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TWIN PRIME CORRELATIONS FROM THE PAIR CORRELATION OF RIEMANN ZEROS

J. P. KEATING AND D. J. SMITH

ABSTRACT. We establish, via a heuristic Fourier inversion calculation, that the Hardy-Littlewood twin prime conjecture is equivalent to an asymptotic formula for the two-point correlation function of Riemann zeros at a height E on the critical line. Previously it was known that the Hardy-Littlewood conjecture implies the pair correlation formula, and we show that the reverse implication also holds. An averaged form of the Hardy-Littlewood conjecture is obtained by inverting the $E \to \infty$ limit of the two-point correlation function and the precise form of the conjecture is found by including asymptotically lower order terms in the two-point correlation function formula.

1. INTRODUCTION

The Riemann zeta-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1.1}$$

when $\text{Re}(s) > 1$ and then by analytic continuation. It has zeros at the negative even integers, and infinitely many zeros lying off the real line which are termed the *non-trivial zeros*. The non-trivial zeros lie in the strip $0 < \text{Re}(s) < 1$, and the Riemann Hypothesis asserts that they all lie on the *critical line* $\text{Re}(s) = 1/2$; that is, they all lie at points $s = 1/2 + iE$, where E is real. The *functional equation* asserts that the zeta function, when multiplied by $\pi^{-s/2}\Gamma(s/2)$, is symmetric with respect to reflection in the critical line [37]. The zeros of the zeta function are of central importance in Number Theory.

There has for a long time been an interest in a conjectural connection between the non-trivial zeros of the Riemann zeta-function and the semi-classical theory of quantum chaotic systems [2, 4, 10, 27]. This is motivated by two observations. First, the explicit formula relating the zeros and the primes closely resembles the semiclassical trace formula connecting classical periodic orbits and quantum energy levels in chaotic systems. Second, Montgomery conjectured that the limiting pair correlation of the zeros coincides with that of the eigenvalues of random complex Hermitian matrices drawn from the Gaussian Unitary Ensemble (GUE) of Random Matrix Theory (RMT) [32], and proved a theorem consistent with this; and it is one of the central conjectures of Quantum Chaos that the energy levels of generic, classically chaotic, non-time-reversal-symmetric systems should, on the scale of the mean level separation, also exhibit GUE statistics in the semiclassical limit.

Odlyzko computed statistics close to the 10^{20}th zero of the zeta function, finding that there is a remarkably close agreement between the numerical data and predictions based on Montgomery's conjecture on the scale of the mean zero spacing, but that there are significant deviations on scales large compared to the mean zero spacing [34]. In [3], Berry wrote down a formula describing Odlyzko's data uniformly. This augments the random-matrix limit with lower order terms which describe the pair correlations at a height E on the critical line and which vanish when $E \to \infty$.

It was shown in [7] that the lower order terms identified by Berry may be expressed in terms of the lowest zeros of the zeta function; that is, there is a resurgent relationship between the pair correlation of the high-lying zeros and the positions of the low-lying zeros. Moreover, an additional contribution from the low-lying zeros not captured in Berry's formula was identified. For a review of these formulae and a comparison with numerical data, see [4]. The results described above concerning the pair correlation of the Riemann zeros have been extended to other principal L-functions [26, 35] and to all n-tuple correlations [5, 6, 35]. For an overview, see [28, 29].

Connections between RMT and the statistical properties of the Riemann zeta-function and other L-functions have subsequently been developed in a number of different directions, including moments [12, 30, 31], their connections with divisor correlations [14–18], and ratios formulae [13]. The connections have led to several alternative heuristic derivations of the pair correlation formula with lower order terms first derived for the zeta function in [7]; see, for example, [8, 9, 19–21].

In his paper [32], Montgomery mentioned heuristics based on the Hardy-Littlewood conjecture concerning the distribution of prime pairs with a given separation (known as *twin primes*) [24] to justify his conjecture that the limiting pair-correlations of the zeta zeros coincide those of GUE eigenvalues. He did not give the details of this calculation, but it has subsequently been repeated with variations several times in the literature, for example in [5, 6, 22, 27]. Goldston and Montgomery [23] proved rigorously that the limiting pair correlation conjecture is equivalent to an asymptotic formula for the variance of the number of primes in short intervals, and Montgomery and Soundararajan [33] proved that this variance formula follows from the Hardy-Littlewood twin prime conjecture, under certain assumptions. The analysis in [23] has since been extended to all L-functions in the Selberg Class [11, 36]. The Hardy-Littlewood twin prime conjecture was also the basis of the original heuristic derivation of the pair correlation formula including lower order terms derived in [7]; that is, the Hardy-Littlewood twin prime conjecture heuristically implies the the pair correlation formula including all lower order terms.

Our purpose in this brief note is to show via a heuristic calculation that the reverse implication is also true, namely that the the pair correlation formula including lower order terms obtained in [7] can be used to derive the Hardy-Littlewood twin prime conjecture. Thus, heuristically, the two formulae are equivalent. (Here by heuristic we mean that the calculation relies on unproved conjectures and does not attempt to estimate the errors associated with various approximations.) We demonstrate this via Fourier

inversion of the pair correlation formula. This calculation extends one reported in [1], where an averaged form of the Hardy-Littlewood twin prime conjecture was obtained from the random-matrix limiting expression for the pair correlations, in a manner similar to [23].

We denote the non-trivial zeros of the Riemann zeta-function by $1/2+iE_n$ and assume the Riemann Hypothesis, so that $E_n \in \mathbb{R} \ \forall n$. Let $d(E) = \sum_n \delta(E - E_n)$ denote the density of zeros at height E on the critical line. The pair correlation function may then be defined by

$$R_2(\epsilon) = \langle d(E - \epsilon/2)d(E + \epsilon/2) \rangle, \tag{1.2}$$

where the angular brackets denote an average with respect to E. It may be expressed in the form

$$R_2(\epsilon) = \bar{d}^2(E) + R_2^{(diag)}(\epsilon) + R_2^{(off)}(\epsilon), \tag{1.3}$$

where

$$\bar{d}(E) = \frac{1}{2\pi} \ln\left(\frac{E}{2\pi}\right) \tag{1.4}$$

is the asymptotic mean density of the zeros, $R_2^{(diag)}(\epsilon)$ represents the contributions from the diagonal terms when the pair correlation function is expressed as a sum over pairs of primes using the explicit formula, and $R_2^{(off)}(\epsilon)$ represents the off-diagonal terms [27].

The diagonal part of the correlation function may be evaluated to give

$$\lim_{E\to\infty} \frac{1}{\bar{d}^2(E)} R_2^{(diag)}\left(\epsilon/\bar{d}(E)\right) = -\frac{1}{2\pi^2\epsilon^2}. \tag{1.5}$$

Montgomery's conjecture is then equivalent to

$$\lim_{E\to\infty} \frac{1}{\bar{d}^2(E)} R_2^{(off)}\left(\epsilon/\bar{d}(E)\right) = \frac{\cos(2\pi\epsilon)}{2\pi^2\epsilon^2}. \tag{1.6}$$

For finite E the diagonal part is given by [4, 7]

$$-\frac{1}{4\pi^2}\left(\frac{d^2}{dw^2}\ln\zeta(1+iw) + \sum_{\substack{p \\ k\geq 0}}(\ln p)^2 k p^{-(1+i\epsilon)(k+1)}\right)_{w=\epsilon} + \text{ c.c.} \tag{1.7}$$

The off-diagonal correlations of Riemann zeros may be expressed in terms of primes through the von Mangoldt function;

$$\Lambda(n) = \begin{cases} \ln p & \text{if } n \text{ is a power of a prime, } p, \\ 0 & \text{otherwise}; \end{cases} \tag{1.8}$$

explicitly [27],

$$R_2^{(off)}(\epsilon) = \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\sum_{h\neq 0}\frac{\Lambda(n+h)\Lambda(n)}{n}\exp(i(\epsilon\ln n - Eh/n)) + \text{c.c.} \tag{1.9}$$

The finite-E expression for the off-diagonal contribution to the pair correlation of Riemann zeros can then be calculated by making use of a conjecture

by Hardy and Littlewood [24] concerning the distribution of pairs of primes. This states that if

$$\alpha(h) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \Lambda(n+h)\Lambda(n), \qquad (1.10)$$

then

$$\alpha(h) = \begin{cases} 2C_2 \displaystyle\prod_{p|h} \frac{p-1}{p-2} & \text{if } h \text{ is even} \\ 0 & \text{otherwise}, \end{cases} \qquad (1.11)$$

where C_2 is the twin prime constant

$$C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2} \right) \approx 0.6601618. \qquad (1.12)$$

Using the Hardy-Littlewood conjecture, the finite-E off-diagonal contribution to the two-point correlation function was evaluated in [7] to give

$$R_2^{(off)}(\epsilon) = \frac{1}{4\pi^2} |\zeta(1+i\epsilon)|^2 \left(\exp(-2\pi i\epsilon\bar{d}(E)) \prod_p \left(1 - \left(\frac{1-p^{-i\epsilon}}{p-1} \right)^2 \right) \right) + \text{c.c.} \qquad (1.13)$$

This formula has since been derived using a number of other approaches [8, 9, 19–21]. It reduces to (1.6) in the limit $E \to \infty$.

The purpose of this note is to demonstrate a dual relationship between the correlations of zeros and primes. An averaged form of the Hardy-Littlewood conjecture will first be determined by making use of the limiting form of the pair correlation of zeros (1.6) and then the full Hardy-Littlewood conjecture will be recovered using finite E corrections to the pair correlation, as captured by (1.13).

We can therefore summarise the connections between the various conjectures discussed here as follows. It was proved by Goldston & Montgomery [23] that Montgomery's conjecture relating to the pair correlation of the Riemann zeros is equivalent to a conjecture concerning the variance of sums of the von Mangoldt function $\Lambda(n)$ over short intervals; Montgomery's conjecture is also heuristically equivalent to an averaged version of the Hardy-Littlewood conjecture for the pair correlations of the von Mangoldt function [1, 27, 32], and the conjecture concerning the variance of sums of the von Mangoldt function over short intervals follows, under certain additional assumptions, from the Hardy-Littlewood conjecture [33]. The Hardy-Littlewood conjecture heuristically implies the Montgomery conjecture with lower order terms, if no averaging of it is employed [7], and what we have shown heuristically here is that the reverse implication holds as well, so these two formulae may be considered as being equivalent.

2. Inverting the 2-point correlation

Replacing $\Lambda(n+h)\Lambda(n)$ in (1.9) by $\alpha(h)$, and approximating the sums by integrals, heuristically, the off-diagonal contribution can be expressed as

$$R_2^{(off)}(\epsilon) = \frac{1}{4\pi^2} \int_1^\infty \int_{-\infty}^\infty \frac{\alpha(h)}{y} \exp(i(\epsilon \ln y - Eh/y)) \, dh \, dy + \text{c.c.}, \qquad (2.1)$$

c.f. [5, 6, 27], and so an expression for $\alpha(h)$ may be determined by Fourier inversion, as follows.

$$\int_{-\infty}^{\infty} R_2^{(off)}(\epsilon) \, d\epsilon = \frac{1}{2\pi} \int_1^{\infty} \int_{-\infty}^{\infty} \frac{\alpha(h)}{y} \delta(\ln y) \exp(-iEh/y) \, dh dy + \text{ c.c} \tag{2.2}$$

$$= \frac{1}{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \alpha(h)\delta(z) \exp(-iEh \exp(-z)) \, dh dz + \text{ c.c} \tag{2.3}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha(h) \exp(-iEh) \, dh. \tag{2.4}$$

Giving the Fourier transform

$$\mathscr{F}[\alpha(h)] = 2\pi \int_{-\infty}^{\infty} R_2^{(off)}(\epsilon) \, d\epsilon, \tag{2.5}$$

and by Fourier inversion,

$$\alpha(h) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_2^{(off)}(\epsilon) \exp(ihE) \, d\epsilon dE. \tag{2.6}$$

3. Averaged form of the Hardy-Littlewood conjecture

In [27] Montgomery's expression for the pair correlation was shown heuristically to follow from knowledge of the average of the function in the Hardy-Littlewood conjecture. Asymptotically as $|h| \to \infty$ the average is given by $1 - \frac{1}{2|h|}$, in the sense that

$$\frac{1}{2|h|} \sum_{H=-h}^{h} \alpha(H) \sim 1 - \frac{\ln|h|}{2|h|} \text{ as } |h| \to \infty. \tag{3.1}$$

The $-\frac{1}{2|h|}$ term in the averaged Hardy-Littlewood conjecture was recovered in [1] by assuming Montgomery's conjecture (the first term does not contribute to the zero correlations), specifically by inverting the off-diagonal contribution of the limiting pair correlation (1.6). For completeness, this calculation is shown here to follow from the inversion formula (2.6).

Substituting the limiting form of the off-diagonal pair correlation gives the integral over ϵ,

$$\frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{1}{\epsilon^2} \cos(2\pi\epsilon\bar{d}(E)) \, d\epsilon = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\epsilon^2} \cos(\epsilon\bar{d}(E)) \, d\epsilon$$

$$= \frac{1}{\pi} \mathscr{F}\left[\frac{1}{\epsilon^2}\right] (\bar{d}(E)). \tag{3.2}$$

This Fourier transform may be deduced by considering the Fourier transform of the triangle function

$$T(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

and by noting that the triangle function may be expressed in terms of the sign function,

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{otherwise,} \end{cases} \tag{3.4}$$

by the linear relation

$$T(x) = \frac{1}{2}(1-x)\text{sgn}(1-x) + \frac{1}{2}(1+x)\text{sgn}(1+x) - x\text{sgn}(x). \tag{3.5}$$

Then

$$\mathscr{F}[T](k) = 2\int_0^1 (1-x)\cos(kx)\,dx = \left(\text{sinc}\left(\frac{k}{2}\right)\right)^2, \tag{3.6}$$

by integrating by parts, where

$$\text{sinc}(x) = \frac{\sin(x)}{x}. \tag{3.7}$$

Then

$$\int_{-\infty}^{\infty} \left(\text{sinc}\left(\frac{k}{2}\right)\right)^2 \exp(ikx)\,dk = \pi(1-x)\text{sgn}(1-x)$$
$$+ \pi(1+x)\text{sgn}(1+x)$$
$$- 2\pi x\text{sgn}(x) \tag{3.8}$$

by Fourier inversion. Expanding the sinc function in terms of exponential functions then gives

$$\int_{-\infty}^{\infty} \left(\text{sinc}\left(\frac{k}{2}\right)\right)^2 \exp(ikx)\,dk = -\mathscr{F}\left[\frac{1}{k^2}\right](1-x)$$
$$- \mathscr{F}\left[\frac{1}{k^2}\right](1+x)$$
$$+ 2\mathscr{F}\left[\frac{1}{k^2}\right](x), \tag{3.9}$$

yielding the identification

$$\mathscr{F}\left[\frac{1}{x^2}\right](k) = -\pi k\,\text{sgn}(k). \tag{3.10}$$

The averaged correlation function is then given by

$$-\mathscr{F}\left[\bar{d}(E)\text{sgn}(\bar{d}(E))\right](h). \tag{3.11}$$

Substituting the expression (1.4) for the mean density and truncating the integral (beyond the truncation range the integrand is highly oscillatory and so the integral there makes a negligible contribution) gives

$$\frac{1}{\pi}\int_0^1 \ln(E)\cos(hE)\,dE \text{ as } |h| \to \infty. \tag{3.12}$$

Integration by parts then yields

$$-\frac{1}{\pi h}\text{Si}(h) \text{ as } |h| \to \infty, \tag{3.13}$$

where the sine integral,

$$\text{Si}(x) = \int_0^x \text{sinc}(t)\,dt, \tag{3.14}$$

has been introduced.

This gives, to leading asymptotic order as $|h| \to \infty$, $-\frac{1}{2|h|}$, since $\lim_{x\to\infty} \mathrm{Si}(x) = \frac{\pi}{2}$; recovering the result found in [1].

4. The full Hardy-Littlewood conjecture

4.1. **Off-diagonal pair correlation.** As reviewed in the introduction, various heuristic calculations [7, 8, 19–21] lead to an expression for the off-diagonal contribution to the two-point correlation function;

$$\frac{1}{2\pi^2} \exp(-2\pi i\epsilon\bar{d}(E))|\zeta(1+i\epsilon)|^2 \prod_p \frac{\left(1 - \frac{1}{p^{1+i\epsilon}}\right)(p^2 - 2p + p^{1-i\epsilon})}{(p-1)^2}. \quad (4.1)$$

Using the Euler product on the 1-line allows us to rewrite this as

$$\frac{1}{2\pi^2} \exp(-2\pi i\epsilon\bar{d}(E))\zeta(1-i\epsilon) \prod_p \left(1 - \frac{1}{(p-1)^2} + \frac{p^{1-i\epsilon}}{(p-1)^2}\right)$$

$$= \frac{1}{2\pi^2} \exp(-2\pi i\epsilon\bar{d}(E))\zeta(1-i\epsilon) \prod_p \left(1 + \frac{p^{1-i\epsilon}\left(1 - \frac{1}{p^{1-i\epsilon}}\right)}{(p-1)^2}\right). \quad (4.2)$$

Euler's totient function $\phi(n)$ is the number of natural numbers less than and coprime to n; and the Möbius function $\mu(n)$ is zero if n is divisible by a square, and -1 to the power of the number of distinct prime factors of n otherwise [25]. Both are multiplicative functions. This allows the products over primes to be changed to sums over natural numbers in the following way.

$$\frac{1}{2\pi^2} \exp(-2\pi i\epsilon\bar{d}(E))\zeta(1-i\epsilon) \prod_p \left(1 + \left(\frac{\mu(p)}{\phi(p)}\right)^2 p^{1-i\epsilon}\left(1 + \frac{\mu(p)}{p^{1-i\epsilon}}\right)\right), \quad (4.3)$$

which, by multiplicativity of ϕ and μ yields

$$\frac{1}{2\pi^2} \exp(-2\pi i\epsilon\bar{d}(E))\zeta(1-i\epsilon) \sum_{n=1}^{\infty} \left(\frac{\mu(n)}{\phi(n)}\right)^2 n^{1-i\epsilon} \prod_{p|n} \left(1 + \frac{\mu(p)}{p^{1-i\epsilon}}\right), \quad (4.4)$$

where the sum is over squarefree values of n due to the presence of $\mu(n)$. Making use of the multiplicativity of μ once more gives

$$\frac{1}{2\pi^2} \exp(-2\pi i\epsilon\bar{d}(E))\zeta(1-i\epsilon) \sum_{n=1}^{\infty} \left(\frac{\mu(n)}{\phi(n)}\right)^2 n^{1-i\epsilon} \sum_{d|n} \frac{\mu(d)}{d^{1-i\epsilon}}. \quad (4.5)$$

Using now the series representation of the Riemann zeta-function on the 1-line results in

$$\frac{1}{2\pi^2} \exp(-2\pi i\epsilon\bar{d}(E)) \sum_{n=1}^{\infty} \left(\frac{\mu(n)}{\phi(n)}\right)^2 n^{1-i\epsilon} \sum_{m=1}^{\infty} \sum_{d|n} \frac{\mu(d)}{(md)^{1-i\epsilon}}. \quad (4.6)$$

Replacing md by l, the sum over m and $d|n$ becomes a sum over l, $d|n$ and $d|l$,

$$\sum_{l=1}^{\infty} \sum_{d|n,d|l} \frac{\mu(d)}{l^{1-i\epsilon}} = \sum_{l=1}^{\infty} \frac{1}{l^{1-i\epsilon}} \sum_{d|(l,n)} \mu(d), \quad (4.7)$$

which, by the indicator property of the the Möbius function [25] gives

$$\sum_{\substack{l=1 \\ (l,n)=1}}^{\infty} \frac{1}{l^{1-i\epsilon}}. \tag{4.8}$$

The two-point correlation function is therefore given by

$$\frac{1}{2\pi^2} \exp(-2\pi i\epsilon \bar{d}(E)) \sum_{n=1}^{\infty} \sum_{\substack{l=1 \\ (l,n)=1}}^{\infty} \left(\frac{\mu(n)}{\phi(n)}\right)^2 \left(\frac{n}{l}\right)^{1-i\epsilon}. \tag{4.9}$$

4.2. **Full inversion.** Substituting into the Fourier inversion formula after substituting the expression (1.4) for the mean density and truncating the sum over l at n (i.e. retaining just the largest, and least oscillatory terms) then gives the approximation

$$\frac{1}{2\pi^2} \sum_{n=1}^{\infty} \sum_{\substack{l=1 \\ (l,n)=1}}^{n} \frac{n}{l} \left(\frac{\mu(n)}{\phi(n)}\right)^2 \int_{-\infty}^{\infty} \exp(ihE) \int_{-\infty}^{\infty} \exp\left(-i\epsilon \ln\left(\frac{En}{2\pi l}\right)\right) d\epsilon dE. \tag{4.10}$$

Integrating over ϵ introduces a delta function, giving

$$\frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{\substack{l=1 \\ (l,n)=1}}^{n} \frac{n}{l} \left(\frac{\mu(n)}{\phi(n)}\right)^2 \int_{0}^{\infty} \delta\left(\ln\left(\frac{En}{2\pi l}\right)\right) \exp(ihE) \, dE, \tag{4.11}$$

where the domain of the logarithm restricts the integral over $E > 0$. Changing variables and truncating the upper limit (neglecting any highly oscillatory contribution to the integral) gives the approximation

$$2 \sum_{n=1}^{\infty} \sum_{\substack{l=1 \\ (l,n)=1}}^{n} \left(\frac{\mu(n)}{\phi(n)}\right)^2 \int_{0}^{1} \delta(\ln w) \exp((2\pi i whl)/n) \, dw, \tag{4.12}$$

which after a further change of variables gives

$$2 \sum_{n=1}^{\infty} \sum_{\substack{l=1 \\ (l,n)=1}}^{n} \left(\frac{\mu(n)}{\phi(n)}\right)^2 \int_{-\infty}^{0} \exp(z)\delta(z) \exp((2\pi i \exp(z)hl)/n) \, dz. \tag{4.13}$$

Finally integrating over $z \leq 0$ captures half the mass of the delta function, resulting in

$$\sum_{n=1}^{\infty} \sum_{\substack{l=1 \\ (l,n)=1}}^{n} \left(\frac{\mu(n)}{\phi(n)}\right)^2 \exp((2\pi i hl)/n). \tag{4.14}$$

Introducing the Ramanujan sum [25],

$$c_n(h) = \sum_{\substack{l=1 \\ (l,n)=1}}^{n} \exp((2\pi i lh)/n), \tag{4.15}$$

this can be written as

$$\sum_{n=1}^{\infty} \left(\frac{\mu(n)}{\phi(n)}\right)^2 c_n(h). \tag{4.16}$$

Ramanujan sums are multiplicative,

$$c_m(l)c_n(l) = c_{mn}(l) \tag{4.17}$$

if $(m,n) = 1$. They also satisfy

$$c_p(n) = \begin{cases} p-1 & \text{if } p|n \\ -1 & \text{otherwise} \end{cases} \tag{4.18}$$

for all primes, p. Using these properties then gives

$$\sum_{n=1}^{\infty} \left(\frac{\mu(n)}{\phi(n)}\right)^2 c_n(h)$$

$$= \prod_p \left(1 + \left(\frac{\mu(p)}{\phi(p)}\right)^2 c_p(h)\right)$$

$$= \begin{cases} 2\prod_{p>2} \left(1 + \left(\frac{\mu(p)}{\phi(p)}\right)^2 c_p(h)\right) & \text{if } h \text{ is even} \\ 0 & \text{otherwise,} \end{cases} \tag{4.19}$$

where

$$\prod_{p>2} \left(1 + \left(\frac{\mu(p)}{\phi(p)}\right)^2 c_p(h)\right)$$

$$= C_2 \prod_{p|h} \left(\frac{p-1}{p-2}\right). \tag{4.20}$$

Thus the exact Hardy-Littlewood twin prime conjecture may be derived heuristically from the pair correlation formula for the Riemann zeros including lower order terms. This is what we set out to demonstrate. As already shown in [7] the reverse implication also holds heuristically. Therefore the two conjectures may be thought of as being heuristically equivalent.

Finally, we again point out that this calculation involves several approximations for which we have not rigorously estimated the errors. It would be interesting to do this because, whilst we believe that the errors are small compared to the leading order expression in the Hardy-Littlewood twin prime conjecture, it remains to demonstrate this explicitly.

5. Acknowledgements

References

[1] N. Argaman, F. M. Dittes, E. Doron, J.P. Keating, A. Kitaev, M. Sieber & U. Smilansky, *Correlations in the actions of periodic orbits derived from quantum chaos*, Phys. Rev. Lett. 71 (1993), 4326–4329.

[2] M.V. Berry. Riemann's zeta function: a model for quantum chaos?. In Quantum chaos and statistical nuclear physics, eds. T H Seligman and H Nishioka, Springer Lecture Notes in Physics No. 263 (1986), 1-17.

[3] M.V. Berry, *Semiclassical formula for the number variance of the Riemann zeros*, Nonlinearity 1 (1988), 399-407.

[4] M. V. Berry & J. P. Keating, *The Riemann zeros and eigenvalue asymptotics*, SIAM Rev. 41 (1999), 236–266.

[5] E. B. Bogomolny & J. P. Keating, *Random matrix theory and the Riemann zeros I: three- and four-point correlations*, Nonlinearity 8 (1995), 1115–1131.

[6] E. B. Bogomolny & J. P. Keating, *Random matrix theory and the Riemann zeros II: n-point correlations*, Nonlinearity 9 (1996), 911–935.

[7] E. B. Bogomolny & J. P. Keating, *Gutzwiller's trace formula and spectral statistics: beyond the diagonal approximation*, Phys. Rev. Lett. **77** (1996), 1472–1475.

[8] E. B. Bogomolny & J. P. Keating, *Two-point correlation function for Dirichlet L-functions*, J. Phys. A 46 (2013), 095202

[9] E. B. Bogomolny & J. P. Keating, *A method for calculating spectral statistics based on random-matrix universality with an application to the three-point correlations of the Riemann zeros*, J. Phys. A 46 (2013), 305203

[10] P. Bourgade & J. P. Keating, *Quantum chaos, random matrix theory, and the Riemann ζ-function*, in Chaos: Poincaré Seminar 2010, Progress in Mathematical Physics vol. 66, edited by B. Duplantier, S. Nonnenmacher and V. Rivasseau (Springer) (2013), 125–168.

[11] H. M. Bui, J. P. Keating & D. J. Smith, *On the variance of sums of arithmetic functions over primes in short intervals and pair correlation for L-functions in the Selberg class*, J. Lond. Math. Soc. 94 (2016), 161-185.

[12] J. B. Conrey, D. W. Farmer, J.P. Keating, M.O. Rubinstein, & N.C. Snaith, *Integral moments of L-functions*, Proc. Lond. Math. Soc. 91 (2005), 33–104.

[13] J. B. Conrey, D. W. Farmer & M. R. Zirnbauer, *Autocorrelation of ratios of L-functions*, Commun. Number Theory Phys. 2 (2008), 593–636.

[14] J. B. Conrey & J. P. Keating, *Moments of zeta and correlations of divisor-sums: I*, Phil. Trans. R. Soc. A 373 (2015), 20140313. arXiv:1506.06842

[15] J. B. Conrey & J. P. Keating, *Moments of zeta and correlations of divisor-sums: II*, in Advances in the Theory of Numbers – Proceedings of the Thirteenth Conference of the Canadian Number Theory Association, Fields Institute Communications (Editors: A. Alaca, S. Alaca & K.S. Williams), 75–85 (2015, Springer); arXiv:1506.06843

[16] J. B. Conrey & J. P. Keating, *Moments of zeta and correlations of divisor-sums: III*, Indagationes Mathematicae 26 (2015), no. 5, 736–747; arXiv:1506.06844

[17] J. B. Conrey and J. P. Keating, *Moments of zeta and correlations of divisor-sums: IV*, Res. Number Theory (2016) 2:24; arXiv:1506.06844

[18] J. B. Conrey & J. P. Keating, *Moments of zeta and correlations of divisor-sums: V*, Proc. Lond. Math. Soc. **118** (2019), 729–752; arXiv:1701.06651

[19] J. B. Conrey & J. P. Keating, *Pair correlation and twin primes revisited*, Proc. R. Soc. A 472 (2016), 20160548; arXiv:1604.06124

[20] J. B. Conrey & J.P. Keating, *Averages of ratios of the Riemann zeta-function and correlations of divisor sums*, Nonlinearity 30 (2017), R67-R80.

[21] J. B. Conrey & N. C. Snaith, *Applications of the L-functions ratios conjectures*, Proc. Lond. Math. Soc. **94** (2007), 594–646.

[22] D. A. Goldston & S. M. Gonek, *Mean value theorems for long Dirichlet polynomials and tails of Dirichlet series*, Acta Arith. 84 (1998), no. 2, 155–192.

[23] D. A. Goldston & H. L. Montgomery, *Pair correlation of zeros and primes in short intervals*, in Analytic number theory and Diophantine problems (Stillwater, OK, 1984), 183–203, *Progr. Math.* 70 (1987), Birkhäuser Boston, Boston, MA.

[24] G. H. Hardy & J. E Littlewood, *Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes*, Acta Math. **44** (1923), 1–70.

[25] G. H. Hardy & Wright, *An Introduction to the Theory of Numbers.*

[26] N. M. Katz & P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy*, American Mathematical Society Colloquium Publications, 45. American Mathematical Society, Providence, Rhode Island (1999).

[27] J. P. Keating, *The Riemann zeta-function and quantum chaology*, In Quantum chaos (Varenna, 1991). Proc. Internat. School of Phys. Enrico Fermi, CXIX (1993), 145–185.

[28] J. P. Keating, *L-functions and the characteristic polynomials of random matrices*, in Recent Perspectives in Random Matrix Theory and Number Theory; editors, F. Mezzadri & N.C. Snaith, pages 251–278. London Mathematical Society Lecture Note Series 322 (Cambridge University Press) (2005).

[29] J. P. Keating, *Random matrices and number theory: some recent themes*, in Stochastic Processes and Random Matrices: Lecture Notes of the Les Houches Summer School: Volume 104, July 2015, edited by Grgory Schehr, Alexander Altland, Yan V. Fyodorov, Neil O'Connell & Leticia F. Cugliandolo (2017).

[30] J. P. Keating & N. C. Snaith, *Random matrix theory and $\zeta(1/2+it)$*, Commun. Math. Phys. 214 (2000), 57–89.

[31] J. P. Keating & N. C. Snaith, *Random matrix theory and L-functions at $s = 1/2$*, Commun. Math. Phys. 214 (2000), 91–110.

[32] H.L. Montgomery, *The pair correlation of zeros of the Riemann zeta-function*, Proc. Symp. Pure Math. **24** (1973), 181–193.

[33] H. L. Montgomery & K. Soundararajan, *Primes in short intervals*, Commun. Math. Phys. 252 (2004), 589–617.

[34] A. M. Odlyzko, *The 10^{20}th zero of the Riemann zeta function and 70 million of its neighbors*, Preprint (1989).

[35] Z. Rudnick & P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Math. J. 81 (1996), 269–322.

[36] D. J. Smith, *Statistics of the zeros of L-functions and Arithmetic correlations*, PhD Thesis, University of Bristol (2016); arXiv:1607.03297

[37] E.C. Titchmarsh, *The Theory of the Riemann Zeta Function*, Oxford University Press (1986).

School of Mathematics, University of Bristol, Bristol, BS8 1TW, UK