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Effect of molecular relaxation on nonlinear evolution of N-waves

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The propagation of an initially antisymmetric disturbance through a relaxing medium in one-dimension is considered. If dissipation and dispersion effects are small compared with the effect of nonlinearity, the disturbance approaches the classic N-wave profile with narrow shocks controlled by relaxation processes. As the N-wave propagates it spreads and decays in amplitude, affecting key balances between competing physical processes. In this paper we analyse the change in the shock structure as the outer solution evolves, using asymptotic analysis supplemented by numerical results. Two numerical schemes are described - an implicit scheme with variable spatial mesh which allows good resolution of the shock structure, and a pseudospectral scheme which is used when multiple relaxation modes are considered. Experimental measurements (Pawlowski et al 2005 and Yuldashev et al 2008) reveal the appearance of a slowly decaying shock tail previously unexplained by analysis of the augmented Burgers equation. In this paper we demonstrate that this phenomenon occurs when one of the relaxation timescales is comparable to the time of pulse duration.

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1. FORMULATION

Accurate predictions of shock overpressure and shock rise-time are important in determining the subjective annoyance of sonic booms produced by supersonic aircraft. For typical acoustic waves, nonlinearity is locally small, but the effect is cumulative, leading to significant deformation over long ranges. For one-dimensional propagation, inclusion of thermoviscosity in addition to nonlinearity leads to the well-known Burgers equation, which can be solved exactly using the Cole-Hopf linearising transformation. When the coefficient of the viscous term is small, wave-steepening occurs leading to a narrow shock region where nonlinearity is balanced by the viscosity. For any disturbance which is initially anti-symmetric and isolated in space, asymptotic analysis involving the method of characteristics demonstrates that the solution approaches an N-wave. For this reason, taking a unit N-wave as the initial condition can be considered a canonical problem for the study of long-range propagation. Asymptotic analysis then predicts that the shock widens before a regime of linear decay is reached,[1] predictions confirmed by numerical solution. However, for prediction of shock amplitude and shock width in realistic applications, the effects of geometric spreading, density stratification and relaxation effects must all be considered.[2] In the atmosphere, the important relaxation mechanisms are associated with oxygen and nitrogen, while long-range underwater acoustics in sea-water are influenced relaxation modes due to the presence of trace concentrations of magnesium sulphate and boric acid

In this paper we focus on the effect of relaxation and consider one-dimensional propagation through a uniform medium. Each relaxation mode is characterised by two parameters – a relaxation time and an effective concentration. Including the effect of n relaxation modes the non-dimensional gives the augmented Burgers equation,

$$p_t + pp_x + \sum_{i=1}^{n} \Delta_i p_x^{(i)} = \epsilon p_{xx}, \qquad \left(1 - \tau_i \frac{\partial}{\partial x}\right) p^{(i)} = -\tau_i p_x. \tag{1}$$

Here $p^{(i)}$ is the partial pressure associated with i-th relaxation mode and τ_i the corresponding non-dimensional relaxation time. When $\tau_i \gg 1$, $p_x^{(i)} \approx p_x$ and we see that Δ_i is the change in the non-dimensional linear sound speed. Conversely, if $\tau_i \ll 1$ then $p^{(i)} = -\tau_i p_x$ and the effect of the relaxation mode is to change the coefficient of the viscous term from ϵ to $\epsilon + \Delta_i \tau_i$. When $\tau_i = O(1)$ the effect of the relaxation term is both dissipative and dispersive, and this is the focus of the current paper. In particular experimental measurements[3,4] reveal a slowly decaying tail behind the main wave – an effect not currently explained by asymptotic analysis. Here we consider a single relaxation mode and by suitable choice of non-dimensionalisation we take as our initial condition the unit N-wave.

2. ASYMPTOTIC ANALYSIS

We consider the solution of (1), with $n = 1$, in the asymptotic limit $\Delta_1, \epsilon \to 0$, with

$$p(x, t = 1) = \begin{cases} x & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

The initial condition is taken at $t = 1$ for algebraic simplicity. The outer solution is given by a spreading N-wave with shocks located at $x = \pm t^{\frac{1}{2}}$ and with amplitude $t^{-\frac{1}{2}}$. The asymptotic analysis is then simplified greatly by rescaling with $P = t^{\frac{1}{2}} p$, $X = t^{-\frac{1}{2}} x$ and $T = t^{\frac{1}{2}}$, so that the leading order outer solution is the unit N-wave. The set of governing equations then becomes,

$$TP_T - (XP)_X + (P^2)_X + 2\Delta P_X^{(1)} = 2\epsilon P_{XX}, \qquad P_X^{(1)} - \mu P^{(1)} = P_X, \qquad \mu = T/\tau.$$

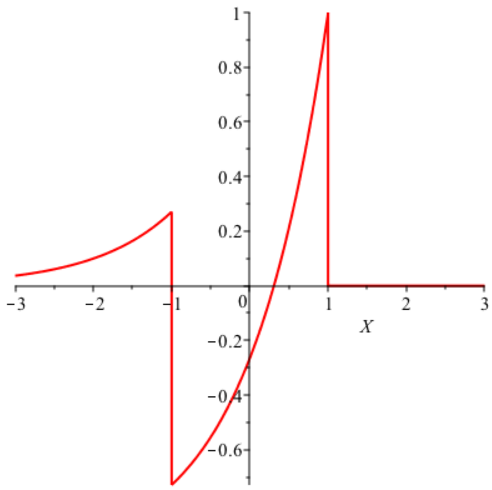

Figure 1: Plot of the leading order partial pressure $P_1^{(1)}(X)$ ***for*** $\mu = 1$

Writing the solution as $P(X,T) = P_1 + \Delta P_2 + \epsilon P_3 + o(\epsilon, \delta)$, outer solutions must be determined for the ranges $X < 1$, $-1 < X < 1$ and $X > 1$, supplemented by inner solutions centred at $X = \pm 1$. Here we focus on the outer solution, but it should be noted that in the absence of relaxation, the ground-breaking asymptotic results of Crighton & Scott[1] for the shock structure of Burgers equation (including geometric effects) can be reproduced in a much more concise fashion. The leading order outer solution P_1 is the unit N-wave and hence the associated partial pressure $P_1^{(1)}$ is readily determined,

$$P_1^{(1)} = \begin{cases} 2\left(\cosh\mu - \dfrac{\sinh\mu}{\mu}\right)\exp(\mu X) & X < -1 \\ \left(1 + \dfrac{1}{\mu}\right)\exp(\mu(X-1)) - \dfrac{1}{\mu} & -1 < X < 1 \\ 0 & X > 0 \end{cases} \tag{2}$$

with unit discontinuities in $P_1^{(1)}$ at $X = \pm 1$, forced by P_1. $P_1^{(1)}(X)$ given by (2) is plotted in figure 1.

Focussing next on the effect of the relaxation, P_2 satisfies

$$T(P_2)_T - (XP_2)_X + 2(P_1P_2)_X + 2(P_1^{(1)})_X = 0.$$

The key features of the resulting solution is that $P_2 = 0$ for $X > 1$, while for $X < -1$ the leading-order correction due to relaxation is

$$P_2(X,T) = 4\tau\mu G(\mu)e^{\mu X}, \qquad G(\mu) = \int_{\mu_0}^{\mu} \frac{\sinh z}{z} dz - \sinh(\mu) + \sinh(\mu_0), \tag{3}$$

where $\mu = T/\tau$ with $\mu_0 = 1/\tau$. For $|X| < 1$, the solution is obtained by the method of characteristics.

3. NUMERICAL RESULTS

In order to validate the asymptotic results, and more importantly identify the possible breakdown in the predictions, two numerical schemes were used. With the rescaling described above, so that at leading

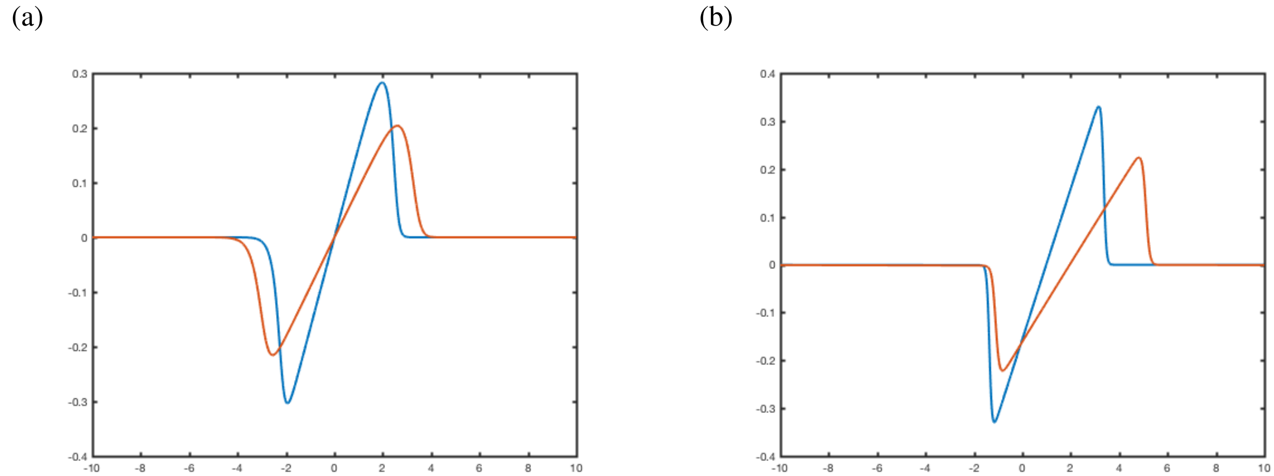

Figure 2: Numerical solution of wave form at $t = 5, 10$ ***for*** $\Delta = 0.2$***,*** $\epsilon = 0.01$ ***with (a)*** $\tau = 0.1$ ***and (b)*** $\tau = 10$***.***

order the shocks are fixed, a variable mesh was defined with points concentrated in the shock regions. The transformed system of PDEs was then solved using an implicit method following that described by Chong.[5]

An alternative is to take discrete Fourier transforms in space of (1), which after eliminating the transform of the partial pressure term gives

$$\widehat{p}_t + \tfrac{1}{2}ik\widehat{p^2} = f(k)\widehat{p}, \qquad f(k) = -\epsilon k^2 - \frac{\Delta\tau k^2}{1 - i\tau k},$$

where $\widehat{p}(k,t) = \mathcal{F}(p)$. This could be advanced forward in time in spectral space, but the stability constraint is relaxed by eliminating the linear terms (which make the system stiff) by means of an integrating factor. Defining $\widehat{P} = e^{-f(k)t}\widehat{p}$ the system becomes

$$\widehat{P}_t + \tfrac{1}{2}ike^{-f(k)t}\mathcal{F}\left(\left\{\mathcal{F}^{-1}(e^{f(k)t}\widehat{P})\right\}^2\right) = 0,$$

which is advanced forward in time using a four-step Runge-Kutta scheme. The two methods gave identical results, however the pseudospectral scheme has the advantage that it can immediately be extended to multiple relaxation modes by adding additional terms to $f(k)$.

We begin by illustrating results for small and large values of the relaxation time τ. In figure 2 (a) when $\tau = 0.1$ it is seen that the propagation is unchanged and the solution is comparable to that found by solving Burgers equation with the relaxation mechanism modifying the viscous parameter. In figure 2 (b) when $\tau = 10$, it is seen that the propagation velocity is increased by $\Delta\tau$, but that the N-wave structure is preserved.

We then consider the $\tau = O(1)$ case which was the subject of the asymptotic analysis of the previous section. In figure 3 numerical results for $\tau = 1$ with $\Delta, \epsilon \ll 1$, are plotted at times $t = 10, 40$.

For $t = 10$ the shock is still approximately symmetric and there is good agreement with the asymptotic prediction (3), except close to the shocks. However when $t = 40$ the displacement of the shock centres away from the weak-shock location, along with shock thickening, is clearly seen and must be accounted for by considering the effect of relaxation on the inner shock solution. However, the prediction of the decaying shock tail which persists over long timescales (or equivalently long propagation distances) is an important result of the asymptotic analysis.

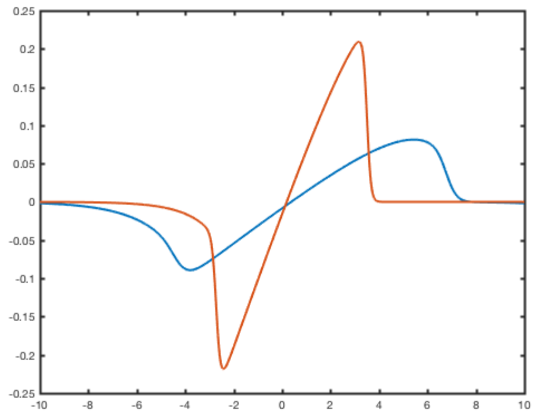

Figure 3: Numerical results for $p(x,t)$ ***at*** $t = 10$ ***(red) and*** $t = 40$ ***(blue) with*** $\tau = 1$***,*** $\Delta = 0.05$***,*** $\epsilon = 0.01$***.***

REFERENCES

[1] Crighton, D. & Scott, J. Asymptotic Solutions of Model Equations in Nonlinear Acoustics Phil. Trans. Roy. Soc. Lond. A 292, 107-134, 1979.

[2] Hammerton, P. Effect of Molecular Relaxation on the Propagation of Sonic Booms Through a Stratified Atmosphere Wave Motion, 33, 359-377, 2001.

[3] Joseph W Pawlowski, David H Graham, Charles H Boccadoro, Peter G Coen, and Domenic J Maglieri. Origins and overview of the shaped sonic boom demonstration program. AIAA paper, 5:2005, 2005.

[4] Yuldashev PV, Averiyanov MV, Khokhlova VA, Ollivier S, and Blanc-Benon Ph. Non- linear spherically divergent shock waves propagating in a relaxing medium. Acoustical Physics, 54(1):3241, 2008.

[5] TH Chong, TH. A variable mesh finite difference method for solving a class of parabolic differential equations in one space variable. SIAM Journal on Numerical Analysis, 15(4):835857, 1978.