})\chi_1(-v^{-1} - v^{-1}x_0\varpi^{l-a_2}) \\ &\cdot \psi(-v^{-1}(1+x_0\varpi^{l-a_2})\varpi^{-a_1} - b_2\varpi^{-a_2})q^{\frac{a_1-l}{2}+s(a_1-l)}, \end{aligned}$$

where $x_0 \in \mathfrak{o}^\times$ is the unique solution to

$$-v^{-1}x^2\varpi^{2l-a_1-a_2} + x((b_1 - v^{-1})\varpi^{l-a_1} + b_2\varpi^{l-a_2}) - b_2 = 0. \tag{3.4.23}$$

□

Corollary 3.4.13. *Let $\pi = \chi_1|\cdot|^s \boxplus \chi_2|\cdot|^{-s}$ be an irreducible principle series representation with $a(\chi_1) > a(\chi_2)$ and $s \in i\mathbb{R}$. One has*

$$W_\pi(g) \leq 2q^{-\frac{t+\max(2l,n)}{2}},$$

as long as n is odd or $b_1b_2 \notin \mathfrak{o}^{2\times}$. If n is even and $b_1b_2 \in \mathfrak{o}^{2\times}$, we get the weaker bound

$$|W_\pi(g_{t,l,v})| \ll q^{\frac{n}{4}-\frac{a_2}{3}}.$$

The implicit constant is bounded by 2 if $\kappa_F = 1$.

Proof. If $l \notin (a_2, a_1)$, then the claim follows directly from the expressions given in Lemma 3.3.9 and 3.4.12.

Thus, we have to deal with the remaining cases. First note that the general upper bound follows trivially from the local bound as long as $\frac{a_2}{2} \leq \kappa_F$. Thus, we can assume without loss of generality that a_2 is large enough. If $a_2 < l < \frac{a_1+a_2}{2}$, the claimed estimate follows from the expression given in Lemma 3.4.12. The same is true for $\frac{a_1+a_2}{2} < l < a_1$ and $l - a_2$.

Finally, we treat the case $l = \frac{a_1+a_2}{2}$. If $a_2 = 1$, we compute

$$W_\pi(g_{-l-a_1,l,v}) = \chi_2(v^{-1})\epsilon(\frac{1}{2},\chi_1^{-1})q^{\frac{l}{2}-1} \sum_{y\in(\mathfrak{o}/\mathfrak{p})^\times} \chi_2(y)\chi_1^{-1}(1+\varpi^{\frac{a_1-1}{2}}y)\psi(v^{-1}y\varpi^{-\frac{a_1+1}{2}})$$
$$= \chi_2(v^{-1})\epsilon(\frac{1}{2},\chi_1^{-1})q^{\frac{l}{2}-1} \sum_{y\in(\mathfrak{o}/\mathfrak{p})^\times} \chi_2(y)\psi\left(\varpi^{-1}\left(\frac{y\beta}{v}+\frac{b_1y^2}{2}\right)\right).$$

The remaining sum can be treated using Weil's bound (3.1.2). One arrives at

$$|W_\pi(g_{-l-a_1,l,v})| \leq 2q^{\frac{l-a_2}{2}}.$$

The remaining cases were covered in the previous lemma. □

Remark 3.4.14. *The condition $b_1b_2 \in \mathfrak{o}^{2\times}$ is sensitive under twists. Thus, there is a twist of π with the same conductor but no degenerate critical point. In other words, we can twist the large value of W_π away. This was not possible for twists of Steinberg where we can always create a degenerate critical point by choosing v accordingly.*

As preparation for the last situation we prove some estimates for

$$K_{l_2} = K(\chi_1\otimes\chi_2,(\varpi^{t+l_2},\varpi^{-l_2}),v\varpi^{-l}) \tag{3.4.24}$$

when $a(\chi_1) = a(\chi_2) = l > 1$.

Lemma 3.4.15 ([4], Lemma 5.11). *Suppose*[6] $a(\chi_1) = a(\chi_1) = l \geq 8\kappa_F$.

First, consider $a(\chi_1\chi_2^{-1}) \leq \lceil\frac{n}{4}\rceil$. If $-2a(\chi_1\chi_2^{-1}) < t < a(\chi_1\chi_2^{-1})$, we have

$$K_{l_2} = \begin{cases} q^{-\frac{n}{4}}\zeta_F(1)\chi_2^{-1}(v)\,\epsilon(\chi_2^{-1},\frac{1}{2})S_{\chi_1\chi_2^{-1}}\left(1,-\frac{b_2}{v}\varpi^{2a(\chi_1\chi_2^{-1})+t},a(\chi_1\chi_2^{-1})\right) \\ \qquad\qquad \text{if } l_2 = -a(\chi_1\chi_2^{-1})-t, \\ q^{-\frac{n}{4}}\zeta_F(1)\chi_1^{-1}(v)\,\epsilon(\chi_1^{-1},\frac{1}{2})S_{\chi_2\chi_1^{-1}}\left(1,-\frac{b_1}{v}\varpi^{2a(\chi_1\chi_2^{-1})+t},a(\chi_1\chi_2^{-1})\right) \\ \qquad\qquad \text{if } l_2 = a(\chi_1\chi_2^{-1}), \\ 0 \qquad\qquad \text{else.} \end{cases}$$

If $t \leq -2a(\chi_1\chi_2^{-1})$, $K_{l_2} \neq 0$ if and only if $l_2 = -\frac{t}{2}$. We find that

$$K_{-\frac{t}{2}} = q^{-\frac{n}{4}}\zeta_F(1)\chi_2^{-1}(v)\,\epsilon(\chi_2^{-1},\frac{1}{2})S_{\chi_1\chi_2^{-1}}\left(1,-\frac{b_2}{v},-\frac{t}{2}\right), \tag{3.4.25}$$

6 The numerical value $8\kappa_F$ is taken for safety reasons. It is obvious from the proof, that when $F = \mathbb{Q}_p$, one can use 1 instead.

for $-2a(\chi_1\chi_2^{-1}) \geq t \geq -2\lceil \frac{n}{4} \rceil$. *In the range* $-2\lceil \frac{n}{4} \rceil > t > -n$ *we find that*

$$K_{-\frac{t}{2}} = \zeta_F(1)^2 q^{-\frac{n}{4}+\frac{t}{4}} \epsilon(\frac{1}{2}, \chi_2^{-1}) \sum_{\pm} \gamma_F(x_\pm, -\frac{t}{2}) \chi_1(x_\pm) \chi_2^{-1}(\varpi^{a_1+\frac{t}{2}} + vx_\pm) \psi(\varpi^{\frac{t}{2}} x_\pm), \tag{3.4.26}$$

where $x_\pm \in \mathfrak{o}^\times$ *are the unique solutions to*

$$vx^2 + x(vb_{\chi_1\chi_2^{-1}} \varpi^{-\frac{t}{2}-a(\chi_1\chi_2^{-1})} + \varpi^{a_1+\frac{t}{2}}) + b_1 = 0.$$

Second, we look at $a(\chi_1\chi_2^{-1}) > \lceil \frac{n}{4} \rceil$. *In the range* $-a(\chi_1\chi_2^{-1}) > t > -2a(\chi_1\chi_2^{-1})$, *the integral* K_{l_2} *is non-zero for* $l_2 = -a(\chi_1\chi_2^{-1}) - t$ *and* $l_2 = a(\chi_1\chi_2^{-1})$. *If* $l_2 = -a(\chi_1\chi_2^{-1}) - t$, *we obtain the expressions*

$$K_{l_2} = \zeta_F(1)^2 q^{-\frac{n}{4}-\frac{a(\chi_1\chi_2^{-1})}{2}} \epsilon(\chi_2^{-1}, \frac{1}{2}) \epsilon(\chi_1^{-1}\chi_2, \frac{1}{2}) \chi_2^{-1}(v) \psi\left(\frac{b_2\varpi^{t+a(\chi_1\chi_2^{-1})}}{vb_{\chi_1\chi_2^{-1}}}\right),$$

for $-a(\chi_1\chi_2^{-1}) > t \geq -a(\chi_1\chi_2^{-1}) - \lceil \frac{n}{4} \rceil$; *and, for* $-a(\chi_1\chi_2^{-1}) - \lceil \frac{n}{4} \rceil > t > -2a(\chi_1\chi_2^{-1})$, *we have*

$$\begin{aligned} K_{l_2} = \zeta_F(1)^2 q^{-\frac{a_1+a(\chi_1\chi_2^{-1})}{2}} \epsilon(\frac{1}{2}, \chi_2^{-1}) \gamma_F(-2b_{\chi_1\chi_2}, a(\chi_1\chi_2^{-1})) \\ \cdot \chi_1(x_0)\chi_2^{-1}(\varpi^{t+a(\chi_1\chi_2^{-1})+a_1} + vx_0)\psi(\varpi^{-a(\chi_1\chi_2^{-1})}x_0), \end{aligned}$$

where $x_0 \in \mathfrak{o}^\times$ *is the unique solution to*

$$vx^2 + x(vb_{\chi_1\chi_2^{-1}} + \varpi^{t+a(\chi_1\chi_2^{-1})+a_1}) + b_1\varpi^{t+2a(\chi_1\chi_2^{-1})} = 0.$$

In the range $-2a(\chi_1\chi_2^{-1}) \geq t > -n$, *the only non-zero situation is* $l_2 = -\frac{t}{2}$. *If* $t \neq -2a(\chi_1\chi_2^{-1})$, *we recover the expression* (3.4.26). *If* $t = -2a(\chi_1\chi_2^{-1})$ *and* $\Delta = (b_{\chi_1\chi_2^{-1}} + \frac{\varpi^{a_1-a(\chi_1\chi_2^{-1})}}{v})^2 - \frac{4b_1}{v} \in \mathfrak{o}^\times$, *then we get* (3.4.26) *as well. However, if* $\Delta \in \mathfrak{p}$ *we encounter degenerate critical points. In this case we still have the upper bound*

$$\left|K_{-\frac{t}{2}}\right| \leq 2\zeta_F(1)^2 q^{-\frac{a_1}{2}+\frac{t}{6}}. \tag{3.4.27}$$

Finally, if $t = -2a_1$ *and* $\Delta = (1 - vb_{\chi_1\chi_2^{-1}}\varpi^{a_1-a(\chi_1\chi_2^{-1})})^2 - 4vb_2 \in \mathfrak{o}^{2\times}$, *we have*

$$\begin{aligned} K_{a_1} = \zeta_F(1)^2 q^{-a_1} \sum_{\pm} \chi_F((x_\pm - b_2)^2 + b_1b_2)\chi_1(x_\pm - b_{\chi_1\chi_2^{-1}}\varpi^{a_1-a(\chi_1\chi_2^{-1})}) \\ \cdot \chi_2(x_\pm)\psi((x_\pm - b_1)\varpi^{-a_1}), \end{aligned}$$

where

$$x_\pm = -\frac{1}{2v} + \frac{b_1 - b_2}{2} \pm \frac{\sqrt{\Delta}}{2v} \in \mathfrak{o}^\times.$$

However, if $\Delta \in \mathfrak{p}$ we have degenerate critical points. In this case we have the upper bound (3.4.27) holds. Note that, if $a(\chi_1\chi_2^{-1}) < a_1$ this can always happen. If, on the other hand, $a(\chi_1\chi_2) < a_1$, then there are critical points if and only if $-1 \in \mathfrak{o}^{2\times}$.

Proof. An important invariant in the following calculations will be $v(b_1 - b_2)$. Note that we have

$$b_1 - b_2 = b_{\chi_1\chi_2^{-1}}\varpi^{a(\chi_1)-a(\chi_1\chi_2^{-1})}.$$

We write S_{l_2} for the set S defined in Lemma 3.1.7 to keep track of the l_2 dependence. We set $a_1 = 2r + \rho$ as usual.

Case I: $a_1 - l_2, a_1 + l_2 + t \geq r$**.** This leads to a very simple structure of S_{l_2}. Indeed the congruence condition can be transformed into

$$x_2 = -\frac{b_2}{vx_1} + \mathfrak{p}^r \text{ and } b_1 - b_2 \in \mathfrak{p}^r.$$

Thus, if $v(b_1 - b_2) \geq r$, then every $x_1 \in (\mathfrak{o}/\mathfrak{p}^r)^\times$ determines a unique x_2. Otherwise S_{l_2} is empty. Even more, by investigating the Gauß sum appearing in Lemma 3.1.7 we observe that the matrix has rank 1. This imposes stronger conditions on x_1 and possibly $b_1 - b_2$. For $-l_2 - t \geq l_2$, we compute

$$\begin{aligned} K_{l_2} &= q^{-2r-\frac{\rho}{2}}\zeta_F(1)^2\chi_2\left(-\frac{b_2}{v}\right)\psi(-b_2\varpi^{-l})\gamma_F(-2b_2,a_1) \\ &\quad \cdot \sum_{x_1\in(\mathfrak{o}/\mathfrak{p}^r)^\times}[\chi_1\chi_2^{-1}](x_1)\psi\left(x_1\varpi^{t+l_2} - \frac{b_2}{vx_1}\varpi^{-l_2}\right) \\ &= q^{-\frac{a_1}{2}}\zeta_F(1)\chi_2^{-1}(v)\,\epsilon(\chi_2^{-1},\frac{1}{2})S_{\chi_1\chi_2^{-1}}\left(1,-\frac{b_2}{v}\varpi^{-2l_2-t},-l_2-t\right). \end{aligned}$$

In the last step we used Remark 3.3.6 to identify the epsilon factor and we ignored the conditions on x_1 that may appear for $\rho = 1$ since these conditions match the critical points of the twisted Kloosterman sum. Finally, let us use some facts concerning the support of twisted Kloosterman sums from Lemma 3.1.8. If $-l_2 - t = a(\chi_1\chi_2^{-1})$, we encounter two cases. First, if $-2l_2 - t = 0$, then $t = -2a(\chi_1\chi_2^{-1})$ and $l_2 = -\frac{t}{2}$. In this case we can encounter degenerate critical points. Otherwise, if $-2l_2 - t > 0$, then $l_2 = -t - a(\chi_1\chi_2^{-1})$. In particular, $t > -2a(\chi_1\chi_2^{-1})$. If $-l_2 - t < a(\chi_1\chi_2^{-1})$, then the twisted Kloosterman sum vanishes. Second, if $-l_2 - t > a(\chi_1\chi_2^{-1}) \geq 1$, then due to the support of twisted Kloosterman sums the only non-zero situation is $t = -2l_2$ and we have square root cancellation.

For $l_2 \leq -l_2 - t$, the same argument with x_1 and x_2 interchanged yields

$$K_{l_2} = q^{-\frac{a_1}{2}}\zeta_F(1)\chi_1^{-1}(v)\,\epsilon(\chi_1^{-1},\frac{1}{2})S_{\chi_2\chi_1^{-1}}\left(1,-\frac{b_1}{v}\varpi^{2l_2+t},-l_2\right).$$

Taking Lemma 3.1.8 into account completes this part of the proof.

Case II.1: $a_1 - l_2 \geq r > a_1 + l_2 + t$. In this situation the set S_{l_2} is slightly more complicated. It is described by the congruences

$$x_2 = -\frac{b_2}{vx_1} + \mathfrak{p}^r \text{ and } x_1\varpi^{a_1+l_2+t} = -b_1 + b_2 + \mathfrak{p}^r.$$

We observe that this implies $a(\chi_1\chi_2^{-1}) > \frac{a_1}{2}$, since otherwise there are no solutions for x_1. Furthermore, S_{l_2} is empty unless $l_2 = -t - a(\chi_1\chi_2^{-1})$. We can parametrise x_1 by

$$x_1 = -b_{\chi_1\chi_2^{-1}} + \alpha\varpi^{a(\chi_1\chi_2^{-1})-r-\rho} \text{ for } \alpha \in \mathfrak{o}/\mathfrak{p}^{a_1-a(\chi_1\chi_2^{-1})}.$$

The degenerate shape of $A_\mathfrak{p}$ further implies

$$x_1\varpi^{a_1+l_2+t} - x_2\varpi^{a_1-l_2} \in -b_1 + b_2 + \mathfrak{p}^{r+\rho}.$$

This automatically imposes some conditions on α. Indeed, we get

$$\alpha \in \begin{cases} \mathfrak{p}^\rho & \text{if } a_1 - l_2 \geq r + \rho, \\ -\frac{b_2}{vb_{\chi_1\chi_2^{-1}}} + \mathfrak{p}^\rho & \text{if } a_1 - l_2 = r. \end{cases}$$

Each choice of x_1 determines x_2 by

$$x_2 = -\frac{b_2}{vx_1} = \frac{b_2}{v}\sum_{j\geq 0}\frac{\alpha^j}{b_{\chi_1\chi_2^{-1}}^{j+1}}\varpi^{ja(\chi_1\chi_2^{-1})-jr}.$$

Note that, if $a(\chi_1\chi_2^{-1}) = a_1$, $\det(A_\mathfrak{p}) \in \mathfrak{o}^\times$ and there is a unique x_1 and $A_\mathfrak{p}$ is non-degenerate. Thus we obtain

$$K_{l_2} = \zeta_F(1)^2 q^{-\frac{n}{4}-\frac{a(\chi_1\chi_2^{-1})}{2}}\epsilon(\chi_2^{-1},\frac{1}{2})\epsilon(\chi_1^{-1}\chi_2,\frac{1}{2})\chi_2^{-1}(-v)\psi\left(\frac{b_2\varpi^{t+a(\chi_1\chi_2^{-1})}}{vb_{\chi_1\chi_2^{-1}}}\right). \quad (3.4.28)$$

Assuming the contrary we use the parametrisation above and write

$$\begin{aligned} K_{l_2} =& \chi_1(-b_{\chi_1\chi_2^{-1}})\chi_2^{-1}\left(vb_{\chi_1\chi_2^{-1}}\right)\psi\left(-b_{\chi_1\chi_2^{-1}}\varpi^{-a(\chi_1\chi_2^{-1})} + \frac{b_2}{vb_{\chi_1\chi_2^{-1}}}\varpi^{t+a(\chi_1\chi_2^{-1})}\right) \\ &\cdot \epsilon(\frac{1}{2},\chi_2^{-1})\zeta_F(1)^2 q^{-a_1+\frac{\rho}{2}} \sum_{\alpha\in\mathfrak{o}/\mathfrak{p}^{a_1-a(\chi_1\chi_2^{-1})-\rho}} [\chi_1\chi_2^{-1}]\left(1-\frac{\alpha}{b_{\chi_1\chi_2^{-1}}}\varpi^{a(\chi_1\chi_2^{-1})-r}\right) \\ &\cdot \psi\left(\alpha\varpi^{-r} + \frac{b_2}{v}\sum_{j\geq 1}\frac{\alpha^j}{b_{\chi_1\chi_2^{-1}}^{j+1}}\varpi^{t+(j+1)a(\chi_1\chi_2^{-1})-jr}\right). \end{aligned}$$

The linear term in the remaining sum is $\frac{b_2\varpi^{2a(\chi_1\chi_2^{-1})+t-r}}{vb_{\chi_1\chi_2^{-1}}^2}$. After observing that $t \leq -2a(\chi_1\chi_2^{-1})$ is excluded by the current assumptions we observe that the quadratic term is dominated by $-b_{\chi_1\chi_2^{-1}}^{-1}2^{-1}\varpi^{a(\chi_1\chi_2^{-1})-2r}$. Thus the Gauß sum is non-zero whenever $t \geq -a(\chi_1\chi_2^{-1}) - r$. We find that (3.4.28) remains true. The case $a_1 - l_2 = r$, corresponding to $t = -a(\chi_1\chi_2^{-1}) - r - \rho$ is slightly different, but the result turns out to be the same.

Case II.2: $a_1 - l_2 < r \leq a_1 + l_2 + t$**.** This case is exactly the same as the previous one. After exchanging roles of χ_1 and χ_2 we find that we must have $a(\chi_1\chi_2^{-1}) > r$ and $l_2 = a(\chi_1\chi_2^{-1})$. Also the rest of the argument remains essentially the same and as long as $t \geq -a(\chi_1\chi_2^{-1}) - r$ one arrives at (3.4.28) with χ_1 and χ_2 interchanged.

Case III: $a_1 + t + l_2, a_1 - l_2 < r$ **and** $t > -2l$**.** We observe that $-l_2 > -a_1$, since otherwise we are in the situation where $t = -2l$. We compute

$$\begin{aligned} K_{l_2} &= \int_{\mathfrak{o}^\times} \chi_1(x_1) G((\varpi^{a_1-l_2} + vx_1)\varpi^{-a_1}, \chi_2)\psi(\varpi^{t+l_2}x_1)d^\times x_1 \\ &= \zeta_F(1)q^{-\frac{a_1}{2}}\epsilon(\frac{1}{2},\chi_2^{-1})\int_{\mathfrak{o}^\times}\chi_1(x_1)\chi_2^{-1}(\varpi^{a_1-l_2}+vx_1)\psi(\varpi^{t+l_2}x_1)d^\times x_1. \end{aligned}$$

Using the p-adic logarithm yields

$$\begin{aligned} K_{l_2} = &\zeta_F(1)^2q^{-\frac{a_1}{2}-\kappa}\epsilon(\frac{1}{2},\chi_2^{-1})\sum_{x\in(\mathfrak{o}/p^\kappa)^\times}\chi_1(x)\chi_2^{-1}(\varpi^{a_1-l_2}+vx)\psi(\varpi^{t+l_2}x) \\ &\cdot\int_{\mathfrak{o}}\psi\Bigg(t(\varpi^{a_1+l_2+t}+\frac{b_1}{x}-\frac{vb_2}{vx+\varpi^{a_1-l_2}})\varpi^{\kappa-a_1} \\ &\qquad -\sum_{j\geq 2}\frac{(-1)^j}{j}t^j\left(\frac{b_1}{x^j}-\frac{b_2v^j}{(vx+\varpi^{a_1-l_2})^j}\right)\varpi^{j\kappa-a_1}\Bigg)dt, \end{aligned}$$

for every $\kappa \geq \kappa_F$. From the linear term we obtain the quadratic congruence

$$vx^2\varpi^{a_1+l_2+t} + x(vb_{\chi_1\chi_2^{-1}}\varpi^{a_1-a(\chi_1\chi_2^{-1})} + \varpi^{2a_1+t}) + b_1\varpi^{a_1-l_2} \in \mathfrak{p}^\kappa,$$

which is necessary for the t-integral to be non-zero. For this congruence to have a solution it is required that at least two of its terms have the same valuation. We consider the corresponding cases. Note that the cases $a_1 + l_2 + t = 2a_1 + t < a_1 - l_2$ and $a_1 + l_2 + t > 2a_1 + t = a_1 - l_2$ can not occur due to our restrictions on t.

Case III.1: $l_2 + t < -l_2$ **and** $a_1 - a(\chi_1\chi_2^{-1}) \leq 2a_1 + t$**.** These assumptions imply that there are solutions only if $l_2 = -t - a(\chi_1\chi_2^{-1})$ and in particular $a(\chi_1\chi_2^{-1}) > r + \rho$. We choose $\kappa = \lfloor\frac{a(\chi_1\chi_2^{-1})}{2}\rfloor$. This is possible if we assume $a_1 > 4\kappa_F$. Under the current assumptions we can truncate the sum in the integral after the second term. We are left with a quadratic Gauß sum. Looking at the entries carefully reveals that there is exactly one admissible x for which we can evaluating the Gauß sum to get

$$\begin{aligned} K_{l_2} = \zeta_F(1)^2q^{-\frac{a_1+a(\chi_1\chi_2^{-1})}{2}}\epsilon(\frac{1}{2},\chi_2^{-1})\gamma_F(-2b_{\chi_1\chi_2}, a(\chi_1\chi_2^{-1})) \\ \cdot\chi_1(x_0)\chi_2^{-1}(\varpi^{a_1-l_2}+vx_0)\psi(\varpi^{t+l_2}x_0), \end{aligned}$$

where $x_0 \in \mathfrak{o}^\times$ is the unique solution to

$$vx^2 + x(vb_{\chi_1\chi_2^{-1}} + \varpi^{a_1-l_2}) + b_1\varpi^{-t-2l_2} = 0.$$

Case III.2: $l_2 + t > -l_2$ **and** $a_1 - a(\chi_1\chi_2^{-1}) \leq 2a_1 + t$**.** This is dual to Case III.1 and can be treated by exchanging the roles of x_1 and x_2. As a result we find that $l_2 = a(\chi_1\chi_2^{-1})$. The expression for K_{l_2} is as expected.

Case III.3: $t = -2l_2 \neq -2a(\chi_1\chi_2^{-1})$**.** If $a(\chi_1\chi_2^{-1}) > r + \rho$, we choose $\kappa = \lfloor \frac{l_2}{2} \rfloor$. This is possible since $l_2 > a(\chi_1\chi_2^{-1}) > \frac{a_1}{2} > 2\kappa_F$. A familiar arguments yields

$$K_{l_2} = \zeta_F(1)^2 q^{-\frac{a_1}{2}+\frac{t}{4}} \epsilon(\frac{1}{2}, \chi_2^{-1}) \sum_{\pm} \gamma_F(x_\pm, -\frac{t}{2}) \chi_1(x_\pm)\chi_2^{-1}(\varpi^{a_1-l_2} + vx_\pm)\psi(\varpi^{t+l_2}x_\pm), \tag{3.4.29}$$

where $x_\pm \in \mathfrak{o}^\times$ are the solutions to the equation

$$vx^2 + x(vb_{\chi_1\chi_2^{-1}}\varpi^{l_2-a(\chi_1\chi_2^{-1})} + \varpi^{a_1-l_2}) + b_1 = 0. \tag{3.4.30}$$

In particular, $x - \pm$ exist if and only if $-vb_1 \in \mathfrak{o}^{2\times}$.

If $a(\chi_1\chi_2^{-1}) \leq r + \rho$, we argue slightly different. We note that the current assumptions imply that $-n < t < -2r - 2\rho$. The quadratic term in the t-integral is given by

$$-\left(\frac{b_1 vx}{x^2(vx + \varpi^{a_1+\frac{t}{2}})^2} + O(\mathfrak{p}) \right) t^2 \varpi^{2\kappa+\frac{t}{2}}.$$

We choose $\kappa = -\lfloor \frac{t}{4} \rfloor$ and take solutions $x_\pm$ of (3.4.30). In particular, we assume that $-vb_1 \in \mathfrak{o}^{2\times}$. Evaluating the remaining Gauß sum yields the same result as in (3.4.29).

Case III.4: $t = -2a(\chi_1\chi_2^{-1})$**.** Note that the current configuration implies $a(\chi_1\chi_2^{-1}) \geq r$. For this situation there might exists degenerate critical points. We have to solve the congruence

$$x^2 + x(b_{\chi_1\chi_2^{-1}} + \frac{\varpi^{a_1-a(\chi_1\chi_2^{-1})}}{v}) + \frac{b_1}{v} \in \mathfrak{p}^{a(\chi_1\chi_2^{-1})-\kappa}.$$

If the discriminant $\Delta = (b_{\chi_1\chi_2^{-1}} + \frac{\varpi^{a_1-a(\chi_1\chi_2^{-1})}}{v})^2 - \frac{4b_1}{v}$ is a unit, we may argue as above. Let us assume $\Delta \in \mathfrak{p}$ and focus on upper bounds. We write $a(\chi_1\chi_2^{-1}) = 2r' + \rho'$, choose $\kappa = r'$ and parametrise

$$x = -\frac{b_{\chi_1\chi_2^{-1}}}{2} - \frac{\varpi^{a_1-a(\chi_1\chi_2^{-1})}}{2v} \pm \frac{Y}{2}\varpi^\delta + \alpha\varpi^{r'+\rho'-\delta} \text{ for } \alpha \in \mathfrak{o}/\mathfrak{p}^{-\rho'+\delta}$$

and

$$Y = \begin{cases} 0 & \text{if } v(\Delta) \geq r' + \rho', \\ Y_0 & \text{if } v(\Delta) < r' + \rho' \text{ and } (\Delta)_0 = Y_0^2, \end{cases} \quad \text{and}$$

$$\delta = \begin{cases} \lfloor \frac{r'+\rho'}{2} \rfloor & \text{if } v(\Delta) \geq r' + \rho', \\ \delta_0 & \text{if } v(\Delta) = 2\delta_0 < r' + \rho'. \end{cases}$$

Reinserting this parametrisation in the x-sum gives

$$K_{l_2} = \zeta_F(1)^2 q^{-\frac{a_1}{2} - r' - \rho' + \delta} \sum_{\pm} \gamma_{\pm} \int_{\mathfrak{o}} \psi\Bigg(A_1 x - \sum_{j \geq 2} \frac{(-1)^j}{j} x^j \left(\frac{b_1}{A} - \frac{b_2 v^j}{(vA + \varpi^{a_1 - a(\chi_1\chi_2^{-1})})^j} \right) \varpi^{j(r'+\rho'-\delta)-a_1} \Bigg) dx$$

for some $A_1 \in F$, some $\gamma_{\pm} \in S^1$ and $A = -\frac{b_{\chi_1\chi_2^{-1}}}{2} - \frac{\varpi^{a_1 - a(\chi_1\chi_2^{-1})}}{2v} \pm \frac{Y}{2}\varpi^{\delta}$. Observe that $A(vA + \varpi^{a_1 - a(\chi_1\chi_2^{-1})}) \in \mathfrak{o}^{\times}$, so that the j-th coefficient satisfies

$$\left(\frac{b_1}{A^j} - \frac{b_2 v^j}{(vA + \varpi^{a_1 - a(\chi_1\chi_2^{-1})})^j} \right) \in \mathfrak{p}^{a_1 - a(\chi_1\chi_2^{-1})}.$$

Furthermore, we check that

$$(vA + \varpi^{a_1 - a(\chi_1\chi_2^{-1})})^j b_1 - A^j b_2 = A^j v^j (b_1 - b_2) + j v^{j-1} A^{j-1} \varpi^{a_1 - a(\chi_1\chi_2^{-1})} b_1 + \mathfrak{p}^{2a_1 - 2a(\chi_1\chi_2^{-1})}.$$

This helps us to check that the second order term is in $\mathfrak{p}^{a_1 - a(\chi_1\chi_2^{-1}) + \delta}$ and the third order term is in $3^{-1}\varpi^{a_1 - a(\chi_1\chi_2^{-1})}\mathfrak{o}^{\times}$. Note that we can truncate the Taylor series latest after the 3rd term. Thus, in the worst case scenario, we obtain the bound

$$|K_{l_2}| \leq 2\zeta_F(1)^2 q^{-\frac{a_1}{2} - \frac{2a(\chi_1\chi_2^{-1})}{3}}.$$

Case IV: $t = -2l$. In this case we will take a very familiar approach. First, we note that $l_2 = t + l_2 = l = a_1$. Thus, the congruences reduce too

$$vx_2^2 + (1 - vb_{\chi_1\chi_2^{-1}}\varpi^{a_1 - a(\chi_1\chi_2^{-1})})x_2 + b_2 \in \mathfrak{p}^r \text{ and } x_1 = x_2 - b_{\chi_1\chi_2^{-1}}\varpi^{a_1 - a(\chi_1\chi_2^{-1})} \in \mathfrak{p}^r.$$

We can solve the remaining quadratic congruence as in many of the previous cases. Its discriminant is given by

$$\Delta = (1 - vb_{\chi_1\chi_2^{-1}}\varpi^{a_1 - a(\chi_1\chi_2^{-1})})^2 - 4vb_2 = 1 - 2v(b_1 + b_2) + v^2 b_{\chi_1\chi_2^{-1}}^2 \varpi^{2a_1 - 2a(\chi_1\chi_2^{-1})}.$$

A short computation modulo $\mathfrak{p}$ shows that $\det A_{\mathfrak{p}} \in \mathfrak{o}^\times$ if and only if $\Delta \in \mathfrak{o}$.

If $\Delta \in \mathfrak{o}^\times$, then obviously $\sharp S_{a_1} \leq 2$ and using Lemma 3.1.7 we can give a satisfying expression for K_{a_1} in terms of $x_\pm \in S_{a_1}$.

From now on we assume that $\Delta \in \mathfrak{p}$. In particular, $A_{\mathfrak{p}}$ is singular, which implies a slightly stronger congruence condition. We define

$$\delta = \begin{cases} \lfloor \frac{r}{2} \rfloor & \text{if } v(\Delta) \geq r, \\ \delta_0 & \text{if } v(\Delta) = 2\delta_0 < r, \end{cases} \qquad Y = \begin{cases} 0 & \text{if } v(\Delta) \geq r, \\ Y_0 & \text{if } (\Delta)_0 = Y_0^2 \text{ and } v(\Delta) < r. \end{cases}$$

Assuming S_{a_1} is non empty, we parametrise it by

$$S_{a_1} = \left\{ \left(A_\pm + \alpha\varpi^{r+\rho-\delta}, B_\pm + \alpha\varpi^{r+\rho-\delta} \right) : \alpha \in \mathfrak{o}/\mathfrak{p}^{\delta-\rho} \right\},$$

for $A_\pm = -\frac{1}{2v} - \frac{b_1-b_2}{2} \pm \frac{Y}{2v}\varpi^\delta$ and $B_\pm = -\frac{1}{2v} + \frac{b_1-b_2}{2} \pm \frac{Y}{2v}\varpi^\delta$.

We proceed by inserting the parametrisation of S_{a_1} in the S_{a_1}-sum. This yields

$$\begin{aligned} &K_{a_1} = \\ &\gamma_F(-2b_1,a_1)\sum_\pm \chi_1(A_\pm)\chi_2(B_\pm)\psi(A_\pm\varpi^{-a_1} + B_\pm\varpi^{-a_1} + vA_\pm B_\pm\varpi^{-a_1})\zeta_F(1)^2 q^{-2r-\frac{\rho}{2}} \\ &\cdot \sum_{\alpha\in\mathfrak{o}/\mathfrak{p}^{\delta-\rho}} \chi_1(1+\frac{\alpha}{A_\pm}\varpi^{r+\rho-\delta})\chi_2(1+\frac{\alpha}{B_\pm}\varpi^{r+\rho-\delta})\psi((1\pm Y\varpi^\delta)\alpha\varpi^{-r-\delta} + v\alpha^2\varpi^{\rho-2\delta}). \end{aligned}$$

As usual we use Lemma 3.1.3 and the p-adic logarithm to deal with the two characters. Observing $\delta \leq \frac{r}{2}$ enables us to truncate the Taylor expansion of the logarithm after the 3rd term. We get

$$\begin{aligned} K_{a_1} = \gamma_F(-2b_1,a_1)&\sum_\pm \chi_1(A_\pm)\chi_2(B_\pm)\psi(A_\pm\varpi^{-a_1} + B_\pm\varpi^{-a_1} + vA_\pm B_\pm\varpi^{-a_1}) \\ &\cdot \zeta_F(1)^2 q^{\delta-a_1-\frac{\rho}{2}} \int_{\mathfrak{o}} \psi\bigg(t(1\pm Y\varpi^\delta + \frac{b_1}{A_\pm} + \frac{b_2}{B_\pm})\varpi^{-r-\delta} \\ &\qquad + t^2(v - \frac{1}{2}(\frac{b_1}{A_\pm^2} + \frac{b_2}{B_\pm^2}))\varpi^{\rho-2\delta} + \frac{t^3}{3}(\frac{b_1}{A_\pm^3} + \frac{b_2}{B_\pm^3})\varpi^{r+2\rho-3\delta}\bigg) dt. \end{aligned}$$

For S_{l_2} to be non-empty it is necessary that $A_\pm, B_\pm \in \mathfrak{o}^\times$. This translates into

$$v \notin \pm(b_1-b_2)^{-1} + \mathfrak{p}.$$

Since

$$A_\pm B_\pm \in \frac{1}{4v^2} - \frac{(b_1-b_2)^2}{4} + \mathfrak{p},$$

we conclude that $A_\pm B_\pm \in \mathfrak{o}^\times$.

Note that if the linear or the quadratic term are units then we have at least square root cancellation. Thus, we are left with showing that the coefficient in front of t^3 is (close to) a unit. The following computations are modulo $\mathfrak{p}$. Indeed, $\Delta \in \mathfrak{p}$ implies

$$1+v^2(b_1-b_2)^2 \in 2v(b_1+b_2)+\mathfrak{p}.$$

We also compute

$$A_\pm^2 \in \frac{b_1}{v}+\mathfrak{p} \text{ and } B_\pm^2 \in \frac{b_2}{v}+\mathfrak{p}.$$

Using this, an easy computation shows

$$\frac{b_1}{A_\pm^3}+\frac{b_2}{B_\pm^3} \in (A_\pm B_\pm)^{-1}+\mathfrak{p} \subset \mathfrak{o}^\times.$$

Thus we are left with an p-adic Airy function and get the bound[7]

$$|K_{a_1}| \leq 2\zeta_F(1)^2 q^{-a_1+\frac{a_1}{6}}.$$

□

Lemma 3.4.16 ([4], Lemma 5.12). *Let $\pi = \chi_1 |\cdot|^s \boxplus \chi_2 |\cdot|^{-s}$ where $s \in i\mathbb{R}$, and $a(\chi_1)=a(\chi_2)$ but $\chi_1 \neq \chi_2$. We put $k=\max(l,a(\chi_1))$.*

If $0<l<n$ and $l \neq \frac{n}{2}$, then

$$W_\pi(g_{t,l,v}) = \gamma_F(b_1b_2)^{k\delta_{k=a(\chi_1)}}\chi_1(x_0-b_{\chi_1\chi_2^{-1}}\varpi^{k-a(\chi_1)})\chi_2(x_0)\psi(x_0\varpi{-}k-b_{\chi_1}\varpi{-}a(\chi_1)),$$

for $t=-k$ and $W_\pi(g_{t,l,v})=0$ otherwise. Here $x_0 \in \mathfrak{o}^\times$ is the unique solution to

$$vx^2\varpi^{k-l}+x(1-vb_{\chi_1\chi_2^{-1}}\varpi^{2k-a(\chi_1)-l})+b_2\varpi^{k-a(\chi_1)}=0.$$

If $l=\frac{n}{2}$,

$$W_\pi(g_{t,l,v}) = \zeta_F(1)^{-2}q^{-\frac{t}{2}}\Big[G(v\varpi^{-l},\chi_1)G(\varpi^{-a(\chi_1^{-1}\chi_2)},\chi_1^{-1}\chi_2)q^{-st} + G(v\varpi^{-l},\chi_2)G(\varpi^{-a(\chi_2^{-1}\chi_1)},\chi_2^{-1}\chi_1)q^{st}\Big],$$

for $-2 \geq t \geq -n_0(\pi)$;

$$W_\pi(g_{t,l,v}) = \zeta_F(1)^{-2}q^{-\frac{t}{2}}\left(q^{s(t+2n_0(\pi))}K_{-t-n_0(\pi)}+q^{-s(t+2n_0(\pi))}K_{n_0(\pi)}\right),$$

for $-n_0(\pi)<t<-2n_0(\pi)$; and

$$W_\pi(g_{t,l,v}) = \zeta_F(1)^{-2}q^{-\frac{t}{2}}K_{-\frac{t}{2}},$$

7 If $F=\mathbb{Q}_p$ we may apply Lemma 3.1.5 to obtain cube root cancellation.

for $-2n_0(\pi) \leq t \leq -n$. *Evaluations for* K_{l_2} *can be found in Lemma 3.4.15. In particular we have*

$$W_\pi(g) \ll_F q^{\frac{n}{12}}.$$

If $a(\chi_1\chi_2) < \frac{n}{2}$ *and* $-1 \notin \mathfrak{o}^{2\times}$, *then we have the stronger bound*

$$|W_\pi(g)| \ll_F 1.$$

If $\kappa_F = 1$, *then the implicit constants can taken to be* 2.

The proof follows our usual strategy and is very similar in spirit to the proof of previous lemmata. Thus we will be very brief.

Proof. If $l = 0$ or $l \geq n$ or $t > -2$, the formulas given in Lemma 3.3.9 can be easily estimated. The cases $0 < l < n$ and $l \neq \frac{n}{2}$ follow directly from Lemma 3.1.7 in a standard manner. Finally, if $l = \frac{n}{2}$, the work has been done in Lemma 3.3.9 and Lemma 3.4.15. □

3.4.4 *Summary*

A consequence of [69, Corollar 2.35] is that

$$|W_\pi(g)| \leq \sqrt{2}q^{\frac{1}{2}\lfloor\frac{n}{2}\rfloor}$$

for all $g \in G(F)$. In this chapter we went through great pain to establish tight bounds for W_π using the $g_{t,l,v}$ coordinates. Here we give a brief summary of our findings.

First of all let us observe that for $l \neq \frac{n}{2}$ and any π we have

$$|W_\pi(g_{t,l,v})| \ll_{F,\epsilon} q^{(\epsilon-\frac{1}{2})(t+\max(n,2l))}.$$

Furthermore, whenever $\pi \neq \chi_1 |\cdot|^s \boxplus \chi_2 |\cdot|^{-s}$ with $a(\chi_1) > a(\chi_2) > 0$ or $l \neq a(\chi_2)$ we can take the implicit constant to be 1 and remove the ϵ. Also note that in conjunction with the support properties of W_π stronger bounds might be possible.

We turn to the transition region. Here we have[8]

$$W_\pi(g_{t,\frac{n}{2},v}) \ll_F \mathcal{E}(\pi,t). \tag{3.4.31}$$

The values of $\mathcal{E}(\pi,t)$ are recorded in the following table.

8 The implicit constant is 2 for $F = \mathbb{Q}_p$.

π	t	$\mathcal{E}(\pi,t)$
supercuspidal	$t=-n,-n_0(\pi)$	$q^{-\frac{n+t}{4}-\frac{t}{12}}$
	$-n<t<-n_0(\pi)$	$q^{-\frac{n+t}{4}}$
χSt or $\chi\lvert\cdot\rvert^s\boxplus\chi\lvert\cdot\rvert^{-s}$	$t=-n$	$q^{\frac{n}{12}}$
	$-n<t$	$q^{-\frac{n+t}{4}}$
$\chi\lvert\cdot\rvert^s\boxplus\lvert\cdot\rvert^{-s}$	$t=-\frac{3}{2}n$	$q^{\frac{n}{4}}$
$\chi\lvert\cdot\rvert^s\boxplus\chi\lvert\cdot\rvert^{-s}$ for $a(\chi_1)>a(\chi_2)>0$	$t=-\frac{n}{2}-m$	$q^{\frac{m}{3}-\frac{n}{12}}$
$\chi\lvert\cdot\rvert^s\boxplus\chi\lvert\cdot\rvert^{-s}$ for $a(\chi_1)=a(\chi_2)>0$	$t=-n$	$q^{\frac{n}{12}}$
	$-n<t<-2n_0(\pi)$	$q^{-\frac{n+t}{4}}$
	$t=-2n_0(\pi)$	$q^{-\frac{n+t}{4}+\frac{n_0(\pi)}{6}}$
	$-n_0(\pi)<t$	$q^{-\frac{n+t}{4}-\frac{2n_0(\pi)+t}{4}}$

Let $t_0(\pi)=\min\{t\colon W_\pi(g_{t,\frac{n}{2},v})\neq 0\}$. The (essential) sharpness of (3.4.31) depends on the existence of degenerate critical points. The following table summarises sufficient and necessary conditions for the existence of such points.

π	$t_0(\pi)$	Condition for degeneracy
$\tilde{\pi}=\omega_\xi$, E/F unramified	$-n$	$\mathrm{Nr}_{E/F}(b_\xi)\in\mathfrak{o}^{2\times}$ (e.g. non twist minimal)
$\tilde{\pi}=\omega_\xi$, E/F ramified	$-n$	n even (i.e. non twist minimal)
χSt or $\chi\lvert\cdot\rvert^s\boxplus\chi\lvert\cdot\rvert^{-s}$	$-n$	none (always exists)
$\chi\lvert\cdot\rvert^s\boxplus\lvert\cdot\rvert^{-s}$	$-\frac{3}{2}n$	no critical point
$\chi_1\lvert\cdot\rvert^s\boxplus\chi_1\lvert\cdot\rvert^{-s}$, $a(\chi_1)>a(\chi_2)>0$	$-\frac{n}{2}-m$	$b_{\chi_1}b_{\chi_2}\in\mathfrak{o}^{2\times}$
$\chi_1\lvert\cdot\rvert^s\boxplus\chi_2\lvert\cdot\rvert^{-s}$, $a(\chi_1)=a(\chi_2)>0$ and $m<\frac{n}{2}$	$-n$	$-1\in\mathfrak{o}^{2\times}$
$\chi_1\lvert\cdot\rvert^s\boxplus\chi_2\lvert\cdot\rvert^{-s}$, $a(\chi_1)=a(\chi_2)>0$ and $m=\frac{n}{2}$	$-n$	none (always exists)

3.5 MISCELLANEOUS INTEGRALS

We will conclude this chapter by evaluating some local integrals that appear in the residual part ot the spectral expansion. More precisely we will calculate the integral

$$I(\chi) = \int_{Z(F)\backslash G(F)} f(g)\chi(\det(g))dg \tag{3.5.1}$$

for several choices of f. In particular, all $f_{\mathfrak{p}}$ that appear in Section 4.7.1 below. These computations have previously appeared in [2, Appendix A].

First, we consider

$$f(g) = \kappa_k(g) = \begin{cases} \omega_\pi(z)^{-1} & \text{for } g = zm \text{ with } z \in Z(F), m \in \mathrm{Mat}_2(\mathfrak{o}), \det(m) = k, \\ 0 & \text{else,} \end{cases}$$

for some k.

Lemma 3.5.1 ([2], Lemma A1). *For $k \geq 0$ we have*

$$\int_{Z(F)\backslash G(F)} \kappa_k(g)\chi(\det(g))dg = \begin{cases} \chi(\varpi^k)vol(X_k) & \text{if } \chi \text{ is unramified,} \\ 0 & \text{else.} \end{cases}$$

Proof. The calculation for unramified χ is straight forward. Thus, we assume that χ is ramified. In this case let us write $X_k = \sqcup_i \alpha_i K$. We clearly have

$$\int_{Z(F)\backslash G(F)} \kappa_k(g)\chi(\det(g))dg = \sum_i \chi(\det(\alpha_i)) \int_{Z(F)\backslash G(F)} \chi(\det(g))\mathbb{1}_K(g)dg.$$

Using the choice of Haar measure and the fact $\mathbb{1}_K(n(x)a(y)k) = \mathbb{1}_{\mathfrak{o}}(x)\mathbb{1}_{\mathfrak{o}^\times}(y)$ yields

$$\begin{aligned} &\int_{Z(F)\backslash G(F)} \kappa_k(g)\chi(\det(g))dg \\ &\qquad = \sum_i \chi(\det(\alpha_i)) \int_{\mathfrak{o}} \int_K \kappa_k(k) \int_{\mathfrak{o}^\times} \chi(y)d\mu^\times(y)d\mu_K(k)d\mu(x) = 0. \end{aligned}$$

This concludes the proof. □

Second, we look at

$$f(g) = \begin{cases} \mathrm{vol}(Z(\mathfrak{o}) \setminus \tilde{K}_0(1))^{-1}\omega_\pi^{-1}(z) & \text{if } g = zk \in Z(F)\tilde{K}_0(1), \\ 0 & \text{else.} \end{cases}$$

Lemma 3.5.2 ([2], Lemma A2). *For a quadratic character χ and unramified ω_π we have*

$$I(\chi) = 1.$$

Proof. We first observe that for each $g \in \tilde{K}_0(1)$ we have $\det(g) \in (\mathfrak{o}^\times)^2 + \varpi\mathfrak{o}$. Thus, if $a(\chi) \leq 1$ we have $\chi(g) = 1$ for all $g \in \tilde{K}_0(1)$. Further, since χ is quadratic, the case $a(\chi) > 1$ can not appear in odd residual characteristic.

□

After this warm up we come to the most interesting case. We consider the truncated matrix coefficient. More precisely we look at

$$f(g) = \Phi'_\pi(g) = \begin{cases} \Phi_\pi(a(\varpi^{-n_1})ga(\varpi^{n_1})) & \text{if } g \in ZK^0, \\ 0 & \text{else,} \end{cases}$$

with

$$K^0 = \begin{cases} K & \text{if } n \text{ is even,} \\ K^0(1) & \text{if } n \text{ is odd.} \end{cases}$$

We obtain the following result.

Lemma 3.5.3 ([2], Lemma A3). *If $\chi^2 = \omega_\pi$, one has*

$$I(\chi) = 0,$$

unless $a(\pi) = 1$. In this case the integral may be non-zero but we still have $I(\chi) \geq 0$.

Before we begin with the proof we recall some properties of Φ_π, which date back to [30]. For any unitary, generic representation π of $G(F)$ we define the matrix coefficient associated to a Whittaker new vector W_π by

$$\Phi_\pi(g) = \langle W_\pi, \pi(g)W_\pi \rangle .$$

Lemma 3.5.4 ([2], Lemma A4). *We have*

$$\Phi_\pi(n(x)g_{t,l,1}) = \sum_{m\in\mathbb{Z}} W_\pi(a(\varpi^m)) \sum_{\mu\in\mathfrak{X}_l} \overline{c_{t+m,l}(\mu)} G(-\varpi^m x, \omega_\pi\mu).$$

Proof. First we use the definition of Φ_π. We arrive at

$$\begin{aligned} \Phi_\pi(n(x)g_{t,l,1}) &= \langle W_\pi, \pi(n(x)g_{t,l,1})W_\pi \rangle \\ &= \int_{F^\times} W_\pi(a(y))\overline{W_\pi(a(y)n(x)g_{t,l,1})} d\mu^\times(y) \\ &= \sum_{m\in\mathbb{Z}} W_\pi(a(\varpi^m)) \int_{\mathfrak{o}^\times} \omega_\pi(v)\overline{W_\pi(a(\varpi^m v)n(x)g_{t,l,1})} d\mu^\times(v). \end{aligned}$$

It is straight forward to check that

$$a(\varpi^m v)n(x)g_{t,l,1} = n(\varpi^m vx)g_{t+m,l,v^{-1}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}}_{\in K_1(n)}.$$

Expanding $W_\pi(g_{t,l,v})$ in its finite Fourier expansion and recalling the definition of the Gauß sum completes the proof. □

Proof of Lemma 3.5.3. Put $b = \max(a(\chi), n)$. Then $\chi \circ \det$ and Φ_π are bi-$K_1(b)$-invariant. Further, we recall

$$\Phi_\pi'(g) = \mathbb{1}_{ZK^\circ}(g)\Phi_\pi(a(\varpi^{-n_1})ga(\varpi^{n_1})).$$

Thus a simple change of variables yields

$$I(\chi) = \int_{Z(F)\backslash G(F)} \chi(\det(g))\Phi_\pi(g)\mathbb{1}_{ZK^\circ}(a(\varpi^{n_1})ga(\varpi^{-n_1}))dg.$$

It is easy to check that $\mathbb{1}_{ZK^\circ}(a(\varpi^{n_1}) \cdot a(\varpi^{-n_1}))$ is bi-$K_0(b)$-invariant. It follows that the hole integrand is bi-$K_0(b)$-invariant, so that we can use [30, Lemma 3.2.4]. This yields

$$I(\chi) = \sum_{l=0}^{b} c_l \sum_{t\in\mathbb{Z}} q^{t+l} \int_F \chi(\varpi^t)\Phi_{\pi'}(n(x)g_{t,l,1})\mathbb{1}_{ZK^\circ}(a(\varpi^{n_1})n(x)g_{t,l,1}a(\varpi^{-n_1}))d\mu(x),$$

for some positive constants c_l. We remark that, since $\omega_{\pi'}$ is trivial on the uniformiser, so is χ. Next we will investigate which restrictions on x, l, and t are imposed by the characteristic function (up to the center). One checks

$$a(\varpi^{n_1})n(x)g_{t,l,1}a(\varpi^{-n_1}) = z \cdot \begin{pmatrix} \varpi^k x & \varpi^{n_1-l+k}x - \varpi^{t+n_1+k} \\ \varpi^{-n_1+k} & \varpi^{k-l} \end{pmatrix}.$$

Here we use the center to force all coefficients to be in $\mathfrak{o}$. This holds for

$$k \geq \max(n_1, l, -v(x), -v(x) - n_1 + l)$$

and suitable t. But we also need to make sure that the determinant is in $\mathfrak{o}^\times$. This implies $t + 2k = 0$.

We now consider n to be even. In this case $K^\circ = K$ and we get the conditions

$$k = n_1, t = -2n_1, l \leq n_1, \text{ and } -v(x) \leq n_1. \tag{3.5.2}$$

After inserting the formula from Lemma 3.5.4 for the matrix coefficient we obtain

$$I(\chi) = \sum_{l=0}^{n_1} c_l q^{l-2n_1} \sum_{m\in\mathbb{Z}} W_\pi(a(\varpi^m)) \sum_{\mu\in\mathfrak{X}_l} \overline{c_{t+m}(\mu)} \int_{\varpi^{-n_1}\mathfrak{o}} G(-\varpi^m x, \omega_\pi\mu)d\mu(x). \tag{3.5.3}$$

Inserting the evaluation of the Gauß sum together with character orthogonality shows that most of the integrals vanish. We are left with

$$I(\chi) = \sum_{l=a(\omega_\pi)}^{n_1} c_l q^{l-2n_1} \sum_{m\in\mathbb{Z}} W_\pi(a(\varpi^m))\overline{c_{m-2n_1,l}(\omega_\pi^{-1})} \cdot \sum_{t\geq 0} q^{-t+n_1} \int_{\mathfrak{o}^\times} G(-\varpi^{m+t-n_1}x, 1) d\mu(x).$$

We have to consider different cases. First, we deal with representations that satisfy $L(s,\pi) = 1$. In this case using (1.3.1) and (1.3.7) yields

$$\begin{aligned} I(\chi) &= \sum_{l=a(\omega_\pi)}^{n_1} c_l q^{l-2n_1} \overline{c_{-2n_1,l}(\omega_\pi^{-1})} \sum_{t\geq 0} q^{n_1-t} \int_{\mathfrak{o}^\times} G(\varpi^{t-n_1}v, 1) d\mu(v) \\ &= \sum_{l=a(\omega_\pi)}^{n_1} c_l q^{l-2n_1} \overline{c_{-2n_1,l}(\omega_\pi^{-1})} \left[\sum_{t\geq n_1} q^{n_1-t}\zeta_F(1) - 1 \right] = 0. \end{aligned}$$

Second, we consider the case $\pi = \chi_1 \boxplus \chi_2$ with $a(\chi_1) > a(\chi_2) = 0$. In this case we have $a(\omega_\pi) = a(\chi_1) = n > 0$. Recall that at the moment we are considering n even. Thus, $a(\omega_\pi) > n_1 \geq 1$. We conclude that $I(\chi) = 0$ since the l-sum is empty. Let us remark, that $\pi = \chi\mathrm{St}$ for unramified χ has conductor 1 and therefore does not need to be considered yet.

We have checked that $I(\chi) = 0$ for even n by considering all necessary types of π. Now let us move on to n odd. In this case $K^\circ = K^0(1)$ and additionally to (3.5.2) the characteristic function forces $v(\varpi^{2n_1-l}x - 1) \geq 1$. This implies

$$l = n_1 \text{ and } x \in \varpi^{-n_1}(1+\varpi\mathfrak{o}).$$

Analogously to (3.5.3) we get

$$\begin{aligned} I(\chi) &= c_{n_1} q^{-n_1} \sum_{m\in\mathbb{Z}} W_\pi(a(\varpi^m)) \sum_{\mu\in\mathfrak{X}_{n_1}} \overline{c_{-2n_1+m,n_1}(\mu)} \int_{\varpi^{-n_1}(1+\varpi\mathfrak{o})} G(-\varpi^m x, \omega_\pi\mu) d\mu(x) \\ &= c_{n_1} \sum_{m\in\mathbb{Z}} W_\pi(a(\varpi^m)) \sum_{\substack{\mu\in\mathfrak{X}_{n_1}, \\ a(\mu\omega_\pi)\leq 1}} \overline{c_{-2n_1+m,n_1}(\mu)} \int_{(1+\varpi\mathfrak{o})} G(-\varpi^{m-n_1}x, \omega_\pi\mu) d\mu(x). \end{aligned}$$

In the last step we used again the Gauß sum evaluation (1.3.1) and orthogonality of characters to remove all μ with $a(\mu\omega_\pi) > 1$.

We have to consider different cases again. First, let us look at π with $L(s,\pi) = 1$. In this case me have $n > 2$, since we assume n odd. By (1.3.7) we get

$$I(\chi) = c_{n_1} \sum_{\substack{\mu\in\mathfrak{X}_{n_1}, \\ a(\mu\omega_\pi)\leq 1}} \overline{c_{-2n_1,n_1}(\mu)} \int_{(1+\varpi\mathfrak{o})} \underbrace{G(-\varpi^{-n_1}x, \omega_\pi\mu)}_{=0} d\mu(x) = 0.$$

Second, let $\pi = \chi_1 \boxplus \chi_2$ with $a(\chi_1) > a(\chi_2) = 0$. If $n = a(\chi_1) > 1$, we immediately have $a(\omega_\pi \mu) > n_1$ for all $\mu \in \mathfrak{X}_{n_1}$. Thus, in these cases, $I(\chi) = 0$. So we can assume $1 = n = n_1 = a(\chi_1)$. Using (1.3.1) and (1.3.7) we have the identity

$$\begin{aligned} I(\chi) = c_1 \mathrm{Vol}(1+\varpi\mathfrak{o}, \mu)\Big(& \sum_{\substack{\mu\in\mathfrak{X}_1,\\ \mu\neq\omega_\pi^{-1}}} \overline{c_{-2,1}(\mu)}\zeta_{F_\mathfrak{p}}(1)q^{-\frac{1}{2}}\epsilon(\frac{1}{2},\omega_\pi^{-1}\mu^{-1})\omega_\pi(-1)\mu(-1) \\ & + \sum_{m\geq 1} \chi_1(\varpi^m)q^{-\frac{m}{2}}\overline{c_{-2+m,1}(\omega_\pi^{-1})} \\ & -\zeta_F(1)q^{-1}\overline{c_{-2,1}(\omega_\pi^{-1})}\Big). \end{aligned}$$

Inserting the expressions for $c_{t,1}(\cdot)$ given in Lemma 3.2.3 yields

$$\begin{aligned} I(\chi) &= c_1 \mathrm{Vol}(1+\varpi\mathfrak{o}, d\mu)\omega_\pi(-1)\left(\sum_{\mu\neq\omega_\pi^{-1}} \zeta_F(1)^2 q^{-1} + \sum_{m\geq 1} q^{-m} + \zeta_F(1)^2 q^{-2}\right) \\ &= c_1 \mathrm{Vol}(1+\varpi\mathfrak{o}, d\mu)\omega_\pi(-1)\left(\zeta_F(1)^2 q^{-1}(q-2) + \zeta_F(1)q^{-1} + \zeta_F(1)^2 q^{-2}\right). \end{aligned}$$

Observe $\omega_\pi(-1) = \chi(-1)^2 = 1$ and deduce that $I(\chi) \geq 0$.

This leaves us with the case $\pi = \chi\mathrm{St}$ for unramified χ. In particular, we have $\omega = \chi^2 = 1$. Thus we are dealing with $\pi = \mathrm{St}$ and we have $a(\pi) = n = n_1 = 1$. We obtain

$$I(\chi) = c_1 \sum_{m\geq 0} q^{-m} \sum_{\mu\in\mathfrak{X}_1} \overline{c_{m-2,1}(\mu)} \int_{1+\varpi\mathfrak{o}} G(-\varpi^{m-1}x, \mu) d\mu(x).$$

Evaluating the Gauß sum reveals

$$\begin{aligned} I(\chi) = c_1 \mathrm{Vol}(1+\varpi\mathfrak{o})\Big(& \sum_{a(\mu)=1} \zeta_F(1)q^{-\frac{1}{2}}\epsilon(\frac{1}{2},\mu^{-1})\mu(-1)\overline{c_{-2,1}(\mu)} \\ & + \sum_{m\geq 1} q^{-m}\overline{c_{m-2,1}(1)} \\ & -\zeta_F(1)q^{-1}\overline{c_{-2,1}(1)}\Big). \end{aligned}$$

Using the evaluation of $c_{t.l}(\cdot)$ given in Lemma 3.2.1 one obtains

$$\begin{aligned} I(\chi) &= c_1 \mathrm{Vol}(1+\varpi\mathfrak{o})\left(\sum_{a(\mu)=1} \zeta_F(1)^2 q^{-1} + \sum_{m\geq 1} q^{-2m} + \zeta_F(1)^2 q^{-2}\right) \\ &= c_1 \mathrm{Vol}(1+\varpi\mathfrak{o})\left(\zeta_F(1)^2 q^{-1}(q-2) + q^{-2}\zeta_F(2) + \zeta_F(1)^2 q^{-2}\right) > 0 \end{aligned}$$

This was the last case to consider and the proof is complete. □

Part III

GLOBAL APPLICATION

In this part we seek to apply the local methods developed so far to the sup-norm problem for automorphic forms. This is a global question and it requires non-trivial arguments to put the local pieces together.

We treat essentially two aspects of the problem. First, we exhibit large values high in the cusp. Here the local to global argument is straight forward solely relying on the uniqueness of Whittaker models. Second, we prove upper bounds for the global sup-norm. The latter requires some beautiful global techniques developed in [20]. Essentially we will be extending the results from [2].

Throughout this section we will deal with an arbitrary number field F of degree n. This base field will be considered as fixed and thus we allow all constants to depend in it. In particular, we will freely discard contributions like 2^n or $\mathcal{N}(\mathfrak{d})$. All corresponding local fields with their associated objects will appear with subscripts.

4

THE SUP-NORM OF CUSPIDAL AUTOMORPHIC FORMS

Classically the sup-norm problem is motivated by quantum chaos. However, in the setting of automorphic forms there are close connections to the theory of L-functions. We will start this chapter by briefly reviewing the classical theory including some previous results. Then we will set up the necessary notation and start to prove Theorem 1.2.1. This proof will occupy the remainder of this thesis. We closely follow [2], making the modifications necessary for the slightly general set-up allowed in Theorem 1.2.1 respectively Theorem 4.8.1 and Theorem 4.8.2 below.

4.1 BACKGROUND

The goal of this section is to introduce the sup-norm problem by briefly painting the classical picture and then giving an overview over previous results on the sup-norm of automorphic forms. Since there is a huge amount of literature on this topic we organise this section according to different aspects.

Classical aspects and local methods

Let (M,g) be a n dimensional, compact Riemannian manifold without boundary. Such a manifold comes naturally with the (positive) Laplace-Beltrami operator $\triangle_g$, whose spectrum, Spec(M), can be seen as an important geometric invariant of M. Since we assume M to be compact, Spec(M) is discrete and has no finite accumulation point. We are interested in understanding the map

$$\begin{aligned} M_\infty \colon \text{Spec}(M) &\to \mathbb{R}, \\ \lambda &\mapsto \sup_{\triangle_g \phi = \lambda\phi} \frac{\|\phi\|_\infty}{\|\phi\|_2}. \end{aligned}$$

This quantity is closely connected to the multiplicity of eigenspaces and the remainder of the *Weyl-law*. Indeed, from the local Weyl-law one can deduce the bound

$$M_\infty(\lambda) \ll_M \lambda^{\frac{n-1}{4}}. \tag{4.1.1}$$

This *local bound* originates from the work of Levitan [58], Avacumovič [6], and Hörmander [44]. It improves upon the trivial bound coming from standard Sobolev estimates.

While the estimate (4.1.1) is sharp in general, as can be seen considering the (euclidean) sphere S_n, it is possibly far from the truth in general. In [77] the authors show that there is a close connection between the true size of M_∞ and the dynamics of the geodesic flow on M. Roughly speaking, they show that if (4.1.1) is sharp for M, then there must be a point $x \in M$ at which a positive amount of initial directions lead to geodesic loops. It turns out that this is almost sufficient [76].

For more background and references concerning the microlocal analysis of eigenfunctions we refer to the very nice survey [91] and the neat book [75].

The estimate (4.1.1) mentioned above relies purely on the local structure of M. However, if M possesses global symmetries considering only the function M_∞ might lead to loss of information. The reason for this is that M_∞ takes the supremum over the whole eigenspace, but it might be enlightening to single out special elements in each eigenspace. This can be illustrated by looking at the sphere S_2. As mentioned earlier, the local bound for M_∞ is sharp in this case. However, VanderKam exploited the presence of Hecke operators to construct an orthonormal basis ϕ_j of Laplace-Beltrami eigenfunctions satisfying

$$\|\phi_j\|_\infty \ll \lambda_j^{\frac{5}{24}}. \tag{4.1.2}$$

Thus it makes sense to consider specific sequences of eigenfunctions and investigate the eigenfunction growth along these.

A similar phenomenon can be observed if M is a compact locally symmetric space. In this situation the commutative ring of invariant differential operator $\mathcal{D}(M)$ is generated by $\mathrm{rank}(M)$ elements. Therefore we can consider functions ϕ on M which are eigenfunctions of all operators in $\mathcal{D}(M)$ simultaneously. Such ϕ are in particular Laplace-Beltrami eigenfunctions and we refer to them as *joint eigenfunctions*. In his famous letter [72] Sarnak observed that (generic) joint eigenfunctions satisfy the improved bound

$$\frac{\|\phi\|_\infty}{\|\phi\|_2} \ll \lambda_\phi^{\frac{n-\mathrm{rank}(M)}{4}}. \tag{4.1.3}$$

This has been further investigated in [60] and [67].

The last two points raise the question about the growth of the L^∞-norm of eigenfunctions along sequences of eigenfunctions. Furthermore, one can try to identify families with similar growth properties. Part of these questions is what we call *the sup-norm prob-*

lem. It is the problem of improving upon the bound (4.1.1) or (4.1.3) depending on the setting.

The purity conjecture

The most general conjecture in this area is the so called *purity conjecture* posed in [72]. It states that, for any sequence $(\phi_j)_j$ of L^2-normalised Hecke-eigenfunctions with regular spectral parameter on a compact, arithmetic, negatively curved Riemannian manifold (without boundary), we have

$$\mathrm{Acc}\frac{\log \|\phi_j\|_\infty}{\log \lambda_j} \subset \frac{1}{4}\mathbb{Z} \cap [0, \frac{n - \mathrm{rank}(M)}{4}).$$

To get a feeling for this conjecture we consider the example $M = \Gamma\backslash\mathcal{H}^{(2)}$, where Γ is a co-compact arithmetic lattice. The local bound implies that the accumulation points that can appear must lie between 0 and $\frac{1}{4}$. We first note that, as stated here, the purity conjecture includes the existence of a sub-local (subconvex) bound. This is because the accumulation point $\frac{1}{4}$ is excluded. Thus, in this particular setting, the purity conjecture is equivalent to the very strong sup-norm bound

$$\|\phi\|_\infty \ll_\epsilon \lambda_\phi^\epsilon \text{ for any } \epsilon > 0.$$

Note that the conjecture does not apply to the sphere S^2. In this case the high dimensional eigenspaces are responsible for the existence of many different basis which make it possible to accommodate for arbitrary accumulation points in the interval $[0, \frac{1}{4}]$. A well known example is the sequence consisting of zonal spherical harmonics, which is responsible for the accumulation point $\frac{1}{4}$. On the other hand, we can look at the orthonormal basis consisting of Hecke eigenfunctions considered by VanderKam. In view of (4.1.2) this sequence only yields accumulation points in the interval $[0, \frac{5}{24}]$. It seems reasonable to believe that this sequence is pure in the sense that it really has accumulation point 0.

Non-compact spaces.

Up to this point we assumed that M is a compact Riemannian manifold without boundary. However, many interesting manifolds fail to be compact. Prominent examples are quotients of the upper half plane $\mathcal{H}^{(2)}$ by congruence subgroups, which are of great interest to number theorists. Thus, one would like to consider the sup-norm problem in this setting as well. But the failure to be compact makes it impossible to define the global sup-norm in general. There are several ways to go around this. First, we can fix a compact set K and consider the *restricted sup-norm*

$$\|\phi\|_{\infty,K} = \sup_{x\in K} |\phi(x)|$$

for smooth functions ϕ. Second, we can exclude the continuous part of the spectrum and restrict ourselves to eigenfunctions that appear in the discrete spectrum. Functions in the discrete spectrum will have essentially compact support so that it is possible to consider the global sup-norm. If $M = \mathrm{SL}_2(\mathbb{Z}) \setminus \mathcal{H}^{(2)}$, then the second approach amounts to studying the sup-norm of classical Maaß cusp forms.

At this stage a word of warning is in order. It is not possible to translate all the results and conjectures from the compact in the non-compact setting. Indeed it has been shown in [24] that the bound (4.1.3) fails for GL_n cusp forms if n is large. Also the purity conjecture does not generalise to the non-compact setting without compromises. Indeed, it has been shown in [7] that certain Eisenstein series on $\mathrm{SL}_k(\mathbb{Z}) \setminus \mathrm{GL}_k(\mathbb{R})/O_k(\mathbb{R})\mathbb{R}^\times$ violate the purity conjecture. In view of these two negative results the global behaviour of eigenfunctions on non-compact manifolds is far from transparent. This warning aside, it is still believed that most classical results, as well as the purity conjecture, carry over to eigenfunctions in the discrete spectrum restricted to a fixed compact set.

Instead of looking at non-compact spaces, it is also possible to pass to suitable compactifications. See [23] for a detailed account on the subject of compactification. Unfortunately, the compactification will usually be a manifold with boundary or even with edges. This leads to the study of elliptic PDE's with boundary condition. For example, on $M = \mathrm{SL}_2(\mathbb{Z}) \setminus \mathcal{H}^{(2)}$ the property 'vanishing at the cusp' would translate to Dirichlet boundary condition on an appropriate compactification. To get intuition for the global behaviour of Maaß cusp forms on non-compact spaces one might have a look at classical results concerning mass concentration of Laplace eigenfunctions with Dirichlet boundary condition. It has been shown in [74] that (4.1.1) still holds for eigenfunctions of the Laplace-Beltrami operator with Dirichlet boundary condition on 2 dimensional Riemannian manifolds with boundary. However, on the disc $\{x \in \mathbb{R}^2 \colon |x| \leq 1\}$, some Laplace eigenfunctions ϕ with Dirichlet boundary condition concentrate within a neighbourhood of the boundary, which is roughly of size $\lambda_\phi^{-\frac{1}{3}}$, [37]. Such Phenomena might explain the growth of Maaß cusp forms on GL_n shown in [24].

The level aspect

Instead of bounding the L^∞-norm of eigenfunctions in terms of the spectral parameter λ, one can bound it using other invariants of M. One instance of this is the volume. To make this precise we consider a sequence of manifolds M_n with $\mathrm{vol}(M_n) \to \infty$ as $n \to \infty$. Then questions about the size of

$$\sup_{\substack{\triangle\phi=\lambda\phi,\\ \lambda\sim T}} \sup_{x\in M_n} \frac{|\phi(x)|}{\|\phi\|_2}$$

in terms of $\mathrm{vol}(M_n)$ arise. This is often referred to as the *level aspect* of the sup-norm problem. The first supremum ensures that the size of the spectral parameter remains comparable throughout the sequence of eigenfunctions under consideration. We can not fix a specific eigenvalue since it might not be contained in the spectrum of all M_n. One can naturally ask the bound to be explicit in both T and the volume. Such estimates are usually called *hybrid bounds*. The name level aspect comes from the fact that one usually considers families of the form $M_n = \Gamma_n \setminus X$ for a fixed Riemann symmetric space X and a sequence of lattices $\Gamma_n \subset \Gamma_0$. The volume of M_n can then be expressed as the 'level' or index of the lattice Γ_n in Γ_0.

One instance of a hybrid bound appears in the works of Blomer and Michel on ellipsoids over number fields, [16, 17]. The sequence of manifolds under consideration comes from Eichler orders in totally definite quaternion algebras over totally real fields. These manifolds have an arithmetic structure, which allows to establish hybrid bounds for Hecke-Laplace eigenfunctions.

Sup-norm bounds in the GL_2-setting

In this section we try to give an overview on sup-norm bounds for eigenfunctions living on quotients of the upper half plane $\mathcal{H}^{(2)}$. We call this the GL_2-setting, since most of the objects appearing here are instances of automorphic forms on GL_2 as defined in [22].

A milestone in the history of the sup-norm problem is the work of Iwaniec and Sarnak [52]. They consider L^2-normalised Hecke-Maaß forms on $\Gamma \setminus \mathcal{H}^{(2)}$, where Γ is a lattice arising from a maximal order in a quaternion algebra. They adapt the amplification method, which is a known tool in the analysis of L-functions, to this setting, and are the first to obtain a sup-norm bound superior to (4.1.1). Indeed they obtain the power saving estimate

$$\|\phi\|_\infty \ll_\epsilon \lambda_\phi^{\frac{5}{24}+\epsilon} \text{ for all } \epsilon > 0. \tag{4.1.4}$$

In the appendix they sketch how to modify their argument to make it work for the non-compact quotient $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathcal{H}^{(2)}$.

Blomer and Holowinsky, in [12], prove a bound in the level aspect. More precisely they show that

$$\|\phi\|_\infty \ll N^{\frac{1}{2}-\frac{1}{37}} \lambda_\phi^{\frac{11}{4}},$$

where ϕ is a Hecke-Maaß newform on $\Gamma_0(N) \setminus \mathcal{H}^{(2)}$ for square-free N. Their theorem holds also for modular forms of positive (non exceptional) weight. Furthermore, they work through the details sketched in the appendix to [52] and derive

$$\|\phi\|_\infty \ll N^{\frac{1}{2}+\epsilon} \lambda_\phi^{\frac{5}{24}+\epsilon}$$

for square-free N. This leads to the hybrid bound

$$\|\phi\|_\infty \ll N^{\frac{1}{2}-\frac{1}{2300}} \lambda_\phi^{\frac{1}{4}-\frac{1}{2300}}$$

for N square-free. It follows an explosion of work on the sup-norm problem for Hecke-Maaß newforms on $\Gamma_0(N) \setminus \mathcal{H}^{(2)}$. The papers [39–41, 81] refine the amplification process and establish increasingly good bounds in the (square-free) level-aspect. This progress then culminates in the hybrid bound

$$\|\phi\|_\infty \ll N^{\frac{1}{3}+\epsilon} \lambda^{\frac{5}{24}+\epsilon} \tag{4.1.5}$$

for square-free N, which was proven in [83]. This combines the best known bounds in level and spectral aspect.

The first step towards removing the square-free condition was made in the paper [71] using classical language. However, soon after A. Saha obtained the state of the art hybrid bound

$$\|\phi\|_\infty \ll (MN_0)^{\frac{1}{2}+\epsilon} N_2^{\frac{1}{3}+\epsilon} \lambda_\phi^{\frac{5}{24}+\epsilon}$$

for a Hecke-Maaß newform ϕ of level $N_0^2 N_2$, where N_2 is square-free, and central character χ with conductor M. This was proven in [70] exploiting the powerful language of automorphic representations.

Note that despite all the effort, the bound in the spectral aspect has been untouched since the groundbreaking work of Iwaniec and Sarnak. Further, it has always been assumed that ϕ is an eigenfunction of all the Hecke operators. Even if strongly believed, it is not known in general, if one can improve the local bound without this assumption. In this direction there is the very interesting work [54] which considers Maaß forms that are eigenfunctions of only finitely many Hecke operators. In this case it turns out that each Hecke operator yields a log-saving upon the local bound.

The sup-norm bounds have also been generalised to the number field setting. One of the cases considered were quotients of the space $\mathcal{H}^{(3)}$ by congruence subgroups. In this situation Blomer, Harcos, and Milièeviè obtained a hybrid bound which is as strong as (4.1.5). As usual they had to restrict themselves to square-free level. Quite recently they extended their work to arbitrary number fields. Together with Maga, [20], they proved a hybrid bound for square-free level. Over totally real fields their bound is of the same strength as (4.1.5). This is the point where our result fits in. We treat Maaß forms of powerful level and with arbitrary central character over number fields using the tools from [70].

There is an obvious version of the sup-norm problem for Eisenstein series. In this scenario it is impossible to consider the global sup-norm. However, one can investigate the size of Eisenstein series restricted to compact sets. The first to look at this situation was Young in his note [89]. He establishes the bound

$$\sup_{z\in K}\left|E(z,\frac{1}{2}+iT)\right| \ll_{K,\epsilon} T^{\frac{3}{8}+\epsilon},$$

for the standard Eisenstein series E on $\mathrm{SL}_2(\mathbb{Z})\setminus\mathcal{H}^{(2)}$. Here K is some fixed compact subset of $\mathcal{H}^{(2)}$. The interesting feature is that the explicit construction of Eisenstein series allows for an optimised amplifier. This leads to an improvement upon the seemingly unbeatable exponent $\frac{5}{24}$ in the spectral aspect. This work has been generalised to Eisenstein series of square-free level in [50]. We extend it further to Eisenstein series over number fields allowing powerful level and central character in [3]. A similar improvement in the eigenvalue aspect can be achieved for dihedral Maaß forms using a related amplifier. This has been worked out in [49].

One can also investigate the size of holomorphic cusp forms on $\mathcal{H}^{(2)}$. In this case the situation is slightly different. This is due to the appearance of non-trivial K-types at infinity. One has to make a slight change in the definition of the sup-norm. We set

$$\|\phi\|_\infty = \sup_{z\in\mathcal{H}^{(2)}} \Im(z)^{\frac{k}{2}}\left|\phi(z)\right|,$$

where k is the weight of ϕ. For compact quotients of $\mathcal{H}^{(2)}$ it has been shown in [33] that,

$$\|\phi\|_\infty \ll k^{\frac{1}{2}-\frac{1}{33}}.$$

Note that the spectral parameter in this case is roughly k^2 and the bound above indeed improves upon the local bound $k^{\frac{1}{2}}$. In the non-compact situation one can use the Fourier-Whittaker expansion at infinity to show

$$k^{\frac{1}{4}-\epsilon} \ll \|\phi\|_\infty \ll k^{\frac{1}{4}+\epsilon}.$$

This has been calculated in [87]. It remains an interesting open question if this bound remains sharp if one restricts the modular form to some fixed compact subset. In the level aspect the bounds derived in [40] carry over to the case of holomorphic modular forms. This is due to the fact that bounds in the level aspect rely purely on information at finite places. For details see [88]. Furthermore, non-integral weight modular forms have been considered in the papers [55, 78, 79].

Most of the results mentioned so far work for forms that in one sense or another correspond to a new vector in the corresponding representation. However, automorphic representations contain many interesting vectors, whose mass distribution can be studied. The paper [48] goes in this direction. Here the authors study other minimal vectors for which they can produce sharp sup-norm bounds using only the Whittaker expansion. Classically these forms live on quotients of the upper half plane by a special class of arithmetic lattices which are different from the classical principal congruence subgroups mentioned above.

Higher rank results

In higher rank spaces the game is to find sequences of joint eigenfunctions along which one can improve the local bound (4.1.3). This is usually achieved by looking at eigenfunctions of Hecke operators and applying an appropriate amplification argument. The first instance of a proper higher rank sup-norm bound which features a power saving in comparison to the local bound was established in [19]. After this breakthrough many more higher rank situation have been studied. For example self dual Maaß forms on $GL(n)$ have been treated in the sequence of papers [14, 15, 43]. The most general results at the moment are [59, 68], which can deal with Maaß forms on a wide variety of reductive groups. All the results mentioned so far only deal with the size of eigenfunctions in the bulk of the space. Only very recently Blomer, Harcos, and Maga established the first global sup-norm result in higher rank, see [9, 10]. These feature explicit exponents which however are worse than expected. Due to the delicate analysis of high rank Whittaker expansions. An very strong saving in the exponent for Hecke-eigenfunctions on S^3 was established in [80].

In the depth aspect the only result to date is the work [47]. However, the author does not consider newforms, but instead looks at certain minimal forms.

4.2 SETTING UP THE SCENE

In this section we are interested in bounding the sup-norm of cuspidal automorphic forms of $G = \mathrm{GL}_2$ over F. More precisely we will study functions

$$\phi \in L^2_0(G(F)\setminus G(\mathbb{A}_F),\omega) \subset L^2(G(F)\setminus G(\mathbb{A}_F),\omega)$$

which are right $K_1(\mathfrak{n})$-invariant for minimal $\mathfrak{n}$ and satisfying some reasonable archimedean restrictions. Indeed, we allow for a mix between Maaß forms and Hilbert modular forms. The associated automorphic representation will be denoted by π_ϕ As explained in [22, p. 4.6] each cuspidal automorphic representation with central character ω can be (uniquely) realised as a closed invariant subspace of $L^2_0(G(F)\setminus G(\mathbb{A}_F),\omega)$. In this way the problem of estimating the sup-norm of ϕ is closely linked to properties of π_ϕ. However, the sup-norm itself is only defined for smooth elements in $L^2_0(G(F)\setminus G(\mathbb{A}_F),\omega)$ and it does not make sense in different realisations of π_ϕ. Therefore we will make the following convention.

Convention 4.2.1 ([2], Convention 1)**.** *Let (π, V_π) be a cuspidal automorphic representation with central character ω_π with an intertwiner $\sigma\colon V_\pi \hookrightarrow L^2_0(G(F)\setminus G(\mathbb{A}_F),\omega_\pi)$. Then the sup-norm of a K-finite vector $v \in V_\pi$ is defined to be*

$$\|v\|_\infty = \frac{\|\sigma(v)\|_\infty}{\|\sigma(v)\|_2}.$$

Let us make some remarks concerning this convention.

- Note that this is indeed well defined. First, we observe that by *multiplicity one* for GL_2 the intertwiner σ is unique up to scaling. However, the scaling does not matter since we L^2-normalise the image. Secondly, K-finiteness ensures that the L^∞-norm of $\sigma(v)$ is defined.

- This convention may seem unnecessary at first. But it gives us the flexibility to realise π in arbitrary models without changing the fixed cusp form whose sup-norm we want to bound.

- The restriction to K-finite vectors shows that we should actually work with the $G(\mathbb{A}_F)$-module underlying π.

Let us describe the structure of the cuspidal automorphic representation π, keeping in mind that we are mainly interested in (almost) spherical Maaß newforms mixed with Hilbert modular forms. We write V_π for the representation space of π. First, note that since (π, V_π) is an cuspidal automorphic representation it is in particular unitary and admissible. For convenience we assume that the central character ω_π of π satisfies $\omega_\pi|_{\mathbb{R}_+} = 1$. This can be achieved by an unramified twist.

In order to describe the automorphic forms we will consider, we specify the underlying representation. To do so we write

$$\pi = \bigotimes_{\nu} \pi_\nu \otimes \bigotimes_{\mathfrak{p}} \pi_{\mathfrak{p}},$$

where $(\pi_{\mathfrak{p}}, V_{\pi,\mathfrak{p}})$ (respectively $(\pi_\nu, V_{\pi,\nu})$) are irreducible representation of $G(F_{\mathfrak{p}})$ (reps $G(F_\nu)$) with central character $\omega_{\pi,\mathfrak{p}}$ (respectively $\omega_{\pi,\nu}$). Note that this decomposition preserves the subspaces of K-finite vectors.

To describe the structure at infinity we decompose the set of archimedean places in $S_\infty = S_{\text{hol}} \sqcup S_{\text{sph}}$ with the only restriction that $S_{\text{hol}} \subset S_{\mathbb{R}}$. For $\nu \in S_{\text{sph}}$ we assume that $\pi_\nu = \chi_1 \boxplus \chi_2$ with

$$\chi_j(y) = |y|_\nu^{it_{\nu,j}} \operatorname{sgn}(y)^{m_{\nu,j}} \qquad \text{if } \nu \text{ is real,}$$
$$\chi_j(re^{i\theta}) = r^{i2t_{\nu,j}} e^{im_{\nu,j}\theta} \qquad \text{else.}$$

Recall the invariants

$$t_\nu = t_{\nu,1} - t_{\nu,2},\ m_\nu = m_{\nu,1} - m_{\nu,2} \text{ and } s_\nu = t_{\nu,1} + t_{\nu,2}.$$

Furthermore, $v \in V_{\pi,\nu}$ is an eigenvector of the Casimir operator with eigenvalue

$$\lambda_\nu = \begin{cases} \frac{1+t_\nu^2}{4} & \text{if } \nu \text{ is real,} \\ 1 + t_\nu^2 & \text{else.} \end{cases}$$

This justifies calling t_ν the spectral parameter of π. At places $\nu \in S_{\text{hol}}$ we assume that $\pi_\nu = \sigma(\chi_1, \chi_2)$. In this case we have the invariants

$$k_\nu = t_{\nu,1} - t_{\nu,2} + 1 \in \mathbb{N} \text{ and } s_\nu = t_{\nu,1} + t_{\nu,2}.$$

Here k_ν is the weight of π_ν and we will usually assume that $k_\nu \geq 2$. For $\nu \in S_{\mathbb{R}} \cap S_{sph}$ we set $k_\nu = m_\nu \mod 2$. In this case $k_\nu \in \{0,1\}$ will be the weight of the archimedean new vector.

Note that all together we must have $\omega_{\pi,\infty}|_{\mathbb{R}^+} = |\cdot|^{\sum_\nu [F_\nu : \mathbb{R}] s_\nu}$. Thus, the assumption $\omega_\pi|_{\mathbb{R}^+} = 1$ implies $\sum_\nu [F_\nu : \mathbb{R}] s_\nu = 0$.

Next we will single out a special element in V_π. Again we do so place by place. If $\nu \in S_{\mathbb{C}}$, we assume π_ν to be spherical, in particular $m_\nu = 0$. In other words, $(\pi_\nu, V_{\pi,\nu})$ contains a K_ν-invariant vector v_ν° which is unique up to scaling. If $\nu \in S_{\mathbb{R}}$, then $v_\nu^\circ \in V_{\pi,\nu}$ will be the unique lowest weight vector. More precisely, this vector is distinguished by assuming that $\pi_\nu(k(\theta))v_\nu^\circ = e^{ik_\nu\theta}v_\nu^\circ$ for all $\theta \in [0, 2\pi)$. Note that $\nu \in S_{\text{sph}} \cap S_{\mathbb{R}}$ then k_ν is either 0 or 1 and we are still dealing with principal series representations. At the non-archimedean places we define $n_{\mathfrak{p}} = a(\pi_{\mathfrak{p}})$ and let $v_{\mathfrak{p}}^\circ \in V_{\pi,\mathfrak{p}}$ be the up to scaling unique $K_{1,\mathfrak{p}}(n_{\mathfrak{p}})$-invariant vector. Globally we define the arithmetic conductor of π to be the ideal $\mathfrak{n} = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}$. Thus V_π contains a unique (up to scaling) vector which is $K_1(\mathfrak{n})_{\text{fin}}$-invariant and has the prescribed transformation behaviour at infinity. The vector

$$v^\circ = \bigotimes_\nu v_\nu^\circ \otimes \bigotimes_{\mathfrak{p}} v_{\mathfrak{p}}^\circ$$

does the job and we will call it the (global) new vector. Under the action of the center $Z(\mathbb{A}_F)$ the new vector v° transforms with resect to the central character ω_π, which has conductor $\mathfrak{m} = \prod_{\mathfrak{p}} \mathfrak{p}^{m_{\mathfrak{p}}}$.

With this restrictions on π in place we observe that

$$\phi_\circ = \sigma(v^\circ) \tag{4.2.1}$$

is a newform over F with central character ω_π. At the places $\nu \in S_{\text{sph}}$ it has Casimir eigenvalue $(\lambda_\nu)_{\nu \in S_{\text{sph}}}$ and it has weight $(k_\nu)_{\nu \in S_{\text{hol}}}$. Furthermore, by our convention

$$\|v^\circ\|_\infty = \frac{\|\phi_\circ\|_\infty}{\|\phi_\circ\|_2}.$$

This is exactly the setting in which we will study sup-norm problem. It is the natural generalisation of classical Maaß wave forms and holomorphic modular forms on the upper half plane $\mathcal{H}^{(2)}$.

By imposing that $\phi_\circ$ is spherical at all complex places we exclude the interesting case of vector valued modular forms. We make this assumption for technical convenience. Indeed, it allows us to ignore issues arising in the amplification process caused by the non-spherical spectral transform. On a smaller note this assumption makes the spherical test vector more natural and the archimedean Whittaker functions easier to handle. With some additional work it should be possible to allow fixed non-trivial K_ν-types or $m_v > 0$ at complex places. However, it seems very difficult to prove bounds which are good in the m_v-aspect. These issues, which are certainly very interesting, lie outside the scope of this thesis.

4.3 LOWER BOUNDS FOR NEWFORMS NEAR THE CUSP

We start by investigating some obstructions to small sup-norms. In our setting there are essentially two such obstructions. One produces large values attained in the bulk of the space. These have been studied already in [52] and in many articles after. Instead we focus on the other one which can be thought of as a resonance phenomenon happening high in the cusp. In particular we follow the approach from [69, 82] to give qualitative lower bounds, which we expect to be sharp. This section can be seen as a supplement to [69] relying on the new local results produced in Part ii.

We start by reproducing [69, Theorem 3.3] in our setting. To this end let us recall that

$$h(\pi_{\mathfrak{p}}) = \frac{\sup_{g\in G(F_{\mathfrak{p}})} |W_{\pi_{\mathfrak{p}}}(g)|}{\|W_{\pi_{\mathfrak{p}}}\|_2},$$

where $W_{\pi_{\mathfrak{p}}}$ is the new vector in the Whittaker model of $\pi_{\mathfrak{p}}$. We collect the ramified pieces together and set

$$h(\pi_{\mathfrak{n}}) = \prod_{\mathfrak{p}|\mathfrak{n}} h(\pi_{\mathfrak{p}}).$$

Proposition 4.3.1. *Let $\phi_\circ$ be a newform of level $\mathfrak{n}$ and arbitrary central character. Then*

$$\frac{\|\phi_\circ\|_\infty}{\|\phi_\circ\|_2} \gg_{F,\epsilon} |k|_{hol}^{\frac{1}{4}-\epsilon} |T|_{sph}^{\frac{1}{6}-\epsilon} \mathcal{N}(\mathfrak{n})^{-\epsilon} h(\pi_{\mathfrak{n}}).$$

Proof. Without loss of generality we assume that $\phi_\circ$ is L^2-normalised. Observe that

$$\sup_{g\in G(\mathbb{A}_F)} |W_{\phi_\circ}(g)| = \sup_{g\in G(\mathbb{A}_F)} \left| \int_{F\backslash\mathbb{A}_F} \phi_\circ(n(x)g)\psi(-x)dx \right|$$
$$\leq \text{Vol}(F\setminus\mathbb{A}_F, \mu_{\mathbb{A}_F}) \sup_{g\in G(\mathbb{A}_F)} |\phi_\circ(g)|.$$

Thus, we reduced the statement to the study of the Whittaker function. Recall the definition

$$\|W_{\phi_\circ}\|_{reg} = L^*(\pi, Ad, 1) \prod_\nu \frac{\zeta_{F_\nu}(2)\|W_{\phi_\circ,\nu}\|_2^2}{\zeta_{F_\nu}(1)} \cdot \prod_{\mathfrak{p}} \frac{\zeta_{F_{\mathfrak{p}}}(2)\|W_{\phi_\circ,\mathfrak{p}}\|_2^2}{\zeta_{F_{\mathfrak{p}}}(1)L(\pi_p, Ad, 1)},$$

from [61, (2.2) and (2.3)], where $L^*(\pi, Ad, 1)$ is the regularised value of the (finite part of the) adjoint square L-function. According to [61, (2.4) and Lemma 2.2.3] we have

$$\|\phi_\circ\|_2^2 = C_F \|W_{\phi_\circ}\|_{reg}^2 \text{ and } L^*(\pi, Ad, 1) \ll_{F,\epsilon} (\mathcal{N}(\mathfrak{n})\, |T|_{sph}\, |k|_{hol})^{\epsilon},$$

for some positive constant C_F depending only on F. Furthermore note that for $\mathfrak{p} \nmid \mathfrak{n}$ we have

$$\frac{\zeta_{F_\mathfrak{p}}(2)\|W_{\phi_\circ,\mathfrak{p}}\|_2^2}{\zeta_{F_\mathfrak{p}}(1)L(\pi_p, Ad, 1)} = |W_{\phi_\circ,\mathfrak{p}}(1)|^2 .$$

Thus, by restricting the range of the supremum, we find

$$\sup_{g\in G(\mathbb{A}_F)} |W_{\phi_\circ}(g)| \gg_{F,\epsilon} (\mathcal{N}(\mathfrak{n})\, |T|_{sph}\, |k|_{hol})^{-\epsilon} \prod_{\nu} \frac{\sup_{g\in G(F_\nu)} |W_{\phi_\circ,\nu}|}{\|W_{\phi_\circ,\nu}\|_2} \cdot \prod_{\mathfrak{p}|\mathfrak{n}} h(\pi_\mathfrak{p}) \frac{\zeta_{F_\mathfrak{p}}(1)^{\frac{1}{2}} L(\pi_p, Ad, 1)^{\frac{1}{2}}}{\zeta_{F_\mathfrak{p}}(2)^{\frac{1}{2}}}.$$

Recalling lower bounds for $\frac{\sup_{g\in G(F_\nu)}|W_{\phi_\circ,\nu}|}{\|W_{\phi_\circ,\nu}\|_2}$ via the transition region of archimedean Whittaker functions. Note that $|W_{\phi_\circ,\nu}|$ is independent of s_ν, so that the lower bound does not dependent on these parameters. Finally, observing that

$$\frac{\zeta_{F_\mathfrak{p}}(1)^{\frac{1}{2}} L(\pi_p, Ad, 1)^{\frac{1}{2}}}{\zeta_{F_\mathfrak{p}}(2)^{\frac{1}{2}}} \asymp 1$$

yields the desired statement. □

Even though the case of high ramification was already treated in the works [69, 82] let us give the following corollary.

Corollary 4.3.2. *Suppose that* $\mathfrak{n} \mid \mathfrak{m}^2$, *then we have*

$$\|\phi_\circ\|_\infty \gg_{F,\epsilon} |k|_{hol}^{\frac{1}{4}-\epsilon} |T|_{sph}^{\frac{1}{6}-\epsilon} \mathcal{N}\left(\frac{\mathfrak{m}^2}{\mathfrak{n}}\right)^{\frac{1}{4}} \mathcal{N}(\mathfrak{n})^{-\epsilon} \|\phi_\circ\|_2.$$

If $\mathfrak{n}$ *is a perfect square, then there exist forms* $\phi_\circ$ *such that*

$$\|\phi_\circ\|_\infty \gg_{F,\epsilon} |k|_{hol}^{\frac{1}{4}-\epsilon} |T|_{sph}^{\frac{1}{6}-\epsilon} \mathcal{N}\left(\frac{\mathfrak{m}^4}{\mathfrak{n}}\right)^{\frac{1}{12}} \mathcal{N}(\mathfrak{n})^{-\epsilon} \|\phi_\circ\|_2.$$

Proof. The result will follow from Proposition 4.3.1 after evaluating $h(\pi_\mathfrak{n})$ locally. We start with the first lower bound, which holds in general and does not depend on the existence of degenerate critical points. Note that

$$h(\pi_{\mathfrak{p}})) \gg 1$$

for all possibilities of $\pi_\mathfrak{p}$. This trivial lower bound is sufficient for all places where $m_p = \frac{n_\mathfrak{p}}{2}$. At the remaining places we find that $\pi_\mathfrak{p} = \chi_{1,\mathfrak{p}} \boxplus \chi_{2,\mathfrak{p}}$ for two characters satisfying $a(\chi_{1,\mathfrak{p}}) > a(\chi_{2,\mathfrak{p}})$. For these representations it follows from Lemma 3.3.5 and Lemma 3.4.12 that

$$h(\pi_{\mathfrak{p})}) \gg q_\mathfrak{p}^{2m_p - n_\mathfrak{p}}.$$

The bound stated above follows by combining these local lower bounds.

The second bound relies on the existence of degenerate critical points and does not hold for every newform. In particular $\mathfrak{n}$ being a square is a necessary condition for this bound to hold. We have to construct local components $\pi_\mathfrak{p}$, which feature degenerate critical points and where $h(\pi_\mathfrak{p})$ is large enough. To do so we let $\pi_\mathfrak{p} = \chi_{1,\mathfrak{p}} \boxplus \chi_{2,\mathfrak{p}}$, where the characters are chosen as follows. If $\mathfrak{p} \mid \mathfrak{n}$ and $2m_\mathfrak{p} = n_\mathfrak{p}$, any characters satisfying $a(\chi_{1,\mathfrak{p}}) = a(\chi_{2,\mathfrak{p}})$ will do the job. At the remaining places $\mathfrak{p} \mid \mathfrak{n}$ we assume that $a(\chi_{1,\mathfrak{p}}) = m_\mathfrak{p}$ and $a(\chi_{1,\mathfrak{p}}) = n_\mathfrak{p} - m_\mathfrak{p}$. Furthermore, these characters must satisfy

$$b_{\chi_{1,\mathfrak{p}}} \cdot b_{\chi_{1,\mathfrak{p}}} \in ((\mathcal{O}_F/\mathfrak{p})^\times)^2,$$

where $b_{\chi_{1,\mathfrak{p}}}$ and $b_{\chi_{2,\mathfrak{p}}}$ are associated to the characters using Lemma 3.1.3. In all these cases our construction ensures the existence of degenerate critical points, see the second table on p.138. Thus the bound (3.4.31) is sharp and we obtain

$$h(\pi_\mathfrak{p}) \gg \mathcal{E}(\pi_\mathfrak{p}, t_0(\pi_\mathfrak{p})) = q_\mathfrak{p}^{\frac{4m_\mathfrak{p} - n_\mathfrak{p}}{12}}.$$

The size of $\mathcal{E}(\pi_\mathfrak{p}, t_0(\pi_\mathfrak{p}))$ can be read off from the first table on page 138.

These observations are insensitive to unitary, unramified twists of the characters $\chi_{1,\mathfrak{p}}$ and $\chi_{2,\mathfrak{p}}$. Thus we find a global cuspidal automorphic representation π with suitable local components by using [65, Theorem 3.2.1]. By construction the newform $\phi_\circ$ associated to π will have the desired properties. □

In the opposite situation, when the central character is not highly ramified, we obtain the following interesting result.

Corollary 4.3.3. *Let $\mathfrak{n}$ be a perfect square such that $-1 \in (\mathcal{O}_F/\mathfrak{n})^{2\times}$. Further, let ω_π be a character of conductor $\mathfrak{m}$ such that $\mathfrak{m} \mid \mathfrak{n}^{\frac{1}{2}}$ then there are newforms $\phi_\circ$ of level $\mathfrak{n}$ and central character ω_π such that*

$$\|\phi_\circ\|_\infty \gg_{F,\epsilon} |k|_{hol}^{\frac{1}{4}-\epsilon} |T|_{sph}^{\frac{1}{6}-\epsilon} \mathcal{N}(\mathfrak{n})^{\frac{1}{12}-\epsilon} \|\phi_\circ\|_2.$$

This is maybe surprising since it provides counter examples to [69, Conjecture 3].

Proof. We start by constructing local representations having the desired behaviour. To do so fix $\mathfrak{p} \mid \mathfrak{n}$ and choose characters $\chi_{1,\mathfrak{p}}, \chi_{2,\mathfrak{p}} \colon F_{\mathfrak{p}}^{\times} \to S^1$ such that

$$a(\chi_{1,\mathfrak{p}}) = a(\chi_{1,\mathfrak{p}}) \text{ and } \chi_{1,\mathfrak{p}}\chi_{1,\mathfrak{p}} = \omega_{\pi,\mathfrak{p}}.$$

This is possible because $a(\omega_{\pi,\mathfrak{p}}) = m_{\mathfrak{p}} \leq \frac{n_{\mathfrak{p}}}{2}$. Furthermore, if $m_{\mathfrak{p}} = \frac{n_{\mathfrak{p}}}{2}$, we make sure that $a(\chi_{1,\mathfrak{p}}\chi_{2,\mathfrak{p}}^{-1}) < \frac{n_{\mathfrak{p}}}{2}$. We set $\pi_{\mathfrak{p}} = \chi_{1,\mathfrak{p}} \boxplus \chi_{2,\mathfrak{p}}$ and $\sigma_{\mathfrak{p}} = \chi_{1,\mathfrak{p}}\chi_{2,\mathfrak{p}}^{-1} \boxplus 1$.

By Lemma 3.3.9 we find that

$$\left| W_{\pi_{\mathfrak{p}}}(g_{-n,\frac{n}{2},v}) \right| = \zeta_{F_{\mathfrak{p}}}(1)^{-2} q_{\mathfrak{p}}^{\frac{n}{2}} \left| K(\chi_{1,\mathfrak{p}} \otimes \chi_{2,\mathfrak{p}}, (\varpi_{\mathfrak{p}}^{-\frac{n}{2}}, \varpi_{\mathfrak{p}}^{-\frac{n}{2}}), v\varpi_{\mathfrak{p}}^{-\frac{n}{2}}) \right|.$$

By construction of $\pi_{\mathfrak{p}}$ and because -1 is a square in $\mathfrak{o}_{\mathfrak{p}}^{\times}$ the Fourier type integral for $K(\chi_{1,\mathfrak{p}} \otimes \chi_{2,\mathfrak{p}}, (\varpi^{-\frac{n}{2}}, \varpi^{-\frac{n}{2}}), \cdot)$ exhibits a degenerate critical point $v_0 \in \mathfrak{o}_{\mathfrak{p}}^{\times}$. Therefore we have

$$h(\pi_{\mathfrak{p}}) \geq \left| W_{\pi_{\mathfrak{p}}}(g_{-n,\frac{n}{2},v_0}) \right| \gg_{F_{\mathfrak{p}}} q_{\mathfrak{p}}^{\frac{n_{\mathfrak{p}}}{12}}.$$

This is the context of Lemma 3.4.15. Note that these observations are not affected by twisting $\chi_{1,\mathfrak{p}}$ and $\chi_{2,\mathfrak{p}}$ with unitary unramified characters.

Thus we can find a global Hecke-character χ_2 satisfying $\chi_2|_{\mathbb{R}_+} = 1$, which has conductor $\mathfrak{m}$ and with local components equal $\chi_{2,\mathfrak{p}}$ up to unramified twist. Furthermore we use [65, Theorem 3.2.1] to construct a cuspidal automorphic form π of level $\mathfrak{n}$, central character $\omega_{\pi}\chi_2^{-2}$ and local components equal to $\sigma_{\mathfrak{p}}$ up to unitary unramified twists of the characters. The result follows by applying Proposition 4.3.1 to $\chi_2\pi$. □

Remark 4.3.4. *Note that we only used large values caused by principal series representations in the corollary above. However, also supercuspidal representations can lead to big peaks of the Whittaker functions. However their behaviour is slightly more subtle to describe and we refer to Section 3.4.4 for precise conditions which force the existence of critical points.*

Finally, let us look at the following neat example which lies exactly at the border to highly ramified behaviour. The interesting point of this example is that it contains an easily constructible family which always exhibits degenerate behaviour.

Corollary 4.3.5. *Let π be a cuspidal automorphic representation of level 1 and let χ be Hecke character of conductor $\mathfrak{m}$. Then we have*

$$\|\chi \otimes \phi_{\circ}\|_{\infty} \gg_{F,\epsilon} |k|_{hol}^{\frac{1}{4}-\epsilon} |T|_{sph}^{\frac{1}{6}-\epsilon} \mathcal{N}(\mathfrak{m})^{\frac{1}{6}-\epsilon} \|\chi \otimes \phi_{\circ}\|_2.$$

Proof. Note that the local components at primes dividing $\mathfrak{m}$ of $\chi\pi$ are of the shape $|\cdot|^{s_1}\chi_{\mathfrak{p}} \boxplus |\cdot|^{s_2}\chi_{\mathfrak{p}}$. Thus we conclude by using Lemma 3.4.10. □

Remark 4.3.6. *Note that the tools developed in Part ii are strong enough to produce precise lower bounds for any cuspidal automorphic representation π as long as one knows its local constitutes. However, in general the statements are combinatorial hard to formulate and therefore we do not give a general statement here. We hope that the examples given above are shedding some light on the general picture.*

4.4 THE GENERATING DOMAIN

In this section we follow [70, Section 3.2] and [2, Section 2] to derive a generating domain for

$$Z(\mathbb{A}_F)G(F)\setminus G(\mathbb{A}_F)/K_1(\mathfrak{n}).$$

From this we deduce that the global sup-norm problem reduces to the study of (twists of) $\phi_\circ$ on very special elements of $G(\mathbb{A}_F)$. The central result of this section is Corollary 4.4.7 below. Note that the newform $\phi_\circ$ as defined above might not transform trivially under the action of $K_1(\mathfrak{n})$ and $Z(\mathbb{A}_F)$. However, $|\phi_\circ|$ does.

4.4.1 *Local preliminaries*

Several steps necessary to deal with powerful level rely on local methods. In this section we briefly recall the ingredients needed from [70] and Part ii. We start by collecting some simple results capturing the behaviour of the invariants defined in (1.3.5).

Lemma 4.4.1 ([2], Lemma 2.1)**.** *Let $g \in K_{\mathfrak{p}}a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}})$. If $n_{\mathfrak{p}}$ is odd, then*

$$n_{1,\mathfrak{p}}(g) = n_{0,\mathfrak{p}} \iff g \in \omega K_{\mathfrak{p}}^0(1)a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}).$$

If $n_{\mathfrak{p}}$ is even, then

$$n_{1,\mathfrak{p}}(g) = n_{0,\mathfrak{p}}.$$

Proof. The first part is a consequence of [70, Lemma 2.2,(2)]. The second part is trivial. □

Lemma 4.4.2 ([70], Lemma 2.3). *Let $n_\mathfrak{p}$ be odd. Further take $k \in K_{0,\mathfrak{p}}(1)$ and*

$$\epsilon_\mathfrak{p} \in \left\{ 1, \begin{pmatrix} 0 & 1 \\ \varpi_\mathfrak{p} & 0 \end{pmatrix} \right\}.$$

Then

$$k\epsilon_\mathfrak{p}\omega a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}}) = \omega k' a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}})\epsilon'_\mathfrak{p} z$$

for $k' \in K_\mathfrak{p}^0(1)$, $z \in Z(F_\mathfrak{p})$ and

$$\epsilon'_\mathfrak{p} = \begin{cases} 1 & \text{if } \epsilon_\mathfrak{p} = 1, \\ \begin{pmatrix} 0 & 1 \\ \varpi_\mathfrak{p}^{n_\mathfrak{p}} & 0 \end{pmatrix} & \text{else.} \end{cases}$$

Proof. The case $\epsilon_\mathfrak{p} = 1$ is very simple. One writes

$$k\epsilon_\mathfrak{p}\omega a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}}) = \omega \underbrace{(\omega^{-1}k\omega)}_{=k'} a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}}).$$

It is a straight forward calculation to check $k' \in K_\mathfrak{p}^0(1)$. In the remaining case we write

$$k\epsilon_\mathfrak{p}\omega a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}}) = \omega \underbrace{(\omega^{-1}k\omega)}_{=k'} a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}})\epsilon'_\mathfrak{p} \begin{pmatrix} -\varpi_\mathfrak{p}^{n_{0,\mathfrak{p}}} & 0 \\ 0 & -\varpi_\mathfrak{p}^{n_{0,\mathfrak{p}}} \end{pmatrix}.$$

As before we have $k' \in K_\mathfrak{p}^0(1)$. To verify the equality one only needs the observation that since $n_\mathfrak{p}$ is odd we have $n_{0,\mathfrak{p}} = n_{1,\mathfrak{p}} - 1$. □

4.4.2 *Finding the generating set*

Our goal is to recreate the argument from [70, Section 3.2] coupled with the results from [20, Section 5]. As one expects this general setting brings the class group and the unit group into the picture. We start with several definitions. For any ideal $\mathfrak{L}$ in $\mathcal{O}_F$ we define

$$\begin{aligned} \eta_\mathfrak{L} &= \prod_{\mathfrak{p}|\mathfrak{L}} \begin{pmatrix} 0 & 1 \\ \varpi_\mathfrak{p}^{n_\mathfrak{p}} & 0 \end{pmatrix} \prod_{\mathfrak{p}\nmid\mathfrak{L}} 1, \\ h_\mathfrak{L} &= \prod_{\mathfrak{p}|\mathfrak{L}} a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}}) \prod_{\mathfrak{p}\nmid\mathfrak{L}} 1, \\ K_\mathfrak{L} &= \prod_{\mathfrak{p}|\mathfrak{L}} K_\mathfrak{p} \prod_{\mathfrak{p}\nmid\mathfrak{L}} \{1\} \subset GL_2(\mathbb{A}_{\text{fin}}), \\ J_\mathfrak{L} &= K_\mathfrak{L} h_\mathfrak{L} \text{ and} \\ \mathcal{J}_\mathfrak{L} &= \{g \in J_\mathfrak{L} \colon n_{1,\mathfrak{p}}(g_\mathfrak{p}) = n_{0,\mathfrak{p}} \forall \mathfrak{p} \mid \mathfrak{L}\}. \end{aligned}$$

Let us make the following little observation.

Lemma 4.4.3 ([2], Lemma 2.3). *For $g \in J_{\mathfrak{L}}$ one has*

$$g \in \mathcal{J}_{\mathfrak{L}} \iff g_{\mathfrak{p}} \in \omega K_{\mathfrak{p}}^0(1) a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}) \text{ for all } \mathfrak{p}|\mathfrak{L} \text{ with } n_{\mathfrak{p}} \text{ odd.}$$

Proof. The proof proceeds by applying Lemma 4.4.1 for each $\mathfrak{p}|\mathfrak{L}$. □

Corollary 4.4.4 ([2], Corollary 2.1). *For $g_{\mathfrak{p}} \in \mathcal{J}_{\mathfrak{p}}$ and $v \in \mathfrak{o}_{\mathfrak{p}}^{\times}$ we have $a(v)g \in \mathcal{J}_{\mathfrak{p}}$.*

Proof. Obviously $a(v)g_{\mathfrak{p}} \in J_{\mathfrak{p}}$. One concludes by using Lemma 4.4.3 and

$$a(v)\omega = \omega \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}.$$

□

In terms of the local invariants we write

$$\mathfrak{n}_0 = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{0,\mathfrak{p}}}, \mathfrak{n}_1 = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{1,\mathfrak{p}}}, \mathfrak{n}_2 = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{1,\mathfrak{p}}-n_{0,\mathfrak{p}}}.$$

Note that $\mathfrak{n}_2$ is square-free and that we have $\mathfrak{n} = \mathfrak{n}_0^2\mathfrak{n}_2$.

Now we want to use the generating domain from [20] for the square-free ideal $\mathfrak{n}_2$. Recall the group

$$K^* = Z(F_\infty)K_\infty \prod_{\mathfrak{p}\nmid\mathfrak{n}_2} Z(F_{\mathfrak{p}})K_{\mathfrak{p}} \prod_{\mathfrak{p}|\mathfrak{n}_2} \langle K_{0,\mathfrak{p}}(1), \begin{pmatrix} 0 & 1 \\ \varpi_{\mathfrak{p}} & 0 \end{pmatrix} \rangle$$

defined in [20, Section 2]. Let $\mathcal{F}(\mathfrak{n}_2)$ be the generating domain for $G(F) \setminus G(\mathbb{A}_F)/K^*$ defined in [20, p. 14]. An element in $\mathcal{F}(\mathfrak{n}_2)$ is of the form

$$\underbrace{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}}_{\in B(F_\infty)} \underbrace{\begin{pmatrix} \theta_i & 0 \\ 0 & 1 \end{pmatrix}}_{=a(\theta_i)},$$

where $|y|_\infty$ is maximal and $\theta_i \in \hat{\mathcal{O}}_F$, $1 \leq i \leq h_F$, is some representative in the class group. Furthermore, we can assume that y is balanced and that $x_\nu \ll 1$ for all ν. This follows as in [20, (5.9)]. We will call such matrices special. Define

$$\mathcal{F}_{\mathfrak{n}_2} = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : \exists i \in \{1, \cdots, h\} \text{ such that } \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} a(\theta_i) \in \mathcal{F}(\mathfrak{n}_2) \right\}.$$

We can write down a generating domain in the spirit of [70, Proposition 3.6].

Proposition 4.4.5 ([2], Proposition 2.1). *For $g \in G(\mathbb{A}_F)$ we find $\mathfrak{L}|\mathfrak{n}_2$ and $1 \le i \le h_F$ such that*

$$g \in Z(\mathbb{A})G(F)\left(a(\theta_i)\mathcal{J}_{\mathfrak{n}} \times \mathcal{F}_{\mathfrak{n}_2}\right)\eta_{\mathfrak{L}}K_1(\mathfrak{n}).$$

The proof follows the steps in [70] exploiting that the fundamental domain $\mathcal{F}(\mathfrak{n}_2)$ from [20] is already given adélically.

Proof. Let $\omega_{\mathfrak{n}}$ be the diagonal embedding of ω in $K_{\mathfrak{n}}$. Then the determinant map

$$\omega_{\mathfrak{n}}h_{\mathfrak{n}}K_1(\mathfrak{n})_{fin}h_{\mathfrak{n}}^{-1}\omega_{\mathfrak{n}}^{-1} \to \prod_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}}^{\times}$$

is surjective. Thus we can apply strong approximation to the element $gh_{\mathfrak{n}}^{-1}\omega_{\mathfrak{n}}^{-1}$ and find $g_\infty t_i \in G(\mathbb{A}_F)$ such that

$$g \in G(F)g_\infty t_i\omega_{\mathfrak{n}}h_{\mathfrak{n}}K_1(\mathfrak{n}).$$

Using the properties of $\mathcal{F}(\mathfrak{n}_2)$ we write $g_\infty t_i = \gamma f z k^*$ with $\gamma \in G(F)$, $zk^* \in K^*$ and $f \in \mathcal{F}(\mathfrak{n}_2)$. By construction of K^* we can assume

$$\begin{aligned} k_{\mathfrak{p}}^* &= \begin{cases} k_{\mathfrak{p}}^* \in K_{\mathfrak{p}} & \text{if } \mathfrak{p} \nmid \mathfrak{n}_2, \\ k_{\mathfrak{p}}'\epsilon_{\mathfrak{p}} \in K_{0,\mathfrak{p}}\epsilon_{\mathfrak{p}} & \text{if } \mathfrak{p} \mid \mathfrak{n}_2, \end{cases} \\ k_v^* &\in K_v. \end{aligned}$$

for $\epsilon_{\mathfrak{p}} \in \left\{1, \begin{pmatrix} 0 & 1 \\ \varpi_{\mathfrak{p}} & 0 \end{pmatrix}\right\}$. Define

$$\mathfrak{L} = \prod_{\mathfrak{p} \text{ s.t. } \epsilon_{\mathfrak{p}} \neq 1} \mathfrak{p}$$

and write

$$g \in Z(\mathbb{A})G(F)f \underbrace{\prod_{\mathfrak{p}\nmid\mathfrak{n}} k_{\mathfrak{p}}^*}_{\in K_1(\mathfrak{n})} \prod_{\mathfrak{p}|\mathfrak{n},\mathfrak{p}\nmid\mathfrak{n}_2} k_{\mathfrak{p}}^*\omega a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}) \prod_{\mathfrak{p}|\mathfrak{n}_2,\mathfrak{p}\nmid\mathfrak{L}} k_{\mathfrak{p}}'\omega a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}) \prod_{\mathfrak{p}|\mathfrak{L}} k_{\mathfrak{p}}'\epsilon_{\mathfrak{p}}\omega a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}})K_1(\mathfrak{n}).$$

Let us treat each product appearing above separately. First, we include the product over $\mathfrak{p} \nmid \mathfrak{n}$ into $K_1(\mathfrak{n})$. Next, we notice that if $\mathfrak{p} \mid \mathfrak{n}$ but $\mathfrak{p} \nmid \mathfrak{n}_2$ then $n_{\mathfrak{p}}$ must be even. Since $k_{\mathfrak{p}}^*\omega \in K_{\mathfrak{p}}$ we apply Lemma 4.4.1 to absorb the second product into $\mathcal{J}_{\mathfrak{n}}$. In the two remaining cases, namely $\mathfrak{p} \mid \mathfrak{n}_2$, $n_{\mathfrak{p}}$ must be odd. First, for $\mathfrak{p} \nmid \mathfrak{L}$ we apply Lemma 4.4.2 to obtain

$$k_{\mathfrak{p}}'\omega a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}) = \omega \underbrace{\hat{k}_{\mathfrak{p}}}_{\in K_{\mathfrak{p}}^{o}(1)} a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}).$$

It follows from Lemma 4.4.3 that also the third product is contained in $\mathcal{J}_{\mathfrak{n}}$. Finally, we use Lemma 4.4.2 and Lemma 4.4.3 again to get

$$k'_{\mathfrak{p}}\epsilon_{\mathfrak{p}}\omega a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}) = \underbrace{\omega \overbrace{\hat{k}_{\mathfrak{p}}}^{K^0_{\mathfrak{p}}(1)} a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}})}_{\in \mathcal{J}_{\mathfrak{n}}} \hat{\epsilon'_{\mathfrak{p}}(z)}.$$

Thus

$$g \in Z(\mathbb{A})G(F)f\mathcal{J}_{\mathfrak{n}} \underbrace{\prod_{\mathfrak{p}|\mathfrak{L}} \epsilon'_p}_{=\eta_{\mathfrak{L}}} K_1(\mathfrak{n}).$$

One concludes the proof by writing $f = pa(\theta_i)$ for a special matrix $p \in \mathcal{F}_{\mathfrak{n}_2}$ and some $i \in \{1 \cdots h_F\}$. □

4.4.3 *The action of $\eta_{\mathfrak{L}}$*

The next step is to understand how $\eta_{\mathfrak{L}}$ acts on $\phi_\circ$ Let us define the character $\omega_\pi^{\mathfrak{L}} = \omega_{\pi,\infty}^{\mathfrak{L}} \prod_{\mathfrak{p}} \omega_{\pi,\mathfrak{p}}^{\mathfrak{L}}$ by

$$\omega_{\pi,\mathfrak{p}}^{\mathfrak{L}}|_{\mathfrak{o}_{\mathfrak{p}}^\times} = \begin{cases} 1 & \text{if } \mathfrak{p} \mid \mathfrak{L}, \\ \omega_{\pi,\mathfrak{p}}|_{\mathfrak{o}_{\mathfrak{p}}^\times} & \text{if } \mathfrak{p} \nmid \mathfrak{L}. \end{cases}$$

We also impose that $\omega_{\pi,\infty}^{\mathfrak{L}}|_{F_{\infty,+}} = 1$. Strong approximation for $\mathbb{A}_F^\times$ shows that there is such a character which is $F^\times$ invariant and unitary.

Let us make some observations. Locally one has

$$\omega_{\pi,\mathfrak{p}}^{-1}\omega_{\pi,\mathfrak{p}}^{\mathfrak{L}}|_{\mathfrak{o}_{\mathfrak{p}}^\times} = \begin{cases} \omega_{\pi,\mathfrak{p}}^{-1}|_{\mathfrak{o}_{\mathfrak{p}}^\times} & \text{if } \mathfrak{p} \mid \mathfrak{L}, \\ 1 & \text{if } \mathfrak{p} \nmid \mathfrak{L}. \end{cases} \tag{4.4.1}$$

Let (π, V_π) be a cuspidal automorphic representation. We define the twisted representation $(\pi^{\mathfrak{L}}, V_\pi)$ by

$$\pi^{\mathfrak{L}}(g) = \omega_\pi^{-1}\omega_\pi^{\mathfrak{L}}(\det(g))\pi(g).$$

This representation is sometimes denoted by $\pi^{\mathfrak{L}} = (\omega_\pi^{-1}\omega_\pi^{\mathfrak{L}})\pi$. The central character of $\pi^{\mathfrak{L}}$ is $\omega_\pi^{-1}(\omega_\pi^{\mathfrak{L}})^2$ and looks locally like

$$\omega_{\pi,\mathfrak{p}}^{-1}(\omega_{\pi,\mathfrak{p}}^{\mathfrak{L}})^2|_{\mathfrak{o}_{\mathfrak{p}}^\times} = \begin{cases} \omega_{\pi,\mathfrak{p}}^{-1}|_{\mathfrak{o}_{\mathfrak{p}}^\times} & \text{if } \mathfrak{p} \mid \mathfrak{L}, \\ \omega_{\pi,\mathfrak{p}}|_{\mathfrak{o}_{\mathfrak{p}}^\times} & \text{if } \mathfrak{p} \nmid \mathfrak{L}. \end{cases} \tag{4.4.2}$$

In particular, the log-conductor of the new central character coincides with the log-conductor of ω_π, namely

$$\mathfrak{m} = \prod_{\mathfrak{p}} \mathfrak{p}^{m_\mathfrak{p}}.$$

Further, we note that this twist does not change the spectral data at ∞. Concerning the conductor of $\pi^{\mathfrak{L}}$ we have the following statement.

Lemma 4.4.6 ([70], Lemma 3.4)**.** *The log-conductor of $\pi^{\mathfrak{L}}$ is $\mathfrak{n}$ and*

$$v^\circ_{\mathfrak{L}} = \pi(\eta_{\mathfrak{L}})v^\circ \tag{4.4.3}$$

is a new vector in $\pi^{\mathfrak{L}}$.

Proof. Note that for $\mathfrak{p} \nmid \mathfrak{L}$ one simply has $\pi^{\mathfrak{L}}_{\mathfrak{p}} = \pi_{\mathfrak{p}}$. However, at the places $\mathfrak{p} \mid \mathfrak{L}$ the representation $\pi^{\mathfrak{L}}_{\mathfrak{p}}$ is equivalent to $\tilde{\pi}_{\mathfrak{p}}$ up to some unramified twist. Here $\tilde{\pi}_{\mathfrak{p}}$ denotes the contragredient representation of $\pi_{\mathfrak{p}}$. Since $a(\pi_{\mathfrak{p}}) = a(\tilde{\pi}_{\mathfrak{p}})$ it suffices to show that the vector given in (4.4.3) has the correct transformation behaviour under $K_1(\mathfrak{n})$.

We proceed place by place. For $\mathfrak{p} \nmid \mathfrak{L}$ and ν there is nothing to do. For $\mathfrak{p} \mid \mathfrak{L}$ we calculate

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{=k_{\mathfrak{p}} \in K_{1,\mathfrak{p}}(n_{\mathfrak{p}})} \begin{pmatrix} 0 & 1 \\ \varpi_{\mathfrak{p}}^{n_{\mathfrak{p}}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \varpi_{\mathfrak{p}}^{n_{\mathfrak{p}}} & 0 \end{pmatrix} \underbrace{\begin{pmatrix} d & c\varpi_{\mathfrak{p}}^{-n_{\mathfrak{p}}} \\ \varpi_{\mathfrak{p}}^{n_{\mathfrak{p}}} b & a \end{pmatrix}}_{=k'_{\mathfrak{p}} \in K_{0,\mathfrak{p}}(n_{\mathfrak{p}})}.$$

It is easy to verify that $k'_{\mathfrak{p}} z(\det(k_{\mathfrak{p}}))^{-1} \in K_{1,\mathfrak{p}}(n_{\mathfrak{p}})$. Therefore, using (4.4.1) and (4.4.2), we have

$$\begin{aligned} \pi^{\mathfrak{L}}_{\mathfrak{p}}(k_{\mathfrak{p}})v^\circ_{\mathfrak{L},\mathfrak{p}} &= \omega^{-1}_{\pi,\mathfrak{p}}(\det(k_{\mathfrak{p}}))\pi_{\mathfrak{p}}(k_{\mathfrak{p}}[\eta_{\mathfrak{L}}]_{\mathfrak{p}})v^\circ_{\mathfrak{p}} \\ &= \omega^{-1}_{\pi,\mathfrak{p}}\big(\det(k_{\mathfrak{p}})\big) \underbrace{\pi_{\mathfrak{p}}\big(z(\det(k_{\mathfrak{p}}))\big)}_{=\omega_{\mathfrak{p}}\left(\det(k_{\mathfrak{p}})\right)} \pi_{\mathfrak{p}}\big([\eta_{\mathfrak{L}}]_{\mathfrak{p}}\big) \underbrace{\big[\pi_{\mathfrak{p}}\big(\underbrace{z(\det(k_{\mathfrak{p}})^{-1})k'_{\mathfrak{p}}}_{\in K_{1,\mathfrak{p}}(n_{\mathfrak{p}})}\big)v^\circ_{\mathfrak{p}}\big]}_{=v^\circ_{\mathfrak{p}}} \\ &= \pi_{\mathfrak{p}}\big([\eta_{\mathfrak{L}}]_{\mathfrak{p}}\big)v^\circ_{\mathfrak{p}} = v^\circ_{\mathfrak{L},\mathfrak{p}}. \end{aligned}$$

□

Observe that $(\pi^{\mathfrak{L}}, V_\pi)$ is also a cuspidal automorphic representation. Furthermore, an intertwiner $\sigma^{\mathfrak{L}}$ to $L^2_0(G(F)\setminus G(\mathbb{A}_F), \omega_\pi^{-1}(\omega_\pi^{\mathfrak{L}})^2)$ is given by

$$[\sigma^{\mathfrak{L}}(v)](g) = \omega_\pi^{-1}\omega_\pi^{\mathfrak{L}}(\det(g))[\sigma(v)](g).$$

This leads us to the definition of the twisted newform $\phi_\circ^{\mathfrak{L}} = \sigma^{\mathfrak{L}}(v_{\mathfrak{L}}^\circ)$. One immediately observes that

$$\phi_\circ(g\eta_{\mathfrak{L}}) = \omega_\pi(\omega_\pi^{\mathfrak{L}})^{-1}(\det(g))\phi_\circ^{\mathfrak{L}}(g).$$

Giving us exactly the ingredient we needed to understand the action of $\eta_{\mathfrak{L}}$ on $\phi_\circ$. We derive the following corollary.

Corollary 4.4.7 ([2], Corollary 2.2). *If $\phi_\circ$ is the newform associated to a cuspidal automorphic representation (π, V_ν) then*

$$\sup_{g\in GL_2(\mathbb{A})} |\phi_\circ(g)| \leq \sup_{\mathfrak{L}|\mathfrak{n}_2} \sup_{1\leq i\leq h_F} \sup_{g\in\mathcal{J}_\mathfrak{n}\times\mathcal{F}_{\mathfrak{n}_2}} \left|\phi_\circ^{\mathfrak{L}}\left(a(\theta_i)g\right)\right|. \quad (4.4.4)$$

We have reduced the sup-norm problem for the newform $\phi_\circ$ to bounding the newforms $\phi_\circ^{\mathfrak{L}}$ on very special matrices. In the following we will fix an arbitrary $\mathfrak{L} \mid \mathfrak{n}_2$, write $\phi = \phi_\circ^{\mathfrak{L}}$ and bound ϕ on $a(\theta_i)(\mathcal{J}_\mathfrak{n} \times \mathcal{F}_{\mathfrak{n}_2})$.

4.5 COUNTING RESULTS

In this section we provide the necessary counting results that will be crucial for later estimates. The first part of this section is taken from [2, Section 3.2] and based on [20]. Here we recall counting results for lattice points in adelic boxes with subtle arithmetic constraints. This will be essential for our treatment of the Whittaker expansion. The second part is dedicated to counting integer matrices. The arguments are extracted [20]. However, we relax the constraints at real places. This is important for our amplification argument as our test function will not necessarily have compact support.

4.5.1 *Counting field elements in boxes*

This subsection is concerned with estimating the number of field elements in different adelic boxes. These can be archimedean boxes or p-adic boxes. The choice of parameters in this sections may seem arbitrary. However, it is well motivated by applications later on.

We start by considering some archimedean boxes. The following argument is almost completely taken from [20]. Take parameters $R_\nu \geq \frac{T_\nu}{2\pi|y_\nu|}$ and an ideal $\imath$. Further, fix $a \in \imath$ such that

$$\mathcal{N}(\imath) \leq \mathcal{N}((a)) \leq \left(\frac{2}{\pi}\right)^{r_1} \sqrt{|d_F|}\mathcal{N}(\imath). \tag{4.5.1}$$

This is possible by [63, Lemma 6.2]. In particular one has $a\imath^{-1} \subset \mathcal{O}_F$.

Define

$$I_\nu(l_\nu) = \begin{cases} \{\xi_\nu \in F_\nu^\times : l_\nu\,|a|\,R_\nu < |\xi_\nu| \leq (l_\nu+1)\,|a|\,R_\nu\} & \text{if } l_\nu \geq 1, \\ \{\xi_\nu \in F_\nu^\times : \ |\xi_\nu| \leq |a|\,R_\nu, -l_\nu \leq \left||\xi_\nu| - \frac{|a|T_\nu}{2\pi|y_\nu|}\right| < -l_\nu + 1\} & \text{if } l_\nu \leq 0 \\ & \quad \text{and } \nu \in S_{sph}, \\ \{\xi_\nu \in F_\nu^\times : \ |\xi_\nu| \leq |a|\,R_\nu, -l_\nu < |\xi_\nu| \leq -l_\nu + 1\} & \text{if } l_\nu \leq 0 \\ & \quad \text{and } \nu \in S_{hol}. \end{cases} \tag{4.5.2}$$

For $\boldsymbol{\ell} \in \mathbb{Z}^{r_1+r_2}$, let $I(\boldsymbol{\ell}) = \prod_\nu I_\nu(l_\nu)$.

Let us start by establishing a simple but crucial property of these sets.

Lemma 4.5.1 ([2], Lemma 3.6)**.** *If $l_\nu < -\lfloor |a|\,R_\nu \rfloor$, then $I_\nu(l_\nu) = \emptyset$.*

Proof. We start with $\nu \in S_{sph}$. Suppose $l_\nu < -\lfloor |a|\,R_\nu \rfloor$. We consider two cases. First, let $|\xi_\nu| > \frac{|a|T_\nu}{2\pi|y_\nu|}$. The two inequalities in the definition of $I_\nu(\cdot)$ yield

$$\frac{|a|\,T_\nu}{2\pi\,|y_\nu|} + \lfloor |a|\,R_\nu \rfloor < |\xi_\nu| \leq |a|\,R_\nu.$$

But the set of such ξ_ν is empty. Second, we assume $|\xi_\nu| \leq \frac{|a|R_\nu}{2\pi}$. This gives

$$|\xi_\nu| < \frac{|a|\,T_\nu}{2\pi\,|y_\nu|} - \lfloor |a|\,R_\nu \rfloor < 0$$

which is also impossible. The case $\nu \in S_{hol}$ is trivial. □

Our next goal is to establish good estimates for $\sharp(I(\boldsymbol{\ell}) \cap a\imath^{-1})$. This will be achieved by a standard volume argument. Choose a fundamental set $\mathcal{P}$ for the lattice $a\imath^{-1} \subset F_\infty$. Without loss of generality we can assume $0 \in \mathcal{P}$. Let D be the diameter of $\mathcal{P}$. It is an elementary fact, see [63], that

$$\mathrm{Vol}(\mathcal{P}) \sim_F \mathcal{N}((a))\mathcal{N}(\imath^{-1}) \asymp_F 1.$$

Further, we define

$$J_\nu(l_\nu) = \begin{cases} \{\xi_\nu \in F_\nu \colon l_\nu \left|a\right| R_\nu - D < |\xi_\nu| \leq (l_\nu + 1) \left|a\right| R_\nu + D\} & \text{if } l_\nu \geq 1, \\ \{\xi_\nu \in F_\nu \colon -l_\nu - D \leq \left| |\xi_\nu| - \frac{|a|T_\nu}{2\pi|y_\nu|} \right| < -l_\nu + 1 + D\} & \text{if } l_\nu \neq 0 \text{ and } \nu \in S_{sph}, \\ \{\xi_\nu \in F_\nu \colon -l_\nu - D < |\xi_\nu| \leq -l_\nu + 1 + D\} & \text{if } l_\nu \neq 0 \text{ and } \nu \in S_{hol}. \end{cases}$$

and $J(\boldsymbol{\ell}) = \prod_\nu J_\nu(l_\nu)$.

Lemma 4.5.2 ([2], Lemma 3.7). *The volume of $J_\nu(l_\nu)$ is given by*

$$Vol(J_\nu(l_\nu)) = \begin{cases} 2\left|a\right| R_\nu + 4D & \text{if } \nu \text{ is real and } l_\nu \geq 1, \\ 4(1+2D) & \text{if } \nu \in S_{sph} \text{ is real and } l_\nu \leq 0, \\ 2(1+2D) & \text{if } \nu \in S_{hol} \text{ is real and } l_\nu \leq 0, \\ \pi(2l_\nu+1)\left|a\right| R_\nu(\left|a\right| R_\nu + 2D) & \text{if } \nu \text{ is complex and } l_\nu \geq 1, \\ 2\frac{|a|T_\nu}{y_\nu}(1+2D) & \text{if } \nu \text{ is complex and } l_\nu \leq 0. \end{cases}$$

Proof. The proof is an elementary volume calculation. □

As consequence of Minkowski-theory we can choose $\mathcal{P}$ such that

$$D \ll \mathcal{N}(a\imath^{-1})^{\frac{1}{n}} \ll_F 1.$$

Therefore

$$\text{Vol}(J(\boldsymbol{\ell})) \ll_F \prod_\nu f_\nu(l_\nu), \tag{4.5.3}$$

for

$$f_\nu(l_\nu) = \begin{cases} |aR_\nu| + 1 & \text{if } \nu \text{ is real and } l_\nu \geq 1, \\ 1 & \text{if } \nu \text{ is real and } l_\nu \leq 0, \\ l_\nu(\frac{|a|T_\nu}{y_\nu} + 1)^2 & \text{if } \nu \text{ is complex and } l_\nu \geq 1, \\ \frac{|a|T_\nu}{y_\nu} + 1 & \text{if } \nu \text{ is complex and } l_\nu \leq 0. \end{cases} \tag{4.5.4}$$

With this at hand we can establish the following counting result.

Lemma 4.5.3 ([2], Lemma 3.8). *One has*

$$\sharp(a\imath^{-1} \cap I(\boldsymbol{\ell})) \ll_F \prod_\nu f_\nu(l_\nu).$$

Proof. By construction of $\mathcal{P}$ we have

$$\sharp(a\imath^{-1}\cap I(\boldsymbol{\ell}))=\frac{\mathrm{Vol}(\bigcup_{q\in a\imath^{-1}\cap I(\underline{l})}(q+\mathcal{P}))}{\mathrm{Vol}(\mathcal{P})}\leq\frac{\mathrm{Vol}(J(\boldsymbol{\ell}))}{\mathrm{Vol}(\mathcal{P})}.$$

One concludes using (4.5.3). □

Furthermore, we will need to count field elements with strong non-archimedean restrictions. We will be able to reduce this problem to the following lemma.

Lemma 4.5.4 ([20], Lemma 7 and Corollary 1)**.** *Let $a\in\mathbb{A}_F^\times$ be any idele, $y\in F_\infty^\times$ and $\mathfrak{m}\subset F$ be a non-zero fractional ideal. Then*

1. $\sharp\{x\in F^\times\colon |x|_\nu\leq|a_\nu|_\nu$ *and* $|x|_\mathfrak{p}\leq|a_\mathfrak{p}|_\mathfrak{p}\}\ll|a|_\mathbb{A}$,
2. $\sharp\{x\in F^\times\colon |x|_\nu\leq|a_\nu|_\nu$ *and* $|x|_\mathfrak{p}=|a_\mathfrak{p}|_\mathfrak{p}\}\ll_\epsilon|a|_\mathbb{A}^\epsilon$,
3. $\sharp\{x\in F^\times\colon |x|_\nu\leq|y_\nu|_\nu$ *and* $x\mathcal{O}_F\subset\mathfrak{m}\}\ll\frac{|y|_\infty}{\mathcal{N}(\mathfrak{m})}$,
4. $\sharp\{x\in F^\times\colon |x|_\nu\leq|y_\nu|_\nu$ *and* $x\mathcal{O}_F=\mathfrak{m}\}\ll_\epsilon\left(\frac{|y|_\infty}{\mathcal{N}(\mathfrak{m})}\right)^\epsilon$.

Define the sets

$$\begin{aligned}
B(R) &= \{x\in F_\infty\colon |x_\nu|\leq R_\nu\},\\
\mathbb{Z}^\mathfrak{n} &= \prod_{\mathfrak{p}|\mathfrak{n}}\{k_\mathfrak{p}\in\mathbb{Z}\colon k_\mathfrak{p}\geq -v_\mathfrak{p}(\imath)\},\\
\mathbb{A}^\imath_{fin} &= \{a\in\mathbb{A}_{fin}\colon v_\mathfrak{p}(a_\mathfrak{p})\geq -v_\mathfrak{p}(\imath)\},\\
C^\imath(\mathbf{k}) &= \{a\in\mathbb{A}^\imath_{fin}\colon v_\mathfrak{p}(a_\mathfrak{p})=k_\mathfrak{p}\quad\forall\mathfrak{p}|\mathfrak{n}\}\text{ and}\\
C^\imath(\mathbf{k},[\mathbf{u}]) &= \{a\in C^\imath(\underline{k})\colon a_\mathfrak{p}=\varpi_\mathfrak{p}^{k_\mathfrak{p}}a'_\mathfrak{p}\text{ with }[a'_\mathfrak{p}]=[u_\mathfrak{p}]\in\mathfrak{o}_\mathfrak{p}^\times/(1+\varpi_\mathfrak{p}^{n_{0,\mathfrak{p}}(g_\mathfrak{p})}\mathfrak{o}_\mathfrak{p})\quad\forall\mathfrak{p}|\mathfrak{n}\}.
\end{aligned}$$

It will be useful to know the volumes of these sets.

Lemma 4.5.5 ([2], Lemma 3.9)**.** *We have*

$$\begin{aligned}
Vol(\mathbb{A}^\imath_{fin},d\mu_{fin}) &= \mathcal{N}(\imath),\\
Vol(C^\imath(\mathbf{k}),d\mu_{fin}) &= \frac{\mathcal{N}(\imath)}{\mathcal{N}([\imath]_\mathfrak{n})}\zeta_\mathfrak{n}(1)\prod_{\mathfrak{p}|\mathfrak{n}}q_\mathfrak{p}^{-k_\mathfrak{p}}\\
Vol(C^\imath(\mathbf{k},[\mathbf{u}])d\mu_{fin}) &= Vol(C^\imath(\mathbf{k},[\mathbf{u}'])d\mu),\\
Vol(C^\imath(\mathbf{k},[\mathbf{u}]),d\mu_{fin}) &= \frac{\mathcal{N}(\imath)}{\mathcal{N}([\imath]_\mathfrak{n})}\prod_{\mathfrak{p}|\mathfrak{n}}q_\mathfrak{p}^{-k_\mathfrak{p}-n_{0,\mathfrak{p}}(g_\mathfrak{p})}.
\end{aligned}\qquad(4.5.5)$$

Proof. This is a standard adelic volume computation done place by place. □

Finally, we are ready to prove the following counting result.

Lemma 4.5.6 ([2], Lemma 3.10). *We have*

$$\sharp((a\imath^{-1}\setminus\{0\})\cap B(R)C^{\imath}(\mathbf{k},[\mathbf{u}]))\ll F_R(\mathbf{k})=1+\frac{|R|_\infty\mathcal{N}(\imath)}{\mathcal{N}(\mathfrak{n}_0(g))\mathcal{N}([\imath]_\mathfrak{n})}\prod_{\mathfrak{p}|\mathfrak{n}}q_\mathfrak{p}^{-k_\mathfrak{p}}$$

uniform in $[\mathbf{u}]$. *Furthermore,*

$$(F^\times\cap B(R)C^{\imath}(\mathbf{k}))\setminus\{0\}=\emptyset$$

for $\prod_{\mathfrak{p}|\mathfrak{n}}q_\mathfrak{p}^{k_p}>|R|_\infty\mathcal{N}(\imath^{-1}[\imath]_\mathfrak{n})$.

Proof. Let S be the set we want to count. If S is empty, we have nothing to show. Thus take $q_0\in S$. Define the shifted set $S'=\frac{1}{q_0}S-1$. Any $x\in S'$ satisfies

$$\begin{aligned}|x|_\nu&\leq 2\left|\frac{R}{q_0}\right|_\nu &&\text{for all }\nu,\\ |x|_\mathfrak{p}&\leq\left|\frac{\varpi_\mathfrak{p}^{-v_\mathfrak{p}(\imath)}}{q_0}\right|_\mathfrak{p} &&\text{for all }\mathfrak{p}\nmid\mathfrak{n}\text{ and}\\ |x|_\mathfrak{p}&\leq\left|\varpi_\mathfrak{p}^{n_{0,\mathfrak{p}}(g_\mathfrak{p})}\right|_\mathfrak{p} &&\text{for all }\mathfrak{p}\mid\mathfrak{n}.\end{aligned}$$

Define the idéle s by $s_\nu=2^{1/[F_\nu:\mathbb{R}]}\frac{R}{q_0}$ and

$$s_\mathfrak{p}=\begin{cases}\frac{\varpi_\mathfrak{p}^{-v_\mathfrak{p}(\imath)}}{q_0} & \text{if }\mathfrak{p}\not|\mathfrak{n},\\ \varpi_\mathfrak{p}^{n_{0,\mathfrak{p}}(g_\mathfrak{p})} & \text{else.}\end{cases}$$

After noting that $0\in S'$ we conclude that

$$\sharp S\leq 1+\sharp\{x\in F^\times\colon\ |x|_\nu\leq|s|_\nu\text{ and }|x|_\mathfrak{p}\leq|s|_\mathfrak{p}\}.$$

To estimate the last set we use Lemma 4.5.4. We obtain

$$\sharp S\leq 1+|s|_{\mathbb{A}_F}.$$

The adelic norm of s is computed using

$$\prod_\nu|q_0|_\nu^{-1}\prod_{\mathfrak{p}\nmid\mathfrak{n}}|q_0|_\mathfrak{p}^{-1}=\prod_{\mathfrak{p}|\mathfrak{n}}|q_0|_\mathfrak{p}=\prod_{\mathfrak{p}|\mathfrak{n}}q_\mathfrak{p}^{-k_\mathfrak{p}}.$$

To prove the second claim we suppose $\prod_{\mathfrak{p}|\mathfrak{n}}q_\mathfrak{p}^{k_p}>|R|_\infty\mathcal{N}(\imath^{-1}[\imath]_\mathfrak{n})$ and define the ideal $\mathfrak{m}=\prod_{\mathfrak{p}|\mathfrak{n}}\mathfrak{p}^{k_p}$. In order to have $q\in C^{\imath}(\mathbf{k})$ one needs $\mathcal{N}((q))\geq\mathcal{N}(\mathfrak{m})\imath[\imath]_\mathfrak{n}^{-1}$. But for $q\in B(R)$ we require $|q|_\infty\leq|R|_\infty$. We conclude by

$$1=|q|_\mathbb{A}=|q|_\infty|q|_{fin}=\frac{|q|_\infty}{\mathcal{N}((q))}\leq\frac{|R|_\infty\mathcal{N}([\imath]_\mathfrak{n})}{\mathcal{N}(\mathfrak{m}\imath)}<1.$$

□

Roughly the same reasoning applies to elements of $\imath^{-1} \cap B(R)$.

Corollary 4.5.7 ([2], Corollary 3.2). *If* $|R|_\infty < \mathcal{N}(\imath)^{-1}$, *then*

$$\imath^{-1} \cap B(R) = \{0\}.$$

4.5.2 *Counting integer matrices*

Throughout this section we fix two ideals $\mathfrak{n}$ and $\mathfrak{q}$. Let

$$\mathcal{P}(L) \subset \{\alpha \in \mathcal{O}_F \colon \mathcal{N}(\alpha) \in [L, 2L]\}$$

for a large parameter L. Later this set will be the basic support of the amplifier and we will impose further restrictions on its elements. Define

$$\Gamma_j(i,l) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F) \colon a, d, \theta_i^{-1} b \in \theta_j,\ a-d \in \theta_j \mathfrak{q},\ c \in \theta_i^{-1}\theta_j \mathfrak{n}\mathfrak{q}, \right.$$
$$\left. \text{and } ad-bc = l \right\}.$$

for $l \in \mathcal{O}_F$. Recall the definition of the generalised upper half space $\mathcal{H}$, (1.3.10), as well as the local point pair invariants u_ν, (1.3.11). Throughout this section we fix a special matrix $n(x)a(y) \in \mathcal{F}_{\mathfrak{n}_2}$ and define the point $P = (P_\nu)_\nu \in \mathcal{H}$ by setting $P_\nu = y_\nu i_\nu + x_\nu$.

Let $\boldsymbol{\delta} = (\delta_\nu)_\nu \in \mathbb{R}_+^{r_1+r_2}$ such that $\delta_\nu \ll 1$ for $\nu \in S_{\text{sph}}$. We consider the subsets of matrices

$$\Gamma_j(i,l,\boldsymbol{\delta}) = \Gamma_j(i,l,\boldsymbol{\delta})^0 \sqcup \Gamma_j^{par}(i,l,\boldsymbol{\delta}) \sqcup \Gamma_j^{gen}(i,l,\boldsymbol{\delta}).$$

Here, we define

$$\Gamma_j(i,l,\boldsymbol{\delta})^0 = \left\{ \gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma_j(i,l) \colon u_\nu(\gamma.P_\nu.P_\nu) \leq \delta_\nu \text{ for all } \nu \right\},$$
$$\Gamma_j(i,l,\boldsymbol{\delta})^{par} = \left\{\gamma \in \Gamma_j(i,l) \colon \operatorname{tr}(\gamma)^2 = 4\det(\gamma) \text{ and } u_\nu(\gamma.P_\nu.P_\nu) \leq \delta_\nu \text{ for all } \nu\right\} \text{ and}$$
$$\Gamma_j(i,l,\boldsymbol{\delta})^{gen} = \left\{\gamma \in \Gamma_j(i,l) \setminus (\Gamma_j(i,l,\boldsymbol{\delta})^0 \sqcup \Gamma_j(i,l,\boldsymbol{\delta})^{par}) \colon u_\nu(\gamma.P_\nu.P_\nu) \leq \delta_\nu \text{ for all } \nu\right\}.$$

The goal of this section is to bound the number of elements in these sets. For notational simplicity we write $\hat{\boldsymbol{\delta}} = (\max(1,\delta_\nu))_\nu$. Note that $\hat{\delta}_\nu \ll 1$ for all $\nu \in S_{\text{sph}}$. In the upcoming estimates we have made no effort to optimise the dependence on $\hat{\boldsymbol{\delta}}$.

We will closely follow [20, Section 10], starting by deriving preliminary estimates coming from the archimedean restriction.

Lemma 4.5.8. *Suppose* $\gamma_\nu = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{pmatrix} \in GL_2(F_\nu)$ *such that* $u(\gamma_\nu P_\nu, P_\nu) \le \delta_\nu$ *and* $a_\nu d_\nu - b_\nu c_\nu = l_\nu$.

Then we have

$$\|c_\nu P_\nu + d_\nu\| = |l_\nu|^{\frac{1}{2}} (1 + O(\sqrt{\delta_\nu})), \tag{4.5.6}$$

$$\|c_\nu P_\nu - a_\nu\| = |l_\nu|^{\frac{1}{2}} (1 + O(\sqrt{\delta_\nu})), \tag{4.5.7}$$

$$|c_\nu y_\nu| \le |2 l_\nu|^{\frac{1}{2}} (1 + \sqrt{\delta_\nu}), \tag{4.5.8}$$

$$|2c_\nu x_\nu - a_\nu + d_\nu| \le 2\, |2 l_\nu|^{\frac{1}{2}} (1 + \sqrt{\delta_\nu}), \tag{4.5.9}$$

$$|a_\nu + d_\nu| \le 2\, |2 l_\nu|^{\frac{1}{2}} (1 + \sqrt{\delta_\nu}), \tag{4.5.10}$$

$$\left| \overline{c_\nu} \frac{l_\nu}{|l_\nu|} y_\nu^2 - c_\nu x_\nu^2 + (a_\nu - d_\nu) x_\nu + b_\nu \right| \le y_\nu\, |2 l_\nu|^{\frac{1}{2}} \sqrt{\delta_\nu}, \tag{4.5.11}$$

$$\left| -c_\nu x_\nu^2 + (a_\nu - d_\nu) x_\nu + b_\nu \right| \le y_\nu\, |2 l_\nu|^{\frac{1}{2}} (2 + 3\sqrt{\delta_\nu}), \tag{4.5.12}$$

$$\left| \Re\left(\frac{2c_\nu x_\nu - a_\nu + d_\nu}{\sqrt{l_\nu}} \right) \right| \le \sqrt{2\delta_\nu} \text{ and} \tag{4.5.13}$$

$$\left| \Im\left(\frac{a_\nu + d_\nu}{\sqrt{l_\nu}} \right) \right| \le \sqrt{2\delta_\nu}. \tag{4.5.14}$$

This lemma is a summary of the inequalities [20, (10.2)-(10.10)]. The proof is taken from [20] and [11]. Note that we slightly modified the argument to allow for general δ_ν and l_ν. Similar inequalities already appeared in [52].

Proof. The starting point is the inequality

$$\delta_\nu \ge u(\gamma_\nu P_\nu, P_\nu) \ge \frac{|\Im(\gamma_\nu P_\nu) - \Im(P_\nu)|^2}{2\Im(\gamma_\nu P_\nu)\Im(P_\nu)} = \frac{1}{2} \left| \frac{|l_\nu|^{\frac{1}{2}}}{\|c_\nu P_\nu + d_\nu\|} - \frac{\|c_\nu P_\nu + d_\nu\|}{|l_\nu|^{\frac{1}{2}}} \right|^2$$

for real as well as complex places ν. This implies (4.5.6). To show the second inequality we observe that $u(\gamma_\nu P_\nu, P_\nu) = u(\gamma_\nu^{-1} P_\nu, P_\nu)$ and apply the inequality above with γ_ν^{-1}. This yields (4.5.7). Inequalities (4.5.8)-(4.5.10) follow directly from (4.5.6) and (4.5.7) by observing

$$|c_\nu y_\nu| = |\Im(c_\nu P_\nu + d_\nu)| \le \|c_\nu P_\nu + d_\nu\|,$$

$$|a_\nu + d_\nu| = |\Re(c_\nu P_\nu + d_\nu) + \Re(c_\nu P_\nu - a_\nu)| \le \|c_\nu P_\nu + d_\nu\| + \|c_\nu P_\nu - a_\nu\|, \text{ and}$$

$$|2c_\nu x_\nu - a_\nu + d_\nu| = |\Re(c_\nu P_\nu + d_\nu) - \Re(c_\nu P_\nu - a_\nu)| \le \|c_\nu P_\nu + d_\nu\| + \|c_\nu P_\nu - a_\nu\|.$$

To prove the remaining inequalities we consider real and complex places separately. If ν is real, we observe that

$$\delta_\nu \ge u(\gamma_\nu P_\nu, P_\nu) = \frac{\|a_\nu P_\nu + b_\nu - c_\nu P_\nu^2 - d_\nu P_\nu\|^2}{2\, |l_\nu|\, \Im(P_\nu)^2}.$$

Equations (4.5.11) and (4.5.13) follow by taking real part and imaginary part in the inequality above. Finally, (4.5.12) follows from (4.5.8) together with (4.5.11). Furthermore, in the real case (4.5.14) is trivial.

If ν is complex, we observe that

$$\delta_\nu \geq \frac{\|a_\nu P_\nu + b_\nu - P_\nu c_\nu P_\nu - P_\nu d_\nu\|^2}{2 l_\nu \Im(P_\nu)^2}.$$

The numerator on the right hand side is a quaternion and considering its complex part yields (4.5.11). Similarly looking at its j- and k-part yields (4.5.13) and (4.5.14). As before (4.5.12) follows from (4.5.8) and (4.5.11). □

Lemma 4.5.9 ([20], Lemma 12). *We have*

$$\sharp\Gamma_j^0(i,l,\boldsymbol{\delta}) \ll_\epsilon (|\hat{\boldsymbol{\delta}}|_\infty |l|_\infty)^\epsilon (1 + |l|_\infty^{\frac{1}{2}} |y|_\infty |\boldsymbol{\delta}|_\infty^{\frac{1}{2}}).$$

Proof. Since $ad = l$, the ideal version of the divisor bound gives up to $\mathcal{N}(l)^\epsilon$ possibilities for the ideals (a) and (d). However, fixing a choice of ideals we observe that (4.5.6), (4.5.7) and Lemma 4.5.4 give

$$\sharp\{(a,d)\} \ll (|\hat{\boldsymbol{\delta}}|_\infty |l|_\infty)^\epsilon$$

choices for a, d. We conclude the proof by counting the number possible b, once a, d are fixed, using (4.5.11) and Lemma 4.5.4. □

Lemma 4.5.10 ([20], Lemma 13). *If P is in the fundamental domain $\mathcal{F}(\mathfrak{n}_2)$ and $|\boldsymbol{\delta}|_\infty \ll \mathcal{N}(l)^{-1} |y|_\infty^{-2}$, then $\Gamma_j^{par}(i,l,\boldsymbol{\delta}) = \emptyset$.*

Since the proof carries over without modifications, we will not reproduce it here.

Lemma 4.5.11 ([20], Lemma 14). *We have*

$$\sharp\Gamma_j^{par}(i,l,\delta) \ll_\epsilon \frac{\mathcal{N}(l)^{1+\epsilon} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{3}{4}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{4}} |\hat{\boldsymbol{\delta}}|_{\mathbb{R}}^{\frac{1}{8}+\epsilon}}{\mathcal{N}(\mathfrak{n}_2)^{1+\epsilon}}.$$

Proof. Observe that $(a+d)^2 = 4(ad - bc)$ implies that $l = ad - bc = \lambda^2$ is a square. At real places (4.5.13) implies that

$$c_\nu x_\nu + d_\nu = (c_\nu x_\nu - \frac{a_\nu}{2} + \frac{d_\nu}{2}) + (\frac{a_\nu}{2} + \frac{d_\nu}{2}) = \lambda_\nu(1 + O(\sqrt{\delta_\nu})).$$

In combination with (4.5.6) this gives

$$(c_\nu y_\nu)^2 = |c_\nu P_\nu + d_\nu|^2 - (c_\nu x_\nu + d_\nu)^2 \ll \lambda_\nu^2 \sqrt{\delta_\nu}.$$

At complex places (4.5.8) yields

$$c_\nu y_\nu \ll |l_\nu|^{\frac{1}{2}}.$$

Because $(a-d)^2+4bc=0$, there is $\mathfrak{a} \mid (a-d)$ such that $\mathcal{N}(\mathfrak{a}) \geq \mathcal{N}((c))^{\frac{1}{2}}$. Thus, according to (4.5.9), (4.5.13) and Lemma 4.5.4 we have

$$\sharp\{a-d\} \ll 1 + \frac{\mathcal{N}(l)^{\frac{1}{2}} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{1}{2}}}{\mathcal{N}((c))^{\frac{1}{2}}},$$

for fixed c. Summing over all admissible choices for c yields

$$\begin{aligned}\sharp\{(c,a-d)\} &\ll \frac{\mathcal{N}(l)^{\frac{1}{2}} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{1}{4}}}{|y|_\infty \mathcal{N}(\mathfrak{n}_2)} + \mathcal{N}(l)^{\frac{1}{2}} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{1}{2}} \sum_{\substack{0\neq c\in\theta_i^{-1}\mathfrak{n}_2,\\ c_\nu \ll l_\nu^{\frac{1}{2}}\delta_\nu^{\frac{1}{4}} y_\nu^{-1} \text{ real},\\ c_\nu \ll l_\nu^{\frac{1}{2}} y_\nu^{-1} \text{ complex}}} \mathcal{N}((c))^{-\frac{1}{2}} \\ &\ll_\epsilon \frac{\mathcal{N}(l)^{\frac{1}{2}} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{1}{4}}}{|y|_\infty \mathcal{N}(\mathfrak{n}_2)} + \frac{\mathcal{N}(l)^{\frac{3}{4}+\epsilon} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{5}{8}+\epsilon}}{|y|_\infty^{\frac{1}{2}+\epsilon} \mathcal{N}(\mathfrak{n}_2)^{1+\epsilon}} \ll \frac{\mathcal{N}(l)^{1+\epsilon} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{3}{4}+\epsilon} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{4}} |\hat{\boldsymbol{\delta}}|_{\mathbb{R}}^{\frac{1}{8}+\epsilon}}{\mathcal{N}(\mathfrak{n}_2)^{1+\epsilon}}.\end{aligned}$$

In the last step we used Lemma 4.5.10.

We conclude by observing that γ is determined by its trace $2\lambda = a+d$, the numbers c, $a-d$, and the condition $(a-d)^2+4bc=0$. □

Remark 4.5.12. *In the notation of [20] we have*

$$M_0(L,j,\boldsymbol{\delta}) = \sum_{\alpha,\beta\in\mathcal{P}(L)} \sharp\Gamma_1^0(i,\alpha^j\beta^j,\boldsymbol{\delta}) \ll |\hat{\boldsymbol{\delta}}|_\infty^\epsilon L^{2+\epsilon}(1+L^j |y|_\infty |\boldsymbol{\delta}|_\infty^{\frac{1}{2}}) \tag{4.5.15}$$

as well as

$$M_2(L,j,\boldsymbol{\delta}) = \sum_{\alpha,\beta\in\mathcal{P}(L)} \sharp\Gamma_1^{par}(i,\alpha^j\beta^j,\boldsymbol{\delta}) \ll \frac{L^{3j+\epsilon} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{3}{4}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{4}} |\hat{\boldsymbol{\delta}}|_{\mathbb{R}}^{\frac{1}{8}+\epsilon}}{\mathcal{N}(\mathfrak{n}_2)}. \tag{4.5.16}$$

This shows that by summing up the individual bounds given here we recover the exact statements from [20, Lemma 12, Lemma 14]. However, in order to bound the number of generic matrices one uses a ingenious lattice point counting trick which is more effective when bundling matrices with comparable determinant together.

We define

$$\begin{aligned} M_{j,i}^{gen}(L,\delta) &= \sum_{L<\mathcal{N}(\alpha)\leq 2L} \sharp\Gamma_j^{gen}(i,\alpha,\delta) \text{ and} \\ M_{j,i}^{gen,\square}(L,\delta) &= \sum_{\substack{L<\mathcal{N}(\alpha)\leq 2L,\\ \alpha \text{ square}}} \sharp\Gamma_j^{gen}(i,\alpha,\delta).\end{aligned}$$

In the notation of [20] this corresponds to

$$M_3(L,1,\delta) \leq M_{1,i}^{gen}(L^2,\delta) + M_{1,i}^{gen}(2L^2,\delta) \text{ and}$$
$$M_3(L,2,\delta) \leq M_{1,i}^{gen,\square}(L^4,\delta) + M_{1,i}^{gen,\square}(2L^4,\delta) + M_{1,i}^{gen,\square}(4L^4,\delta) + M_{1,i}^{gen,\square}(8L^4,\delta).$$

We have the following adaption of [20, Lemma 15] to our setting.

Lemma 4.5.13. *We have*

$$M_{j,i}^{gen}(L,\boldsymbol{\delta}) \ll |\hat{\boldsymbol{\delta}}|_\infty \left(L + \frac{L^{\frac{5}{4}} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{4}}} + \frac{L^2 |\boldsymbol{\delta}|_{\mathbb{R}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{3}{4}}}{\mathcal{N}(\mathfrak{n}_2)} \right) \text{ and}$$
$$M_{j,i}^{gen,\square}(L,\boldsymbol{\delta}) \ll |\hat{\boldsymbol{\delta}}|_\infty^{\frac{1}{2}+\epsilon} L^\epsilon \left(L^{\frac{1}{2}} + \frac{L |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{1}{2}}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}}} + \frac{L^{\frac{3}{2}} |\boldsymbol{\delta}|_{\mathbb{R}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{2}}}{\mathcal{N}(\mathfrak{n}_2)} \right).$$

In particular, we obtain the bounds

$$M_3(L,1,\boldsymbol{\delta}) \ll |\hat{\boldsymbol{\delta}}|_\infty \left(L^2 + \frac{L^{\frac{5}{2}} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{4}}} + \frac{L^4 |\boldsymbol{\delta}|_{\mathbb{R}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{3}{4}}}{\mathcal{N}(\mathfrak{n}_2)} \right) \text{ and}$$
$$M_3(L,2,\boldsymbol{\delta}) \ll |\hat{\boldsymbol{\delta}}|_\infty^{\frac{1}{2}+\epsilon} L^\epsilon \left(L^2 + \frac{L^4 |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{1}{2}}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}}} + \frac{L^6 |\boldsymbol{\delta}|_{\mathbb{R}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{2}}}{\mathcal{N}(\mathfrak{n}_2)} \right).$$

Proof. The number of possible values for c that can contribute to $M_{j,i}^{gen}(L,\boldsymbol{\delta})$ is bounded by

$$\sharp\{c\} \ll \frac{L^{\frac{1}{2}} |\hat{\boldsymbol{\delta}}|_\infty^{\frac{1}{2}}}{|y|_\infty \mathcal{N}(\mathfrak{n}_2)}.$$

Let $M_{j,i}^{gen}(L,\boldsymbol{\delta},c)$ denote the sub-count of $M_{j,i}^{gen}(L,\boldsymbol{\delta})$ which counts only matrices with given c as lower left entry. We further split

$$M_{j,i}^{gen}(L,\boldsymbol{\delta},c) = \sum_{\mathbf{n}} M^*(\mathbf{n}).$$

Here $*$ is an abbreviation for the fixed quintuple $(j,i,L,\boldsymbol{\delta},c)$ and $\mathbf{n} = (n_\nu)_{\nu\in S_\mathbb{C}}$ such that $0 \leq n_\nu \leq \frac{2\pi}{\sqrt{\delta_\nu}}$ and $M^*(\mathbf{n})$ is counting only those matrices γ satisfying

$$n_\nu\sqrt{\delta_\nu} \leq \arg(\det(\gamma_\nu)) < (n_\nu+1)\sqrt{\delta_\nu},$$

for all complex places ν. Without loss of generality we can assume that $M^*(\mathbf{n}) \neq 0$ and fix an element $\gamma_{\mathbf{n}} = \begin{pmatrix} a_{\mathbf{n}} & b_{\mathbf{n}} \\ c & d_{\mathbf{n}} \end{pmatrix}$ contributing to this count. Every other matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which is counted this way, is uniquely determined by the differences

$$a' = a - a_{\mathbf{n}}, \quad b' = b - b_{\mathbf{n}} \text{ and } d' = d - d_{\mathbf{n}}.$$

By construction the determinants $l = \det(\gamma)$ and $l_{\mathbf{n}} = \det(\gamma_{\mathbf{n}})$ satisfy

$$\frac{l_\nu}{|l_\nu|} - \frac{l_{\mathbf{n},\nu}}{|l_{\mathbf{n},\nu}|} \ll \begin{cases} \sqrt{\delta_\nu} & \text{if } \nu \in S_{\mathbb{C}}, \\ 0 & \text{if } \nu \in S_{\mathbb{R}} \end{cases} \text{ and } \frac{\sqrt{l_\nu}}{\sqrt{l_{\mathbf{n},\nu}}} = \left| \frac{l_\nu}{l_{\mathbf{n},\nu}} \right|^{\frac{1}{2}} + O(\sqrt{\delta_\nu})$$

at all places $\nu \mid \infty$ for a suitably chosen branch of the square root. Vanishing at the real places follows from the fact that the determinant is totally positive. Furthermore, since $l, l_{\mathbf{n}}$ and y are assumed to be balanced, we can apply (4.5.8) and (4.5.11) to find

$$(a'_\nu - d'_\nu)x_\nu + b'_\nu \ll L^{\frac{1}{2n}} |y|_\infty^{\frac{1}{n}} \sqrt{\delta_\nu}.$$

According to (4.5.13) and (4.5.14) we have

$$\Re\left(\frac{a'_\nu - d'_\nu}{\sqrt{l_{\mathbf{n},\nu}}}\right), \Im\left(\frac{a'_\nu + d'_\nu}{\sqrt{l_{\mathbf{n},\nu}}}\right) \ll \sqrt{\delta_\nu}.$$

On a smaller note (4.5.9) and (4.5.10) imply

$$\Im\left(\frac{a'_\nu - d'_\nu}{\sqrt{l_{\mathbf{n},\nu}}}\right), \Re\left(\frac{a'_\nu + d'_\nu}{\sqrt{l_{\mathbf{n},\nu}}}\right) \ll \sqrt{\hat{\delta}_\nu}.$$

Next, we decompose

$$M^*(\mathbf{n}) = \sum_{\mathbf{p},\mathbf{q}} M^*(\mathbf{n},\mathbf{p},\mathbf{q}).$$

Here $M^*(\mathbf{n},\mathbf{p},\mathbf{q})$ is the sub-count of $M^*(\mathbf{n})$ counting only elements satisfying

$$p_\nu \sqrt{\delta_\nu} \leq \Im\left(\frac{a'_\nu - d'_\nu}{\sqrt{l_{\mathbf{n},\nu}}}\right) < (p_\nu + 1)\sqrt{\delta_\nu},$$
$$q_\nu \sqrt{\delta_\nu} \leq \Re\left(\frac{a'_\nu + d'_\nu}{\sqrt{l_{\mathbf{n},\nu}}}\right) < (q_\nu + 1)\sqrt{\delta_\nu},$$

for all complex places ν. In particular, the bounds above imply that

$$M^* = \sum_{\substack{\mathbf{n},\mathbf{p},\mathbf{q} \in \mathbb{Z}^{r_2}, \\ n_\nu, p_\nu, q_\nu \ll \sqrt{\delta_\nu}}} M^*(\mathbf{n},\mathbf{p},\mathbf{q}) \tag{4.5.17}$$

Without loss of generality we fix any element $\gamma_{\mathbf{n},\mathbf{p},\mathbf{q}}$ contributing to $M^*(\mathbf{n},\mathbf{p},\mathbf{q})$. Any other matrix γ counted by $M^*(\mathbf{n},\mathbf{p},\mathbf{q})$ is determined by the numbers

$$\tilde{a} = s(a - a_{\mathbf{n},\mathbf{p},\mathbf{q}}), \quad \tilde{b} = s(b - b_{\mathbf{n},\mathbf{p},\mathbf{q}}), \text{ and } \tilde{d} = s(d - d_{\mathbf{n},\mathbf{p},\mathbf{q}}), \tag{4.5.18}$$

where s is a unit satisfying $s_\nu\delta_\nu \asymp |\boldsymbol{\delta}|_\infty^{\frac{1}{2n}}$. By constructions and the same arguments used for estimating expressions involving a', b', d' we obtain

$$(\tilde{a}_\nu - \tilde{d}_\nu)x_\nu + \tilde{b}_\nu \ll L^{\frac{1}{2n}} |y|_\infty^{\frac{1}{n}} |\boldsymbol{\delta}|_\infty^{\frac{1}{2n}},$$

$$\tilde{a}_\nu - \tilde{d}_\nu \ll L^{\frac{1}{2n}} |\boldsymbol{\delta}|_\infty^{\frac{1}{2n}} \text{ and } \tilde{a}_\nu + \tilde{d}_\nu \ll L^{\frac{1}{2n}} s_\nu \cdot \begin{cases} \hat{\delta}_\nu^{\frac{1}{2}} & \text{if } \nu \in S_\mathbb{R}, \\ \delta_\nu^{\frac{1}{2}} & \text{if } \nu \in S_\mathbb{C}. \end{cases}$$

These two bounds combined yield the following key inequality:

$$\|(\tilde{a} - \tilde{d})P + \tilde{b}\| \ll L^{\frac{1}{2n}} |y|_\infty^{\frac{1}{n}} |\boldsymbol{\delta}|_\infty^{\frac{1}{2n}}.$$

From the lattice counting result given in [20, Lemma 6, Part (d)] we deduce that

$$\sharp\{(\tilde{a} - \tilde{d}, \tilde{b})\} \ll 1 + L^{\frac{1}{2}} |y|_\infty |\boldsymbol{\delta}|_\infty^{\frac{1}{2}} \mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}} + L |y|_\infty |\boldsymbol{\delta}|_\infty.$$

Furthermore, we observe that

$$\sharp\{(\tilde{a} + \tilde{d})\} \ll 1 + L^{\frac{1}{2}} |\boldsymbol{\delta}|_\mathbb{C}^{\frac{1}{2}} |\hat{\boldsymbol{\delta}}|_\mathbb{R}^{\frac{1}{2}}.$$

We conclude that

$$M^*(\mathbf{n}, \mathbf{p}, \mathbf{q}) \ll \left(1 + L^{\frac{1}{2}} |\boldsymbol{\delta}|_\mathbb{C}^{\frac{1}{2}} |\hat{\boldsymbol{\delta}}|_\mathbb{R}^{\frac{1}{2}}\right)\left(1 + L^{\frac{1}{2}} |y|_\infty |\boldsymbol{\delta}|_\infty^{\frac{1}{2}} \mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}} + L |y|_\infty |\boldsymbol{\delta}|_\infty\right).$$

In order to finish part one of the proof we consider two cases. First, if $|\boldsymbol{\delta}|_\mathbb{C} > L^{-1} |\boldsymbol{\delta}|_\mathbb{R}^{-1} \mathcal{N}(\mathfrak{n}_2)$, then

$$L^{\frac{1}{2}} |\boldsymbol{\delta}|_\mathbb{C}^{\frac{1}{2}} |\hat{\boldsymbol{\delta}}|_\mathbb{R}^{\frac{1}{2}} \gg 1.$$

We obtain

$$\begin{aligned} M_{j,i}^{gen}(L, \boldsymbol{\delta}) &\ll |\hat{\boldsymbol{\delta}}|_\infty \frac{L^{\frac{1}{2}}}{|y|_\infty \mathcal{N}(\mathfrak{n}_2)} \cdot \frac{L^{\frac{1}{2}} |\boldsymbol{\delta}|_\mathbb{C}^{\frac{1}{2}}}{|\boldsymbol{\delta}|_\mathbb{C}^{\frac{3}{4}}} \cdot \left(1 + L^{\frac{1}{2}} |y|_\infty |\boldsymbol{\delta}|_\infty^{\frac{1}{2}} \mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}} + L |y|_\infty |\boldsymbol{\delta}|_\infty\right) \\ &\ll |\hat{\boldsymbol{\delta}}|_\infty \left(\frac{L}{|y| \mathcal{N}(\mathfrak{n}_2) |\boldsymbol{\delta}|_\mathbb{C}^{\frac{1}{4}}} + \frac{L^{\frac{3}{2}} |\boldsymbol{\delta}|_\mathbb{R}^{\frac{1}{2}} |\boldsymbol{\delta}|_\mathbb{C}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}}} + \frac{L^2 |\boldsymbol{\delta}|_\mathbb{R} |\boldsymbol{\delta}|_\mathbb{C}^{\frac{3}{4}}}{\mathcal{N}(\mathfrak{n}_2)}\right) \\ &\ll |\hat{\boldsymbol{\delta}}|_\infty \left(\frac{L^{\frac{5}{4}} |\boldsymbol{\delta}|_\mathbb{R}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{4}}} + \frac{L^2 |\boldsymbol{\delta}|_\mathbb{R} |\boldsymbol{\delta}|_\mathbb{C}^{\frac{3}{4}}}{\mathcal{N}(\mathfrak{n}_2)}\right). \end{aligned}$$

Second, if $|\boldsymbol{\delta}|_\mathbb{C} \leq L^{-1} |\boldsymbol{\delta}|_\mathbb{R}^{-1} \mathcal{N}(\mathfrak{n}_2)$, we define $\tilde{\boldsymbol{\delta}} = (\tilde{\delta}_\nu)_\nu$ by assuming $|\tilde{\boldsymbol{\delta}}|_\mathbb{R} = |\boldsymbol{\delta}|_\mathbb{R}$ and $|\tilde{\boldsymbol{\delta}}|_\mathbb{C} = \min(16^{r_2}, L^{-1} |\boldsymbol{\delta}|_\mathbb{R}^{-1} \mathcal{N}(\mathfrak{n}_2))$ as well as $\delta_\nu \leq \tilde{\delta}_\nu$. Arguing as before with $\tilde{\boldsymbol{\delta}}$ in place of $\boldsymbol{\delta}$ we find

$$M_{i,j}^{gen}(L, \boldsymbol{\delta}) \leq M_{i,j}^{gen}(L, \tilde{\boldsymbol{\delta}}) \ll |\hat{\boldsymbol{\delta}}|_\infty \left(\frac{L}{|\tilde{\boldsymbol{\delta}}|_\mathbb{C}^{\frac{1}{4}}} + \frac{L^2 |\boldsymbol{\delta}|_\mathbb{R} |\tilde{\boldsymbol{\delta}}|_\mathbb{C}^{\frac{3}{4}}}{\mathcal{N}(\mathfrak{n}_2)}\right) \ll |\hat{\boldsymbol{\delta}}|_\infty \left(L + \frac{L^{\frac{5}{4}} |\boldsymbol{\delta}|_\mathbb{R}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{4}}}\right).$$

This completes the first part of the proof.

We now turn to estimating $M_{j,i}^{gen,\square}(L,\boldsymbol{\delta})$. To this end we recall that every γ, which contributes to the counting satisfies $\det(\gamma)=\lambda^2$ for some $\lambda\in\mathcal{O}_F$. This reveals

$$0\neq(a-d)^2+4bc=(a+d)^2-4\lambda^2=(a+d-2\lambda)(a+d+2\lambda).$$

Thus each tuple $(a-d,b)$ gives rise to $\ll (\left|\hat{\boldsymbol{\delta}}\right|_\infty L)^\epsilon$ possibilities for $a+d$. Therefore we can drop the extra sub-count coming from $\mathbf{q}$ in the argument above. Making the necessary modifications yields

$$\begin{aligned} M_{j,i}^{gen,\square}(L,\boldsymbol{\delta}) &= \sum_{\mathbf{n},\mathbf{p}} M^*(\mathbf{n},\mathbf{p}) \\ &\ll \left|\hat{\boldsymbol{\delta}}\right|_\infty^{\frac{1}{2}+\epsilon} L^\epsilon \frac{L^{\frac{1}{2}}}{|y|_\infty \mathcal{N}(\mathfrak{n}_2)} \cdot |\boldsymbol{\delta}|_{\mathbb{C}}^{-\frac{1}{2}} \cdot (1+L^{\frac{1}{2}}|y|_\infty |\boldsymbol{\delta}|_\infty^{\frac{1}{2}} \mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}} + L|y|_\infty |\boldsymbol{\delta}|_\infty) \\ &\ll \left|\hat{\boldsymbol{\delta}}\right|_\infty^{\frac{1}{2}+\epsilon} L^\epsilon \left(\frac{L^{\frac{1}{2}}}{|\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{2}}} + \frac{L^{\frac{3}{2}} |\boldsymbol{\delta}|_{\mathbb{R}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{2}}}{\mathcal{N}(\mathfrak{n}_2)} \right). \end{aligned}$$

To conclude the proof from here one argues as before. □

Remark 4.5.14. *Adding all the corresponding contributions given above together establishes the useful bound*

$$\sharp\Gamma_j(i,1,\boldsymbol{\delta}) \leq \sharp\Gamma_j^0(i,\alpha,\boldsymbol{\delta}) + \sharp\Gamma_j^{par}(i,\alpha,\boldsymbol{\delta}) + M_{j,i}^{gen,\square}(1,\boldsymbol{\delta}) \ll \left|\hat{\boldsymbol{\delta}}\right|_\infty (1+|y|_\infty |\boldsymbol{\delta}|_\infty^{\frac{1}{2}}).$$

This is a good reality check. In particular, we recover

$$M(L,0,\boldsymbol{\delta}) \ll \left|\hat{\boldsymbol{\delta}}\right|_\infty (1+|y|_\infty |\boldsymbol{\delta}|_\infty^{\frac{1}{2}}),$$

which is the content of [20, Lemma 11]. Following the proof of [20, Lemma 11] directly yields the useful preliminary estimate

$$\sharp\Gamma_j(i,l,\boldsymbol{\delta}) \ll \mathcal{N}(l) \left|\hat{\boldsymbol{\delta}}\right|_\infty (1+|y|_\infty \mathcal{N}(l)^{\frac{1}{2}} |\boldsymbol{\delta}|_\infty^{\frac{1}{2}}).$$

Finally, we recall two more counting results without repeating a proof.

Lemma 4.5.15 ([20], Lemma 16 and Lemma 17)**.** *Let F_0 be the maximal totally real subfield of F. Suppose $\delta_\nu \ll 1$ for all ν, and that $m=[F\colon F_0]\geq 2$. Then we have*

$$M_3(L,1,\boldsymbol{\delta}) \ll L^2 + L^{2m} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{1}{2}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{4}} + \frac{L^{2m+1} |\boldsymbol{\delta}|_{\mathbb{R}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{3}{4}}}{\mathcal{N}(\mathfrak{n}_2)}.$$

Furthermore, $M_3(L,2,\boldsymbol{\delta})=0$ unless

$$1 \ll L^{8(m-1)} |\boldsymbol{\delta}|_{\mathbb{C}}.$$

4.6 ESTIMATES VIA THE WHITTAKER EXPANSION

In this section we consider the Whittaker expansion of cusp forms as in [2, Section 3]. This will provide us with the first upper bounds for the newform $\phi_\circ$. The main result is Proposition 4.6.12 below.

Throughout this section let (π, V_π) be a cuspidal automorphic representation with new vector $v^\circ \in V_\pi$ and associated newform $\phi_\circ = \sigma(v^\circ)$. Without loss of generality we assume that $\phi_\circ$ is L^2-normalised. Further, we fix $g \in \mathcal{J}_\mathfrak{n}$ and $n(x)a(y) \in \mathcal{F}_{\mathfrak{n}_2}$.

4.6.1 *The Whittaker expansion of cusp forms*

Let ψ be the standard additive character of $\mathbb{A}_F$ as defined in (1.3.2). Recall the factorisation (1.3.3). In particular, the conductor of ψ is $\mathfrak{d}^{-1}$.

Having fixed the additive character we define the corresponding global Whittaker function

$$W_{\phi_\circ}(g) = \frac{2^{r_2}}{\sqrt{d_F}} \int_{F\backslash \mathbb{A}_F} \phi_\circ(n(x)g)\psi(-x)d\mu_{\mathbb{A}_F}(x).$$

We want to factor this global function into a product of local functions each of which matches the ones studied in Part ii. To achieve this we have to deal with several technicalities. First, if $\omega_{\pi,\mathfrak{p}}(\varpi_\mathfrak{p}) = |\varpi_\mathfrak{p}|_\mathfrak{p}^{ia_\mathfrak{p}}$, we define $\pi'_\mathfrak{p} = |\cdot|_\mathfrak{p}^{i\frac{a_\mathfrak{p}}{2}} \pi_\mathfrak{p}$. The purpose of this twist is that the central character $\omega'_{\pi_\mathfrak{p}}$ of $\pi'_\mathfrak{p}$ is trivial on the uniformiser. Second, we have to keep in mind that the local constitutes of ψ do not always coincide with the fixed unramified additive characters $\psi_\mathfrak{p}$ and ψ_ν.

Let $W_\mathfrak{p}$ be the Whittaker new vector associated to the representation $\pi'_\mathfrak{p}$ with respect to the character $\psi_\mathfrak{p}$ normalised by $W_\mathfrak{p}(1) = 1$. At infinity we take the local Whittaker function W_ν to be the Whittaker vector associated to v_ν° normalised by $\langle W_\nu, W_\nu \rangle = 1$. This matches the situation in [20] as well as the set-up in Part ii. Having defined these local functions we achieve the factorisation

$$W_{\phi_\circ}(g) = c_{\phi_\circ} \underbrace{\prod_\nu W_\nu(g_\nu)}_{=W_\infty(g_\infty)} \prod_\mathfrak{p} |\det(g_\mathfrak{p})|_\mathfrak{p}^{-i\frac{a_\mathfrak{p}}{2}} W_\mathfrak{p}(a(\varpi_\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{d})})g_\mathfrak{p}).$$

The translation in the finite part comes from the shift in the local additive characters, see (1.3.3). The constant $c_{\phi_\circ}$ arises through the re-normalisation of the local functions.

For $1 \leq i \leq h_F$ and $g \in \mathcal{J}_{\mathfrak{n}}$ we have the well known Whittaker expansion

$$\phi_\circ(a(\theta_i)gn(x)a(y)) \\ = c_{\phi_\circ} \sum_{q\in F^\times} \prod_{\mathfrak{p}} |q\theta_i \det(g_{\mathfrak{p}})|_{\mathfrak{p}}^{-i\frac{a_{\mathfrak{p}}}{2}} W_{\mathfrak{p}}(a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{d})}\theta_i q)g_{\mathfrak{p}})W_\infty(a(q)n(x)a(y)).$$

For convenience we split the local terms in the archimedean part W_∞, the unramified part

$$\lambda_{ur}(q) = \prod_{\mathfrak{p}\nmid\mathfrak{n}} W_{\mathfrak{p}}(a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{d})}\theta_i q)),$$

and the ramified part

$$\lambda_{\mathfrak{n}}(q) = \prod_{\mathfrak{p}|\mathfrak{n}} W_{\mathfrak{p}}(a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{d})}\theta_i q)g_{\mathfrak{p}}).$$

We also collect all the unramified twists together and write $\eta(q) = \prod_{\mathfrak{p}} |q\theta_i \det(g_{\mathfrak{p}})|_{\mathfrak{p}}^{-i\frac{a_{\mathfrak{p}}}{2}}$. Since $|\eta| = 1$ this factor does not influence any of the upcoming estimates.

Let us continue by gathering some properties of $\lambda_{\mathfrak{n}}$ and λ_{ur}. First, we recall the following standard result.

Lemma 4.6.1 ([2], Lemma 3.1). *If $\mathfrak{p} \nmid \mathfrak{n}$, then there are unramified characters $\chi_{1,\mathfrak{p}}$ and $\chi_{2,\mathfrak{p}}$ such that $\pi'_{\mathfrak{p}} = \chi_{1,\mathfrak{p}} \boxplus \chi_{2,\mathfrak{p}}$. In this case we have*

$$W_{\mathfrak{p}}(a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{d})}\theta_i q)) = \begin{cases} 0 & \text{if } v_{\mathfrak{p}}(\theta_i q) + v_{\mathfrak{p}}(\mathfrak{d}) < 0, \\ q_{\mathfrak{p}}^{-(v_{\mathfrak{p}}(\theta_i q)+v_{\mathfrak{p}}(\mathfrak{d}))/2} \frac{\chi_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}})^{v_{\mathfrak{p}}(\theta_i q)+v_{\mathfrak{p}}(\mathfrak{d})+1} - \chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})^{v_{\mathfrak{p}}(\theta_i q)+v_{\mathfrak{p}}(\mathfrak{d})+1}}{\chi_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}}) - \chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})} & \\ \qquad \text{if } v_{\mathfrak{p}}(\theta_i q) + v_{\mathfrak{p}}(\mathfrak{d}) \geq 0. \end{cases}$$

Proof. This follows from [25, Theorem 4.6.4] and [25, Theorem 4.6.5]. □

In particular we find the following support properties of the unramified coefficients.

Corollary 4.6.2 ([2], Corollary 3.1). *If $\lambda_{ur}(q) \neq 0$, then $v_{\mathfrak{p}}(q) \geq -v_{\mathfrak{p}}(\mathfrak{d}) - v_{\mathfrak{p}}(\theta_i)$ for all $\mathfrak{p} \nmid \mathfrak{n}$.*

We can go even further and describe the unramified coefficients in terms the Hecke eigenvalues. To this end we define

$$X_{\mathfrak{p},k} = \{m \in \mathrm{Mat}_2(\mathfrak{o}_{\mathfrak{p}})\colon v_{\mathfrak{p}}(\det(m)) = k\},$$

for $\mathfrak{p} \nmid \mathfrak{n}$ and $k \in \mathbb{N}$. The local new vector $v_{\mathfrak{p}}^\circ$ is an eigenvector of the operator $\pi_{\mathfrak{p}}(\mathbb{1}_{X_{\mathfrak{p},k}})$ and we denote its eigenvalue by $\lambda(\mathfrak{p}^k)$. For any ideal $\mathfrak{a}$ co-prime to $\mathfrak{n}$ we define the global Hecke operator by $T(\mathfrak{a}) = \prod_{\mathfrak{p}|\mathfrak{a}} \pi_{\mathfrak{p}}(\mathbb{1}_{X_{\mathfrak{p},v_{\mathfrak{p}}(\mathfrak{a})}})$. It is clear that the global new vector v° and therefore also the newform $\phi_\circ$, is an eigenvector of this operator with eigenvalue $\lambda(\mathfrak{a}) = \prod_{\mathfrak{p}|\mathfrak{a}} \lambda(\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{a})})$. We can now make a connection between λ_{ur} and the Hecke eigenvalues $\lambda(\cdot)$. It is important to notice that we follow the normalisation of [25, Section 4.6] which differs from the one used in [70] and [20].

Lemma 4.6.3 ([2], Lemma 3.2). *We have*

$$\lambda_{ur}(q) = \frac{\lambda\left(\frac{(q)\theta_i\mathfrak{d}}{[(q)\theta_i\mathfrak{d}]_\mathfrak{n}}\right)}{\mathcal{N}\left(\frac{(q)\theta_i\mathfrak{d}}{[(q)\theta_i\mathfrak{d}]_\mathfrak{n}}\right)}.$$

Proof. The proof proceeds locally by showing

$$\lambda(\mathfrak{p}^k) = q_\mathfrak{p}^k W_\mathfrak{p}(a(\varpi^k)) \text{ for } \mathfrak{p} \nmid \mathfrak{n}.$$

This can be done by induction using [25, Proposition 4.6.4, Proposition 4.6.6] and Lemma 4.6.1. □

Next we turn towards the ramified components $\lambda_\mathfrak{n}$.

Lemma 4.6.4 ([2], Lemma 3.3). *If $\lambda_\mathfrak{n}(q) \neq 0$, then $v_\mathfrak{p}(q) \geq -v_\mathfrak{p}(\theta_i) - v_\mathfrak{p}(\mathfrak{d}) - n_{0,\mathfrak{p}} - m_{1,\mathfrak{p}}(g_p)$ for all $\mathfrak{p} \mid \mathfrak{n}$.*

Although notation differs this is essentially [70, Lemma 3.11].

Proof. Since $g \in \mathcal{J}_\mathfrak{n}$ we have $g_p \in K_\mathfrak{p} a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}})$ and $n_{1,\mathfrak{p}}(g_\mathfrak{p}) = n_{0,\mathfrak{p}}$. But $W_\mathfrak{p}(a(\varpi_\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{d})}\theta_i q)g_\mathfrak{p}) \neq 0$ so that [70, Proposition 2.11,(1)] implies[1]

$$v_\mathfrak{p}(\theta_i q) + v_\mathfrak{p}(\mathfrak{d}) \geq -n_{1,\mathfrak{p}}(g_p) - m_{1,\mathfrak{p}}(g_p).$$

Note that we used Corollary 4.4.4 to include $a(v')$ into $g_\mathfrak{p}$ for $v' \in \mathfrak{o}_\mathfrak{p}^\times$ where $\theta_i q = v'\varpi_\mathfrak{p}^{v_\mathfrak{p}(\theta_i q)}$. □

Later on it will make sense to view $\lambda_\mathfrak{n}$ as a locally constant function on the adéles in an obvious way. It will then be crucial to determine sets on which this function is constant.

Lemma 4.6.5 ([2], Lemma 3.4). *Let $\mathfrak{p}|\mathfrak{n}$ and $u_1, u_2 \in \mathfrak{o}_\mathfrak{p}^\times$ such that $u_1 - u_2 \in \varpi_\mathfrak{p}^{n_{0,\mathfrak{p}}(g_\mathfrak{p})}\mathfrak{o}_\mathfrak{p}$. Then*

$$\left|W_{\pi_\mathfrak{p}}(a(\varpi_\mathfrak{p}^k u_1)g_\mathfrak{p})\right| = \left|W_{\pi_\mathfrak{p}}(a(\varpi_\mathfrak{p}^k u_2)g_\mathfrak{p})\right|.$$

This is essentially [70, Lemma 3.12].

1 Note that in the notation of [70] we have $q(g_\mathfrak{p}) = n_{0,\mathfrak{p}} + m_{1,\mathfrak{p}}(g_\mathfrak{p})$.

Proof. The proof of this little lemma goes back to the decomposition (1.3.4) and the fact that $|W_{\pi_{\mathfrak{p}}}|$ is well defined by its values on $g_{t,l,v}$.

First, let us write

$$g_{\mathfrak{p}} = zng_{t,l,v}k.$$

One observes that

$$a(\varpi_{\mathfrak{p}}^{k}u_1)g_{\mathfrak{p}} = zn'g_{t+k,l,vu_1^{-1}}k'.$$

By doing the same for u_2 we observe, that the claimed equality follows when

$$[vu^{-1}] = [vu_2^{-1}] \in \mathfrak{o}_{\mathfrak{p}}^{\times}/(1+\varpi_{\mathfrak{p}}^{n_0(g_{\mathfrak{p}})}\mathfrak{o}_{\mathfrak{p}}).$$

The last condition leads to $u_1 - u_2 \in \varpi_{\mathfrak{p}}^{n_0(g_{\mathfrak{p}})}\mathfrak{o}_{\mathfrak{p}}$. □

Combining the support properties from Lemma 4.6.4 and Corollary 4.6.2 we derive

$$|\phi_{\circ}(a(\theta_i)gn(x)a(y))| \leq |c_{\phi_{\circ}}| \sum_{q\in\imath^{-1}} |\lambda_{ur}(q)\lambda_{\mathfrak{n}}(q)W_{\infty}(a(qy))|. \tag{4.6.1}$$

Here

$$\imath = \mathfrak{n}_0\mathfrak{m}_1(g)\mathfrak{d}\prod_{\mathfrak{p}}\mathfrak{p}^{v_{\mathfrak{p}}(\theta_i)} \text{ and } \mathfrak{m}_1(g) = \prod_{\mathfrak{p}}\mathfrak{p}^{m_{1,\mathfrak{p}}(g_{\mathfrak{p}})}. \tag{4.6.2}$$

It is easy to deal with the constant $c_{\phi_{\circ}}$.

Lemma 4.6.6 ([2], Lemma 3.5)**.** *We have*

$$c_{\phi_{\circ}} \ll_{F,\epsilon} \left(\mathcal{N}(\mathfrak{n})\,|T|_{\infty}\right)^{\epsilon}.$$

Proof. As in [69] we observe

$$c_{\phi_{\circ}}^2 \ll_F L^{-1}(1,\pi,Ad)^{-1}\prod_{\nu}\langle W_{\nu},W_{\nu}\rangle^{-1} = L(1,\pi,Ad)^{-1}.$$

It is a well known fact that $L(1,\pi,Ad) \gg (\mathcal{N}(\mathfrak{n})\,|T|_{\infty})^{\epsilon}$. Thus

$$c_{\phi_{\circ}} \ll (\mathcal{N}(\mathfrak{n})\,|T|_{\infty})^{\epsilon}.$$

□

Before continuing we fix a parameter $R = (R_{\nu})_{\nu}$ and define the box

$$B(R) = \prod_{\nu}\{\xi_{\nu} \in F_{\nu}\colon\ |\xi_{\nu}| \leq R_{\nu}\}.$$

This box will be used to truncate the Whittaker expansion. Mostly we will work with fixed parameters R depending on y and the spectral parameters of $\phi_\circ$. However, if not otherwise stated we allow for more flexibility.

Applying the Hölder inequality together with $1 = |q|_{\mathbb{A}_F} = |q|_{fin}\,|q|_\infty$ yields

$$|\phi_\circ(a(\theta_i)gn(x)a(y))| \leq |c_{\phi_\circ}| \underbrace{\left(\sum_{q\in\imath^{-1}\cap B(R)} |q|_\infty^{-2}\,|W_\infty(a(qy))|^4 \right)^{\frac{1}{4}}}_{=S_1(R)} \cdot \underbrace{\left(\sum_{q\in\imath^{-1}\cap B(R)} \mathcal{N}(q)^{\frac{2}{3}}\,|\lambda_{ur}(q)\lambda_{\mathfrak{n}}(q)|^{\frac{4}{3}} \right)^{\frac{3}{4}}}_{=S_2(R)} + |c_{\phi_\circ}|\,\mathcal{E}. \tag{4.6.3}$$

with

$$\begin{aligned} S_1(R) &= \left(\sum_{q\in\imath^{-1}\cap B(R)} |q|_\infty^{-2}\,|W_\infty(a(qy))|^4 \right)^{\frac{1}{4}}, \\ S_2(R) &= \left(\sum_{q\in\imath^{-1}\cap B(R)} \mathcal{N}(q)^{\frac{2}{3}}\,|\lambda_{ur}(q)\lambda_{\mathfrak{n}}(q)|^{\frac{4}{3}} \right)^{\frac{3}{4}}, \text{ and} \\ \mathcal{E} &= \sum_{q\in\imath^{-1},q\notin B(R)} |\lambda_{ur}(q)\lambda_{\mathfrak{n}}(q)W_\infty(qy)|\,. \end{aligned}$$

We will estimate each one of these three quantities in the upcoming subsections.

4.6.2 *The sum $S_1(R)$*

In this section we will treat the sum $S_1(R)$. Before we start let us record some explicit expressions for the functions W_ν. The following is taken from Section 2.2, in particular Lemma 2.2.2 and Lemma 2.3.4.

If $\nu \in S_{sph} \cap S_{\mathbb{R}}$ and $k_\nu = 0$, then we have

$$|W_\nu(a(\xi_\nu))| = \frac{\sqrt{2\,|\xi_\nu|_\nu}\,\left|K_{\frac{it_\nu}{2}}(2\pi\,|\xi_\nu|)\right|}{\sqrt{\pi}\,\left|\Gamma(\frac{1+it_\nu}{2})\right|}.$$

If $\nu \in S_{sph}$ and $k_\nu = 1$, then we obtain

$$|W_\nu(a(\xi_\nu))| = 2\,|\xi_\nu|_\nu \frac{\left|K_{\frac{it_\nu+1}{2}}(2\pi\,|\xi_\nu|) + \mathrm{sgn}(\xi_\nu)K_{\frac{it_\nu-1}{2}}(2\pi\,|\xi_\nu|)\right|}{\left|\Gamma(1+\frac{it_\nu}{2})\right|}.$$

If $\nu \in S_{hol}$, then

$$|W_\nu(a(\xi_\nu))| = \begin{cases} \frac{(4\pi)^{\frac{k-1}{2}}}{\sqrt{2}|\Gamma(k)|^{\frac{1}{2}}} \xi_\nu^{\frac{k}{2}} e^{-2\pi\xi_\nu} & \text{if } \xi_\nu > 0, \\ 0 & \text{else.} \end{cases} \tag{4.6.4}$$

Finally, if $\nu \in S_\mathbb{C}$, then

$$|W_\nu(a(\xi_\nu))| = \frac{\sqrt{8\,|\xi_\nu|_\nu}\,|K_{it_\nu}(4\pi\,|\xi_\nu|)|}{|\Gamma(1+it_\nu)|} \tag{4.6.5}$$

Note that up to an absolute constant, which arises through different measure normalisations, our expressions agree with those given in [20, p. 19]. For the cases when $k_\nu \neq 0$ one compares our results to the un-normalised expressions given in [69].

Due to the transition region of the archimedean Whittaker function this argument requires

$$R_\nu = \begin{cases} \frac{T_\nu + T_\nu^{\frac{1}{3}+\epsilon}}{2\pi|y_\nu|} \asymp \frac{T_\nu}{y_\nu} & \text{if } \nu \in S_{sph}, \\ \frac{k_\nu - 1}{4\pi y_\nu} \asymp \frac{k_\nu}{y_\nu} & \text{if } \nu \in S_{hol}. \end{cases} \tag{4.6.6}$$

Note that in view of Corollary 4.5.7 the sum S_1 is empty if $|R|_\infty < \mathcal{N}(\imath)^{-1}$. Therefore we assume

$$|R|_\infty \asymp \frac{|T|_{sph}\,|k|_{hol}}{|y|_\infty} \gg_F \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{-1}$$

throughout this section. Let us fix $a \in \imath$ as in (4.5.1). Further recall that

$$I_\nu(l_\nu) = \begin{cases} \left\{\xi_\nu \in F_\nu^\times : \xi_\nu \leq |a|\,R_\nu, -l_\nu \leq \left||\xi_\nu| - \frac{|a|T_\nu}{2\pi|y_\nu|}\right| < -l_\nu + 1\right\} & \text{if } \nu \in S_{sph}, \\ \{\xi_\nu \in F_\nu^\times : \} & \text{if } \nu \in S_{hol}. \end{cases}$$

for $l_\nu \leq 0$.

In course of the following estimate we need good estimates on the Whittaker functions.

Lemma 4.6.7. *For $\nu \in S_{sph}$ and $|\xi_\nu| \leq R_\nu$ we have*

$$W_\nu(a(\xi_\nu)) \ll \left|\frac{\xi_\nu}{T_\nu}\right|_\nu^{\frac{1}{2}} \min\left(T_\nu^{\frac{1}{6}}, T_\nu^{\frac{1}{4}}\,|2\pi\,|\xi_\nu| - T_\nu|^{-\frac{1}{4}}\right).$$

In particular, if $\xi_\nu \in I_\nu(l_\nu)$ we have

$$W_\nu(a(a^{-1}\xi_\nu y_\nu)) \ll \left|\frac{\xi_\nu}{aR_\nu}\right|^{\frac{1}{2}} g_\nu(l_\nu),$$

where

$$g_\nu(l_\nu) = \begin{cases} \min\left(T_\nu^{\frac{1}{6}}, \left|\frac{aT_\nu}{l_\nu y_\nu}\right|^{\frac{1}{4}}\right), & \text{if } \nu \in S_{sph}, \\ \frac{2^{\frac{k}{2}-1}}{|\Gamma(k-1)|^{\frac{1}{2}}}(2\pi\left|l_\nu y_\nu a^{-1}\right|)^{\frac{k-1}{2}} e^{-2\pi|l_\nu y_\nu a^{-1}|} & \text{if } \nu \in S_{hol}. \end{cases}$$

Whenever $k_\nu = 0$ this agrees with the bounds given in [20].

Proof. The proof for the cases $\nu \in S_{hol}$ is straight forward. In the remaining cases we combine explicit formulae for W_ν given above with the estimates for the K-Bessel functions given in Corollary 2.1.7. □

We are now ready to prove the following estimate.

Lemma 4.6.8 ([2], Lemma 3.11). *We have*

$$S_1 \ll_F |R|_\infty^{-\frac{1}{2}} \prod_{\nu\in S_{sph}} \left(|T_\nu|_\nu^{\frac{1}{6}} + |a|_\nu^{\frac{1}{4}} \left|\frac{T_\nu}{y_\nu}\right|_\nu^{\frac{1}{4}}\right)^{1+\epsilon} \prod_{\nu\in S_{hol}} \left(|k_\nu|_\nu^{\frac{1}{4}} + \left|\frac{a}{y_\nu}\right|_\nu^{\frac{1}{4}} |k_\nu|_\nu^{\frac{3}{8}}\right).$$

where R is fixed as specified above.

Proof. First, we shift the sum by a. This gives

$$S_1^4 = |a|_\infty^2 \sum_{q\in a\imath^{-1}\cap B(|a|R)} |q|_\infty^{-2} \left|W_\infty(a(qa^{-1}y))\right|^4.$$

Then we partition $B(|a|\, R)$ using the boxes defined in (4.5.2). In each box we exploit Lemma 4.6.7 to get

$$S_1^4 \ll |a|_\infty^2 \sum_{\substack{\mathbf{l}\in\mathbb{Z}^{\sharp\{\nu\}},\\ -\lfloor aR_\nu\rfloor\leq l_\nu\leq 0}} \sharp(I(\underline{\underline{k}}) \cap a\imath^{-1}) \prod_\nu \left|a^{-1}\right|_\nu^2 |R_\nu|_\nu^{-2} g_\nu(l_\nu)^4.$$

Inserting the result from Lemma 4.5.3 yields

$$S_1^4 \ll |R|_\infty^{-2} \prod_\nu \sum_{l_\nu=0}^{\lfloor|a|R_\nu\rfloor} g_\nu(-l_\nu)^4 f_\nu(-l_\nu).$$

To estimate the remaining sums we use ideas of [20]. We treat each place separately, starting with $\nu \in S_{sph}$ real. One obtains

$$\begin{aligned} \sum_{k_\nu=0}^{\lfloor|a|R_\nu\rfloor} g_\nu(-k_\nu)^4 f_\nu(-k_\nu) &= T_\nu^{\frac{2}{3}} + \sum_{k_\nu=1}^{\lfloor|a|R_\nu\rfloor} \frac{|a|\, T_\nu}{|y_\nu|\, k_\nu} \\ &\ll \left(|T_\nu|_\nu^{\frac{2}{3}} + \frac{|a|\, T_\nu}{|y_\nu|}\right)^{1+\epsilon}. \end{aligned}$$

Similarly one treats the complex places:

$$\begin{aligned}\sum_{k_\nu=0}^{\lfloor |a| R_\nu \rfloor} g_\nu(-k_\nu)^4 f_\nu(-k_\nu) &\leq T_\nu^{\frac{2}{3}} \left(|a| \frac{T_\nu}{|y_\nu|} + 1\right) + \sum_{k_\nu=1}^{\lfloor |a| R_\nu \rfloor} \left(|a| \frac{T_\nu}{|y_\nu|} + 1\right) \frac{|a| \, T_\nu}{|y_\nu| \, k_\nu} \\ &\ll \left(|a| \frac{T_\nu}{|y_\nu|} + 1\right) \left(T_\nu^{\frac{2}{3}} + |a| \frac{T_\nu}{|y_\nu|}\right)^{1+\epsilon} \ll \left(T_\nu^{\frac{2}{3}} + |a| \frac{T_\nu}{|y_\nu|}\right)^{2+\epsilon} \\ &\ll \left(T_\nu^{\frac{4}{3}} + |a|^2 \frac{T_\nu^2}{y_\nu^2}\right)^{1+\epsilon}.\end{aligned}$$

Finally, we deal with $\nu \in S_{hol}$ following [87]. We note that $g_\nu(l_\nu)$ is monotone increasing. Furthermore, it reaches its maximum at $l_\nu = aR_\nu$. Thus we can estimate

$$\begin{aligned}&\sum_{k_\nu=0}^{\lfloor |a| R_\nu \rfloor} g_\nu(-k_\nu)^4 f_\nu(-k_\nu) \\ &\ll \frac{2^{2k-4}}{\Gamma(k-1)^2} \left(\left| \frac{a}{2\pi y_\nu} \right|_\nu \int_0^\infty x^{2(k-1)} e^{-4x} dx + \left(\frac{k-1}{2}\right)^{2(k-1)} e^{-2(k-1)} \right) \\ &\ll \left| \frac{a}{y_\nu} \right|_\nu |k_\nu|_\nu^{\frac{3}{2}} + |k_\nu|_\nu .\end{aligned}$$

In the final step we eliminated the Γ-factors using Stirling's formula.

Putting everything together gives

$$S_1 \ll_F |R|_\infty^{-\frac{1}{2}} \prod_{\nu \in S_{sph}} \left(|T_\nu|_\nu^{\frac{1}{6}} + |a|_\nu^{\frac{1}{4}} \left| \frac{T_\nu}{y_\nu} \right|_\nu^{\frac{1}{4}} \right)^{1+\epsilon} \prod_{\nu \in S_{hol}} \left(\left| \frac{a}{y_\nu} \right|_\nu^{\frac{1}{4}} |k_\nu|_\nu^{\frac{3}{8}} + |k_\nu|_\nu^{\frac{1}{4}} \right).$$

□

Corollary 4.6.9 ([2], Corollary 3.3). *If we assume*

$$|y_\nu|_\nu \asymp \begin{cases} \left|a^3 T_\nu\right|_\nu^{\frac{\log(|y|_\infty)}{\log(|a^3|_\infty |T|_{sph} |k^{3/2}|_{hol})}} & \text{if } \nu \in S_{sph} \\ \left|a^3 k_\nu^{3/2}\right|_\nu^{\frac{\log(|y|_\infty)}{\log(|a^3|_\infty |T|_{sph} |k^{3/2}|_{hol})}} & \text{if } \nu \in S_{hol} \end{cases} \tag{4.6.7}$$

for all ν, *then we obtain*

$$S_1 \ll \frac{|y|_\infty^{\frac{1}{2}}}{|T|_{sph}^{\frac{1}{2}} |k|_{hol}^{\frac{1}{2}}} \left(|T|_{sph}^{\frac{1}{6}} |k|_{hol}^{\frac{1}{4}} + \mathcal{N}(\mathfrak{n}_0 \mathfrak{m}_1(g))^{\frac{1}{4}} \frac{|T|_{sph}^{\frac{1}{4}} |k|_{hol}^{\frac{3}{8}}}{|y|_\infty^{\frac{1}{4}}} \right)^{1+\epsilon}.$$

Furthermore, this can be always achieved after multiplying with a suitable unit.

Proof. We consider two cases. First, assume $|y|_\infty \leq (|a^3|_\infty |T|_{sph} |k|_{hol}^{\frac{3}{2}})^{\frac{1}{3}}$. Then the balancing assumption implies $|y_\nu|_\nu \ll |a|_\nu |T_\nu|_\nu^{\frac{1}{3}}$ for all $\nu \in S_{sph}$ and $|y_\nu|_\nu \ll |a|_\nu |k_\nu|_\nu^{\frac{1}{2}}$ for all $\nu \in S_{hol}$. Therefore we have

$$\left| \frac{aT_\nu}{y_\nu} \right|_\nu^{\frac{1}{4}} \gg |T_\nu|_\nu^{\frac{1}{6}} \text{ and } \left| \frac{ak_\nu^{\frac{3}{2}}}{y_\nu} \right|_\nu^{\frac{1}{4}} \gg |k_\nu|_\nu^{\frac{1}{4}}$$

respectively.

Secondly, if $|y|_\infty \geq (|a^3|_\infty |T|_{sph} |k|_{hol}^{\frac{3}{2}})^{\frac{1}{3}}$, one argues analogously to obtain

$$\left|\frac{aT_\nu}{y_\nu}\right|_\nu^{\frac{1}{4}} \ll |T_\nu|_\nu^{\frac{1}{6}}$$

for $\nu \in S_{sph}$ and

$$\left|\frac{ak_\nu^{\frac{3}{2}}}{y_\nu}\right|_\nu^{\frac{1}{4}} \ll |k_\nu|_\nu^{\frac{1}{4}}$$

otherwise.

Recalling that $|a|_\infty \ll_F \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))$ completes the proof. □

4.6.3 *The sum $S_2(R)$*

In this section we will estimate the sum $S_2(R)$ by reducing it to well known averages of Hecke eigenvalues and local Whittaker functions.

Lemma 4.6.10 ([2], Lemma 3.12). *We have*

$$S_2(R) \ll (|T|_\infty \mathcal{N}(\mathfrak{n}))^\epsilon |R|_\infty^{\frac{1}{4}+\epsilon} \left(\frac{\mathcal{N}(\mathfrak{n}_0)^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{m}_1(g))^{\frac{1}{4}}} + |R|_\infty^{\frac{1}{2}} \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac{1}{4}} \right).$$

Proof. We start by defining

$$I(\mathfrak{m}) = \{q \in \imath^{-1} | \, |q|_\nu \leq |R_\nu|_\nu, (q) = \mathfrak{m}\}.$$

Using Lemma 4.5.4 we observe that

$$\sharp I(\mathfrak{m}) \ll_\epsilon |R|_\infty^\epsilon \mathcal{N}(\mathfrak{m})^{-\epsilon}. \tag{4.6.8}$$

In particular, if $\mathcal{N}(\mathfrak{m}) \gg |R|_\infty$, then $I(\mathfrak{m})$ must be empty.

By Lemma 4.6.3 we have

$$\begin{aligned} S_2(R)^{\frac{4}{3}} &= \sum_{\substack{\mathfrak{m}\subset \imath^{-1},\\ \mathcal{N}(\mathfrak{m})\ll |R|_\infty}} \mathcal{N}(\mathfrak{m})^{\frac{2}{3}} \mathcal{N}([\mathfrak{m}\imath]_\mathfrak{n})^{\frac{4}{3}} \frac{\left|\lambda\left(\frac{\mathfrak{m}\imath}{[\mathfrak{m}\imath]_\mathfrak{n}}\right)\right|^{\frac{4}{3}}}{\mathcal{N}(\mathfrak{m}\imath)^{\frac{4}{3}}} \sum_{q\in I(\mathfrak{m})} |\lambda_\mathfrak{n}(q)|^{\frac{4}{3}} \\ &= \mathcal{N}(\imath)^{-\frac{2}{3}} \sum_{\substack{\mathfrak{m}_1|\mathfrak{n}^\infty,\\ \mathcal{N}(\mathfrak{m}_1)\ll \mathcal{N}(\imath)|R|_\infty}} \mathcal{N}(\mathfrak{m}_1)^{\frac{2}{3}} \sum_{\substack{(\mathfrak{m}_2,\mathfrak{n})=1,\\ \mathcal{N}(\mathfrak{m}_2)\ll \frac{\mathcal{N}(\imath)|R|_\infty}{\mathcal{N}(\mathfrak{m}_1)}}} \frac{|\lambda(\mathfrak{m}_2)|^{\frac{4}{3}}}{\mathcal{N}(\mathfrak{m}_2)^{\frac{2}{3}}} \sum_{q\in I(\imath^{-1}\mathfrak{m}_1\mathfrak{m}_2)} |\lambda_\mathfrak{n}(q)|^{\frac{4}{3}}. \end{aligned}$$

At this stage we apply the Hölder inequality to the $\mathfrak{m}_2$-sum. This yields

$$S_2(R)^{\frac{4}{3}} = \mathcal{N}(\imath)^{-\frac{2}{3}} \sum_{\substack{\mathfrak{m}_1|\mathfrak{n}^\infty, \\ \mathcal{N}(\mathfrak{m}_1)\ll\mathcal{N}(\imath)|R|_\infty}} \mathcal{N}(\mathfrak{m}_1)^{\frac{2}{3}} S_{\text{ur}}\left(\frac{\mathcal{N}(\imath)\,|R|_\infty}{\mathcal{N}(\mathfrak{m}_1)}\right)^{\frac{1}{3}} \cdot \left(\sum_{\substack{(\mathfrak{m}_2,\mathfrak{n})=1, \\ \mathcal{N}(\mathfrak{m}_2)\ll\frac{\mathcal{N}(\imath)|R|_\infty}{\mathcal{N}(\mathfrak{m}_1)}}} \left(\sum_{q\in I(\imath^{-1}\mathfrak{m}_1\mathfrak{m}_2)} |\lambda_{\mathfrak{n}}(q)|^{\frac{4}{3}} \right)^{\frac{3}{2}} \right)^{\frac{2}{3}}.$$

Here

$$S_{\text{ur}}(X) = \sum_{\substack{(\mathfrak{m},\mathfrak{n})=1, \\ \mathcal{N}(\mathfrak{m})\leq X}} \frac{|\lambda(\mathfrak{m})|^4}{\mathcal{N}(\mathfrak{m})^2}.$$

It is well known that

$$S_{\text{ur}}(X) \ll_{F,\epsilon} (|T|_\infty \mathcal{N}(\mathfrak{n}))^\epsilon X^{1+\epsilon}.$$

This was proved in [42] over $\mathbb{Q}$ but to proof generalises without complications.

Using Jensen's inequality and exploiting (4.6.8) shows that

$$S_2(R)^{\frac{4}{3}} \ll (|T|_\infty |R|_\infty \mathcal{N}(\mathfrak{n}))^\epsilon \mathcal{N}(\imath)^{-\frac{1}{3}} |R|_\infty^{\frac{1}{3}} \cdot \sum_{\substack{\mathfrak{m}_1|\mathfrak{n}^\infty, \\ \mathcal{N}(\mathfrak{m}_1)\ll\mathcal{N}(\imath)|R|_\infty}} \mathcal{N}(\mathfrak{m}_1)^{\frac{1}{3}+\epsilon} \left(\sum_{\substack{(\mathfrak{m}_2,\mathfrak{n})=1, \\ \mathcal{N}(\mathfrak{m}_2)\ll\frac{\mathcal{N}(\imath)|R|_\infty}{\mathcal{N}(\mathfrak{m}_1)}}} \sum_{q\in I(\imath^{-1}\mathfrak{m}_1\mathfrak{m}_2)} |\lambda_{\mathfrak{n}}(q)|^2 \right)^{\frac{2}{3}}. \quad (4.6.9)$$

We will continue to analyse the $\mathfrak{m}_2$-sum. For notational sake we define

$$S_{\text{ram}} = \sum_{\substack{(\mathfrak{m}_2,\mathfrak{n})=1, \\ \mathcal{N}(\mathfrak{m}_2)\ll\frac{\mathcal{N}(\imath)|R|_\infty}{\mathcal{N}(\mathfrak{m}_1)}}} \sum_{q\in I(\imath^{-1}\mathfrak{m}_1\mathfrak{m}_2)} |\lambda_{\mathfrak{n}}(q)|^2.$$

In order to use the notation from Section 4.5.1 we set

$$\mathbf{k}(\mathfrak{m}) = (v_{\mathfrak{p}}(\mathfrak{m}))_{\mathfrak{p}|\mathfrak{n}}.$$

By the local definition of $\lambda_{\mathfrak{n}}$ we can view it as a function on $\mathbb{A}_{fin}^{\imath}$. Lemma 4.6.5 implies that this function is constant on the sets $C^{\imath}(\mathbf{k},[\mathbf{u}])$. Therefore we have

$$\begin{aligned} S_{\text{ram}} &= \sum_{q\in\imath^{-1}\cap B(R)C^{\imath}(\mathbf{k}(\mathfrak{m}_1\imath^{-1}))} |\lambda_{\mathfrak{n}}(q)|^2 \\ &= \sum_{[\mathbf{u}]\in\prod_{\mathfrak{p}|\mathfrak{n}}\mathfrak{o}_{\mathfrak{p}}^{\times}/(1+\varpi_{\mathfrak{p}}^{n_{0,\mathfrak{p}}}\mathfrak{o}_{\mathfrak{p}})} \sum_{q\in\imath^{-1}\cap B(R)C^{\imath}(\mathbf{k}(\mathfrak{m}_1\imath^{-1}),[\mathbf{u}])} |\lambda_{\mathfrak{n}}(q)|^2 \\ &= \sum_{[\mathbf{u}]\in\prod_{\mathfrak{p}|\mathfrak{n}}\mathfrak{o}_{\mathfrak{p}}^{\times}/(1+\varpi_{\mathfrak{p}}^{n_{0,\mathfrak{p}}}\mathfrak{o}_{\mathfrak{p}})} \frac{\sharp(\imath^{-1}\cap B(R)C^{\imath}(\mathbf{k}(\mathfrak{m}_1\imath^{-1}),[\mathbf{u}]))}{\mathrm{Vol}(C^{\imath}(\mathbf{k}(\mathfrak{m}_1\imath^{-1}),[\mathbf{u}]),d\mu)} \int_{C^{\imath}(\mathbf{k}(\mathfrak{m}_1\imath^{-1}),[\mathbf{u}])} |\lambda_{\mathfrak{n}}(q)|^2\, d\mu_{\text{fin}}(q). \end{aligned}$$

Lemma 4.5.6 and (4.5.5) reveal

$$S_{\text{ram}} \ll \frac{\mathcal{N}(\mathfrak{m}_1)\mathcal{N}(\mathfrak{n}_0(g))}{\mathcal{N}(\imath)} F_R(\mathbf{k}(\mathfrak{m}_1\imath^{-1})) \int_{C^{\imath}(\mathbf{k}(\mathfrak{m}_1\imath^{-1}))} |\lambda_{\mathfrak{n}}(q)|^2\, d\mu_{\text{fin}}(q). \tag{4.6.10}$$

The integral appearing here can be estimated using the local result [70, Proposition 2.11]. This is done as follows:

$$\begin{aligned} \int_{C^{\imath}(\mathbf{k})} |\lambda_{\mathfrak{n}}(q)|^2\, d\mu_{\text{fin}}(q) &= \prod_{\mathfrak{p}\nmid\mathfrak{n}} \int_{\varpi_{\mathfrak{p}}^{-v_{\mathfrak{p}}(\imath)}\mathfrak{o}_{\mathfrak{p}}} 1 d\mu_{\mathfrak{p}} \prod_{\mathfrak{p}|\mathfrak{n}} \int_{\varpi_{\mathfrak{p}}^{k_{\mathfrak{p}}}\mathfrak{o}_{\mathfrak{p}}^{\times}} \left| W_{\mathfrak{p}}(a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{d})}\theta_i q)g_{\mathfrak{p}})\right|^2 d\mu_{\mathfrak{p}}(q) \\ &= \frac{\mathcal{N}(\imath)}{\mathcal{N}([\imath]_{\mathfrak{n}})}\zeta_{\mathfrak{n}}(1)^{-1}\prod_{\mathfrak{p}|\mathfrak{n}} q_{\mathfrak{p}}^{-k_{\mathfrak{p}}} \int_{\mathfrak{o}_{\mathfrak{p}}^{\times}} \left| W_{\mathfrak{p}}(a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{d})+v_{\mathfrak{p}}(\theta_i)+k_{\mathfrak{p}}}q)g_{\mathfrak{p}})\right|^2 d\mu_{\mathfrak{p}}^{\times}(q) \\ &\ll \mathcal{N}(\mathfrak{n})^{\epsilon}\frac{\mathcal{N}(\imath)}{\mathcal{N}([\imath]_{\mathfrak{n}})}\zeta_{\mathfrak{n}}(1)^{-1}\prod_{\mathfrak{p}|\mathfrak{n}} q_{\mathfrak{p}}^{-\frac{1}{2}(v_{\mathfrak{p}}(\mathfrak{d})+v_{\mathfrak{p}}(\theta_i)+n_0+m_1(g_{\mathfrak{p}})+3k_{\mathfrak{p}})} \\ &= \mathcal{N}(\mathfrak{n})^{\epsilon}\frac{\mathcal{N}(\imath)}{\mathcal{N}([\imath]_{\mathfrak{n}})^{\frac{3}{2}}}\zeta_{\mathfrak{n}}(1)^{-1}\prod_{\mathfrak{p}|\mathfrak{n}} q_{\mathfrak{p}}^{-\frac{3k_{\mathfrak{p}}}{2}}. \end{aligned}$$

Note that here we crucially rely on $g_{\mathfrak{p}} \in \mathcal{J}_{\mathfrak{n}}$ in order to apply the upper bounds for the local integrals. Inserting this estimate in our expression for S_{ram} we get

$$S_{\text{ram}} \ll \zeta_{\mathfrak{n}}(1)^{-1}\mathcal{N}(\mathfrak{n}_0(g))\mathcal{N}(\mathfrak{m}_1)^{-\frac{1}{2}}F_R(\mathbf{k}(\mathfrak{m}_1\imath^{-1})).$$

The result from Lemma 4.5.6 yields

$$S_{\text{ram}} \ll \zeta_{\mathfrak{n}}(1)^{-1}\left(\frac{\mathcal{N}(\mathfrak{n}_0(g))}{\mathcal{N}(\mathfrak{m}_1)^{\frac{1}{2}}} + \frac{|R|_{\infty}\mathcal{N}(\imath)}{\mathcal{N}(\mathfrak{m}_1)^{\frac{3}{2}}}\right).$$

From (4.6.9) we deduce

$$S_2(R) \ll (|T|_{\infty}|R|_{\infty}\mathcal{N}(\mathfrak{n}))^{\epsilon}\frac{|R|_{\infty}^{\frac{1}{4}}}{\mathcal{N}(\imath)^{\frac{1}{4}}}\left(\sqrt{\mathcal{N}(\mathfrak{n}_0(g))} + \sqrt{|R|_{\infty}\mathcal{N}(\imath)}\right) \cdot \left(\sum_{\substack{\mathfrak{m}_1|\mathfrak{n}^{\infty},\\ \mathcal{N}(\mathfrak{m}_1)\ll\mathcal{N}(\imath)|R|_{\infty}}} \mathcal{N}(\mathfrak{m}_1)^{\epsilon}\right)^{\frac{4}{3}}.$$

Using the Rankin-trick we have

$$\sum_{\substack{\mathfrak{m}_1 | \mathfrak{n}^\infty, \\ \mathcal{N}(\mathfrak{m}_1) \ll \mathcal{N}(\imath)|R|_\infty}} \mathcal{N}(\mathfrak{m}_1)^\epsilon \ll \mathcal{N}(\mathfrak{n})^\epsilon \, |R|_\infty^\epsilon .$$

□

4.6.4 *The error $\mathcal{E}$*

For R as in (4.6.6) we will roughly prove that the error can be absorbed in the main-contribution. More precisely we have the following lemma.

Lemma 4.6.11 ([2], Lemma 3.13)**.** *Under the balancing assumption (4.6.7) we have*

$$\mathcal{E} \ll (|R|_\infty \mathcal{N}(\mathfrak{n}))^\epsilon \Big(|T|_{sph}^{\frac{1}{6}} \, |k|_{hol}^{\frac{1}{4}} \mathcal{N}(\mathfrak{n}_0)^{\frac{1}{2}} + |T|_\infty^{\frac{1}{6}} \, |k|_{hol}^{\frac{1}{4}} \, |R|_\infty^{\frac{1}{4}} \mathcal{N}(\mathfrak{n}_0 \mathfrak{m}_1(g))^{\frac{1}{4}} + |k|_{hol}^{\frac{1}{8}} \, |R|_\infty^{\frac{1}{2}} \mathcal{N}(\mathfrak{n}_0 \mathfrak{m}_1(g))^{\frac{1}{2}} \Big)$$

Proof. For $S \subset \{\nu\}$ we define

$$\begin{aligned} R'(\mathbf{l}) &= \begin{cases} (l_\nu + 1) R_\nu & \text{if } \nu \in S, \\ R_\nu & \text{else,} \end{cases} \\ I_S(\mathbf{l}) &= \prod_{\nu \in S} I_\nu(l_\nu), \\ B_S(R) &= \prod_{\nu \notin S} \{\xi_\nu \in F^\times \,|\, |\xi_\nu| \leq R_\nu\}. \end{aligned}$$

For $l_\nu \geq 1$ and $\nu \in S_{sph}$ we use the exponential decay of the K-Bessel function, made precise in Corollary 2.1.7, to bound

$$\left|a^{-1} q\right|_\nu^{-2} \left|W_\nu(a(a^{-1} q y_\nu))\right|^4 \ll_A l_\nu^{-A} \, |R_\nu|_\nu^{-2} . \tag{4.6.11}$$

for $q \in I_\nu(l_\nu)$ and any $A \geq 2$. It can be checked by explicit computations that the same bound holds for $\nu \in S_{hol}$.

We now decompose $\mathcal{E}$ as follows

$$\mathcal{E} \leq \sum_{\emptyset \neq S \subset \{\nu\}} \sum_{\mathbf{l} \in \mathbb{N}^{\sharp S}} \left(\sum_{q \in a\imath^{-1} \cap I_S(\mathbf{l}) \times B_S(|a|R)} \left|a^{-1} q\right|_\infty^{-2} \left|W_\infty(a^{-1} q y)\right|^4 \right)^{\frac{1}{4}} S_2(R'(\mathbf{l})).$$

Again we included the shift by a only in the archimedean part. Note that by Corollary 4.5.7 below the sum $S_2(R'(\mathbf{l}))$ vanishes if $\prod_{\nu\in S}|l_\nu+1|_\nu\,|R|_\infty<\mathcal{N}(\imath)^{-1}$. We can add the condition $\prod_{\nu\in S}|l_\nu+1|_\nu\geq|R|_\infty^{-1}\mathcal{N}(\imath)^{-1}$ to the sum over $\mathbf{l}$.

First, note that Lemma 4.6.10 is general enough to deal with the non-archimedean part of the sum. To deal with the archimedean part we use the same approach as in Section 4.6.2. In particular taking (4.6.11) into account we have

$$\begin{aligned}
&\sum_{q\in a\imath^{-1}\cap I_S(\mathbf{l})\times B_S(|a|R)}|qa^{-1}|_\infty^{-2}\,|W_\infty(a(qa^{-1}y))|^4\\
&=\sum_{\substack{\mathbf{l^c},\\ -\lfloor|a|R_\nu\rfloor\leq l_\nu\leq 0\ \forall\nu\notin S}}\ \sum_{q\in a\imath^{-1}\cap I(\mathbf{l}\times\mathbf{l^c})}|qa^{-1}|_\infty^{-2}\,|W_\infty(a(qa^{-1}y))|^4\\
&\ll|R|_\infty^{-2}\prod_{\nu\in S}|l_\nu|_\nu^{-A}f_\nu(k_\nu)\prod_{\nu\notin S}\ \sum_{\substack{\mathbf{k^c},\\ -\lfloor|a|R_\nu\rfloor\leq l_\nu\leq 0\ \forall\nu\notin S}}g_\nu(l_\nu)^4f_\nu(l_\nu)\\
&\ll|R|_\infty^{-2}\prod_{\nu\in S}|l_\nu|_\nu^{-A}f_\nu(l_\nu)\prod_{\substack{\nu\notin S,\\ \nu\in S_{sph}}}\left(|T_\nu|_\nu^{\frac{2}{3}}+|aR_\nu|_\nu\right)^{1+\epsilon}\prod_{\substack{\nu\notin S,\\ \nu\in S_{hol}}}\left(|k_\nu|_\nu+|aR_\nu|_\nu\,|k_\nu|_\nu^{\frac12}\right).
\end{aligned}$$

We obtain

$$\begin{aligned}
\mathcal{E}\ll\Big(|R|_\infty\mathcal{N}(\mathfrak{n})\Big)^\epsilon&\sum_{\emptyset\neq S\subset\{\nu\}}\ \sum_{\substack{\mathbf{l}\in\mathbb{N}^{\sharp S},\\ \prod_{\nu\in S}|l_\nu+1|_\nu\\ \geq|R|_\infty^{-1}\mathcal{N}(\imath)^{-1}}}\left(\frac{|R|_\infty^{-\frac14}\mathcal{N}(\mathfrak{n}_0)^{\frac14}}{\mathcal{N}(\mathfrak{m}_1(g))^{\frac14}}+|R|_\infty^{\frac14}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac14}\right)\\
&\cdot\prod_{\substack{\nu\notin S,\\ \nu\in S_{sph}}}\left(|T_\nu|_\nu^{\frac16}+|aR_\nu|_\nu^{\frac14}\right)\prod_{\substack{\nu\notin S,\\ \nu\in S_{hol}}}\left(|k_\nu|_\nu^{\frac14}+|aR_\nu|_\nu^{\frac14}\,|k_\nu|_\nu^{\frac18}\right)\prod_{\nu\in S}|l_\nu|_\nu^{-\frac{A-2}{4}}f_\nu(l_\nu)^{\frac14}.
\end{aligned}$$

Inserting the definition of f_ν from (4.5.4) and using the balancing assumption as in the proof of Corollary 4.6.9 yields

$$\begin{aligned}
\mathcal{E}\ll&\sum_{\emptyset\neq S\subset\{\nu\}}\ \sum_{\substack{\mathbf{l}\in\mathbb{N}^{\sharp S},\\ \prod_{\nu\in S}|l_\nu+1|_\nu\geq|R|_\infty^{-1}\mathcal{N}(\imath)^{-1}}}\Bigg[\left(\frac{|R|_\infty^{-\frac14}\mathcal{N}(\mathfrak{n}_0)^{\frac14}}{\mathcal{N}(\mathfrak{m}_1(g))^{\frac14}}+|R|_\infty^{\frac14}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac14}\right)^{1+\epsilon}\\
&\cdot\left(|T|_{sph}^{\frac16}\,|k|_{hol}^{\frac14}+|R|_\infty^{\frac14}\,|k|_{hol}^{\frac18}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac14}\right)\prod_{\nu\in S}l_\nu^{[F_\nu:\mathbb{R}]-\frac{A-6}{4}}\,|R_\nu|_\nu^{\frac14}\Bigg]\\
\ll&\,(|R|_\infty\mathcal{N}(\mathfrak{n}))^\epsilon\sum_{\emptyset\neq S\subset\{\nu\}}\ \sum_{\substack{\mathbf{l}\in\mathbb{N}^{\sharp S},\\ \prod_{\nu\in S}|l_\nu+1|_\nu\\ \geq|R|_\infty^{-1}\mathcal{N}(\imath)^{-1}}}\Bigg[\Bigg(|R|_\infty^{-\frac14}\frac{|T|_{sph}^{\frac16}\,|k|_{hol}^{\frac14}\mathcal{N}(\mathfrak{n}_0)^{\frac14}}{\mathcal{N}(\mathfrak{m}_1(g))^{\frac14}}+|k|_{hol}^{\frac18}\mathcal{N}(\mathfrak{n}_0)^{\frac12}\\
&+|T|_{sph}^{\frac16}\,|k|_{hol}^{\frac14}\,|R|_\infty^{\frac14}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac14}+|k|_{hol}^{\frac18}\,|R|_\infty^{\frac12}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac12}\Bigg)\prod_{\nu\in S}l_\nu^{[F_\nu:\mathbb{R}]-\frac{A-6}{4}}\Bigg].
\end{aligned}$$

Finally, we use the condition in the $\mathfrak{l}$-sum to remove the factor $|R|^{-\frac{1}{4}}$. We drop any unnecessary condition on $\mathfrak{l}$ and end up with

$$\mathcal{E} \ll (|R|_\infty \mathcal{N}(\mathfrak{n}))^\epsilon \Big(|T|_{sph}^{\frac{1}{6}} |k|_{hol}^{\frac{1}{4}} \mathcal{N}(\mathfrak{n}_0)^{\frac{1}{2}} + |T|_{sph}^{\frac{1}{6}} |k|_{hol}^{\frac{1}{4}} |R|_\infty^{\frac{1}{4}} \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac{1}{4}} + |k|_{hol}^{\frac{1}{8}} |R|_\infty^{\frac{1}{2}} \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac{1}{2}} \Big) \cdot \sum_{\emptyset \neq S \subset \{\nu\}} \sum_{\mathfrak{l} \in \mathbb{N}^{\sharp S}} |l_\nu|^{\frac{[F_\nu:\mathbb{R}]}{2} - \frac{A-6}{4}}.$$

By taking $A \geq 2$ big enough it is no problem to estimate the remaining l_ν-sums. □

4.6.5 *The final Whittaker bound*

It remains to put all the pieces together to prove an upper bound for $\phi_\circ$.

Proposition 4.6.12 ([2], Proposition 3.1). *Let $\phi_\circ = \sigma(v^\circ)$ for some cuspidal automorphic representation (π, V_π) with new vector v°. For $g \in \mathcal{J}_\mathfrak{n}$ we have*

$$|\phi_\circ(a(\theta_i)gn(x)a(y))| \ll_{F,\epsilon} \left(\frac{|T|_{sph} |k|_{hol} \mathcal{N}(\mathfrak{n})}{|y|_\infty} \right)^\epsilon \left(|T|_{sph}^{\frac{1}{6}} |k|_{hol}^{\frac{1}{4}} \mathcal{N}(\mathfrak{n}_0)^{\frac{1}{2}} + \frac{|T|_{sph}^{\frac{5}{12}} |k|_{ho}^{\frac{1}{2}}}{|y|_\infty^{\frac{1}{4}}} \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac{1}{4}} + \frac{|T|_{sph}^{\frac{1}{2}} |k|_{hol}^{\frac{5}{8}}}{|y|_\infty^{\frac{1}{2}}} \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac{1}{2}} \right).$$

Proof. As in [20, (8.7)] we can assume that y is balanced in the sense of (4.6.7). Further, note that if $|R|_\infty = \frac{|T|_{sph}|k|_{hol}}{|y|_\infty} < \mathcal{N}(\imath)^{-1}$, it follows from Corollary 4.5.7 and (4.6.3) that

$$|\psi(a(\theta_i)gn(x)a(y))| \leq |c_{\phi_\circ}| \, \mathcal{E}.$$

In this case we get the desired bound from Lemma 4.6.11 and 4.6.6.

If $|R|_\infty \geq \mathcal{N}(\imath)^{-1}$, the main contribution will obviously come from $S_1S_2(R)$. From Corollary 4.6.9 and Lemma 4.6.10 we get

$$\begin{aligned} S_1S_2(R) &\ll_{F,\epsilon} (|R|_\infty \mathcal{N}(\mathfrak{n}))^\epsilon |R|_\infty^{-\frac{1}{4}} \left(|T|_{sph}^{\frac{1}{6}} |k|_{hol}^{\frac{1}{4}} + \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac{1}{4}} |R|_\infty^{\frac{1}{4}} |k|_{hol}^{\frac{1}{8}} \right)^{1+\epsilon} \\ &\quad \cdot \left(\frac{\mathcal{N}(\mathfrak{n}_0)^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{m}_1(g))^{\frac{1}{4}}} + |R|_\infty^{\frac{1}{2}} \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac{1}{4}} \right) \\ &\ll_F (|R|_\infty \mathcal{N}(\mathfrak{n}))^\epsilon \Big(|R|_\infty^{-\frac{1}{4}} \frac{|T|_{sph}^{\frac{1}{6}} |k|_{hol}^{\frac{1}{4}} \mathcal{N}(\mathfrak{n}_0)^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{m}_1(g))^{\frac{1}{4}}} + |k|_{hol}^{\frac{1}{8}} \mathcal{N}(\mathfrak{n}_0)^{\frac{1}{2}} \\ &\qquad + |T|_{sph}^{\frac{1}{6}} |k|_{hol}^{\frac{1}{4}} |R|_\infty^{\frac{1}{4}} \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac{1}{4}} + |k|_{hol}^{\frac{1}{8}} |R|_\infty^{\frac{1}{2}} \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{\frac{1}{2}} \Big). \end{aligned}$$

We can use $|R|_\infty \geq \mathcal{N}(\imath)^{-1} \gg \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{-1}$ get rid of the factor $|R|_\infty^{-\frac{1}{4}}$. According to Lemma 4.6.11 the error is under control. This concludes the proof. □

4.7 AMPLIFICATION

The central result of this section is the so called amplified pre-trace inequality. From this we will derive suitable bounds for $\phi_\circ$ in the bulk.

More precisely we will define an integral operator which approximates a spectral projector on the subspace of $L^2(G(F)\setminus G(\mathbb{A}_F))$ generated by $\phi_\circ$. A geometric estimation of the kernel will yield the desired estimate.

Let (π, V_π) be a cuspidal automorphic representation with new vector v° and associated newform $\phi_\circ = \sigma(v^\circ)$. Throughout this section we fix a square-free ideal $\mathfrak{q}$ such that all the units that are quadratic residues modulo $\mathfrak{q}$ are indeed contained in $(\mathcal{O}_F^\times)^2$. We will further assume that $(\mathfrak{q}, 2\mathfrak{n}\theta_1\cdots\theta_{h_F}) = 1$. Let us construct this ideal once and for all.

Lemma 4.7.1 ([2], Lemma 5.1)**.** *There is an absolute constant $A > 0$ depending only on F such that for any $\mathfrak{n}$ there is an ideal $\mathfrak{q}$ satisfying the following two properties.*

- *We have $C \leq \mathcal{N}(\mathfrak{q}) \ll \log(\mathcal{N}(\mathfrak{n}))^A$, where C is the absolute constant to be chosen later.*
- *If x is a quadratic residue modulo $\mathfrak{q}$, then $x \in (\mathcal{O}_F^\times)^2$.*
- *If $\mathfrak{a}^2 = (\alpha)$, then we can choose the generator α such that it is a square mod $\mathfrak{q}$.*

Proof. We will construct $\mathfrak{q}$ by putting

$$\mathfrak{q} = \prod_{\substack{u \in \mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2, \\ [u]\neq[1]}} \mathfrak{q}_u.$$

For suitably small prime ideals $\mathfrak{q}_u$ which are inert in $F(\sqrt{u}) : F$ and split completely in $F(\sqrt{v}) : F$ for all $[u] \neq [v]$. It is clear that if this construction is possible $\mathfrak{q}$ will have the desired properties.

For $u \in \mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2$ non-trivial, we look at the quadratic extension $F(\sqrt{u}) : F$. The Galois group is abelian and consists of two elements, say $Gal(F(\sqrt{u})|F) = \{1, \sigma_u\}$. Since we are dealing with a quadratic extension we know that a prime $\mathfrak{p}$ of F is inert in $F(\sqrt{u})$ if and only if the Artin-symbol satisfies

$$\left(\frac{F(\sqrt{u}) : F}{\mathfrak{p}}\right) = \sigma_u.$$

On the other hand, $\mathfrak{p}$ splits completely if

$$\left(\frac{F(\sqrt{v}):F}{\mathfrak{p}}\right)=1.$$

We thus consider the tower $K=\prod_v F(\sqrt{v})-F(\sqrt{u})-F$. There is an integer l such that $Gal(K|F)=(\mathbb{Z}/2\mathbb{Z})^l$ and without loss of generality we can assume that for all $\sigma\in Gal(K|F)$ we have

$$\sigma|_{Gal(F(\sqrt{u})|F)}=\mathrm{pr}_1(\sigma).$$

We define $\tilde{\sigma}_u=(\sigma_u,1,\cdots,1)$.

The Chebotarev set

$$P_{K|F}(\tilde{\sigma}_u)=\left\{\mathfrak{p}\text{ unramified in }K\colon\ \left(\frac{K:F}{\mathfrak{p}}\right)=\tilde{\sigma}_u\right\}$$

contains exactly all the primes of F that are inert in $F(\sqrt{u})$ and split completely in $F(\sqrt{v})$ for the remaining v. The rest of the proof is concerned with the problem of choosing $\mathfrak{q}_u$ as small as possible.

To do so we make several definitions. First, we define

$$[\mathfrak{n}]_u=\prod_{\substack{\mathfrak{p}|\mathfrak{n},\\ \mathfrak{p}\in P_{K|F}(\tilde{\sigma}_u)}}\mathfrak{p}.$$

Further, we number $P_{K|F}(\tilde{\sigma}_u)=\{\mathfrak{p}_{u,1},\mathfrak{p}_{u,2},\cdots\}$ such that $\mathcal{N}(\mathfrak{p}_{u,1})\leq\mathcal{N}(\mathfrak{p}_{u,2})\leq\cdots$.

Consider two cases. First, if $\mathfrak{p}_{u,1}\nmid[\mathfrak{n}]_u$ then we take $\mathfrak{q}_u=\mathfrak{p}_{u,1}$. By a version of Linnik's theorem for Chebotarev sets [90] we have

$$\mathcal{N}(\mathfrak{q}_u)\ll_F 1.$$

Second, we consider the worst case

$$[\mathfrak{n}]_u=\mathfrak{p}_{u,1}\cdot\ldots\cdot\mathfrak{p}_{u,k-1}.$$

Here we define $\mathfrak{q}_u=\mathfrak{p}_{u,k}$. It is clear that we only need to show $\mathcal{N}(\mathfrak{q}_u)\ll\log(\mathcal{N}([\mathfrak{n}]_u))^A$. But this follows from elementary calculations using Chebotarev's density theorem [63, Theorem (13.4), Chapter VII].

It is obvious that we can assume $C\leq\mathcal{N}(\mathfrak{q})$. $\square$

4.7.1 *Amplification and the spectral expansion*

Let $\phi = \phi_\circ^{\mathfrak{L}} = \sigma^{\mathfrak{L}}(v_{\mathfrak{L}}^\circ)$. By Corollary 4.4.7 it is enough to consider $\phi(g)$ for

$$g = a(\theta_i)g'n(x)a(y), \text{ for } n(x)a(y) \in \mathcal{F}_{\mathfrak{n}_2}, g' = kh_{\mathfrak{n}} \in \mathcal{J}_{\mathfrak{n}}.$$

We further define $\phi' = \phi(\cdot h_{\mathfrak{n}})$. This function is $K_1'(\mathfrak{n}) = h_{\mathfrak{n}}K_1(\mathfrak{n})_{\text{fin}}h_{\mathfrak{n}}^{-1}$ invariant and can be considered as an element of the Hilbert space

$$L^2(X) = L^2(G(F) \setminus G(\mathbb{A}_F)/K_{\mathbb{C}}K_1'(\mathfrak{n}), \omega_\pi) \subset L^2(G(F) \setminus G(\mathbb{A}_F), \omega_\pi).$$

Furthermore, we put $w^\circ = \pi^{\mathfrak{L}}(h_{\mathfrak{n}})v_{\mathfrak{L}}^\circ$. Then $\phi' = \sigma^{\mathfrak{L}}(w^\circ)$. We will bound ϕ' on elements $g = a(\theta_i)g'n(x)a(y)$ with $g' \in K_{\mathfrak{n}}h_{\mathfrak{n}}^{-1}$ and $n(x)a(y) \in \mathcal{F}_{\mathfrak{n}_2}$. For notational simplicity we interchange the roles of π and $\pi^{\mathfrak{L}}$. In other words, without loss of generality, we can work with π in all what follows now.

Next we define the kernel function which will be used to construct the approximate spectral projector mentioned earlier. We do this place by place and immediately give some basic properties.

Let $\nu \in S_{\mathbb{C}}$ or $\nu \in S_{\mathbb{R}} \cap S_{\text{sph}}$ such that $k_\nu = 0$. Define

$$f_\nu(z_\nu g_\nu) = \omega_{\pi,\nu}(z_\nu)^{-1}k_\nu\big(u_\nu(g_\nu.i_\nu, i_\nu)\big),$$

for k_ν as in [20, Lemma 10]. Let us recall the bound

$$k_\nu(u) \ll \min(|T_\nu|_\nu, |T_\nu|_\nu^{\frac{1}{2}} |u_\nu|_\nu^{-\frac{1}{4}}), \text{ for } u \geq 0.$$

Furthermore, $\text{supp}(k_\nu) \subset [0,1]$. By uniqueness of the spherical vector we have

$$R(f_\nu)w_\nu^\circ = c_\nu(\pi_\nu)w_\nu^\circ.$$

The number $c_\nu(\pi_\nu)$ depends only on the equivalence class of π_ν and is given by the spherical transform of f_ν at π_ν. By a suitable parametrisation of spherical representations of $G(F_\nu)$ one relates this to the classical Selberg/Harish-Chandra transform of k_ν. Therefore we have[2]

$$c_\nu(\pi_\nu) = h_\nu\left(\frac{t_\nu}{2}\right) \gg 1 \tag{4.7.1}$$

by [20, Lemma 10].

2 The factor $\frac{1}{2}$ appears due to our different normalisation of the spectral parameter compared to [20].

For $\nu \in S_{\text{hol}}$ we define

$$f_\nu(g) = d_{\pi_\nu} \frac{\overline{\langle \pi_\nu(g) w_\nu^\circ, w_\nu^\circ \rangle}}{\langle w_\nu^\circ, w_\nu^\circ \rangle}.$$

Since we are assuming $k_\nu \geq 3$ this function is integrable. We observe that

$$\pi_\nu(f_\nu) w_\nu^\circ = c_\nu(\pi_\nu) w_\nu^\circ = w_\nu^\circ. \tag{4.7.2}$$

Even more, acting on V_{π_ν}, it is exactly the orthogonal projection on $\mathbb{C} w_\nu^\circ$. On the other hand, $\pi_\nu(f_\nu)|V_{\pi'} \equiv 0$ whenever $\pi' \ncong \pi_\nu$. In particular, $\pi_\nu(f_\nu)$ defines an positive operator. Ultimately, by [56, Theorem 14.5] we have

$$f_\nu(g) = \begin{cases} \frac{k-1}{4\pi} \frac{\det(g)^{\frac{k}{2}} (2i)^k}{(-b+c+(a+d)i)^k} & \text{if } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\nu^+, \\ 0 & \text{else.} \end{cases}$$

In particular, for $\det(g) > 0$ we have

$$|f_\nu(g)| = \frac{k-1}{4\pi} \tilde{u}_\nu(g)^{-\frac{k}{2}}, \text{ for } \tilde{u}_\nu(g) = \frac{\|i - g.\bar{i}\|^2}{4\Im(i)\Im(g.i)}.$$

The remaining archimedean places are those $\nu \in S_{\mathbb{R}} \cap S_{\text{sph}}$ where $k_\nu = 1$.[3] Here we define the function f_ν by

$$f_\nu \left(z(\lambda) k(\theta_1) \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} k(\theta_2) \right) = \omega_{\pi,\nu}(\lambda) e^{i(\theta_1+\theta_2)} k_\nu \left(\frac{1}{2}(x^2 - 2 + x^{-2}) \right),$$

for $t > 0$, $z(\lambda) \in Z(\mathbb{R})$and $k_{\theta_1}, k(\theta_2) \in O(2)$. First of all note that by construction and [65, Proposition 7.5.1] we have

$$\pi_\nu(f_\nu) w_\nu^\circ = \underbrace{h_\nu \left(\frac{t_\nu}{2} \right)}_{=c(\pi_\nu) \gg 1} w_\nu^\circ.$$

Furthermore, $\pi(f_\nu)$ always projects projects on the weight 1 subspace of V_π and is a positive operator by construction of h_ν. Finally note that $|f_\nu(g)| = |k_\nu(u_\nu(g.i_\nu, i_\nu))|$. This is because

$$u_\nu \left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} .i_\nu, i_\nu \right) = \frac{x^2}{2} - 1 + \frac{x^{-2}}{2}.$$

3 Note that a similar test function would work to project on a weight k_ν vector in principal series representations. Thus we could also deal with more general Maaß forms. However, we will not do so and stick to newforms, which in our context includes the condition of being lowest weight at infinity.

For $\mathfrak{p}|\mathfrak{n}$ we define

$$f_{\mathfrak{p}}(g_{\mathfrak{p}}) = |\det(g_{\mathfrak{p}})|^{ia_{\mathfrak{p}}}\, \Phi'_{\pi'_{\mathfrak{p}}}(g_{\mathfrak{p}}).$$

Where $\Phi'_{\pi'_{\mathfrak{p}}}$ is the truncated matrix coefficient defined in [70, Section 2.4]. By construction (see [70, Proposition 2.13]) there is $\delta_{\pi'_{\mathfrak{p}}} \gg q_{\mathfrak{p}}^{-n_{1,\mathfrak{p}}-m_{1,\mathfrak{p}}}$ such that

$$R_{\mathfrak{p}}(f_{\mathfrak{p}})w_{\mathfrak{p}}^{\circ} = \int_{Z(F_{\mathfrak{p}})\backslash G(F_{\mathfrak{p}})} f_{\mathfrak{p}}(g)\pi_{\mathfrak{p}}(g)w_{\mathfrak{p}}^{\circ}d\mu_{\mathfrak{p}}(g) = \delta_{\pi'_{\mathfrak{p}}}w_{\mathfrak{p}}^{\circ}.$$

Let us remark that

$$\begin{aligned} |f_{\mathfrak{p}}(g)| &\leq 1 \text{ for all } g \in G(F_{\mathfrak{p}}), \\ \operatorname{supp}(f_{\mathfrak{p}}) &= \begin{cases} Z(F_{\mathfrak{p}})K_{\mathfrak{p}} & \text{if } n_{\mathfrak{p}} \text{ is even,} \\ Z(F_{\mathfrak{p}})K_{\mathfrak{p}}^0(1) & \text{else.} \end{cases} \end{aligned}$$

For $\mathfrak{p}|\mathfrak{q}$ define

$$\tilde{K}_{0,\mathfrak{p}}(1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{0,\mathfrak{p}}(1) \colon a-d \in \varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}} \right\}$$

and put

$$f_{\mathfrak{p}}(g_{\mathfrak{p}}) = \begin{cases} \operatorname{vol}(Z(\mathfrak{o}_{\mathfrak{p}}) \setminus \tilde{K}_{0,\mathfrak{p}}(1))^{-1}\omega_{\pi_{\mathfrak{p}}}^{-1}(z) & \text{if } g_{\mathfrak{p}} = zk \in Z(F_{\mathfrak{p}})\tilde{K}_{0,\mathfrak{p}}(1), \\ 0 & \text{else.} \end{cases}$$

Since $w_{\mathfrak{p}}^{\circ}$ is $K_{\mathfrak{p}}$-fixed, we see:

$$\begin{aligned} R_{\mathfrak{p}}(f_{\mathfrak{p}})w_{\mathfrak{p}}^{\circ} &= \int_{Z(F_{\mathfrak{p}})\backslash G(F_{\mathfrak{p}})} f_{\mathfrak{p}}(g)\pi_{\mathfrak{p}}(g)w_{\mathfrak{p}}^{\circ}d\mu_{\mathfrak{p}}(g) \\ &= \operatorname{vol}(Z(\mathfrak{o}_{\mathfrak{p}}) \setminus \tilde{K}_{0,\mathfrak{p}}(1))^{-1} \int_{Z(\mathfrak{o}_{\mathfrak{p}})\backslash \tilde{K}_{0,\mathfrak{p}}(1)} \omega_{\pi_{\mathfrak{p}}}(z)^{-1}\pi_{\mathfrak{p}}(zk)w_{\mathfrak{p}}^{\circ}d\mu_{\mathfrak{p}}(zk) = w_{\mathfrak{p}}^{\circ}. \end{aligned}$$

We also have the estimate

$$|f_{\mathfrak{p}}| \leq [K_{\mathfrak{p}} : \tilde{K}_{0,\mathfrak{p}}(1)] \ll q_{\mathfrak{p}}^{2+\epsilon}.$$

The remaining places will be treated at once. Set $S_{\text{ur}} = \{\mathfrak{p} \colon (\mathfrak{p}, \mathfrak{q}\mathfrak{n}) = 1\}$ and define the unramified Hecke algebra

$$\mathcal{H}_{\text{ur}} = \langle \{\kappa_{\text{ur}} = \otimes_{\mathfrak{p}\in S_{\text{ur}}}\kappa_{\mathfrak{p}} \colon \kappa_{\mathfrak{p}} \in \mathcal{C}_c^{\infty}(G(F_{\mathfrak{p}}), \omega_{\pi_{\mathfrak{p}}}) \text{ such that } \kappa_{\mathfrak{p}}(K_{\mathfrak{p}}gK_{\mathfrak{p}}) = \kappa_{\mathfrak{p}}(g)\}\rangle_{\mathbb{C}}.$$

Due to [25, Theorem 4.6.1] this is a commutative algebra. To an integral ideal $\mathfrak{c}$ we associate the special element

$$\kappa_{\mathfrak{c}} = \otimes_{\mathfrak{p}\in S_{\text{ur}}}\kappa_{\mathfrak{p},v_{\mathfrak{p}}(\mathfrak{c})} \in \mathcal{H}_{\text{ur}}$$

where

$$\kappa_{\mathfrak{p},k}(g) = \begin{cases} \omega_{\pi_{\mathfrak{p}}}(z)^{-1} & \text{for } g = z \in Z(F_{\mathfrak{p}})X_{\mathfrak{p},k}, \\ 0 & \text{else.} \end{cases}$$

This is well defined since the central character is unramified at the places under consideration. The function $\kappa_{\mathfrak{p},k}$ is constructed such that $\pi(\mathbb{1}_{X_{\mathfrak{p},k}}) = R(\kappa_{\mathfrak{p},k})$. Therefore we have for $w_{\text{ur}}^{\circ} = \otimes_{\mathfrak{p}\in S_{\text{ur}}} w_{\mathfrak{p}}^{\circ}$ that

$$R(\kappa_{\mathfrak{c}})w_{\text{ur}}^{\circ} = \lambda_{\pi}(\mathfrak{c})w_{\text{ur}}^{\circ}.$$

Fix a large parameter L such that $\mathcal{N}(q) \ll (\log L)^{A}$ for some constant A. We define the sets

$$\begin{aligned} \mathcal{P}_{\mathfrak{q}} &= \{\mathfrak{a}\colon \mathfrak{a} = (\alpha) \text{ for } \alpha \in F_{+}^{\times} \cap (1+\mathfrak{q})\} \text{ and} \\ \mathcal{P}(L) &= \{\alpha \in \mathcal{O}_F\colon (\alpha) \in \mathcal{P}_{\mathfrak{q}} \text{ such that } \mathcal{N}(\alpha) \in [L, 2L] \text{ and } ((\alpha), \mathfrak{n}) = 1\}/\sim. \end{aligned}$$

In the last definition we wrote $\alpha \sim \beta$ for the equivalence relation $(\alpha) = (\beta)$. We identify $\mathcal{P}(L)$ with a suitable fundamental domain for $\sim$. We can arrange that $\alpha_v \asymp L^{[F:\mathbb{Q}]}$ for all ν and all $\alpha \in \mathcal{P}(L)$. This is the set on which our amplifier will be supported.

We are choosing two sequences of amplifiers

$$x_1 = (x_{\alpha})_{\alpha\in\mathcal{P}(L)} \text{ and } x_2 = (x_{\alpha^2})_{\alpha\in\mathcal{P}(L)}.$$

Define the quantities

$$\|x\|_{1,\pi} = \left|\sum_{\alpha} \frac{x_{\alpha}\lambda_{\pi}((\alpha))}{\sqrt{\mathcal{N}(\alpha)}}\right| \text{ and } \|x\|_2 = \sum_{\alpha} \left|x_{\alpha}^2\right|.$$

Here x can be any sequence defined for all $\alpha \in \mathcal{O}_F$ but $x_{\alpha} = 0$ for all but finitely many α. In particular, if $x = x_j$, for $j = 1, 2$, we are only summing over α^j with $\alpha \in \mathcal{P}(L)$.

The unramified test function will be

$$f_{\text{ur}} = \left(\sum_{\alpha\in\mathcal{P}(L)} \frac{x_{\alpha}\kappa_{\alpha}}{\sqrt{\mathcal{N}(\alpha)}}\right)\left(\sum_{\alpha\in\mathcal{P}(L)} \frac{x_{\alpha}\kappa_{\alpha}}{\sqrt{\mathcal{N}(\alpha)}}\right)^{*} + \left(\sum_{\alpha\in\mathcal{P}(L)} \frac{x_{\alpha^2}\kappa_{\alpha^2}}{\sqrt{\mathcal{N}(\alpha^2)}}\right)\left(\sum_{\alpha\in\mathcal{P}(L)} \frac{x_{\alpha^2}\kappa_{\alpha^2}}{\sqrt{\mathcal{N}(\alpha^2)}}\right)^{*}.$$

This defines an operator $R(f_{\text{ur}})$ such that

$$R(f_{\text{ur}})w_{\text{ur}}^{\circ} = \underbrace{\left[\|x_1\|_{1,\pi}^2 + \|x_2\|_{1,\pi}^2\right]}_{=c_{\text{ur}}} w_{\text{ur}}^{\circ} = c_{\text{ur}}w_{\text{ur}}^{\circ}.$$

Note that since x_1 and x_2 have disjoint support we find that

$$c_{\text{ur}} \geq \frac{1}{2}\|x_1+x_2\|_{1,\pi}^2. \tag{4.7.3}$$

On the other hand, we can linearise f_{ur} to obtain

$$f_{\text{ur}} = \sum_{\substack{(\mathfrak{b},\mathfrak{n}\mathfrak{q})=1,\\ \mathcal{N}(\mathfrak{b})\leq 16L^4}} (y_1(\mathfrak{b})+y_2(\mathfrak{b}))\frac{\kappa_{\mathfrak{b}}}{\sqrt{\mathcal{N}(\mathfrak{b})}}. \tag{4.7.4}$$

To do so we use the Hecke-relation and obtain

$$\begin{aligned} y_1(\mathfrak{b}) &= \sum_{\substack{\alpha,\beta\in\mathcal{P}(L),\\ (\alpha\beta)=\mathfrak{a}^2\mathfrak{b},\\ \mathfrak{a}\mid(\alpha,\beta)}} \omega_{\pi_{(\beta)\mathfrak{a}^{-1}}}^{-1}((\beta)\mathfrak{a}^{-1})x_\alpha\overline{x_\beta} \text{ and } y_2(\mathfrak{b}) \\ &= \sum_{\substack{\alpha,\beta\in\mathcal{P}(L),\\ (\alpha\beta)^2=\mathfrak{a}^2\mathfrak{b},\\ \mathfrak{a}\mid(\alpha,\beta)^2}} \omega_{\pi_{(\beta)^2\mathfrak{a}^{-1}}}^{-1}((\beta)^2\mathfrak{a}^{-1})x_{\alpha^2}\overline{x_{\beta^2}}. \end{aligned}$$

In particular, we have

$$y_1(1)+y_2(1) = \|x_1+x_2\|_2.$$

Suppose x_α and x_{α^2} are supported on (α) prime. These coefficients are very similar in spirit to the coefficients w_m in [20, (9.16)]. Indeed, in this case we have

$$y_1(\mathfrak{a})+y_2(\mathfrak{a}) = \begin{cases} \sum_{\alpha'\in\mathcal{P}(L)}|x_{\alpha'}|^2+|x_{\alpha'^2}|^2 & \text{if } \mathfrak{a}=1 \\ x_{\alpha_1}\overline{x_{\alpha_2}}+\delta_{\alpha_1=\alpha_2}\omega_{\pi_{(\alpha_1)}}^{-1}(\varpi_{(\alpha_1)})x_{\alpha_1^2}\overline{x_{\alpha_2^2}} & \text{if } \mathfrak{a}=(\alpha_1)(\alpha_2) \text{ for } \alpha_1,\alpha_2\in\mathcal{P}(L), \\ x_{\alpha_1^2}\overline{x_{\alpha_2^2}} & \text{if } \mathfrak{a}=(\alpha_1)^2(\alpha_2)^2 \text{ for } \alpha_1,\alpha_2\in\mathcal{P}(L) \\ 0 & \text{else.} \end{cases} \tag{4.7.5}$$

One compares this to [69, p. 28] and [20, (9.16)] and notes the similarity.

Combining everything we define

$$f = \otimes_\nu f_\nu \otimes_{\mathfrak{p}\mid\mathfrak{q}\mathfrak{n}} f_{\mathfrak{p}} \otimes f_{\text{ur}}.$$

Associated to this function there is the integral operator

$$\begin{aligned} R(f)\colon L^2(G(F)\setminus G(\mathbb{A}_F),\omega_\pi) &\to L^2(G(F)\setminus G(\mathbb{A}_F),\omega_\pi) \\ \phi &\mapsto \left[x\mapsto \int_{Z(\mathbb{A}_F)\setminus G(\mathbb{A}_F)} f(g)\phi(gx)dg\right]. \end{aligned}$$

We have

$$R(f)\phi' = \sigma^{\mathfrak{L}}\left(\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(g)\pi^{\mathfrak{L}}(g)w^{\circ}dg\right) = c_{\text{ur}}\prod_{\nu} c_{\nu}(\pi_{\nu})\prod_{\mathfrak{p}|\mathfrak{n}}\delta_{\pi'_{\mathfrak{p}}}\phi'.$$

The corresponding automorphic kernel is given by

$$K_f(g_1,g_2) = \sum_{\gamma\in Z(F)\backslash G(F)} f(g_1^{-1}\gamma g_2).$$

The spectral expansion of K_f will enable us to bound the sup-norm of ϕ' in terms of the geometric definition of K_f. Let us work out the spectral expansion in detail. The construction of f ensures that the spectral expansion of K_f will only feature automorphic forms which are sufficiently similar to ϕ'. In order to make this more precise we say that $\Psi \in L_0^2(X)$ is compatible with π if it satisfies the following:

- Ψ is contained in an irreducible subspace $V_{\pi_\Psi} \subset L_0^2(X)$ and corresponds to a pure tensor in the decomposition $\pi_\Psi = (\otimes_{\mathfrak{p}}\pi_{\Psi,\nu}) \otimes (\otimes_{\mathfrak{p}}\pi_{\Psi,\mathfrak{p}})$.
- Ψ is spherical at all places $p \nmid \mathfrak{q}\mathfrak{n}$ and the conductor of π_ψ contains $\mathfrak{n}$.
- For all $\nu \in S_{\mathbb{C}}$ the representation $\pi_{\Psi,\nu}$ is spherical. If $\nu \in S_{\text{hol}}$, then $\pi_{\Psi,\nu} \cong \pi_\nu$ and Ψ_ν is the lowest weight vector. Furthermore, if $\nu \in S_{\text{sph}} \cap S_{\mathbb{R}}$, then Ψ_ν is of weight k_ν.
- For each $\mathfrak{p} \mid \mathfrak{n}$ we have $\pi_{\Psi,\mathfrak{p}}(f_{\mathfrak{p}})\Psi_{\mathfrak{p}} = \delta_{\pi'_{\mathfrak{p}}}\Psi_{\mathfrak{p}}$.

We choose an orthonormal basis $\mathcal{A}_0(\pi)$ spanning the space of all functions compatible with π. Obviously we can arrange that $\phi' \in \mathcal{A}_0(\pi)$. Similarly we can choose the orthonormal basis $\mathcal{B}_{\tilde{\mathbf{H}}}$ such that it contains a subset $\mathcal{B}_{\tilde{\mathbf{H}}}(\pi)$ which spans the subspace of functions satisfying the points above.

Finally, we define, for any tuple $\mathbf{t} = (t_\nu)_{\nu\in S_{\text{sph}}}$, the function

$$h(\mathbf{t}) = \prod_{\nu\in S_{\mathbb{C}}} h_\nu(t_\nu) \prod_{\nu\in S_{\mathbb{R}}\cap S_{\text{sph}}} h_\nu(\frac{t_\nu}{2}).$$

The tuple $\mathbf{t}_\Psi$ is the tuple of spectral parameters of Ψ at places $\nu \in S_{\text{sph}}$.

With this at hand we prove the following spectral expansion.

Proposition 4.7.2. *For any $g \in G(\mathbb{A}_F)$ we have*

$$0 \leq \sum_{\Psi\in\mathcal{A}_0(\pi)} \frac{\|x_1+x_2\|^2_{1,\pi_\psi}}{\mathcal{N}(\mathfrak{n}_1)\mathcal{N}(\mathfrak{m}_1)} h(\mathbf{t}_\Psi)\,|\Psi(g)|^2$$
$$+\frac{1}{4\pi}\sum_{\Psi\in\mathcal{B}_{\tilde{\mathbf{H}}}}\int_{-\infty}^{\infty}\frac{\|x_1+x_2\|^2_{1,\pi_\psi(iy)}}{\mathcal{N}(\mathfrak{n}_1)\mathcal{N}(\mathfrak{m}_1)} h(iy+\mathbf{t}_\Psi)\,|E_\Psi(iy,g)|^2\,dy + \mathcal{D} \ll K_f(g,g).$$

Here $\mathcal{D} \geq 0$ is the contribution of the residual spectrum, which vanishes if $\mathfrak{n}$ is not square-free. Note that $S_{hol} \neq \emptyset$ implies $\mathcal{B}_{\tilde{\mathbf{H}}}(\pi) = \emptyset$ and $\mathcal{D} = 0$.[4]

Proof. We decompose

$$K_f = K_{\text{cusp}} + K_{\text{sp}} + K_{\text{cont}}$$

and deal with each piece separately.

We begin with the cuspidal part. By fixing a basis $\mathcal{B}_{\text{cusp}}$ containing $\mathcal{A}_0(\pi)$ for $L^2_0(X)$ consisting of $R(F)$ eigenfunctions. This is possible by a standard multiplicity one argument. For $\Psi \in \mathcal{B}_{\text{cusp}}$ let $c(\Psi)$ be the associated $R(f)$-eigenvalue. Then we obtain

$$K_{\text{cusp}}(h,g) = \sum_{\Psi \in \mathcal{B}_{\text{cusp}}} \langle K_{\text{cusp}}(\cdot, g), \Psi \rangle_{L^2(X)} \Psi(h) = \sum_{\Psi \in \mathcal{B}_{\text{cusp}}} \overline{c(\Psi)\Psi(g)}\Psi(h).$$

By construction of f and [70, Corollary 2.16] it is clear that $c(\Psi) = 0$ for $\Psi \in \mathcal{B}_{\text{cusp}} \setminus \mathcal{A}_0(\pi)$. On the other hand, if $\Psi \in \mathcal{A}_0(\pi)$, then

$$c(\Psi) = \delta_{\pi'} c_{\Psi,\text{ur}} \prod_{\nu} c_\nu(\pi_{\Psi,\nu}).$$

In particular, $c(\Psi) \geq 0$ and $c_{\Psi,\text{ur}} \gg \|x_1 + x_2\|^2_{1,\pi_\Psi}$. Furthermore, according to [70, Proposition 2.13] we have $\delta_{\pi'} \gg \mathcal{N}(\mathfrak{n}_1)^{-1}\mathcal{N}(\mathfrak{m}_1)^{-1}$. This concludes the analysis of the cuspidal part.

The argument for the continuous part is quite similar. Using the theory of Eisenstein series we have the expansion

$$K_{cont}(h,g) = \frac{1}{4\pi} \sum_{\Psi_1,\Psi_2 \in \mathcal{B}_{\tilde{\mathbf{H}}}} \int_{-\infty}^{\infty} \langle R(f)\Psi_2(iy), \Psi_1(iy) \rangle_{\tilde{\mathbf{H}}(iy)} E_{\Psi_1}(iy,h)\overline{E_{\Psi_2}(iy,g)}dy, \tag{4.7.6}$$

see [35, (5.21)]. We can argue as before by choosing $\mathcal{B}_{\tilde{\mathbf{H}}}$ carefully. To complete the analysis of the continuous part one again investigates the $R(f)$-eigenvalues of $\Psi \in \mathcal{B}_{\tilde{\mathbf{H}}}$.

Finally, we treat the residual part of the spectrum. We start from the spectral expansion of K_{sp}. This reads

$$K_{\text{sp}}(h,g) = \frac{1}{\text{Vol}(Z(\mathbb{A}_F)G(F) \setminus G(\mathbb{A}_F))} \sum_{\chi^2 = \omega_\pi} \chi(\det(h))\overline{\chi(\det(g))} \cdot \int_{Z(\mathbb{A}_F)\setminus G(\mathbb{A}_F)} f(x)\chi(\det(x)))dx.$$

4 Similarly one can see that there is no contribution of the residual or continuous part if $\pi_\mathfrak{p}$ is supercuspidal for some $\mathfrak{p}$. However, we will not use this fact.

Since the character χ factors and also f is almost a pure tensor the last integral factors in the local integrals

$$I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = \int_{Z(F_{\mathfrak{p}})\backslash G(F_{\mathfrak{p}})} f_{\mathfrak{p}}(g)\chi_{\mathfrak{p}}(\det(g))dg \text{ if } \mathfrak{p}|\mathfrak{n}\mathfrak{q}$$

and the unramified part $I_{\text{ur}}(\chi_{\text{ur}})$. By Lemma 3.5.1 it is clear that $I_{\text{ur}}(\chi_{\text{ur}}) \geq 0$. The lemma follows from the evaluation of the integrals $I_{\mathfrak{p}}(\chi_{\mathfrak{p}})$ given in Lemma 3.5.2 and 3.5.3. □

By dropping all the unnecessary terms in the previous result and combining it with the definition of K_f one concludes

$$|\phi'(g)|^2 \ll \frac{\mathcal{N}(\mathfrak{n}_1\mathfrak{m}_1)}{\|x_1+x_2\|_{1,\pi}^2} \sum_{\gamma\in Z(F)\backslash G(F)} \left|f(g^{-1}\gamma g)\right|. \tag{4.7.7}$$

This gives an upper bound for ϕ' in terms of the geometry of $G(F)$ and the test function f as long as the amplifier is chosen properly. We will estimate this further in the next section.

4.7.2 *Estimating the geometric expansion*

In this subsection we estimate the right hand side of (4.7.7). This will ultimately lead to good control on $\phi_\circ$ in the bulk. Define

$$k(\gamma) = \prod_{\nu} k_\nu(\gamma) = \prod_{v\in S_{\text{hol}}} \left(\frac{u_v(\gamma_v\overline{P_v}, P_v)}{2}\right)^{-\frac{k}{2}} \prod_{v\in S_{\text{sph}}} k_v(u_v(\gamma_v P_v, P_v))$$

with $P_v = n(x_v)a(y_v).i_v$. We prove the following preliminary result which is an adaption of [29, Lemma 1].

Lemma 4.7.3. *For $v \in S_{hol} \subset S_{\mathbb{R}}$ and $k \geq 2r+2$ we have*

$$|k_v(\gamma)| \leq \begin{cases} \frac{k-1}{4\pi} & \textit{always,} \\ C_r\epsilon^{-1-r} & \textit{if } u(\gamma P_v, P_v) > \epsilon. \end{cases}$$

Proof. From the definition of $\|\cdot\|$ we compute

$$\|\gamma\overline{P}_v - P_v\|^2 = \|\gamma P_v - P_v\|^2 + 4\Im(\gamma P_v)\Im(P_v).$$

We conclude that

$$|k_v(\gamma)| = \frac{k-1}{4\pi}(\frac{u(\gamma P_v, P_v)}{2}+1)^{-\frac{k}{2}}.$$

The general bound follows by dropping the u-term due to positivity. The second bound follows from:

$$|k_v(\gamma)| \leq \frac{k-1}{4\pi}\left(\frac{\epsilon}{2}\right)^{-r}(\frac{\epsilon}{2}+1)^{r-\frac{k}{2}} \leq \frac{2^r}{2\pi}\frac{k-1}{k-2r}\epsilon^{-r-1}.$$

In the last step we applied Bernoulli's inequality. □

With this at hand we can return to the estimation of the geometric expansion of K_f.

Proposition 4.7.4. *Take* $(\mathfrak{q},\mathfrak{n})=1$ *and*

$$g = a(\theta_i)g'n(x)a(y) \text{ with } g' \in K_{\mathfrak{n}}h_{\mathfrak{n}}^{-1} \text{ and } n(x)a(y) \in \mathcal{F}_{\mathfrak{n}_2}.$$

Further, assume that the sequence x_2 is supported on α^2 for $\alpha \in \mathcal{P}(L)$ which are principal prime ideals. We have

$$\begin{aligned} K_f(g,g) \ll L^{\epsilon}\mathcal{N}(\mathfrak{q})^{2+\epsilon}\Bigg[&\|x_1\|_{\infty}^2\Bigg(L\,|T|_{sph}^{1+\epsilon}\,|k|_{hol}^{1+\epsilon}+L^2\,|T|_{sph}^{\frac{1}{2}+\epsilon}\,|k|_{hol}^{\frac{1}{2}+\epsilon}\,|y|_{\infty} \\ &+L^{\frac{3}{2}}\frac{|T|_{sph}^{\frac{1}{2}+\epsilon}\,|T|_{\mathbb{C}}^{\frac{1}{2}}\,|k|_{hol}^{\frac{3}{4}+\epsilon}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{4}}}+L^3\frac{|T|_{sph}^{\frac{1}{2}+\epsilon}\,|k|_{hol}^{\frac{1}{4}+\epsilon}}{\mathcal{N}(\mathfrak{n}_2)}\Bigg) \\ &+\|x_2\|_{\infty}^2\Bigg(L\,|T|_{sph}^{1+\epsilon}\,|k|_{hol}^{1+\epsilon}+L^2\,|T|_{sph}^{\frac{1}{2}+\epsilon}\,|k|_{hol}^{\frac{1}{2}+\epsilon}\,|y|_{\infty} \\ &+L^{2}\frac{|T|_{sph}^{\frac{1}{2}+\epsilon}\,|T|_{\mathbb{C}}^{\frac{1}{2}}\,|k|_{hol}^{\frac{1}{2}+\epsilon}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}}}+L^4\frac{|T|_{sph}^{\frac{1}{2}+\epsilon}\,|k|_{hol}^{\frac{1}{4}+\epsilon}}{\mathcal{N}(\mathfrak{n}_2)}\Bigg)\Bigg]. \end{aligned}$$

In particular, after dividing by L^2, putting $\|x_1\|_{\infty}=\|x_1\|_{\infty}=1$ and ignoring the k-contribution, we recover the formula on the bottom of [20, page 37].

Proof. We begin by inserting the linearisation of f_{ur} given in (4.7.4) into (4.7.7). This yields

$$K_f(g,g) \quad\leq \sum_{0\neq\mathfrak{b}\subset\mathcal{O}_F}\frac{|y_1(\mathfrak{b})+y_2(\mathfrak{b})|}{\sqrt{\mathfrak{b}}}\sum_{\gamma\in Z(F)\backslash G(F)}\left|\kappa_{\mathfrak{b}}\prod_{\mathfrak{p}|\mathfrak{q}\mathfrak{n}}f_{\mathfrak{p}}\right|(g'^{-1}a(\theta_i^{-1})\gamma a(\theta_i)g')\,|k(\gamma)|$$

Let us analyse the support of $f_{\mathfrak{p}}$ and $\kappa_{\mathfrak{b}}$ place by place. At this point we will exploit the special structure of g.

First, if $\mathfrak{p}\nmid\mathfrak{n}$, we have $g'_{\mathfrak{p}}=1$. This case consists of two sub cases. Namely,

$$a(\theta_i^{-1})\gamma a(\theta_i) \in \begin{cases} Z(F_{\mathfrak{p}})\tilde{K}_{0,\mathfrak{p}}(1) & \text{if } \mathfrak{p}\mid\mathfrak{q}, \\ Z(F_{\mathfrak{p}})K_{\mathfrak{p}}a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{b})})K_{\mathfrak{p}} & \text{else.} \end{cases}$$

If $\mathfrak{p} \mid \mathfrak{n}$, then we use Lemma 4.4.3 to see that $g'_{\mathfrak{p}} \in \omega K^0_{\mathfrak{p}}(1)$ if $p \mid \mathfrak{n}_2$ and $g'_{\mathfrak{p}} K_{\mathfrak{p}}$ otherwise. Using the support property of $f_{\mathfrak{p}}$ we conclude that

$$a(\theta_i^{-1})\gamma a(\theta_i) \in \begin{cases} Z(F_{\mathfrak{p}})K_{\mathfrak{p}} & \text{if } \mathfrak{p} \nmid \mathfrak{n}_2, \\ Z(F_{\mathfrak{p}}) \underbrace{\omega K^0_{\mathfrak{p}}(1)\omega^{-1}}_{=K_{0,\mathfrak{p}}(1)} & \text{if } \mathfrak{p} \mid \mathfrak{n}_2. \end{cases}$$

We can choose a representative $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $\gamma =\in Z(F)\setminus G(F)$ such that $a, b, c, d \in \mathcal{O}_F$ and $(a, b, c, d) = \theta_j$ for some $1 \leq j \leq h_F$. We arrive at the following conditions:

$$a, d, \theta_i^{-1}b \in \theta_j \mathcal{O}_F, \quad c \in \theta_i^{-1}\theta_j \mathfrak{n}\mathfrak{q}, \quad a - d \in \theta_j \mathfrak{q} \text{ and } (ad - bc) = \theta_j^2 \mathfrak{b}.$$

In particular, we note that $\theta_j^2\mathfrak{b}$ must be a principal ideal, say $(\alpha) = \theta_j^2\mathfrak{b}$. Whenever $y_1(\mathfrak{b}) + y_2(\mathfrak{b})$ contributes to the sum we must have $(\alpha) = (\gamma\beta)\theta_j^2\mathfrak{a}^{-2}$ for some ideal $\mathfrak{a} \mid (\gamma, \beta)$ and $\gamma, \beta \in \mathcal{P}(L)$. Thus, by construction of $\mathfrak{q}$, we can choose α such that it is a quadratic residue mod $\mathfrak{q}$. Further let us note that $ad - bc \in a^2 + \mathfrak{q}$. Thus, again referring to the construction of $\mathfrak{q}$, we get the identity $ad - bc = w^2\alpha$. However, multiplying a, b, c, d by a unit $w \in \mathcal{O}_F^\times$ does not change the first conditions, so that we can assume $w = 1$. Arranging the sums accordingly we obtain

$$K_f(g, g) \ll \mathcal{N}(\mathfrak{q})^{2+\epsilon} \sum_{j=1}^{h_F} \mathcal{N}(\theta_j) \sum_{0 \neq \alpha \in \theta_j^2/\sim} \frac{\left| y_1((\alpha)\theta_j^{-2}) + y_2((\alpha)\theta_j^{-2}) \right|}{\sqrt{\mathcal{N}(\alpha)}} \sum_{\gamma \in \Gamma_j(i,\alpha)} |k(\gamma)|$$

with

$$\Gamma_j(i, \alpha) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F) \colon a, d, \theta_i^{-1} \in \theta_j, a - d \in \theta_j \mathfrak{q}, c \in \theta_i^{-1}\theta_j \mathfrak{n}\mathfrak{q}, ad - bc = \alpha \right\}.$$

Following the strategy in [20] we associate to each γ a dyadic vector $\boldsymbol{\delta} = (\delta_\nu)_\nu = (2^{p_\nu})_\nu$ by choosing p_ν in $\mathbb{Z}$ minimal such that

$$\delta_\mu = 2^{p_\nu} \geq \begin{cases} \max(T_\nu^{-2}, u_\nu(\gamma P_\nu, P_\nu)) & \text{if } \nu \in S_{\text{sph}}, \\ \max(k_\nu^{-1}, u_\nu(\gamma P_\nu, P_\nu)) & \text{if } \nu \in S_{\text{hol}}. \end{cases}$$

At the places $\nu \in S_{\text{sph}}$ we argue as in [20] and at the remaining places we use Lemma 4.7.3. This leads us to the estimate

$$k(\gamma) \ll |T|_{\text{sph}}^{\frac{1}{2}} |\boldsymbol{\delta}|_{\text{sph}}^{-\frac{1}{4}} |\boldsymbol{\delta}|_{\text{hol}}^{-1} |\hat{\boldsymbol{\delta}}|_{\text{hol}}^{-1-r} \tag{4.7.8}$$

for $T_\nu^{-2} \leq \delta_\nu \leq 4$ for $\nu \in S_{\text{sph}}$ and $k_\nu^{-1} \leq \delta_\nu$ for $\nu \in S_{\text{hol}}$, which are exactly those δ which will contribute to the γ-sum. This will be our replacement for [20, (9.22)]. Sorting the matrices $\gamma \in \Gamma_j(i,*)$ according to $\boldsymbol{\delta}$ we get the bound

$$K_f(g,g) \ll \mathcal{N}(\mathfrak{q})^{2+\epsilon} \sum_{j=1}^{h_F} \mathcal{N}(\theta_j) \sum_{\boldsymbol{\delta}} |T|_{\text{sph}}^{\frac{1}{2}} |\boldsymbol{\delta}|_{\text{sph}}^{-\frac{1}{4}} |\boldsymbol{\delta}|_{\text{hol}}^{-1} |\hat{\boldsymbol{\delta}}|_{\text{hol}}^{-1-r}$$
$$\cdot \sum_{0 \neq \alpha \in \theta_j^2/\sim} \frac{\left| y_1((\alpha)\theta_j^{-2}) + y_2((\alpha)\theta_j^{-2}) \right|}{\sqrt{\mathcal{N}(\alpha)}} \sharp\{\gamma \in \Gamma_j(i,\alpha) \colon u_\nu(\gamma.P_\nu, P_\nu) \leq \delta_\nu \text{ for all } \nu\}. \quad (4.7.9)$$

We continue the estimation term by term. Starting from the sum

$$S_1^{deg}(\delta) = \sum_{j=1}^{h_F} \mathcal{N}(\theta_j) \sum_{0 \neq \alpha \in \theta_j^2/\sim} \frac{\left| y_1((\alpha)\theta_j^{-2}) \right|}{\sqrt{\mathcal{N}(\alpha)}} \left(\sharp\Gamma_j(i,\alpha,\boldsymbol{\delta})^0 + \sharp\Gamma_j(i,\alpha,\boldsymbol{\delta})^{par} \right)$$

we use Lemma 4.5.9 and 4.5.11 to estimate the matrix count. This yields

$$S_1^{deg}(\delta)$$
$$\ll |\hat{\boldsymbol{\delta}}|_\infty^\epsilon L^\epsilon \sum_{n,m \in \mathcal{P}(L)} |x_n x_m| \sum_{\mathfrak{a}|(n,m)} \left(\frac{\mathcal{N}(\mathfrak{a})}{\sqrt{\mathcal{N}(nm)}} + |y|_\infty |\boldsymbol{\delta}|_\infty^{\frac{1}{2}} + \frac{\sqrt{\mathcal{N}(nm)}}{\mathcal{N}(\mathfrak{a})} \frac{|\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{3}{4}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{4}} |\hat{\boldsymbol{\delta}}|_{\mathbb{R}}^{\frac{1}{8}}}{\mathcal{N}(\mathfrak{n}_2)} \right)$$
$$\ll |\hat{\boldsymbol{\delta}}|_\infty^\epsilon L^\epsilon \sum_{n,m \in \mathcal{P}(L)} \sum_{\mathfrak{a}|(n,m)} |x_n x_m| \frac{\mathcal{N}(\mathfrak{a})}{\sqrt{\mathcal{N}(nm)}}$$
$$+ \|x_1\|_1^2 \left(|\hat{\boldsymbol{\delta}}|_\infty^\epsilon |\boldsymbol{\delta}|_\infty^{\frac{1}{2}} |y|_\infty L^\epsilon + \frac{|\hat{\boldsymbol{\delta}}|_{\mathbb{R}}^{\frac{1}{8}+\epsilon} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{3}{4}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{n}_2)} L^{1+\epsilon} \right). \quad (4.7.10)$$

To estimate the remaining sum we view x as a function on ideals by setting $x_{\mathfrak{a}} = 0$ for non-principal ideals $\mathfrak{a}$. The following estimate is standard.

$$\sum_{n,m \in \mathcal{P}(L)} |x_n x_m| \sum_{\mathfrak{a}|(n,m)} \frac{\mathcal{N}(\mathfrak{a})}{\sqrt{\mathcal{N}(nm)}} = \sum_{\mathfrak{l}} \sum_{\substack{\mathcal{N}(\mathfrak{a}),\mathcal{N}(\mathfrak{b}) \leq L/\mathcal{N}(\mathfrak{l}), \\ (\mathfrak{a},\mathfrak{b})=1}} \sum_{\mathfrak{c}|\mathfrak{l}} |x_{\mathfrak{l}\mathfrak{a}} x_{\mathfrak{l}\mathfrak{b}}| \frac{\mathcal{N}(\mathfrak{c})}{\mathcal{N}(\mathfrak{l})\sqrt{\mathcal{N}(\mathfrak{a}\mathfrak{b})}}$$
$$\ll L^\epsilon \sum_{\mathfrak{l}} \sum_{\mathcal{N}(\mathfrak{a}),\mathcal{N}(\mathfrak{b}) \leq L/\mathcal{N}(\mathfrak{l})} \frac{|x_{\mathfrak{l}\mathfrak{a}}|^2}{\mathcal{N}(\mathfrak{b})} \ll L^\epsilon \|x_1\|_2.$$

This yields

$$S_1^{deg}(\delta) \ll |\hat{\boldsymbol{\delta}}|_\infty^\epsilon L^\epsilon \|x_1\|_2 + \|x_1\|_1^2 \left(|\hat{\boldsymbol{\delta}}|_\infty^\epsilon |\boldsymbol{\delta}|_\infty^{\frac{1}{2}} |y|_\infty L^\epsilon + \frac{|\hat{\boldsymbol{\delta}}|_{\mathbb{R}}^{\frac{1}{8}+\epsilon} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{3}{4}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{n}_2)} L^{1+\epsilon} \right).$$

The contribution of the generic matrices can be handled as follows.[5] We estimate

$$
\begin{aligned}
S_1^{gen}(\boldsymbol{\delta}) &\ll \sum_{\alpha,\beta\in\mathcal{P}(L)} |x_\alpha x_\beta| \sum_{\mathfrak{a}|(\alpha,\beta)} \frac{\mathcal{N}(\mathfrak{a})}{\sqrt{\mathcal{N}(\alpha\beta)}} \sharp\Gamma_*^{gen}(i,\frac{\alpha\beta}{\mathfrak{a}^2},\boldsymbol{\delta}) \\
&= \sum_{\mathfrak{a}} \sum_{\substack{(\mathfrak{b},\mathfrak{c})=1,\\ \mathfrak{ab},\mathfrak{ac}\in\mathcal{P}(L)}} \frac{|x_{\mathfrak{ab}}x_{\mathfrak{ac}}|}{\sqrt{\mathcal{N}(\mathfrak{bc})}} \sharp\Gamma_*^{gen}(i,\mathfrak{bc},\boldsymbol{\delta}) \\
&\ll \sum_{\mathfrak{a}} \frac{\mathcal{N}(\mathfrak{a})}{L} \sup_{\substack{(\mathfrak{b},\mathfrak{c})=1,\\ \mathfrak{ab},\mathfrak{ac}\in\mathcal{P}(L)}} |x_{\mathfrak{ab}}x_{\mathfrak{ac}}| \left(M_{*,i}^{gen}(\frac{L^2}{\mathcal{N}(\mathfrak{a}^2)},\boldsymbol{\delta}) + M_{*,i}^{gen}(\frac{2L^2}{\mathcal{N}(\mathfrak{a}^2)},\boldsymbol{\delta}) \right) \\
&\ll \left|\hat{\boldsymbol{\delta}}\right|_\infty \|x_1\|_\infty^2 \sum_{\mathcal{N}(\mathfrak{a})\leq 2L} \left(\frac{L}{\mathcal{N}(\mathfrak{a})} + \frac{L^{\frac{3}{2}} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{a})^{\frac{3}{2}}\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{4}}} + \frac{L^3 |\boldsymbol{\delta}|_{\mathbb{R}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{3}{4}}}{\mathcal{N}(\mathfrak{a})^3\mathcal{N}(\mathfrak{n}_2)} \right) \\
&\ll \left|\hat{\boldsymbol{\delta}}\right|_\infty L^\epsilon \|x_1\|_\infty^2 \left(L + \frac{L^{\frac{3}{2}} |\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{4}}} + \frac{L^3 |\boldsymbol{\delta}|_{\mathbb{R}} |\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{3}{4}}}{\mathcal{N}(\mathfrak{n}_2)} \right).
\end{aligned}
$$

Finally, we deal with the contribution of the $y_2(\alpha)$'s. To this end we observe that, due to our assumption

$$
|y_2(\mathfrak{a})| \leq \begin{cases} \|x_2\|_2 & \text{if } \mathfrak{a}=(1), \\ \|x_2\|_\infty^2 & \text{if } \mathfrak{a}=(\alpha)^2(\beta)^2 \text{ or } \mathfrak{a}=(\alpha)^2 \text{ for } \alpha,\beta\in\mathcal{P}(L), \\ 0 & \text{else.} \end{cases}
$$

In particular, the only contribution comes from principal ideals. We arrive at

$$
\begin{aligned}
S_2(\boldsymbol{\delta}) &\ll \|x_2\|_\infty^2 \frac{M_0(L,1,\delta)+M_2(L,1,\delta)+M_{*,i}^{gen,\square}(L^2,\delta)+M_{*,i}^{gen,\square}(2L^2,\delta)}{L} \\
&\quad + \|x_2\|_\infty^2 \frac{M(L,2,\delta)}{L^2} + \|x_2\|_2 \sharp\Gamma_*(i,1,\delta) \\
&\ll \|x_2\|_\infty^2 \left|\hat{\boldsymbol{\delta}}\right|_\infty^{\frac{1}{2}+\epsilon} L^\epsilon \left(L + L^2 |y|_\infty |\delta|_\infty^{\frac{1}{2}} + \frac{L^2 |\delta|_{\mathbb{R}}^{\frac{1}{2}}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}}} + \frac{L^4 |\delta|_{\mathbb{R}}^{\frac{3}{4}}}{\mathcal{N}(\mathfrak{n}_2)} \right) \\
&\quad + \|x_2\|_2 \left|\hat{\boldsymbol{\delta}}\right|_\infty (1+|y|_\infty |\delta|_\infty^{\frac{1}{2}}).
\end{aligned}
$$

All together we obtain

$$
K_f(g,g) \ll \mathcal{N}(\mathfrak{q})^{2+\epsilon} \sum_{\boldsymbol{\delta}} |T|_{\text{sph}}^{\frac{1}{2}} |\boldsymbol{\delta}|_{\text{sph}}^{-\frac{1}{4}} |\boldsymbol{\delta}|_{\text{hol}}^{-1} \left|\hat{\boldsymbol{\delta}}\right|_{\text{hol}}^{-1-r} \left(S_1^{deg}(\boldsymbol{\delta}) + S_1^{gen}(\boldsymbol{\delta}) + S_2(\boldsymbol{\delta}) \right).
$$

5 Here we exploit the 'averaged counting'. This leads to the appearance of an L^∞ norm of x_1 instead of L^1 and L^2 averages. However, the counting in this setup proves to be much more efficient and de-compensates for this caveat.

Performing the δ-sum yields

$$K_f(g,g) \ll L^{\epsilon}\mathcal{N}(\mathfrak{q})^{2+\epsilon}\Bigg[\|x_1\|_\infty^2\Big(L\,|T|_{sph}^{1+\epsilon}\,|k|_{hol}^{1+\epsilon} + L^{\frac{3}{2}}\frac{|T|_{sph}^{\frac{1}{2}+\epsilon}\,|T|_{\mathbb{C}}^{\frac{1}{2}}\,|k|_{hol}^{\frac{3}{4}+\epsilon}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{4}}} + L^3\frac{|T|_{sph}^{\frac{1}{2}+\epsilon}\,|k|_{hol}^{\epsilon}}{\mathcal{N}(\mathfrak{n}_2)}\Big)$$
$$+\|x_1\|_2\,|T|_{sph}^{1+\epsilon}\,|k|_{hol}^{1+\epsilon} + \|x_1\|_1^2\Big(|T|_{sph}^{\frac{1}{2}+\epsilon}\,|k|_{hol}^{\frac{1}{2}+\epsilon}\,|y|_\infty + L\frac{|T|_{sph}^{\frac{1}{2}+\epsilon}\,|k|_{hol}^{\frac{1}{4}+\epsilon}}{\mathcal{N}(\mathfrak{n}_2)}\Big)$$
$$+\|x_2\|_\infty^2\Big(L\,|T|_{sph}^{1+\epsilon}\,|k|_{hol}^{1+\epsilon} + L^2\,|T|_{sph}^{\frac{1}{2}+\epsilon}\,|k|_{hol}^{\frac{1}{2}+\epsilon}\,|y|_\infty$$
$$+L^2\frac{|T|_{sph}^{\frac{1}{2}+\epsilon}\,|T|_{\mathbb{C}}^{\frac{1}{2}}\,|k|_{hol}^{\frac{1}{2}+\epsilon}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}}} + L^4\frac{|T|_{sph}^{\frac{1}{2}+\epsilon}\,|k|_{hol}^{\frac{1}{4}+\epsilon}}{\mathcal{N}(\mathfrak{n}_2)}\Big)\Bigg].$$

The claimed result follows after transforming all the $x_{1,2}$-dependence in L^∞-norms and dropping some redundant terms. □

We will end this section by proving another estimate for the geometric side in a more specific situation.

Proposition 4.7.5. *Let $F^{\mathbb{R}}$ be the maximal totally real subfield of F and suppose that $[F\colon F^{\mathbb{R}}] = m \geq 2$. Take $(\mathfrak{q},\mathfrak{n}) = 1$ and*

$$g = a(\theta_i)g'n(x)a(y) \text{ with } g' \in K_{\mathfrak{n}}h_{\mathfrak{n}}^{-1} \text{ and } n(x)a(y) \in \mathcal{F}_{\mathfrak{n}_2}.$$

Further assume that the sequence x_1 and x_2 is supported on α^2 for $\alpha \in \mathcal{P}(L)$ which are principal prime ideals. If $S_{hol} = \emptyset$, we have

$$K_f(g,g) \ll \|x_1+x_2\|_\infty^2 L^{\epsilon}\mathcal{N}(\mathfrak{q})^{2+\epsilon}\left(L\,|T|_\infty + L^2\,|T|_\infty^{\frac{1}{2}}\,|y|_\infty + |T|_\infty^{\frac{1}{2}}\,L^{2m-1} + \frac{|T|_\infty^{\frac{1}{2}}\,L^{2m}}{\mathcal{N}(\mathfrak{n})^{\frac{1}{2}}}\right).$$

Proof. Since $S_{hol} = \emptyset$ we have $|\hat{\delta}|_\infty \ll 1$. Further we can exploit the special shape of y given in (4.7.5). With this at hand we follow the proof of Proposition 4.7.4 until (4.7.9). Here we estimate everything trivially arriving at

$$K_f(g,g) \leq \|x_1+x_2\|_\infty^2\mathcal{N}(\mathfrak{q})^{2+\epsilon}\sum_{\delta}\frac{|T|_\infty^{\frac{1}{2}}}{|\delta|^{\frac{1}{4}}}\left(LM(L,0,\delta) + \frac{M(L,1,\delta)}{L} + \frac{M(L,2,\delta)}{L^2}\right).$$

We use Remark 4.5.14 to estimate

$$M(K,0,\delta) \ll 1 + |y|_\infty\,|\delta|_\infty^{\frac{1}{2}}.$$

By Remark 4.5.12 and Lemma 4.5.15 we obtain

$$M(L,1,\delta) \ll L^{2+\epsilon} + L^{3+\epsilon}\,|y|_\infty\,|\delta|_\infty^{\frac{1}{2}} + \frac{L^{3+\epsilon}\,|\delta|_{\mathbb{R}}^{\frac{3}{4}}\,|\delta|_{\mathbb{C}}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{n}_2)} + L^{2m+\epsilon}\,|\delta|_{\mathbb{R}}^{\frac{1}{2}}\,|\delta|_{\mathbb{C}}^{\frac{1}{4}}$$
$$+\frac{L^{2m+1+\epsilon}\,|\delta|_{\mathbb{R}}\,|\delta|_{\mathbb{C}}^{\frac{3}{4}}}{\mathcal{N}(\mathfrak{n}_2)}.$$

Finally, we use Remark 4.5.12 and Lemma 4.5.13 to obtain

$$\begin{aligned} M(L,2,\boldsymbol{\delta}) &\ll L^{2+\epsilon}+L^{4+\epsilon}|y|_{\infty}|\boldsymbol{\delta}|_{\infty}^{\frac{1}{2}}+\frac{L^{4+\epsilon}|\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{1}{2}}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}}}+\frac{L^{6+\epsilon}|\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{3}{4}}|\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{n}_2)} \\ &\ll L^{2+\epsilon}+L^{4+\epsilon}|y|_{\infty}|\boldsymbol{\delta}|_{\infty}^{\frac{1}{2}}+\frac{L^{2m+2+\epsilon}|\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{1}{2}}|\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}}}+\frac{L^{6+\epsilon}|\boldsymbol{\delta}|_{\mathbb{R}}^{\frac{3}{4}}|\boldsymbol{\delta}|_{\mathbb{C}}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{n}_2)}. \end{aligned}$$

In the last step we artificially inserted the factor $|\boldsymbol{\delta}|_{\mathbb{C}}$ by using Lemma 4.5.15. These counting results allow us to execute the $\boldsymbol{\delta}$-sum and obtain the desired result. □

4.8 THE TWO MAIN SUP-NORM THEOREMS

We are finally ready to prove our main theorems. We start by a general upper bound with no assumptions on the base field F.

Theorem 4.8.1 ([2], Theorem 1.1)**.** *Let (π, V_π) be a cuspidal automorphic representation with conductor $\mathfrak{n}$ and spectral parameter $(t_\nu)_{\nu\in S_{sph}}$ and weight $(k_\nu)_{\nu\in S_{hol}}$. And let v° be a new vector of π. Then*

$$\begin{aligned} \|v^\circ\|_\infty \quad \ll_{F,\epsilon} \quad (|T|_{sph}|k|_{hol}\mathcal{N}(\mathfrak{n}))^{\epsilon}\mathcal{N}(\mathfrak{n}_0)^{\frac{1}{2}}\mathcal{N}(\mathfrak{m})^{\frac{1}{2}}\Big(|T|_{sph}^{\frac{5}{12}}|k|_{hol}^{\frac{7}{16}}\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{3}} \\ +|T|_{sph}^{\frac{1}{4}}|T|_{\mathbb{C}}^{\frac{1}{4}}|k|_{hol}^{\frac{1}{4}}\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{4}}\Big). \end{aligned}$$

Proof. By Corollary 4.4.7 it is enough to consider $\phi(g)=\phi_\circ^{\mathfrak{L}}(g)$ for some $\mathfrak{L}\mid\mathfrak{n}$. Further, we fix $1\leq i\leq h_F$ and restrict ourselves to $g=a(\theta_i)g'h_{\mathfrak{n}}n(x)a(y)$ with $n(x)a(y)\in\mathcal{F}_{\mathfrak{n}_2}$ and $g'h_{\mathfrak{n}}\in\mathcal{J}_{\mathfrak{n}}$.

We start by deriving a bound via amplification which will be strong in the bulk. We choose the following amplifier:

$$x_\alpha = \begin{cases} \frac{|\lambda_\pi((\alpha))|}{\lambda_\pi((\alpha))} & \text{if } \alpha\in(\mathcal{P}(L)) \text{ is a principal prime ideal,} \\ 0 & \text{else.} \end{cases}$$

Similarly, we chose

$$x_{\alpha^2} = \begin{cases} \frac{|\lambda_\pi((\alpha)^2)|}{\lambda_\pi((\alpha)^2)} & \text{if } \alpha\in(\mathcal{P}(L)) \text{ is a principal prime ideal,} \\ 0 & \text{else.} \end{cases}$$

Note that we have $\|x_1\|_\infty=\|x_2\|_\infty=1$ and

$$\|x_1+x_2\|_{1,\pi}\gg\sharp\{\alpha\in\mathcal{P}(L)\colon(\alpha)\text{ is a prime ideal}\}\gg L^{1-\epsilon}.$$

Where we used the fact that[6]

$$\frac{|\lambda_\pi(\mathfrak{p})|}{\sqrt{\mathcal{N}(\mathfrak{p})}} + \frac{|\lambda_\pi(\mathfrak{p}^2)|}{\mathcal{N}(\mathfrak{p})} \gg 1.$$

With this at hand we use (4.7.7) together with Proposition 4.7.4 to obtain

$$|\phi'(g)| \ll |L|_\infty^\epsilon \mathcal{N}(\mathfrak{n})^\epsilon \mathcal{N}(\mathfrak{n}_1\mathfrak{m}_1)^{\frac{1}{2}} \Bigg(\frac{|T|_{sph}^{\frac{1}{2}} |k|_{hol}^{\frac{1}{2}}}{L^{\frac{1}{2}}} + \frac{L\,|T|_{sph}^{\frac{1}{4}} |k|_{hol}^{\frac{1}{8}}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}}} + \frac{|T|_{sph}^{\frac{1}{4}} |T|_{\mathbb{C}}^{\frac{1}{4}} |k|_{hol}^{\frac{1}{4}}}{\mathcal{N}(\mathfrak{n}_2)^{\frac{1}{4}}} + |T|_{sph}^{\frac{1}{4}} |k|_{hol}^{\frac{1}{4}} |y|_\infty^{\frac{1}{2}} \Bigg).$$

Taking $L = |T|_{sph}^{\frac{1}{6}} |k|_{hol}^{\frac{1}{4}} \mathcal{N}(\mathfrak{n}_2)^{\frac{1}{3}}$ and inquiring $|y|_\infty \le |T|_{sph}^{\frac{1}{3}} |k|_{hol}^{\frac{3}{8}} \mathcal{N}(\mathfrak{n}_2)^{-\frac{1}{3}}$ produces the stated bound.

If $|y|_\infty > |T|_{sph}^{\frac{1}{3}} |k|_{hol}^{\frac{3}{8}} \mathcal{N}(\mathfrak{n}_2)^{-\frac{1}{3}}$, then Proposition 4.6.12 yields

$$|\phi(g)| \ll_{F,\epsilon} (|T|_{sph} |k|_{hol} \mathcal{N}(\mathfrak{n}))^\epsilon \Bigg(|T|_{sph}^{\frac{1}{6}} |k|_{hol}^{\frac{1}{4}} \mathcal{N}(\mathfrak{n}_0)^{\frac{1}{2}} + |T|_{sph}^{\frac{1}{3}} |k|_{hol}^{\frac{7}{16}} \mathcal{N}(\mathfrak{n}_2)^{\frac{1}{6}} \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1)^{\frac{1}{2}} \Bigg).$$

This concludes the proof. □

The previous theorem features a contribution containing $|T|_{\mathbb{C}}^{\frac{1}{2}}$ which is the local bound. Thus, we do not achieve subconvexity in full generality. To deal with this caveat the author's of [20] came up with a very sophisticated counting argument specifically for non-totally real fields. The second theorem, generalising [20, Theorem 2], relies on this counting and achieves subconvexity in every setting. However, the exponents are not as good in general. Unfortunately we also have to assume that $S_{hol} = \emptyset$. This is due to our inability to adjust the modified counting results accordingly.

Theorem 4.8.2 ([2], Theorem 1.2). *Let F be number field with maximal totally real subfield $F^{\mathbb{R}}$ such that $[F : F^{\mathbb{R}}] = m \geq 2$. Assume that $S_{hol} = \emptyset$. For a cuspidal automorphic representation (π, σ) with conductor $\mathfrak{n}$ and spectral parameter $(t_\nu)_{\nu \in S_{sph}}$ we have*

$$\|v^\circ\|_\infty \ll_{F,\epsilon} (|T|_\infty \mathcal{N}(\mathfrak{n}))^\epsilon |T|_\infty^{\frac{1}{2} - \frac{1}{8m-4}} \mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2} - \frac{1}{8m-4}} \mathcal{N}(\mathfrak{n}_0)^{\frac{1}{2}} \mathcal{N}(\mathfrak{m})^{\frac{1}{2}}$$

where v° is a new vector.

Proof. We start by using Corollary 4.4.7 to reduce the problem as far as possible. Observe that for $|y|_\infty > |T|_\infty^{\frac{1}{4}}$ the estimate in Proposition 4.6.12 gives the upper bound

6 The same trick is used in [20, (9.17)]. But recall that our Hecke-eigenvalues are normalised differently.

$\mathcal{N}(\mathfrak{n})^{\epsilon}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1)^{\frac{1}{2}}|T|_{\infty}^{\frac{3}{8}+\epsilon}$. Therefore we assume that $|y|_{\infty} < |T|_{\infty}^{\frac{1}{4}}$. Using Proposition 4.7.5 with amplifier as in the proof above and with

$$L = \min\left((|T|_{\infty}\mathcal{N}(\mathfrak{n}_2))^{\frac{1}{4m-1}}, C\,|T|_{\infty}^{\frac{1}{4m-4}}\right)$$

yields uniform bound

$$|\phi'_{\circ}(g)| \ll_{F,\epsilon} (\mathcal{N}(\mathfrak{n}_2)\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1)\,|T|_{\infty})^{\frac{1}{2}+\epsilon}\left(|T|_{\infty}^{-\frac{1}{8m-8}} + (|T|_{\infty}\mathcal{N}(\mathfrak{n}_2))^{-\frac{1}{8m-4}}\right)\|\phi'_{\circ}\|_2.$$

If $|T|_{\infty}^{-\frac{1}{8m-8}} \geq \mathcal{N}(\mathfrak{n}_2)^{-\frac{1}{4}}$, we can use Theorem 4.8.1 to get a better bound. This leads to

$$\frac{\sigma(v^{\circ})(g)}{\|\sigma(v^{\circ})\|_2} \ll_{F,\epsilon} (\mathcal{N}(\mathfrak{n}_2)\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1)\,|T|_{\infty})^{\frac{1}{2}+\epsilon} \cdot\left(\min(|T|_{\infty}^{-\frac{1}{8m-8}}, \mathcal{N}(\mathfrak{n}_2)^{-\frac{1}{4}}) + (|T|_{\infty}\mathcal{N}(\mathfrak{n}_2))^{-\frac{1}{8m-4}}\right).$$

One concludes by interpolation as in [20]. □

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