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Barrier Graphs and Extremal Questions on Line, Ray, Segment, and Hyperplane Sensor Networks

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ABSTRACT

A sensor network is typically modeled as a collection of spatially distributed objects with the same shape, generally for the purpose of surveilling or protecting areas and locations. In this dissertation we address several questions relating to sensors with linear shapes: line, line segment, and rays in the plane, and hyperplanes in higher dimensions.

First we explore ray sensor networks in the plane, whose *resilience* is the number of sensors that must be crossed by an agent traveling between two known locations. The coverage of such a network is described by a particular tripartite graph, the *barrier graph* of the network. We show that barrier graphs are perfect (Berge) graphs and have a rigid neighborhood structure due to the rays' geometry.

We introduce two extremal problems for networks in the plane made of line sensors, line segment sensors, or ray sensors, which informally ask how well it is possible to simultaneously protect k locations with n (line/ray/segment)-shaped sensors from intruders. The first question allows any number of intruders, while the second assumes there is a lone intruder. We show these are questions to be answered separately, and provide complete answers for $k = 2$ in both cases. We provide asymptotically tight answers for question (1) when $k = 3, 4$ and the locations are in convex position. We also provide asymptotic lower bounds for question (1) for any k.

Finally, we generalize these extremal problems to d dimensions. For the d-dimensional version of question (1) we provide asymptotic lower and upper bounds for any combination of k and d, though these bounds do not meet.

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1.1 Introduction

Consider a pair of points α and β in the plane, and an arrangement R of rays. If there is no path, unobstructed by rays, from α to β, then we say the rays of R *form a barrier* for α and β, and that the rays represent a *ray barrier network*.

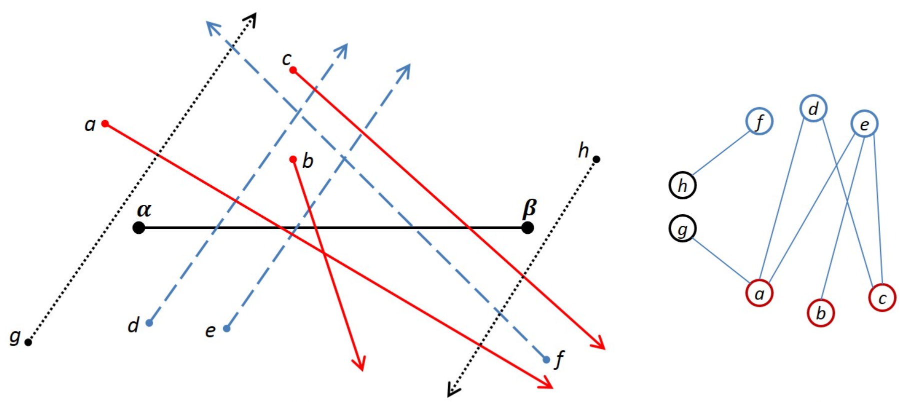
In this chapter, we will explore some natural questions about such a network and structures that arise from its study. An initial question is: given a particular ray arrangement R and point pair α and β, can one efficiently determine whether R forms a barrier? Naturally following this, we may ask which subsets of R are smallest so that removal of any such subset leaves an unobstructed path, and moreover how to find one of these, or how to find its size.

An important observation of Kirkpatrick, Yang, and Zilles (KYZ) [9] was that R forms a barrier if and only if some pair of rays contained in it forms a barrier alone. This pairwise information yields a graph, the ***barrier graph*** for R, α, and β, with rays as vertices and an edge for any pair forming a barrier.

Definition 1.1. *Let α and β be distinct points. A pair of rays r_1 and r_2 forms a* **barrier** *for α and β if as an arrangement they separate the plane into two disjoint cells, one of which contains α while the other contains β.*

Definition 1.2. *Let R be a collection of rays not containing α or β. The* **barrier graph** *for R, α, and β is the graph $G = (V, R)$ with vertices $V = R$ and an edge $(x, y) \in E$ for any $x, y \in R$ such that x and y form a barrier for α and β.*

As discussed by [9] and in section 1.2, these are tripartite graphs, an example of which is seen in Figure 1.1. Note that the straight segment $\overline{\alpha\beta}$ in Figure 1.1a is used to compute the graph (Figure 1.1b) but in no way implies that a path from α to β needs to be straight. A barrier forces *all* paths from α to β to cross at least one ray.



(a) (b)

Figure 1.1: (a) An arrangement of rays with points α and β, and (b) their barrier graph.

Starting with a graph G, we can ask whether G is a barrier graph, i.e., whether there is some arrangement R of rays, together with points α and β, so that G is the corresponding barrier graph. If the answer is yes, we say that the tuple $\langle R, \alpha, \beta \rangle$ is a *realization* of G. However, because any arrangement of rays with points α and β can be rotated, scaled, and translated so that α and β are any other convenient pair of points without changing the corresponding barrier graph, we will usually just refer to the arrangement R as the realization of G, where α and β are understood.

Kirkpatrick and Bereg [2] introduced the notion of a sensor network's *resilience*, which is the minimum number of sensors whose removal permits a path (not necessarily straight) between the chosen points. Since barriers are formed by pairs of rays [9], a ray-barrier network's resilience is the size of a minimum vertex cover of the barrier graph. One can also ask whether knowing the size of a minimum vertex cover without having a realization in hand can tell us whether a realization *could* be found, which we explore in Section 1.4.

In general graphs, the minimum vertex cover problem is NP-hard [8], but as KYZ have shown, the ray barrier resilience (and in particular, a set of rays which witnesses the resilience) of an arrangement of rays can be found efficiently by making use of geometric information about the arrangement in addition to the barrier graph; this suggests quite strongly that the underlying geometric structure greatly limits the graphs which can arise as a barrier graph. KYZ's algorithm takes $O(n^2m)$ steps, where n is the number of rays and m is the number of barriers they form.

It is ultimately the question of which graphs can be realized as barrier graphs which will be the main focus of this chapter. We will first show that any barrier graph belongs to the special class of graphs called perfect graphs, a class for which some algorithmically difficult problems become efficiently solvable. Then we will show how to find a realization for some of the usual classes of bipartite graphs, including complete bipartite graphs, paths, even-length cycles and trees. In addition, and in spite of how natural these constructions for many well-recognized bipartite graphs are, it turns out that few bipartite graphs are barrier graphs, which we show in Section 1.3.

1.2 The Tripartite Structure of Barrier Graphs

One of the key observations of KYZ is that not only does a collection of ray sensors yield a barrier graph, but this graph must be tripartite. In order to show this, they constructed a 3-coloring of a barrier graph as follows (see the example in Figure 1.1). Consider any arrangement of rays and a pair of points α, β – the start and target points – and rotate the realization so that the segment $\overline{\alpha\beta}$ is horizontal. Rays will be colored red if they intersect $\overline{\alpha\beta}$ from above, blue if they intersect $\overline{\alpha\beta}$ from below, and black if they don't intersect $\overline{\alpha\beta}$.

KYZ showed that with this coloring, barriers can only form from intersecting pairs of differently colored rays. Furthermore, if red and blue rays intersect they always form a barrier, but if one of the rays is black, then a barrier is only formed when the intersection is on the same side of the supporting line of $\overline{\alpha\beta}$ as the ancho r of the non-black ray. Since edges in the barrier graph are exactly these barrier pairs, the colors give three sets of vertices connected by edges only to elements of a differently colored set. We will refer to this coloring for every barrier graph as if it were part of the definition of the graph.

Viewing barrier graphs through the lens of this coloring provides a glimpse of how the graph structure depends on the rays' geometry, and allows us to find properties that prevent a graph from being a barrier graph, starting with the following proposition.

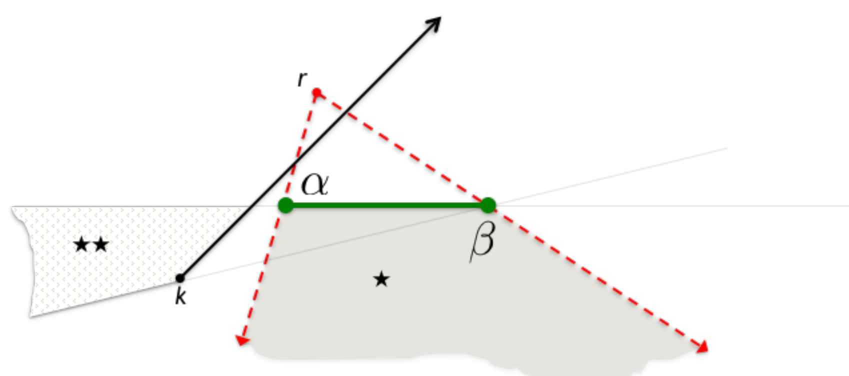


Figure 1.2: A barrier graph may not contain an induced red-black-blue path. The bold segment is the segment $\overline{\alpha\beta}$. The region labeled $\star$ cannot contain the anchor of k. The anchor of b must be located in the region labeled $\star\star$ as, otherwise, it will either not be blue or it will intersect k above $\ell(\overline{\alpha\beta})$. This forces b to also intersect r.

Proposition 1.3. *Let G be a barrier graph, and fix a realization with a particular set of rays and the path endpoints α and β. Then there is no induced length 2 path $r - k - b$ where r is red, k is black, and b is blue.*

Proof. Fix a realization of G, so that the coloring of the vertices of the barrier graph is fixed, and suppose $r - k - b$ is an induced path of length 2 in G, with r red, k black, and b blue.

Because r is red, it must be anchored above the segment in bold $\overline{\alpha\beta}$, and must also intersect $\overline{\alpha\beta}$. Therefore, its ray r lies somewhere in the wedge bounded by the dashed lines in Figure 1.2. Then, since k forms a barrier with r, it cannot be placed in the shaded region labeled $\star$ for, otherwise, it would either intersect $\overline{\alpha\beta}$ (thus being blue instead of black) or intersect r below $\overline{\alpha\beta}$ (thus not forming a barrier with r).

k can either start above or below the supporting line of $\overline{\alpha\beta}$, which we will call ℓ, as long as k intersects r above ℓ and crosses ℓ so that a proper intersection with b is possible. We proceed with the case that has the anchor of k below ℓ and its supporting line to the left of $\overline{\alpha\beta}$ (see Figure 1.2). The argument for each of the other possibilities is similar.

Since b must intersect k below ℓ to form an edge with it, it must be anchored in the region labeled $\star\star$, since were it anchored elsewhere, it would either not intersect $\overline{\alpha\beta}$ (and thus not be blue) or not intersect k below ℓ. But any blue ray anchored in $\star\star$ would intersect r, which means that the vertices r, k, and b would induce a triangle instead of a path. So there could not have been such an induced subgraph. $\square$

Proposition 1.4. *If G is a barrier graph, then any induced subgraph is a barrier graph.*

Proof. This follows immediately from the fact that any pair of rays forms a barrier (or does not) independently of all the other rays in the collection. □

Proposition 1.3 will be our most basic tool to connect the geometry of ray sensor arrangements to the graph structure of their barrier graphs. Proposition 1.4 is more graph focused, but together with Proposition 1.3 leads to the following stronger structural result about barrier graphs.

Proposition 1.5. *No graph containing a chordless cycle of odd order* ≥ 5 *is a barrier graph.*

Proof. Let $G = (V, E)$ be an odd cycle graph with $|V| \geq 5$ and with no chords, and suppose G is a barrier graph. Fix any realization of G, and let R, B, and K be the sets of red, blue, and black vertices respectively. R, B, and K are all nonempty since otherwise G is bipartite, which odd cycles are not.

For subsets $C, D \subseteq V$, define $C_D := \{c \in C \mid \mathcal{N}(c) \subseteq D\}$, i.e. those elements of C whose neighborhood consists only of elements of D.

Suppose every element of K has two neighbors of the same color. Then, K is the disjoint union $K_R \cup K_B$. If we define $R' = R \cup K_B$ and $B' = B \cup K_R$, then R' and B' partition V into disjoint sets.

There are no edges between two elements of R' or between two elements of B', so we have found a bipartition of G, a contradiction.

But then there is an element $k \in K$ with differently colored neighbors $r \in R$ and $b \in B$. So, $r - k - b$ is an induced subgraph (since G is longer than a triangle and is chordless), which contradicts, by Proposition 1.3, that G was a barrier graph.

Thus, odd cycles of order ≥ 5 are not barrier graphs and, by Proposition 1.4, no graph with such an induced subgraph can be a barrier graph. $\square$

While Proposition 1.5 prevents certain tripartite graphs from being barrier graphs, it also leads to an interesting connection to perfect graphs. Recall that a graph is perfect if the chromatic number and clique number of every induced subgraph are equal.

The strong perfect graph theorem [5], of Chudnovsky, Robertson, Seymour and Thomas, gives an alternate structural characterization of perfect graphs by showing that they are the same as the so-called Berge graphs. A Berge graph is a graph which has no odd hole (a hole with an odd number of vertices) or odd antihole with length longer than 3 as an induced subgraph.

From Proposition 1.5 we have that barrier graphs have no odd hole of size ≥ 5. Now, the cycle C_5 has itself as its graph complement, so this also rules out size-5 antiholes in any barrier graph. If we can also rule out antiholes of larger odd size in any barrier graph, then we will have shown that barrier graphs are perfect.

Lemma 1.6. *If G is a barrier graph, then G has no antiholes of size $2k + 1$ for any $k \geq 2$.*

Proof. We may exclude odd antiholes of size 5 by the comment above that C_5 is its own complement, and so is an antihole.

First, we'll show that antiholes of size $2k+1$ have chromatic number $k+1$. Label the vertices of an antihole H of size $2k+1$ as $v_1, \ldots, v_{2k+1}$ so that each v_i and v_{i+1} are adjacent in the cycle which is the graph complement of H. Then v_i and v_{i+1} are not adjacent in H, so we can assign the color j to each vertex $2j$ and $2j-1$, for $1 \leq j \leq k$. This leaves one vertex not in a color pair, which needs its own color. So we have used $k+1$ colors to properly color H.

Now, fix any proper coloring of H. If v_1 is colored c, then because v_1 is adjacent to all other vertices except v_2 and v_{2k+1}, only these two may also be colored c. Moreover, these are adjacent, and so only one of them may be colored c. This argument applies to every vertex, for whatever its color is, and thus each color may appear on at most two vertices.

Therefore, the $2k+1$ vertices of H require at least $k+1$ colors, and the chromatic number of an antihole with $2k+1$ vertices is $k+1$. Since barrier graphs are tripartite, they cannot have induced subgraphs with chromatic number greater than 3, so for $k \geq 2$, barrier graphs have no antiholes. □

Corollary 1.7. *Every barrier graph is a perfect graph.*

Perfect graphs have various properties that distinguish them from tripartite graphs. First, unlike tripartite graphs, we already have algorithms to recognize perfect graphs efficiently [4]. Furthermore, many algorithmically hard problems, such as graph coloring, maximum clique, and maximum independent set have polynomial time algorithms for the class of perfect graphs [6].

We can take advantage of these algorithms in various ways. For instance, since S is an independent set for $G(V, E)$ if and only if $V \setminus S$ is a vertex cover, it follows that a polynomial time algorithm for computing the size of a maximum independent set of a perfect graph can be used, verbatim, to compute the resilience of a barrier graph, also in polynomial time.

This approach provides us with an compelling alternative to the algorithm in KYZ that is also efficient, but does not require explicit use of or access to the geometric information. One could receive a barrier graph and compute its resilience in polynomial time without knowledge of the location and orientation of the sensors involved, and therefore one could compute the resilience of the entire equivalence class of ray barrier arrangements which realize a particular barrier graph. This is impossible with the KYZ algorithm, which makes heavy use of the underlying geometric information.

With the knowledge that barrier graphs are tripartite and perfect, a natural question is whether the converse is true: are all perfect tripartite graphs barrier graphs?

A natural class of perfect tripartite graphs is the class of bipartite graphs, which have both clique number and chromatic number equal to two. In Section 1.3 we will investigate the rigid structure of neighborhoods in barrier graphs, and will show that, in fact, few bipartite graphs *are* barrier graphs.

1.3 The Rigidity of Barrier Graphs

1.3.1 Stabbing Rays and Segments

The next few results connect the geometry of a barrier graph's realization to the kinds of neighborhoods that can appear in the graph. To this end, we give a bound of $O(n^3)$ on the number of subsets of a set of n rays or line segments that could be stabbed by another ray. These next lemmas are general geometric results, but will be particularly helpful in addressing the neighborhood structure of barrier graphs.

Below, we will use the standard geometric dual transformation $\mathcal{D}$ in two dimensions, which associates points in the Euclidean plane with non-vertical lines in the plane. For each point $p = (p_x, p_y) \in \mathbb{R}^2$ we associate a line $\ell = \mathcal{D}(p)$ given by the equation $y = p_y - p_x x$, and for each such line we write $p = \mathcal{D}(\ell)$.

For sets of points P or sets of lines L, we define $\mathcal{D}[P] = \{\mathcal{D}(p) : p \in P\}$ and similarly $\mathcal{D}[L] = \{\mathcal{D}(\ell) : \ell \in L\}$.

An important basic property of $\mathcal{D}$ is that it reverses signed distance along the y-axis between points $p = (p_1, p_2)$ and lines $\ell : y = \ell_y - \ell_x x$:

$$p_y - (\ell_y - \ell_x p_x) = -(\ell_y - (p_y - p_x \ell_x))$$

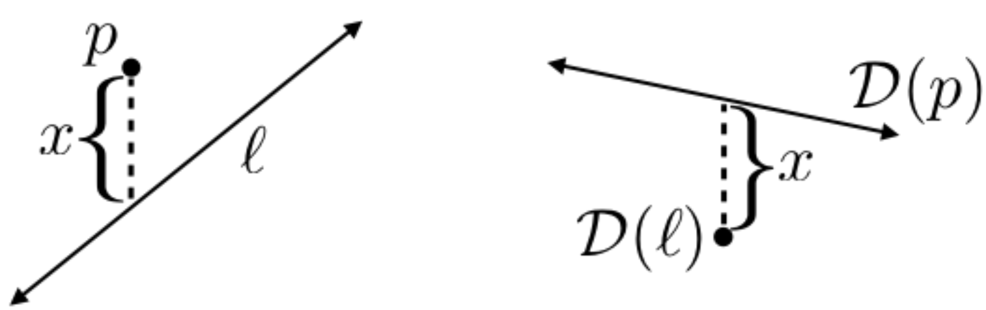


Figure 1.3: The vertical distance (x) between a point and line is the same as the vertical distance between their respective duals, though the role of the point and the line is reversed

This property implies both that D preserves the above/below order and also preserves incidence between points and lines: if point p is above (below) line ℓ, then $\mathcal{D}(p)$ is above (below) $\mathcal{D}(\ell)$, and if p and ℓ intersect, then so do $\mathcal{D}(p)$ and $\mathcal{D}(\ell)$.

The duals of certain kinds of point sets will be particularly useful for us: the dual $\mathcal{D}[\overline{ab}]$ of a (non-vertical) segment $s = \overline{ab}$, between points a and b in the plane, is a region of the plane consisting of the union of the points lying on each line $\mathcal{D}(p)$ for $p \in s$; this region looks like a bowtie, or a double-wedge – a pair of infinite wedges with a common apex (see Figure 1.4).

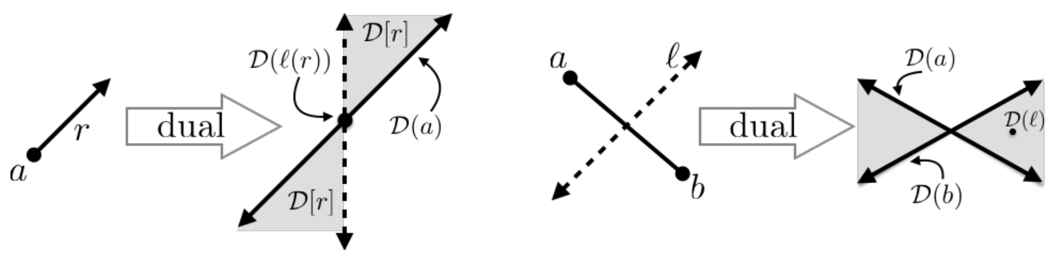


Figure 1.4: Dual transformation of a ray (left) and of a segment (right).

Every dual line $\mathcal{D}(p)$ of a point $p \in s$ contains the dual of the supporting line of s, and this point is where the two wedges of the bowtie meet. This follows from the preservation of incidence between points and lines.

The sides of both wedges in $\mathcal{D}[s]$ are just the dual lines of the segment's endpoints a and b, since the points between these on s have intermediate x-values as well (a vertical segment will, then, just be an infinite strip bounded by parallel lines).

Now, if a line ℓ intersects s at a point q, then since $q \in s$ it must be that $\mathcal{D}(q)$ lies somewhere in the bowtie region $\mathcal{D}[s]$. If we are considering the intersection of a line ℓ with multiple segments $s_1, \ldots, s_n$, since all of the intersection points lie on the line ℓ their duals must all intersect the point $\mathcal{D}(\ell)$, and therefore the point $\mathcal{D}(\ell)$ must lie within each bowtie region corresponding to an intersected segment.

The dual of a ray r is a similar bowtie-like region, bounded by the dual of the anchor of r and by a vertical line through the dual of r's supporting line, and the same reasoning as above means that the dual of a line intersecting multiple rays lies in the intersection of the dual regions of all the rays.

Because we will most often compare duals of objects to duals of other objects, we will sometimes refer to a "primal" plane and a "dual" plane, both of which are just copies of $\mathbb{R}^2$.

The fact that the duals of segments and of rays can each be summarized by two bounding lines is one we will take repeated advantage of. In the following lemma, we begin to formalize the connection between the locations of points and the locations of dual objects.

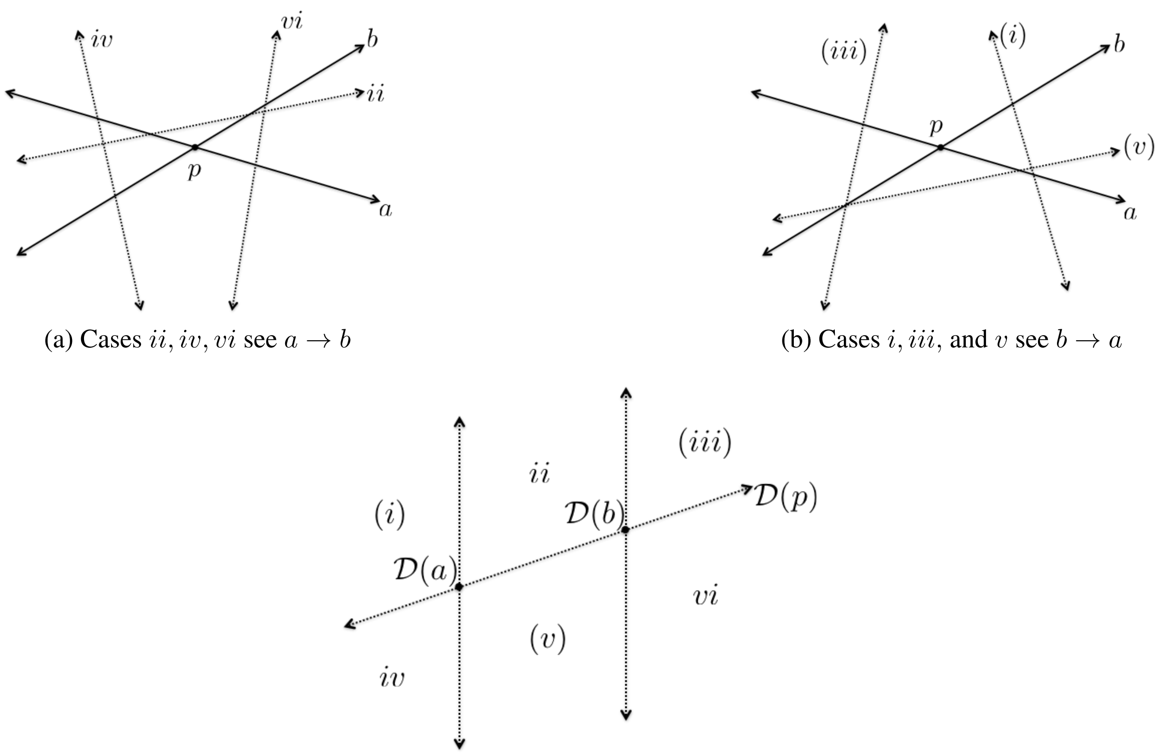
Lemma 1.8. *Let Y be a set consisting of r distinct rays and s distinct line segments in the plane. Then there exists a set Y' of lines that divide the dual plane into regions so that any two points lying in the same region correspond to two lines in the primal plane intersecting the same elements of Y in the same left-right order. Furthermore, Y' can be chosen so that*

$$|Y'| \leq \binom{r+s}{2} + 2(r+s).$$

Proof. Let Y' consist of the following types of lines in the dual plane:

(I) For each ray in Y, the dual line of the anchor and a vertical line through the dual of the supporting line, and for each line segment in Y, the dual of the endpoints (these are the "bow-tie" bounding lines).

(II) For each pair of elements in Y, the dual line of the intersection point of their supporting lines (if they intersect).



(a) Cases ii, iv, vi see $a \to b$

(b) Cases i, iii, and v see $b \to a$

(c) 6 regions in the dual plane

Figure 1.5: Lines of (a) intersect a, then b from left to right, and lines of (b) intersect b, then a from left to right. All lines which intersect a and b are one of these 6 types, and their duals are in the corresponding regions of (c), determined by Type II lines.

The intuition behind this lemma is that regions from Type I lines tell us which elements of Y a primal line intersects, and regions from Type II lines tell us in which order (from left to right) this occurs. The latter information is important for distinguishing between subsets of those elements stabbed by a line can be stabbed by a ray or segment lying within that line.

Let $n := s + r$. Clearly there are at most $\binom{n}{2} = \binom{s+r}{2}$ points of intersection between elements of Y (and hence at most $\binom{s+r}{2}$ lines of Type II), and at most $2(r + s)$ lines of Type I.

Now, consider the partition $\mathcal{P}$ of the dual plane defined by Y'. The *regions* of this partition are simply the k-faces of the partition, i.e., the points of intersection of these lines, the open line segments connecting two points, and the open regions in the plane created by removal of these points and segments. We claim this partition has the desired property.

Consider an arbitrary pair of points x, y in the same region of $\mathcal{P}$. First note that the position of x (or y) with respect to Type I lines determines which line segments and rays of Y the dual of x (or y) intersects. This is precisely determined by the bowtie regions of Figure 1.4. That is, within a region all points are duals of lines which intersect the same elements of Y. Now, fix a pair of elements $a, b \in Y$ which are both intersected by the duals lines of x and y. We show they are intersected in the same left-right order. It suffices to assume that a and b are both lines (possibly replacing a ray or segment with their supporting line.)

The order in which a line intersects two others is determined by both the relative order of the slopes of the lines and whether the first line passes above or below the intersection point of the other two. This is illustrated in Figure 1.5 and is determined by the position of the line's dual with respect to Type I & II lines. In particular, Type II lines carry the data on whether the first primal line passes above or below, while position with respect to the Type

I lines (beyond simply carrying information about whether or not the primal line intersects the involved ray or segments) carries the information about the slope.

Thus, by our construction of $\mathcal{P}$, being in the same region of $\mathcal{P}$ implies that x and y have the same relation with $\mathcal{D}(a)$ and $\mathcal{D}(b)$, and hence $\mathcal{D}(x)$ and $\mathcal{D}(y)$ intersect a and b in the same left-right order. Note that a point which lies within one of the line segments might not intersect one or the other (or both) of a and b, or might intersect both at the same time. Nonetheless, the information required to determine this is captured by the region.

Since a and b are arbitrary, this means that $\mathcal{D}(x)$ and $\mathcal{D}(y)$ intersect all elements of Y in the same order as desired. $\square$

Let us suppose Y has r rays and s line segments, where $r + s = n$, and consider the partition of the dual plane offered by Lemma 1.8. Consider a line in the primal plane. It intersects at most n elements of Y in some order, and hence there are at most $2n$ subsets that a ray can intersect.

We have just shown that given two lines, if the duals of those lines fall into the same region of the dual plane in our partition, they intersect the same elements of Y in the same order. Thus the subsets of Y potentially stabbed by segments of a line are completely determined by the region of the dual plane. For such a region A, we denote this set of subsets by $\mathcal{R}_Y(A)$, or $\mathcal{R}(A)$ when Y is understood. In this language, our observation above is that $|\mathcal{R}(A)| \leq 2n$.

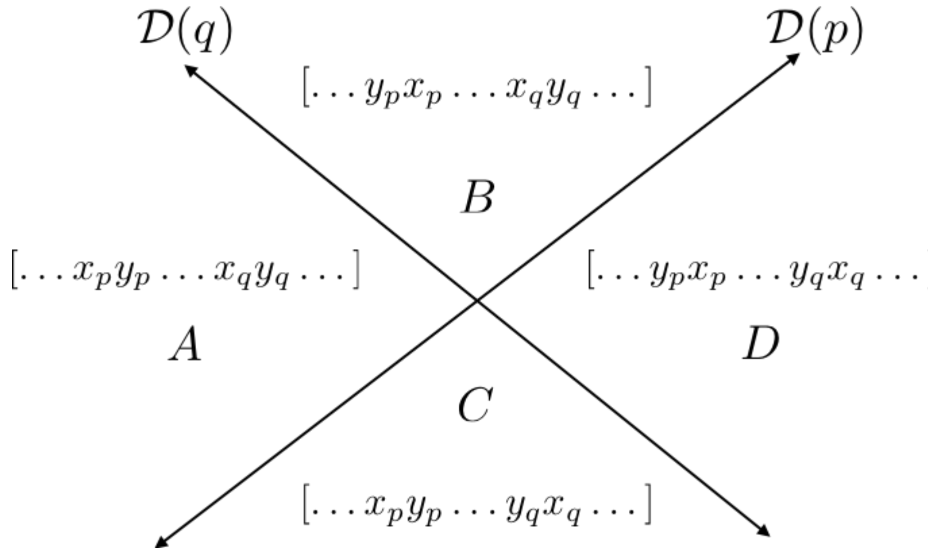


Figure 1.6: Two intersecting Type II lines in the dual plane and the left-right order in which the surrounding regions intersect the corresponding elements of Y

Lemma 1.9. *Let $\ell_1 = \mathcal{D}(p)$ and $\ell_2 = \mathcal{D}(q)$ be two intersecting Type II lines in the dual plane, and let A, B, C, and D be the regions adjacent to the intersection of ℓ_1 and ℓ_2 (see Figure 1.6). Then*

$$\mathcal{R}(D) \subseteq \mathcal{R}(A) \cup \mathcal{R}(B) \cup \mathcal{R}(C).$$

Proof. Note that p and q are the intersection points for pairs $x_p, y_p \in Y$ and $x_q, y_q \in Y$, so the difference between dual points in $\mathcal{R}(A)$ and in $\mathcal{R}(B)$, or for any pair of these regions, is only in the left-right order in which (the supporting lines of) x_p, y_p, x_q, and y_q are intersected.

Suppose that the dual of a point in the region A intersects elements of Y in the left-right order $[\dots, x_p, y_p, \dots, x_q, y_q, \dots]$, B intersects them as $[\dots, y_p, x_p, \dots, x_q, y_q, \dots]$, C intersects them as $[\dots, x_p, y_p, \dots, y_q, x_q, \dots]$, and D intersects $[\dots, y_p, x_p, \dots, y_q, x_q, \dots]$.

Since a ray intersects elements of Y according to a prefix or a suffix of the order lines intersect within a region, the containment above is clear. $\square$

What Lemma 1.9 gives us is a tool to avoid over-counting subsets of Y. It says that the four regions around any crossing of Type II lines cannot all correspond to distinct subsets that can be stabbed by a ray.

Lemma 1.10. *Let Y be a set of n rays or line segments in the plane. Let S_Y denote the set of all subsets $A \subseteq Y$ so that there exists a ray intersecting exactly the elements of A, and no other elements of Y. Then*

$$|S_Y| \leq 6n^3.$$

Proof. We will again use the Type I and Type II descriptions of the lines of the dual-plane partition in Lemma 1.8.

For adjacent 2-faces A and B separated by a Type I line, we have $|R(A) \setminus R(B)| \leq n$. This follows as two points on different sides of a Type I line in the dual plane correspond to one line intersecting that element of Y, and one line not intersecting it. On the other hand for adjacent 2-faces A and B separated by a Type II line, $|R(A) \setminus R(B)| \leq 2$ since passing through such a line either induces a transposition of the order of two order-adjacent intersected elements of Y or has no effect at all.

We now proceed to bound the total number of subsets encountered. Partition the dual plane first by the Type I lines, and then refine this partition by including the Type II lines. We will first (carefully) fix any spanning tree of the 2-faces in the dual plane according to the Type I partition. This tree is shown in red in Figure 1.7 (a). We then fix a root, and note that the total number of subsets encountered by rays within the 2-faces is at most $2n$

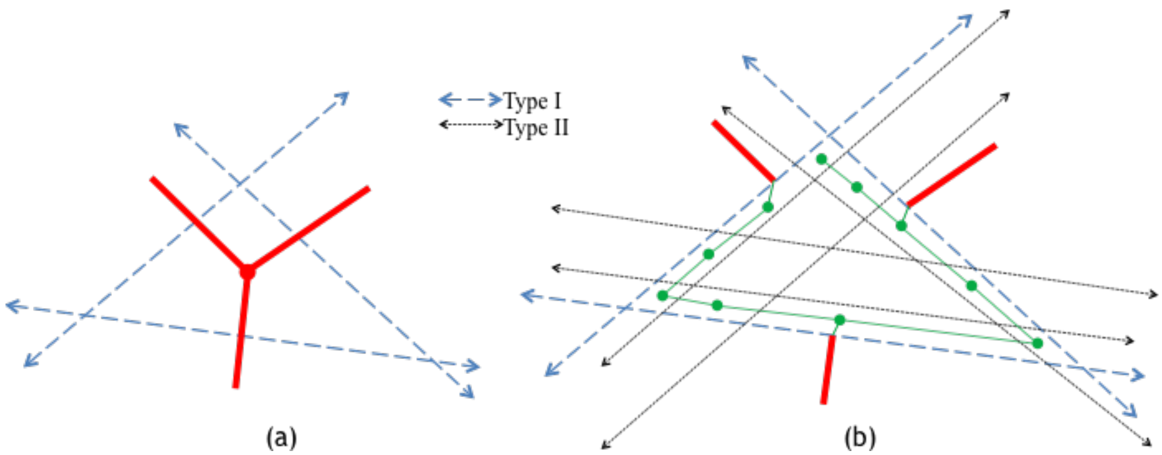


Figure 1.7: (a) The primary spanning tree of 2-faces for Type I lines, which are blue and dashed, and (b) the secondary tree obtained by refining it for a particular 2-face using Type II lines, which are black and dotted.

for the root, plus n times the number of Type I lines crossed by the spanning tree, plus a yet-undetermined amount for the Type II crossings.

In order to do this as efficiently as possible, we do the following: First, consider the regions of the dual plane determined by the (at most) $2n$ different Type I (bow-tie) lines. There are at most $\binom{2n+1}{2} + 1$ such regions, so the primary spanning tree has at most $\binom{2n+1}{2}$ edges.

Next, we consider the regions defined by all the lines together. This refines the Type I partition of the plane. Within each 2-face A according to the Type I partition, we claim that once we account for $\mathcal{R}(B)$ for all refined 2-faces B inside of A and sharing a boundary with A, then we have also accounted for $\mathcal{R}(C)$ for any other refined 2-face C inside of A.

This follows from iterated applications of Lemma 1.9. Indeed, suppose some subset of faces including the boundary faces are accounted for. Then the union of the remaining 2-faces (if any) consist of a collection of polygonal faces. Each of these must have a convex vertex around which three faces are already accounted for, and applying Lemma 1.9 to those faces, one easily observes that all four incident faces are accounted for.

We use this information to refine the (red) spanning tree so that it also connects these Type II boundary 2-faces by first connecting each set of border 2-faces with a path, and then connecting these paths at any boundary 2-face for each edge in the original spanning tree. This spanning tree crosses Type I lines at most $\binom{2n+1}{2}$ times, and in general has one less than the number of 2-faces as edges.

Now we need to count the number of these boundary 2-faces. For each intersection of a Type I line with a type II line, there are four such 2-faces, but since each border 2-face has at least two Type II lines bordering it, each is counted at least twice. So there are at most $2(2n)\binom{n}{2}$ of these border 2-faces.

Thus the total number of subsets encountered within the 2-faces is at most

$$
\begin{aligned}
2n + 2 \cdot \big(2n^2(n-1)\big) + n \cdot \binom{2n+1}{2} \\
= 6n^3 - 3n^2 + 2n \leq 6n^3. \qquad (1.1)
\end{aligned}
$$

Finally, we must determine the contribution from lower dimensional regions in the dual plane. A point in the dual plane which occurs in a Type II line corresponds in the primal plane to a line through the intersection point. It is clear these account for fewer subsets of S_Y as if a segment contains one of the intersecting lines it contains both. But also, these subsets are easily seen to be realizable by lines occurring in the neighboring 2-faces of the dual plane. Thus, for the purposes of differentiating between corresponding collections of subsets of Y, we can ignore points lying on Type II lines.

Similarly, we can ignore points lying on Type I lines: points in the dual plane occurring in these lines correspond to lines through the 'endpoints' (where a point at infinity counts as an endpoint for rays) and the subsets realized by segments of these lines are captured in the neighboring 2-faces.

Thus the bound (1.1) suffices to complete the proof of the lemma. $\square$

1.3.2 Rigidity

With the results of the previous section in hand, it is finally possible to state and prove our main structural theorem about barrier graphs. This next result strongly limits adjacencies within a barrier graph to any fixed subset. In a general graph, when a subset of vertices of size t is fixed, other vertices may potentially have any of the 2^t different adjacencies within the subset. This result states that in a barrier graph G, for any subset $Y \subseteq V(G)$, only polynomially many subsets of Y can appear as the neighborhood of a vertex outside of Y.

This is a strong rigidity theorem in the sense that it implies there are few distinct neighborhoods of vertices outside of Y contained in Y and, instead, there must be many vertices whose neighbors in Y are the same. Quite remarkably, this is true even for fairly large sets Y (for an n vertex graph G; there must be many neighborhood clones even into sets of size nearly $n^{1/3}$). The proof, however, follows quite easily from Lemma 1.10.

Together, Lemmas 1.8 and 1.10 support the following theorem, which provides that the number of subsets of sets Y of n rays that can be stabbed by another ray is $O(n^3)$.

Theorem 1.11 (Barrier Rigidity)**.** *Suppose G is a barrier graph, and $Y \subseteq V(G)$ with $|Y| = t$. Let*

$$S_Y = \{\mathcal{N}_Y(y) : y \in V(G) \setminus Y\}.$$

Then

$$|S_Y| \le 12t^3.$$

Proof. Let $f(t) := 6t^3$.

Fix a realization of G, and vertex $y \in V(G) \setminus Y$. The vertex y corresponds to a ray in the realization. Note that $\mathcal{N}_Y(y)$ is determined by the intersection of that ray, with rays and line segments determined by Y.

In particular, if y is red in our coloring, then the neighborhood of y is determined by its intersections with the blue rays determined by Y and with the parts of black rays above the line $\alpha\beta$. Suppose there are r red, b blue, and k black colored vertices in Y. By Lemma 1.10, there are at most $f(b+k)$ subsets of Y that can arise from such intersections. If y is instead blue, by similar reasoning we see there are most $f(r+k)$ subsets of Y which could be the neighborhood of y.

Similarly, if y is colored black, the neighborhood of y is determined by the intersection of the y-ray with the line segments given by cutting off red rays above the $\alpha\beta$-line and blue rays below the $\alpha\beta$ line, giving a bound of $f(r+b)$.

In total, we arrive at an upper bound on the number of subsets of Y occurring as some vertex's neighborhood of

$$f(r+b) + f(r+k) + f(k+b) = f(t-k) + f(t-b) + f(t-r),$$

where we are subject to the constraint that $r + b + k = t$. Convexity of f implies that this quantity is also convex, and so is maximized when $r = t$, $b = 0$, and $k = 0$, yielding a bound of $2f(t) = 12t^3$ as claimed. $\square$

Theorem 1.11 implies strong conditions on the neighbors of vertices within a barrier graph. This is sufficient to show that barrier graphs are rare amongst bipartite graphs, and hence amongst tripartite graphs as well. Indeed, we show an even stronger theorem: barrier graphs are rare for any size of bipartite graph, even when one of the sides of the bipartite graph is not large.

Theorem 1.12. *Suppose G is chosen uniformly at random from all bipartite graphs with bipartition (A, B) satisfying $16 \leq |A| \leq |B|$, and $|A| + |B| = n$. Then the probability that G is a barrier graph is at most $2^{16}e^{-n2^{-17}} = o(1)$.*

Proof. Fix (arbitrarily) $t \geq 16$ vertices in A, and call these t vertices A', so that $A' \subseteq A$. We will prove that if n is sufficiently large, with probability at least $1 - 2^t e^{-n2^{-(t+1)}}$,

$$|\{\mathcal{N}_{A'}(y) : y \in B\}| = 2^t.$$

In other words, we will show that with high probability, every subset of A' is exactly the neighborhood of some element of B. On the other hand, our Barrier Rigidity Theorem (Theorem 1.11) tells us that the number of subsets of A' which could be the neighborhood of any vertex in B is no more than $12t^3$, which by our choice of t is smaller than 2^t.

Let Z denote the number of subsets of A' which do *not* appear as a neighborhood of a vertex in B. In this language, what Theorem 1.11 says is that if G is a barrier graph, then

$Z \geq 2^t - 12t^3$. In particular, when t is at least 16, if G is a barrier graph then Z is certainly at least 1.

Now, for any $y \in B$, the probability that any fixed subset of A' is the neighborhood of y is just $\frac{1}{2^t}$ because in a uniformly random bipartite graph each edge is present with probability $\frac{1}{2}$, and therefore the probability that any fixed subset of A' is *not* the neighborhood of y is just $1 - \frac{1}{2^t}$.

Hence, for $S \subseteq A'$, the probability that S is not $\mathcal{N}_{A'}(y)$ for all $y \in B$ is $(1 - \frac{1}{2^t})^{|B|}$, since these events are independent. Let I_S be the indicator variable for when S is not the neighborhood of any y; then Z is just the sum $\sum_{S \subseteq A'} I_S$.

By linearity of expectation,

$$\begin{aligned}
\mathbb{E}[Z] &= \mathbb{E}\left[\sum_{S \subseteq A'} I_S\right] \\
&= \sum_{S \subseteq A'} \mathbb{E}[I_S] \\
&= \sum_{S \subseteq A'} \left(1 - \frac{1}{2^t}\right)^{|B|} \\
&= 2^t \left(1 - \frac{1}{2^t}\right)^{|B|}.
\end{aligned}$$

Since $|A| \leq |B|$ and $|A| + |B| = n$, we have $\mathbb{E}[Z] = 2^t \left(1 - \frac{1}{2^t}\right)^{|B|} \leq 2^t \left(1 - \frac{1}{2^t}\right)^{n/2}$.

Because Z is nonnegative and integer-valued, we can use Markov's inequality to bound the probability that it is nonzero: $\mathbb{P}(Z \geq 1) \leq \mathbb{E}[Z]$.

Putting together these last two inequalities and using the identity $(1+a)^b \leq e^{ab}$ we find that:

$$\begin{aligned}\mathbb{P}(Z \geq 1) &\leq 2^t \left(1 - \frac{1}{2^t}\right)^{n/2} \\ &\leq 2^t e^{-(n2^{-t})/2} \\ &= 2^t e^{-n2^{-(t+1)}}.\end{aligned}$$

This probability can thus be made arbitrarily small by choosing large enough n, which implies that the probability that G is a barrier graph is also arbitrarily small. $\square$

An almost immediate corollary of this is the following:

Corollary 1.13. *The probability that a random bipartite graph with n vertices is $o(1)$.*

The main reason the corollary isn't immediate is that choosing a bipartite graph on n vertices uniformly at random in general is not quite as easy as it is when the bipartition is fixed.

Proof. Let G be chosen uniformly at random from the set of (unlabeled) bipartite graphs on n vertices. We say that G admits an s-decomposition if there exists $A, B \subseteq V(G)$ with $|A| = s$ and such that (A, B) is a bipartition of G. Note that G admits an s-decomposition iff it admits an $(n-s)$-decomposition.

Let $\mathcal{A}_s$ denote the event that G admits an s-decomposition and $\mathcal{B}$ denote the event that G is a barrier graph. The content of Theorem 1.12 is that

$$\mathbb{P}(\mathcal{B}|\mathcal{A}_s) \le 2^t e^{-n2^{-(t+1)}},$$

if $s \ge t$, where $t = 16$ is as in Theorem 1.12.

On the other hand, there is some $0 < s \le \lfloor n/2 \rfloor$ so that $G \in \mathcal{A}_s$, because there must be some bipartition witnessing that G is bipartite, and we may take s to always be the size of the smaller half of the partition.

Thus, summing over possible values of s, we have $\mathbb{P}(\mathcal{B}) \le \sum_{s=1}^{\lfloor n/2 \rfloor} \mathbb{P}(\mathcal{B} \cap \mathcal{A}_s)$, with inequality instead of equality because the events $\mathcal{A}_s$ are not disjoint.

Rewriting this and using the inequality that follows from Theorem 1.12, we have

$$\begin{aligned}
\mathbb{P}(\mathcal{B}) &\le \sum_{s=1}^{\lfloor n/2 \rfloor} \mathbb{P}(\mathcal{B} \cap \mathcal{A}_s) \\
&= \sum_{s=1}^{\lfloor n/2 \rfloor} \mathbb{P}(\mathcal{B}|\mathcal{A}_s)\mathbb{P}(\mathcal{A}_s) \\
&= \sum_{s=1}^{t-1} \mathbb{P}(\mathcal{B}|\mathcal{A}_s)\mathbb{P}(\mathcal{A}_s) + \sum_{s=t}^{\lfloor n/2 \rfloor} \mathbb{P}(\mathcal{B}|\mathcal{A}_s)\mathbb{P}(\mathcal{A}_s) \\
&\le \sum_{s=1}^{t-1} \mathbb{P}(\mathcal{A}_s) + 2^t e^{-n2^{-(t+1)}} \sum_{s=t}^{\lfloor n/2 \rfloor} \mathbb{P}(\mathcal{A}_s) \\
&\le \sum_{s=1}^{t-1} \mathbb{P}(\mathcal{A}_s) + \frac{n}{2} 2^t e^{-n2^{-(t+1)}} \\
&\le \sum_{s=1}^{t-1} \mathbb{P}(\mathcal{A}_s) + o(1),
\end{aligned}$$

Finally, we estimate $\mathbb{P}(\mathcal{A}_s)$ for $s < 16$ rather crudely. There are $2^{s(n-s)}$ labeled, bipartite graphs with partite sets sized s and $n - s$, and so there are at most this many unlabeled bipartite graphs admitting s-decompositions.

For the same reason, there are at least $2^{\lfloor n/2 \rfloor^2}$ labeled bipartite graphs with n vertices. Since $n! \leq n \log_2(n)$, this means there are at least $2^{\lfloor n/2 \rfloor^2}/n! \geq 2^{\lfloor n/2 \rfloor^2 - n \log_2 n} \geq 2^{n^2/4 - n \log_2 n}$ unlabeled bipartite graphs with n vertices.

Thus, $\mathbb{P}(\mathcal{A}_s) \leq \dfrac{2^{s(n-s)}}{2^{n^2/4 - n \log_2(n)}} = o(1)$ for $s < 16$, and therefore $\mathbb{P}(\mathcal{B}) = o(1)$. $\square$

1.4 Resilience of Barrier Graphs

Recall that the *resilience* of a sensor network is the number of sensors whose removal from the network prevents that network from providing its coverage. In the case of the ray-barrier sensor networks considered in this chapter, a network provides coverage if every path from the designated starting location α to the target location β crosses at least one ray sensor.

This idea of the resilience of a sensor network was originally explored in sensor networks whose sensors were modeled as discs [2], which is useful, for example, in cell-tower networks to ensure customers have service unless some unlikely number of towers fail simultaneously. In disc sensor networks, where the network's coverage of an area is represented by the area overlapping at least one sensor, there is not as clear an abstraction to a graph as there is with ray-sensor barriers; because of the correspondence between ray-sensor networks and barrier graphs, the resilience of the network *is* the resilience of the graph.

Thus we can also consider the resilience that a network *would* have, assuming that it realizes a given graph. Consider the class of graphs with a fixed minimum vertex cover size r (which, for the purposes of this discussion, we call resilience even if the graph is not realizable). For which r, if any, are *all* graphs of resilience r realizable? In the general case this proves to be a fairly uninteresting question as the 5 cycle, for instance, has resilience 3 and is not realizable. However, the question seems much more interesting (and much less trivial) when restricted to the class of bipartite graphs.

In particular, it is not difficult to prove the following theorems, which we will prove later in this chapter.

Theorem 1.14. *All bipartite graphs of resilience 2 are realizable as barrier graphs*

Theorem 1.15. *All bipartite graphs of resilience 3 are realizable as barrier graphs.*

The realizations proving these theorems, which we leave in Section 1.5.1, are both natural and simple. On the other hand, Theorem 1.12 shows that most bipartite graphs of large enough fixed resilience are *not* realizable, so clearly there is some largest resilience r such that all bipartite graphs of resilience r are realizable. Hence we propose the following question:

Question 1. *What is the minimum resilience $r_\star$ such that there exists a bipartite graph of resilience $r_\star$ that is not realizable as a barrier graph?*

We have shown that $r_\star$ exists, and that $4 \leq r_\star \leq 16$, where the upper bound on $r_\star$ follows from Theorem 1.12, and ultimately from Theorem 1.11. Determining $r_\star$, or even tightening these bounds, would be an interesting step in our understanding of bipartite barrier graphs. Some optimization of the bounds in Lemma 1.10 and Theorem 1.11 can reduce the upper bound on $r_\star$ slightly, but a substantial tightening seems to require a new idea.

1.5 Realizing Particular Barrier Graphs

Recall that a realization of a graph as a barrier graph is an arrangement A of rays together with two points α and β. There are many barrier graphs for which a realization is not difficult to find. In this section, we present some classes of graphs and schemes for realizing them. Several of these classes are bipartite, but several others are identified instead by the size of their minimum vertex cover.

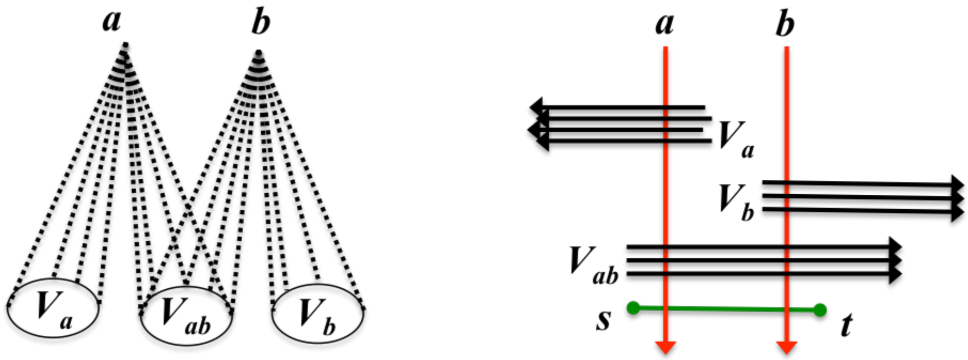
Note that while many of these constructions are natural, they are usually not unique. We begin by showing how to realize barrier graphs of given small resilience through proving the two theorems stated in Section 1.4.

1.5.1 Realizing Graphs of Resilience 2 and 3

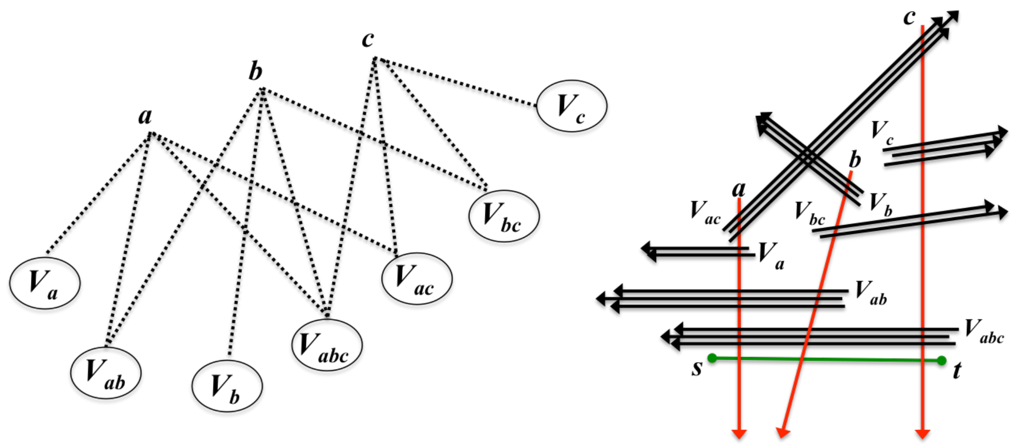
Recall that we refer to the minimum size of a vertex cover of a graph as the ***resilience*** of that graph.

A connected graph can, to some extent, be addressed in terms of vertex covers and the subgraphs they induce, because the edges of vertices not in the cover exactly pick out a subset of the vertex cover. When the vertex cover is small, therefore, the set of possible descriptions of neighborhoods in the graph also becomes small, as do the neighborhoods themselves.

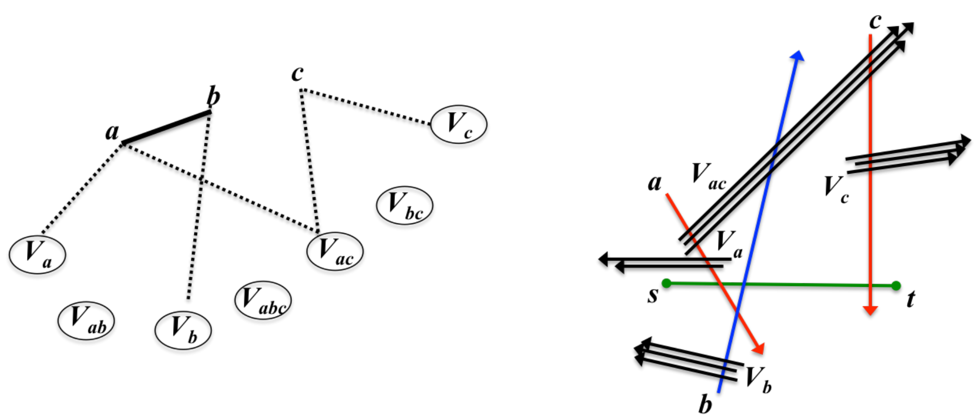
For graphs of small resilience, we can use this idea to consider all possible cases at once and develop a general construction that works for any graph of that resilience. To that end, we begin with the simplest case that is not a star graph: resilience 2.



(a) General resilience 2 bipartite graph with independent-set vertex cover and its realization



(b) General resilience 3 bipartite graph with independent-set vertex cover and its realization



(c) General resilience 3 bipartite graph with one edge between elements of the vertex cover, and its realization. (V_{ab}, V_{abc}, and V_{bc} from (b) shown for context; they are empty)

Figure 1.8: General bipartite graphs with resilience 2 or 3 (left) and their realizations (right)

Theorem 1.14. *All bipartite graphs of resilience 2 are realizable as barrier graphs*

Proof. Let $G = (V, E)$ be a resilience 2 bipartite graph, and $\{a, b\}$ a vertex cover. For $S \subseteq \{a, b\}$, let $V_S \subseteq V$ be the set of vertices in $V \setminus \{a, b\}$ adjacent to exactly those vertices in S. Note that the V_S partition $V \setminus \{a, b\}$.

Each V_S is an independent set, and there are no edges between these sets, because all edges are incident to either a or b.

If $(a, b) \in E$, then $V_{\{a,b\}}$ is empty, because otherwise G contains a triangle and is not bipartite. In this case, G is just a tree, and can be realized as shown in Section 1.5.3.

If $(a, b) \notin E$, G may be realized as in Figure 1.8a.

$V_\emptyset$ is realized by any collection of $|V_\emptyset|$ black rays that intersect no other rays, and is not pictured in the figure. $\square$

Theorem 1.15. *All bipartite graphs of resilience 3 are realizable as barrier graphs*

Proof. Fix a resilience 3 bipartite graph $G = (V, E)$ and a vertex cover $\{a, b, c\}$. As before, for $S \subseteq \{a, b, c\}$ let V_S be the set of vertices in $V \setminus \{a, b, c\}$ adjacent to exactly the vertices in S.

There are only three graphs that can be induced by the vertex cover up to isomorphism: the vertex cover $\{a, b, c\}$ could be an independent set (realization shown in Figure 1.8b); the graph induced by $\{a, b, c\}$ has one edge (realization shown in Figure 1.8c); or the graph induced by $\{a, b, c\}$ is a path. The realization for the third case is identical to the one in Figure 1.8c, except that the ray for c must be extended so that it also intersects the ray for b.

Note that in the case where the graph induced by $\{a, b, c\}$ has one edge (a, b), if $V_{ac} = V_{\{a,c\}}$ is not empty then any vertices in V_{bc} would induce a triangle, and so V_{ac} and V_{bc} are not both nonempty. V_{ab} and V_{abc} are empty in this case for the same reason.

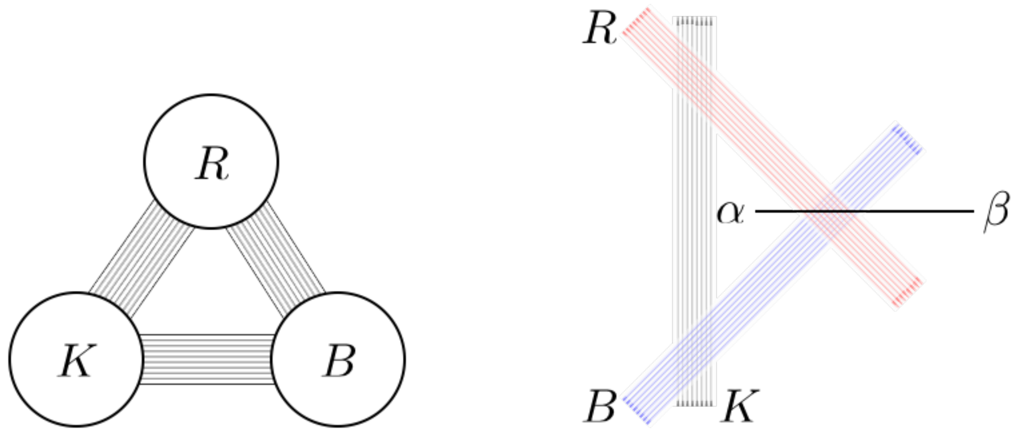
Again, $V_{\emptyset}$ is realized by any collection of $|V_{\emptyset}|$ black rays that intersect no other rays, and is not pictured in the figure. □

1.5.2 Realizing Complete Bi/Tripartite Graphs, Paths, and Cycles

A complete tripartite is perhaps the most natural barrier graph to realize (see Figure 1.9).

Because induced subgraphs of barrier graphs are also barrier graphs, the construction for any complete bipartite graph is built from that for a tripartite barrier graph with two partite sets of the same size, and discarding the third partite set.

A natural construction for realizing a path graph is just as easy, and only uses two colors: alternate blue and red rays that each intersect two consecutive rays of the opposite color, and stop once you've reached as many rays as vertices in the graph (See Figure 1.10).

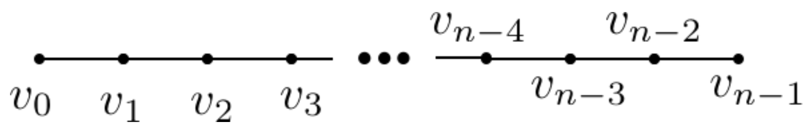


(a) (b)

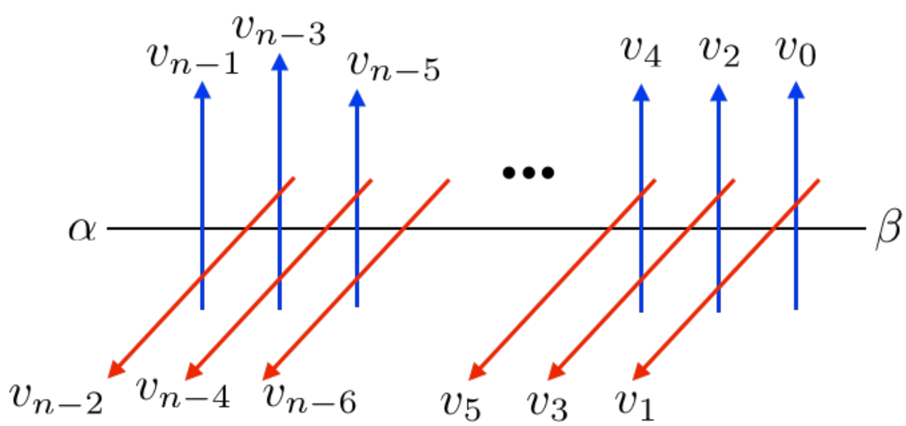
Figure 1.9: A realization (b) of a given complete tripartite graph (a).

To realize a cycle of even length, a natural construction is (not surprisingly) very similar to that for a path. The main trick is to designate a particular vertex to connect the ends of a path into a cycle. As in Figure 1.11, given an even cycle graph on vertices $v_0, v_1, \ldots, v_{n-1}$, with edges $(v_i, v_{i+1}) \forall i$ and (v_0, v_{n-1}), we build a path on the (odd-number many) vertices $v_1, \ldots, v_{n-1}$, and extend the anchor of one of the ends of this path so that there is a straight line that goes above each of the anchors of the path endpoints and also above α. Placing a ray for v_0 of the opposite color from the path endpoints along this line completes the cycle.

Perhaps most interesting about the simplicity of this construction for arbitrarily long even-length cycles is the fact that there *cannot be* a construction for *odd-length* cycles longer than a triangle, as we saw in Proposition 1.5 of Section 1.2.

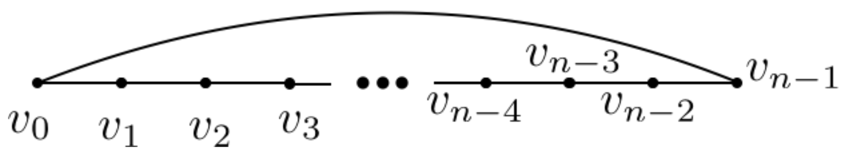


(a)

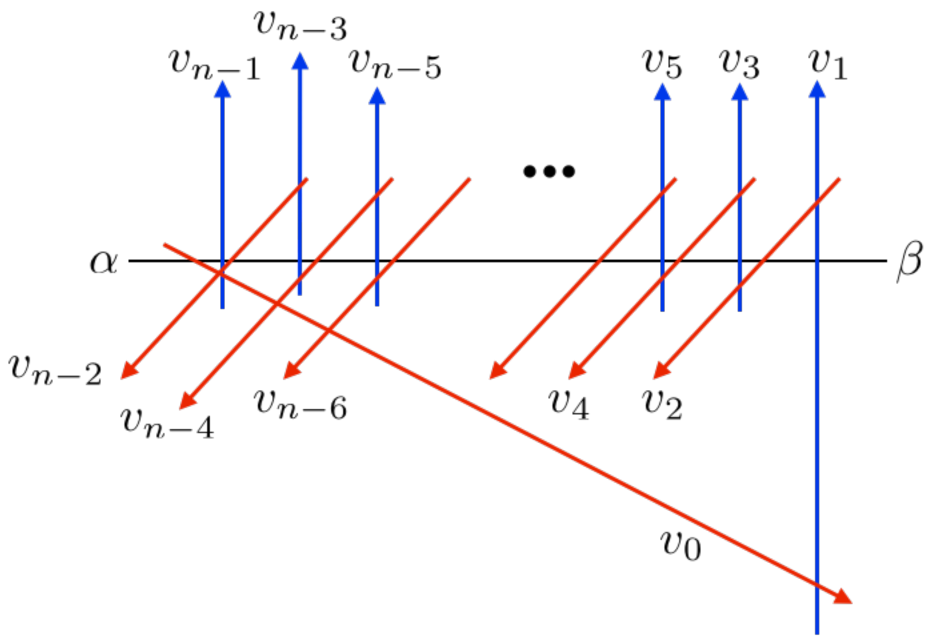


(b)

Figure 1.10: A realization (b) of a given path graph (a) with an even number n of vertices. The realization for odd n is the same but without one of the endpoint blue rays.



(a)



(b)

Figure 1.11: A realization (b) of a given even-length cycle graph (a).

1.5.3 Realizing Trees

To see how to find a realization of an arbitrary tree, we will first show how to find a realization of an arbitrary full k-ary tree. A k-ary tree of ***height*** h is a rooted tree where the root has degree k, every other non-leaf vertex has degree $k+1$, and where the root is distance h from every leaf. We will denote the k-ary tree of height h by T_k^h.

Every tree T is an induced subgraph of T_k^h, where $k = \max_{v \in V(T)} \deg(v)$ and $h = \max_{u,v \in V(T)} d(u,v)$, where $d(u,v)$ is the length of the shortest path in T with endpoints u and v. To see this, designate any vertex r of T as the root and assign it to the root of T_k^h and its children arbitrarily to children of the root of T_k^h. Then, for each neighbor of r, assign each of their respective unassigned neighbors (on the second level) to children of their assigned vertices, and so on. Each step like this is possible because k was chosen to be large enough to accommodate any degree in T and h was chosen to be large enough that the process doesn't reach a leaf of T_k^h until it reaches the end of the longest path in T, and therefore no vertex with unassigned children is ever assigned to a leaf of T_k^h.

By Proposition 1.4, induced subgraphs of barrier graphs are also barrier graphs; given a realization for T_k^h we may simply remove rays for vertices not assigned by the process above to arrive at a realization for T.

The construction for a realization of T_k^h is recursive, and proceeds by levels in the tree (i.e. by distance from the root). For brevity, we will refer to a vertex and its corresponding ray (once chosen) as if they were the same object. Also, for convenience, we will number the vertices of the ℓ^{th} level from 1 to k^ℓ so that the children of each vertex are numbered consecutively, and children of lower-number vertices in level ℓ have lower number than children of higher-number vertices in level ℓ. Then we may refer to the i^{th} vertex of level ℓ as $v_i^{(\ell)}$.

In the process below, many choices of angle are immaterial; it is the relative location of intersections that matter. Because the process cuts convex polygons into multiple longer and thinner convex polygons in which later recursive steps will work, the description of the process is easier to follow when drawn as in the figures.

First, place a red ray for the root r. The root has k neighbors, so for each neighbor $v_i^{(1)}$ anchor a blue ray along r at even intervals from $\overline{\alpha\beta}$. These rays should not intersect before crossing $\overline{\alpha\beta}$. Each $v_i^{(1)}$ has k children ($v_j^{(2)}$), which will be made with red rays anchored along $v_i^{(1)}$. These red rays are placed so that they intersect the root ray r in the same left-right order that they intersect their parent ray $v_i^{(1)}$. See Figure 1.14 for an illustration of the rays for these first two levels of a 3-ary tree.

Note that placing each of these level-2 rays $v_j^{(2)}$ leaves a convex quadrilateral below it, which is bounded by its parent, by r, by itself, and by its next-highest-index sibling, if it has one. If $v_j^{(2)}$ has no next-highest-index sibling, the convex quadrilateral is completed by a line through the following two points: the intersection of the next-highest-index sibling of the parent of $v_j^{(2)}$ and a point further from r along the parent of $v_j^{(2)}$. For such a quadrilateral, call the side defined by $v_j^{(2)}$ itself the "***left***" side, and the side opposite this the "***right***" side, and call the other two sides the "***top***" and "***bottom***" sides (one is always below the other). For examples of such quadrilaterals, see Figure 1.12.

A quadrilateral like the one above, whose left side is the ray for red (resp. blue) leaf vertex v_j^i, is called the ***Extending Quadrilateral*** for $v_j^{(i)}$ if its bottom side is red (resp. blue), its top side is blue (resp. red), and any ray anchored on its left side (along $v_j^{(i)}$)

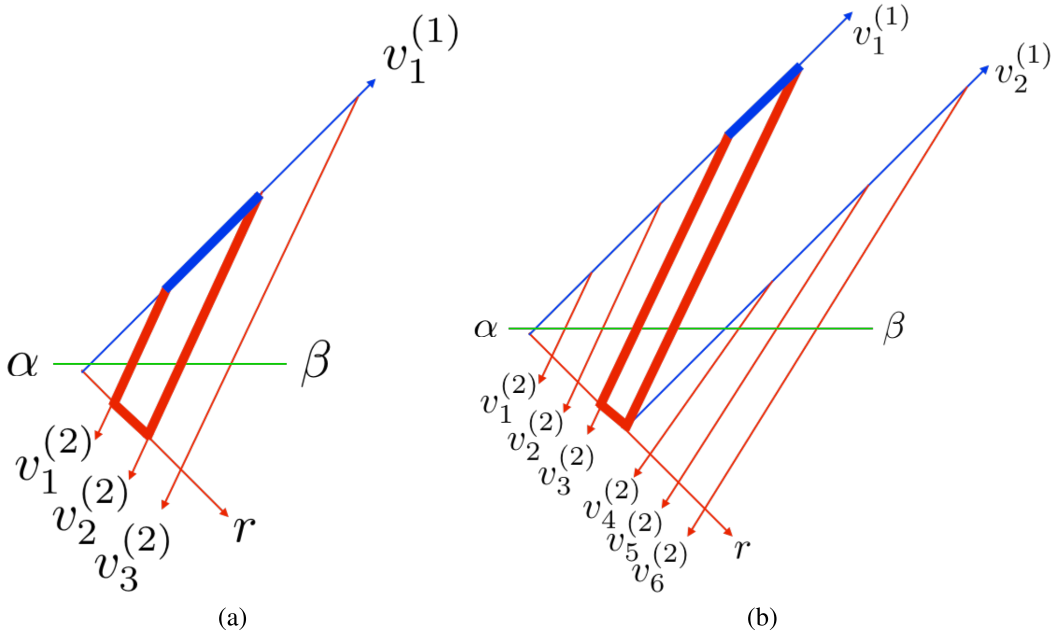


Figure 1.12: A normal Extending Quadrilateral (a) and one for the highest-index child of $v_1^{(1)}$ in a 3-ary tree of height 2 (b).

and intersecting its top or bottom intersects only one color of ray (other than the ray it is anchored along).

For example, the Extending Quadrilateral for $v_2^{(2)}$ in Figure 1.14 is bounded by $v_2^{(2)}$ itself, its sibling $v_3^{(2)}$, its parent $v_1^{(1)}$, and the root r. On the other hand, in the same figure the convex region corresponding to $v_3^{(2)}$ is bounded by itself, the root, its parent, and $v_2^{(1)}$, which is its parent's sibling.

The first tool we will need for our recursive construction is that it is safe to extend the construction by anchoring rays for a vertex v's children along v within this quadrilateral.

Invariant 1.1 (Extendable Tree Realization)**.** *Let A be a set of rays that realizes the tree T_k^h. A has the* Extendable Tree Realization *invariant if there is a distinct Extending Quadrilateral for each leaf v of T_k^h.*

Lemma 1.16. *Let A be an Extendable Tree Realization for T_k^h, where $h \geq 2$. Then there is a set of rays $A' \supseteq A$ that is an Extendable Tree Realization for T_k^{h+1}.*

Proof. Let $v_j^{(i)}$ be the ray of any leaf of T_k^h and let Q be its Extending Quadrilateral.

Suppose $v_j^{(i)}$ is red. Because Q is an Extending Quadrilateral, each of the k children $u_1, \ldots, u_k$ of $v_j^{(i)}$ in T_k^{h+1} can be created by adding a blue ray anchored along $v_j^{(i)}$ above the segment $\overline{\alpha\beta}$ and through the top (blue) edge of Q; these rays intersect only red rays

In each case the anchor represents the only intersection of the new blue ray with a red one, and thus the only barrier created by the new ray is with it's parent $v_j^{(i)}$, so that we preserve the tree structure of the realized barrier graph. Since this can be done to add children to each leaf of T_k^h, this process will successfully produce a realization of T_k^{h+1}.

Each of the new rays forms the left side of a new quadrilateral, which has as its bottom side a segment of $v_j^{(i)}$, and as its top side a segment of the top side of Q. The right side of this quadrilateral is either $v_{j+1}^{(i)}$ (if $j < k$), or an auxiliary segment like the one in Figure 1.12b.

These new quadrilaterals are Extending Quadrilaterals Q_t for their corresponding rays u_t, the children of $v_j^{(i)}$ (See Figure 1.13). This is true for the following reasons:

- Their top sides are sub-segments of the top side of Q (an Extending Quadrilateral) and the only new rays above this top side are also blue rays, so rays anchored along a child and through the top of its quadrilateral will only intersect other blue rays.

- Their bottom sides are the red ray $v_j^{(i)}$ itself, and because of the angle of u_t, rays pointing downward and anchored along u_t can intersect only the rays below Q, which are guaranteed to all be of the same color (red).

For a demonstration of placing such rays, see Figure 1.14. If instead $v_j^{(i)}$ is blue, repeat the same argument, but exchange the roles of red and blue, and above and below. $\square$

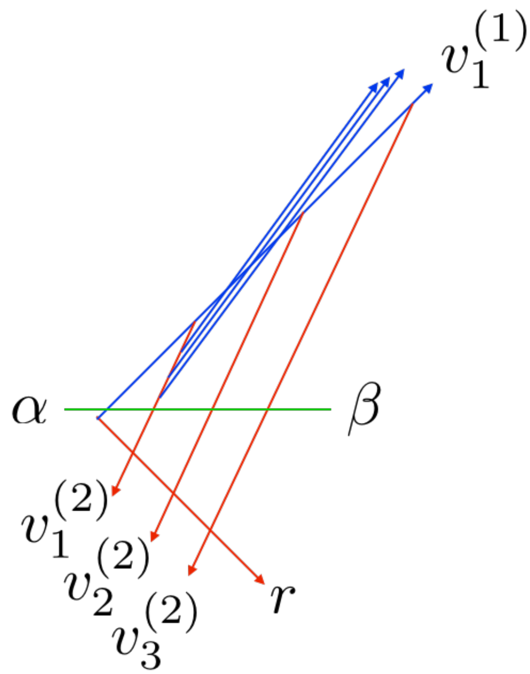


Figure 1.13: Adding the (blue) children of the ray $v_1^{(2)}$ within an Extending Quadrilateral in a way that leaves Extending Quadrilaterals for each child.

Lemma 1.16 shows that if there is any Extendable Tree Realization for T_k^2, then there is one for T_k^h for any h; the lemma required that $h \geq 2$ simply because otherwise the rays do not form the needed quadrilaterals.

The description of how to construct a realization of T_3^2 shown in Figure 1.14 is easily changed to T_k^2 for any k, and so we have shown that any T_k^h is realizable, and therefore any tree is realizable.

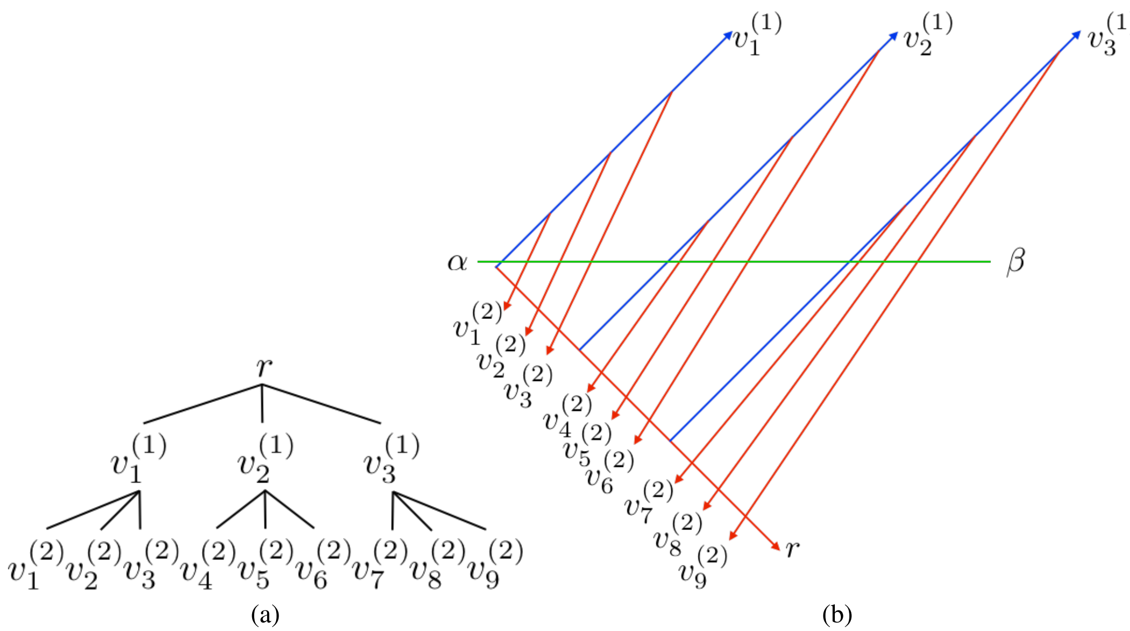


Figure 1.14: A realization (b) of a 3-ary height-2 tree with root r (a).

1.5.4 Further Realization Questions

One notable aspect of all the realizations of bipartite graphs in this section is that none of them requires three colors of rays, but of course there are alternative realizations of the same graphs that use three colors. For example, in any path, one can exchange the endpoints with black rays and still realize the path.

This raises the question of whether all bipartite barrier graphs can be realized with rays of only two colors, or whether there exists a bipartite barrier graph for which every realization uses rays of three colors. Obviously if our standard coloring of an arrangement of rays only requires two colors, then the corresponding graph is bipartite, but the opposite implication (though intuitive) is not as obvious, and if true requires proof. Nonetheless, this is our suspicion:

Conjecture 1.17. *If a bipartite graph G is realizable, then there is a realization for G requiring only two ray colors.*

One approach to this question (and others) would be to develop an algorithm which produces a realization of a given graph, if one exists, and reports failure if this is not possible. Analyzing this algorithm might lead to a constructive proof based on assumptions about the input graph. So far, such an algorithm is not known, nor is the complexity of this problem known.

Conjecture 1.18. *There is an algorithm that takes any tripartite graph G as input and either produces a realization of G or a certificate that G is not a barrier graph.*

1.6 Acknowledgements

The results in this chapter, through Section 1.5.1, are based on the following article: Kirk Boyer, Paul Horn, and Mario A. Lopez. On the barrier graph of an arrangement of ray sensors. *Discrete Appl. Math.*, 225:11–21, 2017.

CHAPTER 2: EXPOSING POINTS

2.1 Exposure and Joint Exposure

In this chapter, we describe two new notions of network coverage with corresponding extremal questions, initially in the plane. We then generalize their geometry to any number of dimensions, and begin to address these questions in that domain as well.

Initially, we consider networks whose sensor geometry is just like those from Chapter 1: arrangements of rays in the plane. Networks whose sensors are line segments or lines have very similar properties, as we will see in Lemma 2.5, and because the questions we will address are more naturally framed in terms of line sensors, we will begin this chapter by describing line sensor networks.

Consider a set L of n lines and a set P of k points in the plane. We will say L and P are in general position if no line of L contains any point of P, and if no line of L is parallel to a line through any two points of P; we assume all sets of points and lines are in general position. Note that this means we do allow lines of L to be parallel to each other.

L partitions the plane into convex cells such that only 2-cells (polygonal regions) contain points of P. As a sensor network protecting the point locations in P, we may consider that locations are protected by line sensors in L from intruders if they are surrounded on all sides by sensors. This means that a point $p \in P$ is ***protected*** if the 2-cell containing it is bounded, and that L ***protects*** p.

Similarly, a point $p \in P$ is ***exposed*** by L if it is in an unbounded cell, and L ***exposes*** P if all $p \in P$ are exposed.

An ***exposing set*** $X \subseteq L$ is one such that $L \setminus X$ exposes P. In other words, X is an exposing set of sensors if removing it from L leaves all points exposed. Relatedly, a ***witness set*** is a set W of k rays, one anchored at each point of P. A witness set W intersects some subset of sensors $E_L(W) \subseteq L$, which is necessarily an exposing set (written $E(W)$ when L is understood), and we say that W ***witnesses*** $E(W)$.

In addition, for every exposing set X there is a witness set that witnesses X, and usually there are many.

Lemma 2.1. *If $L \setminus X$ exposes P, then there exists a witness set W with $E(W) \subseteq X$.*

Proof. Consider a point $p \in P$. It suffices to show that there is a witness ray anchored at p completely contained in c, the unbounded cell of the plane partition with respect to $L \setminus X$ that contains p. If $X = L$ then every ray has this property, so suppose $X \neq L$.

The boundary of c contains at least one unbounded ray r. Since c is convex, the ray r_p, anchored at p and parallel to r, does not intersect any element of X, for any line in X that r_p hits would also hit r, which would contradict that r is entirely contained in the boundary.

Then the set of rays $W = \{r_p : p \in P\}$ witnesses that X is an exposing set, since by construction no ray in W intersects any element of X. $\square$

Such a witness set can now conveniently be called a witness set *of* X. Note that Lemma 2.1 implies that if X is an exposing set for a set of lines L, and L' is another set of lines, then $X \cup L'$ is an exposing set for $L \cup L'$.

We are now ready to introduce our first extremal question.

Question 2. *Given k designated target locations and allowing the placement of n line barriers, what number of those barriers is always sufficient and sometimes necessary to remove so that all target locations are exposed?*

With regard to this question we will establish a general lower bound of the form $\frac{k}{k+1} + O(1)$; we prove that such a bound is tight for $k \in [3]$, where we use the conventional notation $[n] = \{1, 2, \ldots, n\}$. We also prove that our lower bound is tight for $k = 4$ provided the points are in convex position. This first question models a situation where there are as many intruders as target locations, because two locations may be exposed while there is no sensor-avoiding path between them.

It turns out that Question 2 has the same answer whether the network consists of line segments, rays, or infinite lines (see Proposition 2.5), but interestingly, this does not hold when there is only one intruder who must reach all locations, which is represented in the following question:

Question 3. *Given k designated target locations and allowing the placement of n **ray** barriers, what number of those barriers is always sufficient and sometimes necessary to remove so that a **single** unbounded region contains **all** of the targets?*

We find the answer to Question 3 quite interesting. While in the $k = 1$ case this is identical to the answer to Question 2 above (and hence, asymptotically is $\frac{n}{2}$), we show that for $k = 2$ the answer differs and is actually asymptotically equal to $\frac{3}{4}n$ rather than $\frac{2}{3}n$.

The notation we introduce below for the answer to Question 2 emphasizes line barriers because of the relative ease of analysis, but additionally, this is the type of barrier for which these questions are most naturally generalized to higher dimensions; a line is a hyperplane in $\mathbb{R}^2$, and hyperplane sensors are what we will examine in $d \geq 2$ dimensions.

Definition 2.2. *For a finite set $P \subseteq \mathbb{R}^d$ of k points and finite set L of n lines in $\mathbb{R}^2$, $\lambda(P, L)$ is the size of the smallest exposing set $X \subseteq L$ that exposes P.*

The problem of computing $\lambda(P, L)$ is clearly identical to finding a witness set witnessing a minimum size exposing set in L. Since $\lambda(P, L)$ is the size of a subset of L, it is an integer satisfying $0 \leq \lambda(P, L) \leq n$, and we may define the following.

Definition 2.3.

$\lambda(P, n) = \max\{\lambda(P, L) : L \textit{ is a set of } n \textit{ lines in } \mathbb{R}^2\}$

$\lambda(k, L) = \max\{\lambda(P, L) : P \textit{ is a set of } k \textit{ points in } \mathbb{R}^2\}$

One may view $\lambda(P, n)$ as communicating the smallest exposing set size in the "best" set of n lines for protecting the fixed point set P. In other words, $\lambda(P, n)$ is the number of lines which it is always sufficient to remove from any set of n lines to expose the points in P.

Similarly, $\lambda(k, L)$ may be viewed as communicating the size of the smallest exposing set for the "hardest to protect" set of points for a fixed set L of lines.

Definition 2.4.

$$\begin{aligned}
\lambda_k(n) &= \max\{\lambda(P, n) : P \subseteq \mathbb{R}^2 \textit{ and } |P| = k\} \\
&= \max_{|P|=k, |L|=n} \lambda(P, L) \\
&= \max\{\lambda(k, L) : L \text{ a set of } n \text{ lines in } \mathbb{R}^2\}
\end{aligned}$$

First, this means that removing $\lambda_k(n)$ lines from any set of n lines is always sufficient to expose any set of k points.

Second, this means that it is sometimes necessary to remove $\lambda_k(n)$ lines from a set of n in order to expose a set of k points, because there exists a pair P^*, L^* such that $\lambda(P^*, L^*) = \lambda_k(n)$.

Hence, the value of $\lambda_k(n)$ is the answer to Question 2.

Note that the above implies that if removing λ' lines from a set of n is a always sufficient to expose any set of k points, then $\lambda_k(n) \leq \lambda'$; we thus may compute upper bounds for $\lambda_k(n)$ by focusing on sufficiency. The above also means that if it is sometimes necessary to remove λ'' lines from a set of n to expose some set of k points, then $\lambda'' \leq \lambda_k(n)$; we then may demonstrate that λ'' is a lower bound for $\lambda_k(n)$ by exhibiting a particular set P of k points and a corresponding set L of n lines such that $\lambda(P, L) = \lambda''$.

When we consider only points in convex position, we define $\widetilde{\lambda}_k(n)$ analogously to $\lambda_k(n)$, so that $\widetilde{\lambda}_k(n)$ is the smallest number so that every set of n lines protecting a set of k points *in convex position* in $\mathbb{R}^2$ has a smallest exposing set of size at most $\widetilde{\lambda}_k(n)$. We also define $r_k(n)$, $\widetilde{r}_k(n)$, $s_k(n)$, and $\widetilde{s}_k(n)$ to be the analogous values for rays and for segments, respectively, instead of lines.

Proposition 2.5. $\lambda_k(n) = r_k(n) = s_k(n) \geq \widetilde{\lambda}_k(n) = \widetilde{r}_k(n) = \widetilde{s}_k(n)$.

Proof. Fix a collection P of k points, and let L be a set of n lines. Choose a set of rays R so that each ray in R has a distinct line of L as its supporting line, and so that each ray intersects all lines intersected by its supporting line. Choose a set of segments S supported by lines in L similarly.

Suppose $L' \subseteq L$ is an exposing set of minimal size. Let $R' \subseteq R$ be the set of rays whose supporting lines are in L', and let $S' \subseteq S$ be similar for segments. Then P is also exposed by $R \setminus R'$ and $S \setminus S'$, because each point of P is in an unbounded region according to the partition induced by $L \setminus L'$, which is a refinement of the partitions according to $R \setminus R'$ and $S \setminus S'$.

On the other hand, the supporting lines of an exposing set of rays (or of segments) are witnessed by the same witness set that witnesses the rays. Since P was arbitrary, this means the minimal size of an exposing set for any collection of k points is the same whether we are using line barriers, ray barriers, or segment barriers. $\square$

We mention that these questions, while motivated by the topic of ray sensor networks, are natural questions in combinatorial geometry regarding how well a collection of rays can 'protect' a set of k points. As mentioned above, the first question is agnostic to whether the barriers considered are rays, lines, or segments, but for the second question, rays are the only interesting type of barrier, in the following sense.

With n line sensors and more than one point, by placing all lines between some pair of points one forces a single intruder to cross all of them, and so the function for $k \geq 2$ points would be identically n. On the other hand, using n segment sensors, a single intruder has no more difficulty than k intruders, since a segment arrangement always has a single unique unbounded region; therefore, for segment sensors the answers to Question 2 and to Question 3 are the same.

For these reasons we will explore the answer to Question 3 separately and through networks of ray sensors in Section 2.3.

2.2 Point Exposing Bounds in 2 dimensions

We ultimately seek the values of $\lambda_k(n)$, $\widetilde{\lambda}_k(n)$ as functions of n, for fixed k. In lieu of being able to directly calculate these functions, we find upper and lower bounds for them, and make these bounds as tight as possible.

We note in the next lemma that for fixed k, adding one barrier to any set of barriers can increase the size of the smallest exposing set by at most one, and of course cannot reduce the size of the smallest exposing set.

Lemma 2.6. $0 \leq \lambda_k(n+1) - \lambda_k(n) \leq 1$

Next we prove an asymptotic lower bound for $\lambda_k(n)$ for all k. We then prove asymptotically matching upper bounds for $k = 1, 2, 3$, and 4.

As remarked in Section 2.1, to find upper bounds we must describe what is always sufficient, and so the upper bound proofs have a common flavor, in that we choose a small collection of canonical witness sets W as a function of the set of points P, show that each $E(W)$ is always an exposing set for P, and moreover that its size is small enough. In spite of this common flavor, extending these tight upper bounds to larger values of k as with the lower bounds has proven elusive.

To find lower bounds we must describe a point set and line set as a function of k and n for which removing enough lines is necessary, and this summarizes our approach to lower bounds in the following theorem.

Theorem 2.7. $\lambda_k(n) \geq \widetilde{\lambda}_k(n) \geq \lceil \frac{k}{k+1} n \rceil - k$

Proof. Let P be any set of k points in the plane, convex or not. We construct a set of lines so that every exposing set has at least the desired size.

First suppose that $n = a(k+1)$. Choose a different slopes that are not the slope between any pair of points in P. For each chosen slope t, place $k+1$ line barriers so that each pair of these barriers closest to each other has a single point between them (see Figure 2.1 for examples). These slope t barriers we will call t-barriers.

Consider an exposing set X of minimal size, and let W be a witness set as guaranteed by Lemma 2.1. We proceed to lower-bound $|E(W)|$, and hence $|X|$.

Fix a slope t among the chosen slopes and let $k'(t)$ denote the number of rays in W of that slope. Note that t induces an order on P and the t-barriers, and each ray anchored at $p \in P$ not parallel to t intersects all t-barriers either above or below p.

From this it's easy to see that rays in W cross at least $k - k'(t)$ t-barriers.

Thus

$$|X| \geq |E(W)| \geq \sum_t (k - k'(t)) = \sum_t k - \sum_t k'(t) \geq ak - k = \frac{k}{k+1} n - k.$$

If instead $n = a(k+1) + b$, just add the additional $b < k+1$ barriers in a group of parallel barriers with slope different from the rest of the groups. The argument above applies to this group as well, except that we only have b total barriers instead of $k+1$, so

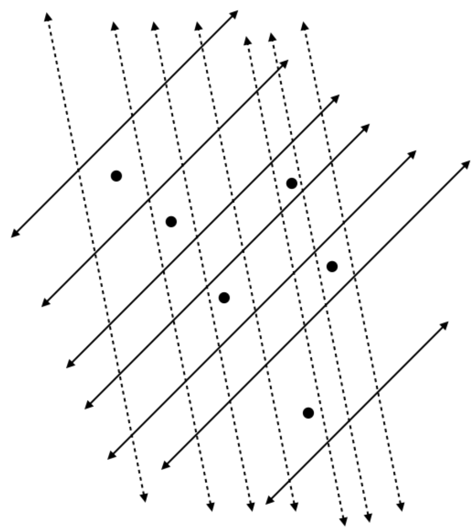


Figure 2.1: Two sets of 7 t-barriers for a set of 6 points.

the barriers from this group crossed by witness rays is now $b - k'(t)$. This implies

$$|E(W)| \geq ak + b - k = \frac{k}{k+1}(n-b) + b - k \geq \left\lceil \frac{k}{k+1}n \right\rceil - k.$$

□

Because the proof of Theorem 2.7 did not assume convexity of the k points, it holds as a lower bound both for $\lambda_k(n)$ and $\widetilde{\lambda}_k(n)$.

The next lemma is a tool we will use for many of our proofs of upper bounds for $\lambda_k(n)$.

Lemma 2.8. *Suppose P is a set of any number of points and L a set of lines such that at most one line intersects the convex hull of P (and no line of L contains a point of P). Then there is an exposing set in L of size at most $\left\lfloor \frac{1}{2}(|L|-1) \right\rfloor$.*

Proof. Suppose there is no line through the convex hull of P. Then all points lie in the same convex region of the partition. Pick an arbitrary point p in that region, and line

$\ell \in L$. Consider the two rays from p parallel to ℓ in opposite directions. One of these rays witnesses an exposing set of size at most $\left\lfloor \frac{1}{2}(|L|-1) \right\rfloor$, by the pigeonhole principle.

If there is a line $\ell \in L$ through the convex hull, the points in P are split into two regions. The proof in this case proceeds similarly, taking rays starting at two arbitrary points, one in each of the two regions, parallel to ℓ. $\square$

Theorem 2.9. $\lambda_1(n) = \lfloor \frac{1}{2}(n-1) \rfloor$.

Proof. Let L be any set of n lines in general position with respect to some target point p. By Lemma 2.8, there is an exposing set of size at most $\left\lfloor \frac{1}{2}(n-1) \right\rfloor$. Thus, $\lambda_1(n) \leq \left\lfloor \frac{1}{2}(n-1) \right\rfloor$.

For the improved lower bound, fix n and a point p. Let L be any set of n lines with their perpendiculars through p uniformly distributed around p. If n is odd, then every ray anchored at p and its opposite together intersect at least $n-1$ lines of L, because they could be parallel to one line. Each of these rays thus witnesses an exposing set of size at least $\frac{1}{2}(n-1) - 1 = \left\lfloor \frac{1}{2}(n-1) \right\rfloor$.

If n is even, each ray anchored at p intersects at least $\frac{1}{2}(n-2) = \left\lfloor \frac{1}{2}(n-1) \right\rfloor$ lines of L, because they can be parallel to at most 2 lines of L. So, every ray anchored at p witnesses an exposing set of size no smaller than $\left\lfloor \frac{1}{2}(n-1) \right\rfloor$. $\square$

Note that while the lower bound in Theorem 2.9 is larger than that of Theorem 2.7, they are asymptotically the same.

For $k > 1$, by removing most lines separating points and thereby placing them in the same region, we can essentially reduce to the case of Theorem 2.9, as captured by the following lemma.

Theorem 2.10. $\lambda_2(n) = \widetilde{\lambda}_2(n) = \frac{2}{3}n - O(1)$.

Proof. Let $L = L_I \cup L_E$ be any set of n lines not containing points p and q, where L_I are the lines that pass between p and q (the "interior" lines), and L_E are the rest (the "exterior" lines). Let $|L| = n$ and $|L_I| = i$, so that $|L_E| = n - i$.

L_E is clearly an exposing set of size $n-i$, witnessed by the rays along the line through p and q pointing outward. On the other hand, L_E along with any one element $\ell \in L_I$ satisfies the conditions of Lemma 2.8, so there is an exposing set for $L_E \cup \{\ell\} = L \setminus (L_I \setminus \{\ell\})$ of size at most $\left\lfloor \frac{1}{2}(|L_E| + 1 - 1) \right\rfloor = \left\lfloor \frac{1}{2}(n-i) \right\rfloor$. This exposing set, together with the rest of L_I, is an exposing set for L of size at most $i - 1 + \left\lfloor \frac{1}{2}(n-i) \right\rfloor$.

For fixed i, the smaller of these two exposing sets is clearly no smaller than $\lambda_2(n)$. This implies

$$\begin{aligned}
\lambda_2(n) &\leq \max_{\substack{0 \leq i \leq n \\ i \in \mathbb{Z}}} \min \left\{ n - i, i - 1 + \left\lfloor \frac{1}{2}(n-i) \right\rfloor \right\} \\
&\leq \max_{\substack{0 \leq i \leq n \\ i \in \mathbb{R}}} \min \left\{ n - i, i - 1 + \left\lfloor \frac{1}{2}(n-i) \right\rfloor \right\} \\
&\leq \max_{\substack{0 \leq i \leq n \\ i \in \mathbb{R}}} \min \left\{ n - i, i - 1 + \frac{1}{2}(n-i) \right\}.
\end{aligned}$$

This maximum occurs when the two bounds are equal, and hence at $i_* = \frac{n+2}{3}$. So, $\lambda_2(n) \leq n - i_* = n - \frac{n+2}{3} = \frac{2}{3}(n-1)$. Since $\lambda_2(n)$ is integer-valued, this means $\lambda_2(n) \leq \left\lfloor \frac{2}{3}(n-1) \right\rfloor$. From this and the bound from Theorem 2.7, the result follows. $\square$

In fact, $\lambda_2(n) = \lfloor \frac{2}{3}(n-1) \rfloor$ exactly, but the proof of the tighter lower bound is more complicated to describe than illuminating (although not difficult).

Theorem 2.11. $\lambda_3(n) = \widetilde{\lambda}_3(n) = \frac{3}{4}n - O(1)$.

Proof. Let a, b, and c be any three points, listed in clockwise order. Consider the lines through each pair of these, with the rays at their ends labeled a_1, a_2, b_1, b_2, c_1, and c_2, in clockwise order according to the nearest point, as in Figure 2.2.

Let L be any set of n lines not containing a, b, or c, and separate L into "interior" and "exterior" lines L_I and L_E as before, according to whether they intersect the convex hull of $\{a, b, c\}$ or not, respectively. As before, let $|L| = n$ and $|L_I| = i$, so that $|L_E| = n - i$.

Let A_1 be those lines of L_I intersecting a_1, B_2 be lines of L_I intersecting b_2, and so on. For lines of L_I that do not intersect any of these, include them with A_1 as well.

Letting $L_1 := A_1 \cup B_1 \cup C_1$ and $L_2 := A_2 \cup B_2 \cup C_2$, we have that L_I is the disjoint union of L_1 and L_2, since a line intersecting any two of these labeled ends does not separate any of a, b, or c. Then either $|L_1| \leq \frac{i}{2}$ or $|L_2| \leq \frac{i}{2}$; without loss of generality, assume $|L_1| \leq \frac{i}{2}$.

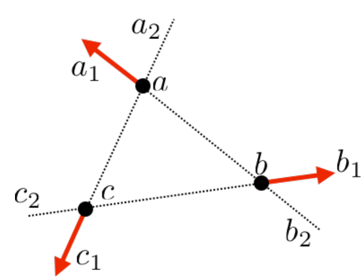


Figure 2.2: A canonical witness set of three rays for a set of three points.

$L_E \cup L_1$ is an exposing set, as witnessed by $W_1 = \{a_1, b_1, c_1\}$, and has size at most $n - \frac{i}{2}$.

For a second bound, we again note that $L_E \cup \{\ell\}$ satisfies the conditions of Lemma 2.8 for any $\ell \in L_I$, and so there is an exposing set for $L_E \cup \{\ell\} = L \setminus (L_I \setminus \{\ell\})$ of size at most $\left\lfloor \frac{1}{2}(n-i) \right\rfloor$, which together with the rest of L_I yields an exposing set of size at most $i - 1 + \left\lfloor \frac{1}{2}(n-i) \right\rfloor$.

As before, the minimum of these two upper bounds is maximized when they are equal, and hence when $i_* = \frac{n}{2} + 1$. Thus $\lambda_3(n) \leq n - i_*/2 = n - \frac{1+n/2}{2} = \frac{3}{4}n - \frac{1}{2}$, which implies (since $\lambda_3(n)$ is integer-valued) that $\lambda_3(n) \leq \lfloor \frac{1}{4}(3n-2) \rfloor = \frac{3}{4}n - O(1)$.

From Theorem 2.7 we have that $\lceil \frac{3}{4}n \rceil - 3 \leq \lambda_3(n)$, so the result follows. □

Theorem 2.12. $\widetilde{\lambda}_4(n) = \left\lceil \frac{4}{5}n \right\rceil - O(1)$.

Proof. Let a, b, c, and d be points in convex position, and suppose L is a set of n lines containing none of the four points. Let L_I and L_E be the set of lines in L which cross the convex hull of the points ("internal") and the set which do not ("external"), respectively. Again, let $|L| = n$ and $|L_I| = i$, so that $|L_E| = n - i$.

As before, using Lemma 2.8 yields an exposing set for $L_E \cup \{\ell\}$ (where $\ell \in L_I$) which together with the rest of L_I is an exposing set of size at most $i - 1 + \lfloor \frac{1}{2}(n-i) \rfloor$.

Consider the diagram in Figure 2.3. Group the rays anchored at the points and pointing outward along the line between each pair of points, as in the figure. Every line of L_I avoids intersecting one of these groups entirely, so let R, G, and B be the sets of lines that avoid the red, blue, and green rays respectively (the dotted, dashed, and solid rays). If a line in L doesn't intersect any of the rays, include it along with R.

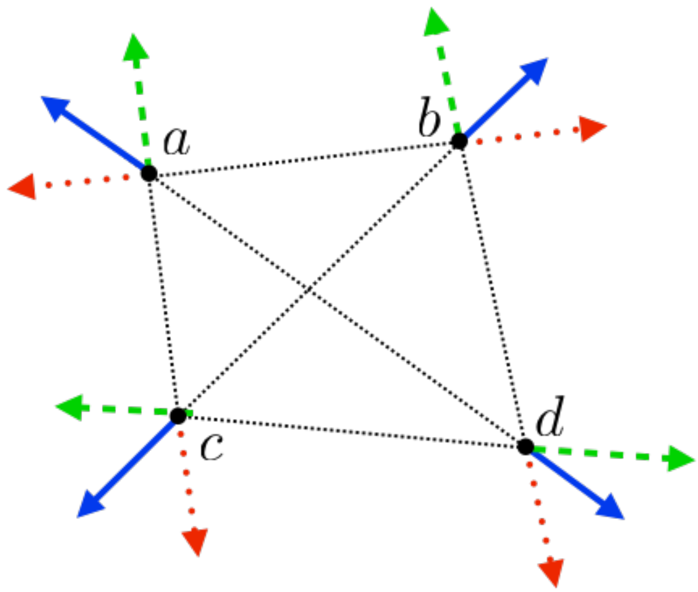


Figure 2.3: The three canonical witness sets for 4 points in convex position; witness sets are grouped by dash type.

Then $L_I = R \cup G \cup B$ and by the pigeonhole principle one of R, G, or B (say, R) must contain at least $\frac{1}{3}i$ lines; therefore, $G \cup B$ contains at most $\frac{2}{3}i$ lines. But $L_E \cup G \cup B$ is an exposing set, witnessed by the red rays, and thus $\widetilde{\lambda}_4(n) \le n - i + \frac{2}{3}i = n - \frac{1}{3}i$.

Together these bounds give $\widetilde{\lambda}_4(n) \le \max_i \min \begin{cases} i - 1 + \lfloor \frac{1}{2}(n-i) \rfloor \\ n - \frac{1}{3}i \end{cases}$.

As before, the minimum of these upper bounds is maximized when they are equal, and hence when $i_* = \frac{3}{5}(n+2)$.

This implies $\widetilde{\lambda}_4(n) \le \lfloor n - \frac{1}{3}(\frac{3}{5}(n+2)) \rfloor = \lfloor \frac{2}{5}(2n-1) \rfloor \le \lfloor \frac{4}{5}n \rfloor$. Together with the result of Theorem 2.7, we have $\lceil \frac{4}{5}n \rceil - 4 \le \widetilde{\lambda}_4(n) \le \lfloor \frac{4}{5}n \rfloor$, so that $\widetilde{\lambda}_4(n) = \lceil \frac{4}{5}n \rceil - O(1)$. $\square$

2.3 Jointly Exposing Multiple Points in the Plane

In this section we examine Question 3 for networks of ray sensors, expanding the idea of exposing points to the idea of exposing a set of points simultaneously, or jointly.

Definition 2.13. *Let P be a set of k points in the plane, and Y a set of n rays. Y partitions the plane into connected regions that are either convex or are the result of removing a finite number of line segments connected to the border of a convex region. We say the points of P are* **jointly exposed** *by Y, or that the partition* **jointly exposes** *them if they are contained in the same region and that region is unbounded.*

A **joint exposing set** *for a set of rays (resp. segments) Y is a subset $X \subseteq Y$ such that the plane partition induced by $Y \setminus X$ jointly exposes the points of P. Similarly, a* **joint witness set** *is a set of rays, one anchored at each point of P, with a common direction.*

$\boldsymbol{R_k(n)}$ *is the smallest integer so that for any k points, any set of n rays has a joint exposing set of size $R_k(n)$.*

Observation 2.1. *A joint witness set W induces a joint exposing set $J(W)$ composed of those rays it intersects.*

This observation highlights a similarity to the situation for Question 2 – that we may examine joint exposing sets through the lens of joint witness sets. Indeed, our lower bound argument in this section will still take the form of a construction: an arrangement of rays which requires at least a particular fraction of them to be part of any joint exposing set. We provide and argue bounds for such a construction for $R_2(n)$ in the majority of this section, but the (asymptotically) matching upper bound is relatively simple and, as before, relies on finding canonical joint witness sets, as we argue in the next theorem.

In arguing each bound we will make use of some properties of the ***barrier graph*** (see Chapter 1), which aids in analyzing the case of finding a path between two points by removing some of a set of ray barriers. The notion of the barrier graph for ray barrier networks is crucial to the problem of finding a joint exposing set for two points. Indeed, removal of any joint exposing set yields an unobstructed path between the exposed points, and hence its rays must form a vertex cover of the associated barrier graph (although typically not a minimal one).

Theorem 2.14. $R_2(n) \leq \frac{3}{4}n$

Proof. Let p and q be points in the plane, and Y any collection of n rays not containing either point. Without loss of generality, suppose p and q lie on a common horizontal line ℓ and p is left of q. Partition Y into four subsets (see Figure 2.4): Y_{left}, those rays whose supporting line intersects ℓ left of p; Y_{right}, those rays whose supporting line intersects ℓ right of q; Y_{above}, those rays themselves intersecting the segment $\overline{pq}$ from above; and Y_{below}, those rays themselves intersecting $\overline{pq}$ from below. Include rays parallel to $\overline{pq}$ along with Y_{left}.

As argued by [9], Y separates the points p and q if and only if some pair of rays in it does. Moreover, two rays from Y_{above} cannot be in such a pair, nor can two rays from Y_{below} or two rays from $Y_{\text{left}} \cup Y_{\text{right}}$; in addition, if $a \in Y_{\text{above}}, b \in Y_{\text{below}}$, then a and b separate p from q if and only if they intersect, while if $c \in Y_{\text{above}} \cup Y_{\text{below}}$ and $d \in Y_{\text{left}} \cup Y_{\text{right}}$, then c and d separate the two points if and only if they intersect on the same side of ℓ as the anchor of c.

Clearly, then, each of $Y_{\text{above}}, Y_{\text{below}}, Y_{\text{left}}$, and Y_{right} is on its own insufficient to separate p from q without a ray from one of the other sets, and the plane partition induced by any one of them jointly exposes p and q. Thus, each of $(Y \setminus Y_{\text{above}})$, $(Y \setminus Y_{\text{below}})$, $(Y \setminus Y_{\text{left}})$, and $(Y \setminus Y_{\text{right}})$ is a joint exposing set for p and q.

On the other hand, summing the sizes of these four exposing sets yields $(n - |Y_{left}|) + (n - |Y_{right}|) + (n - |Y_{above}|) + (n - |Y_{below}|) = 3n$, so at least one of them has size at most $\frac{3}{4}n$, and is thus an exposing set of the desired size, yielding the result. $\square$

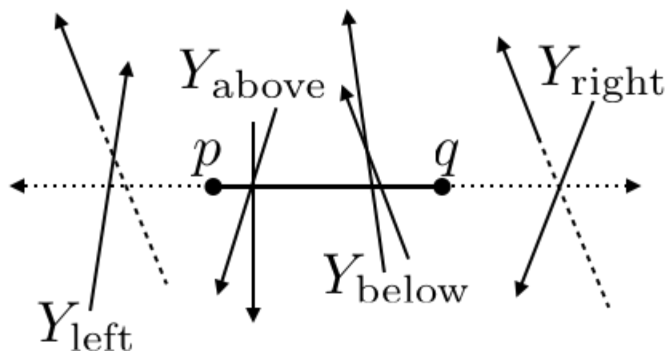


Figure 2.4: The partition of the set Y of rays in Theorem 2.14.

We now give a construction yielding a lower bound for $R_2(n)$ that asymptotically matches the upper bound of Theorem 2.14.

Theorem 2.15. $R_2(n) = \frac{3}{4}n - O(1)$.

To prove Theorem 2.15 we will provide a general construction of a set Y of n rays around a pair of points for the case when $n = 8m$, which only has large joint exposing sets of size at least $6m - O(1) = \frac{3}{4}n - O(1)$.

Fix points p, q at $(-\delta, 0)$ and $(\delta, 0)$, and fix points a and b at $(-\delta', 0)$ and $(\delta', 0)$ for some $0 < \delta' < \delta < 1$ to be precisely determined later. a and b are the points to be protected.

Note that a subset of Y is a joint exposing set for a and b if and only if there is some point t that is jointly exposed along with a and b, which is in an unbounded region of the plane partition due to Y (so that t is initially exposed without removing any ray in Y). This point can be thought of as marking a target region in which to jointly expose a and b.

The anchors of all rays in Y will be located on the unit circle, as follows (see Figure 2.5):

- $8m$ anchor locations are chosen by placing $4m$ of them uniformly around the top half of the unit circle and $4m$ uniformly around the bottom half and each half indexed $1, 2, \ldots, 4m$ from left to right.
- P, a set of $2m$ rays intersecting the point p, where m are anchored uniformly around the top of the unit circle in locations $1, 4, 7, \ldots,$ and m are anchored uniformly around the bottom of the unit circle antipodal to these (so, in locations $4m, 4m-3, \ldots$).
- Q, a set of $2m$ rays similarly placed intersecting q, in locations $2, 5, 8, \ldots,$ on the top of the unit circle and $4m-1, 4m-4, \ldots,$ around the bottom.
- M, a set of $4m$ rays that intersect the origin. These anchors are placed in pairs, where the distance between members of a pair is δ'', and so that the midpoints of the pairs are at locations $3, 6, 9, \ldots,$ on the top of the unit circle and $4m-2, 4m-5, \ldots$ around the bottom. Note that this placement gives antipodal rays of M the same supporting line; if one perturbs the anchors around the unit circle a small enough amount, the argument will still follow.

The rays of $P, Q,$ and M are placed so that no anchor is on the x-axis, and so that no two rays are parallel.

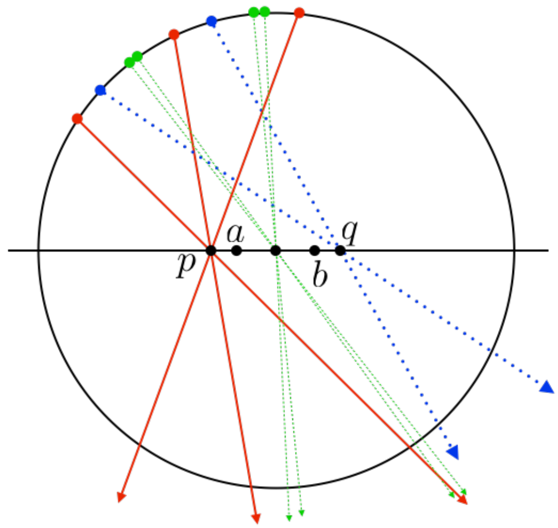


Figure 2.5: The relative arrangement of some of the rays in P (solid), Q (dotted), and M (dashed).

Since a, b, and t must have unobstructed paths to each other when jointly exposed, the corresponding joint exposing set forms a vertex cover of each of the barrier graphs $G_{a,b}$ and $G_{t,b}$ for the point pairs (a, b) and (t, b), respectively. Note that it also forms a vertex cover for the barrier graph $G_{t,a}$, however as any joint vertex cover of the first two graphs has this property, we will not explicitly exploit it.

We define a graph G as the graph with Y as vertices and all edges from both $G_{a,b}$ and $G_{t,b}$, A vertex cover of G thus is a vertex cover of each of $G_{a,b}$ and $G_{t,b}$, and a minimum vertex cover of G is therefore a smallest set of rays whose removal from Y jointly exposes a and b.

A choice of δ, which is the distance from p or q to the origin, determines a circular permutation of the anchors in Y and their intersections with the unit circle (two per ray, one at the anchor and one not). There is some value of δ such that any smaller positive value induces the same permutation, so choose this or any smaller positive value for δ.

In particular, this choice ensures that as one traverses the top of the unit circle in a clockwise fashion, one witnesses the tail of a ray in P followed by an anchor of a ray from P, then a ray's anchor from Q followed by the tail of a ray from Q, and then a pair of ray tails from M and a pair of anchors from M (the tails each contain an anchor), after which the pattern repeats. The same pattern is observed traversing the bottom of the unit circle in a clockwise fashion, except the order of consecutive anchors and tails from P or Q is reversed. This fact is key to understanding neighborhoods in G and thus how vertex covers of G are formed.

After fixing such a δ, there is some circle centered at the origin that intersects the interior of each of the unbounded regions of the plane as partitioned by Y, and without loss of generality we can always choose t to be on this circle, in any of these regions' interiors.

Choose $0 < \delta' < \delta$, which controls the distance from a and b to the origin, so that the triangle with vertices a, b, and t contains at most one anchor from a ray of Y. Finally, choose δ'' so that each pair anchored near the same designated location intersects exactly the same set of other rays in Y. Ideally, we would place these paired rays in the exact same location, but general position forbids this, so we use this δ'' to get as close as is needed.

The following lemma connects joint exposure of a and b to vertex covers of G.

Lemma 2.16. *If C is a minimum vertex cover of G, then $|C| \leq R_2(n)$.*

Our goal, then, is to show that all vertex covers of $G := G_{a,b} \cup G_{t,b}$ contain at least $6m - O(1)$ vertices, for any choice of t.

Fix some point t in an unbounded region of the partition according to Y, and suppose without loss of generality that t is in the top half plane. The line through the origin and t

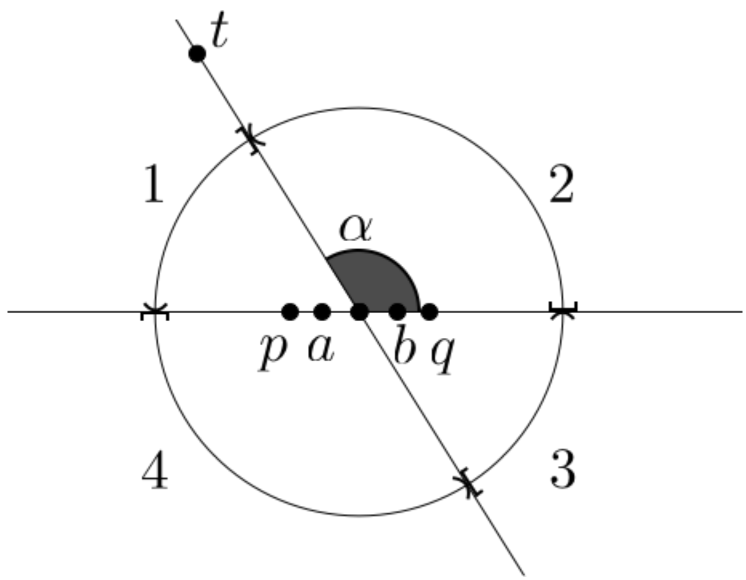


Figure 2.6: The angle α from the positive x-axis to the line through t and the midpoint of $\overline{ab}$ and the partition of the unit circle into half-open arcs which group the rays anchored on it.

forms an angle α with the positive x-axis, normalized to be between 0 and 1, where $\alpha = 1$ corresponds with π radians (See Figure 2.6). This line and the x-axis together divide the unit circle into four regions, labeled 1 through 4 as in the figure.

These regions partition the rays in P, Q, and M further into four subsets each, according to the region containing the ray's anchor. For the rays anchored in region i, call these subsets $P^i(\alpha)$, $Q^i(\alpha)$, and $M^i(\alpha)$, or P^i, Q^i, and M^i when α is clear from context.

We create an auxiliary graph whose vertices are the twelve sets P^i, Q^i, and M^i, which we will call $G_\alpha = (V_\alpha, E_\alpha)$. G_α has an edge between a pair $A, B \in V_\alpha$ if and only if G has an edge in $A \times B$. A vertex cover of G_α then corresponds to a vertex cover of G by simply taking the union of the vertices of G_α. We will refer to the cardinalities of the vertices in V_α (which are sets of rays) as their *weights*. The weights of these vertices are as follows, where the $-O(1)$ terms account for rounding.

$$|P^1| = |P^3| = |Q^1| = |Q^3| = \alpha m - O(1)$$
$$|P^2| = |P^4| = |Q^2| = |Q^4| = (1-\alpha)m - O(1)$$
$$|M^1| = |M^3| = 2\alpha m - O(1)$$
$$|M^2| = |M^4| = 2(1-\alpha)m - O(1)$$

First, notice that each $A \in V_\alpha$ is an independent set of G, and moreover that many pairs $A, B \in V_\alpha$ do not share an edge, a fact recorded by dots in the modified adjacency matrix in Table 2.7.

	P^1	P^2	P^3	P^4	Q^1	Q^2	Q^3	Q^4	M^1	M^2	M^3	M^4
P^1	$\cdot$	$\cdot$	$\cdot$	c	h	$\cdot$	$\cdot$	$\cdot$	h	$\cdot$	h	c
P^2	$\cdot$	$\cdot$	$\cdot$	c	c	h	$\cdot$	h	c	h	$\cdot$	c
P^3	$\cdot$	$\cdot$	$\cdot$	c	h	$\cdot$	h	$\cdot$	h	$\cdot$	h	c
P^4	c	c	c	$\cdot$	$\cdot$	h	c	h	c	c	c	h
Q^1	h	c	h	$\cdot$	$\cdot$	$\cdot$	c	$\cdot$	h	c	c	$\cdot$
Q^2	$\cdot$	h	$\cdot$	h	$\cdot$	$\cdot$	c	$\cdot$	$\cdot$	h	c	h
Q^3	$\cdot$	$\cdot$	h	c	c	c	$\cdot$	c	c	c	h	c
Q^4	$\cdot$	h	$\cdot$	h	$\cdot$	$\cdot$	c	$\cdot$	$\cdot$	h	c	h
M^1	h	c	h	c	h	$\cdot$	c	$\cdot$	$\cdot$	$\cdot$	c	c
M^2	$\cdot$	h	$\cdot$	c	c	h	c	h	$\cdot$	$\cdot$	c	c
M^3	h	$\cdot$	h	c	c	c	h	c	c	c	$\cdot$	c
M^4	c	c	c	h	$\cdot$	h	c	h	c	c	c	$\cdot$

Figure 2.7: The modified adjacencies of G_α, where $\cdot$ represents a non-edge, c represents a complete edge, and h represents a half edge.

The rest of the pairs in G_α form edges that correspond to specific types of subgraphs of G, which we will show in the next lemma. Recall that a *half graph* with $2x$ vertices is a bipartite graph with vertices $u_1, \ldots, u_x$ and $v_1, \ldots, v_x$ that has an edge (u_i, v_j) whenever $i \leq j$. A *doubled half graph* is the the result of duplicating the vertices on one side of a half graph, including adjacencies.

Lemma 2.17. *Each auxiliary edge $(A, B) \in E_\alpha$ induces a subgraph of G that is either a complete bipartite graph, a half graph, or a doubled half graph.*

Proof. Let $r_1, r_2 \in Y$.

If r_1 and r_2 are both anchored on the top unit semicircle, then (r_1, r_2) is an edge of G if and only if they intersect above the x-axis, because then they will either separate a from b or t from b. This means that the G_α auxiliary edges $(P^2, Q^1), (P^2, M^1)$, and (Q^1, M^2), correspond to complete bipartite graphs in G.

For pairs of rays from any of the auxiliary edges (P^1, Q^1), (P^1, M^1), (P^2, Q^2), (P^2, M^2), (Q^1, M^1), and (Q^2, M^2), whether or not they share an edge in G depends on their relative order around the unit circle (See Figure 2.8 for the case of (M^2, Q^2) as an example). In the Figure, a fixed ray $M_j^2 \in M^2$ forms a barrier with each $Q_i^2 \in Q^2$ counterclockwise of it and none clockwise of it, which means its neighborhood in Q^2 is strictly contained in the neighborhood of $M_{j+1}^2 \in M^2$ (a ray clockwise from M_j^2 in M^2). Because of this interaction, these auxiliary edges all correspond to half subgraphs or doubled half subgraphs of G, where edges involving M^i are doubled half subgraphs and the rest are just half subgraphs.

If instead $r_1, r_2 \in Y$ are both anchored on the bottom unit semicircle, whether they form a barrier is still related to whether their intersection is on the same side (below) of the x-axis as their anchors, but now this is not enough because the infinite wedge formed by this interaction could contain all of a, b, and t, or could contain only t (separating it from a and b). Otherwise, the same kinds of observations are in play. So, the auxiliary edges (P^3, P^4), (P^3, M^4), (P^4, Q^3), (P^4, M^3), (Q^3, Q^4), (Q^3, M^4), (Q^4, M^3), and (M^3, M^4) all correspond to complete bipartite subgraphs of G, and the auxiliary edges (P^3, Q^3), (P^4, Q^4), (P^3, M^3), (P^4, M^4), (Q^3, M^3), and (Q^4, M^4) all correspond to half subgraphs or doubled half subgraphs of G. Again, the half subgraphs involving M^i on only one side are the doubled half graphs.

Next, suppose $r_1, r_2 \in Y$ are anchored on different sides of the x-axis. As before, they form a barrier if and only if the infinite wedge they form contains a nonempty proper subset of $\{a, b, t\}$. Although the wedges formed by ray pairs from the same auxiliary edge don't all contain the same subset because the presence of an edge in G only records whether there is some pair of separated points, the same kinds of subgraphs are still formed. In this group, the edges (P^1, P^4), (P^1, M^4), (P^2, P^4), (P^2, M^4), (P^4, M^1), (P^4, M^2), (Q^1, Q^3), (Q^1, M^3), (Q^2, Q^3), (Q^2, M^3), (Q^3, M^1), (Q^3, M^2), (M^1, M^3), (M^1, M^4), (M^2, M^3), and (M^2, M^4), all represent complete bipartite subgraphs of G. The remaining edges represent half-graphs or doubled half-graphs, and these are: (P^1, M^3), (P^2, Q^4), (P^3, Q^1), (P^3, M^1), (P^4, Q^2), (Q^2, M^4), and (Q^4, M^2).

This accounts for all edges in G_α. $\square$

With Lemma 2.17 in hand, we will refer to edges of G_α as either *complete edges* (for complete bipartite subgraphs of G) or *half edges* (for both half subgraphs and doubled half subgraphs of G), and we will refer to the subgraph of G_α induced by the complete edges as G^c_α, while the subgraph induced by the half edges is G^h_α.

A modified adjacency matrix of G_α is shown in Table 2.7, indicating complete edges with a c, half edges with an h, and non-edges with a dot. G_α itself is shown in Figure 2.9.

With the next lemma we connect these edge labelings to vertex covers; in any vertex cover of our barrier graph G, at least one end of each complete edge of G_α is included as a subset of the cover. What this means is that vertex covers of G can be viewed as extensions of vertex covers of the auxiliary graph G^c_α by taking the union of the sets in the cover of G^c_α and adding rays appearing in the ends of edges of G^h_α.

Lemma 2.18. *Let C be a vertex cover of G, and let $(A, B) \in E_\alpha$ be a complete edge. Then either $A \subseteq C$ or $B \subseteq C$.*

Proof. Suppose $A \nsubseteq C$. Then $\exists x \in A \setminus C$. x is adjacent to every element of B since (A, B) is complete, so $B \subseteq C$, because C is a vertex cover of the edges of G. $\square$

In light of Lemma 2.18, we can bound from below the size of all vertex covers of G by beginning with vertex covers of G^c_α and showing that any extension to a vertex cover of G includes at least $6m - O(1)$ rays in total.

The only vertex covers of G that do not immediately satisfy the target lower bound of $\frac{3}{4}n - O(1) = 6m - O(1)$ are thus those that extend vertex covers of G^c_α whose weight is not already at least $6m - O(1)$; without loss of generality, we may extend minimal covers of G^c_α. Each minimal cover C^c of G_α is the complement of a maximal independent set in G^c_α, $V^c_\alpha \setminus C^c$. Moreover, it is from $V^c_\alpha \setminus C^c$ that we find rays to extend the cover to a minimal vertex cover of G.

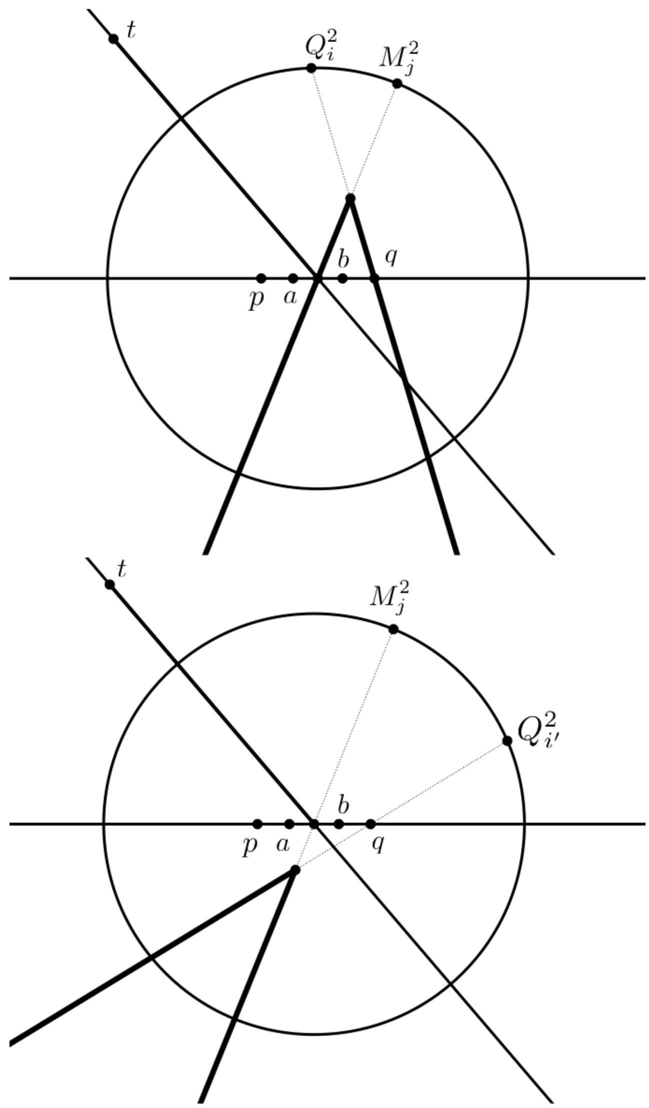


Figure 2.8: Rays $Q_i^2 \in Q^2$ that are counterclockwise from $M_j^2 \in M^2$ form a barrier between b and t and between a and b (left), whereas rays $Q_{i'}^2$ that are clockwise from M_j^2 do not (right). This is what makes the edge (Q^2, M^2) in G_α a half graph, which is doubled because each M_j^2 is paired with another ray with the exact same barrier graph adjacencies.

For such a minimal cover C^c of G^c_α, the rays we add to get a minimum vertex cover of G are in the complement $V^c_\alpha \setminus C^c$, which is a maximal independent set in G^c_α.

For any α there are 5 maximal independent sets in G^c_α with weight more than $2m+O(1)$, which means their complements, vertex covers of G^c_α, have weight less than $6m - O(1)$. This was verified by an exhaustive enumeration of the vertex covers of this graph using the igraph package in R.

Code is available at https://github.com/Kirkules/extremal_problems_ray_sensors. These independent sets are pictured in Figure 2.10.

In the next lemma we show that a minimum vertex cover of G cannot take too few vertices from both sides of a half edge, which further allows us to bound from below the size of a vertex cover of G extending a cover of G^c_α.

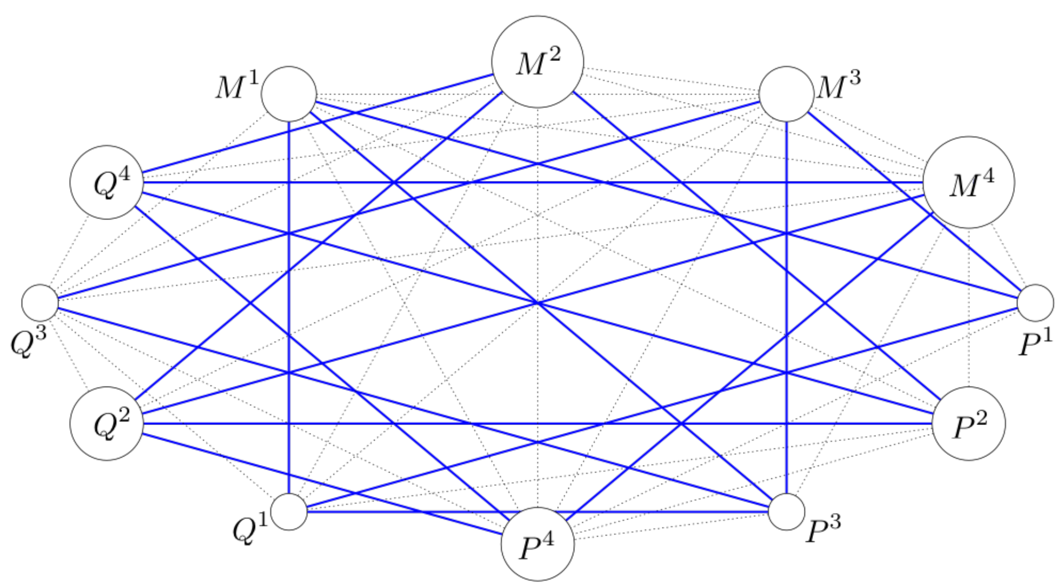


Figure 2.9: G_α with half edges shown as solid lines, and complete edges shown as dashed lines, and relative vertex weights visualized by vertex radius.

Lemma 2.19. *Fix t (and thus α) and a vertex cover C of the barrier graph G. If (M^t, B) is a half edge of G_α (for any $B \in V_\alpha$), then $|B \cap C| \geq (1-\epsilon)|B| - O(1)$, where ϵ depends on B.*

Proof. The doubled half graph between M^t and B labels M^t as $\{m_1, \ldots, m_{|M^t|}\}$, where $i < j \implies \mathcal{N}_B(m_i) \subseteq \mathcal{N}_B(m_j)$.

Let r be the largest index so that $m_r \notin C$. Since the rays in M^t are paired so that pairs have the same neighborhood, i.e. $\mathcal{N}_B(m_{2i-1}) = \mathcal{N}_B(m_{2i})$ for each i.

Because each consecutive pair has a neighborhood larger by 1 than the last, and because r may be the smaller index of its pair, $\deg_B(m_r) \geq \frac{r}{2} - 1$. (We subtract 1 because the first pair, m_1 and m_2, may have no neighbors in B, depending on α.)

Since C is a cover of all edges in G and $m_r \notin C$, we have that $\mathcal{N}_B(m_r) \subseteq C$. This means that $|B \cap C| \geq \deg_B(m_r) \geq \frac{r}{2} - 1$, and we may write:

$$
\begin{aligned}
|B \cap C| &\geq \frac{r}{2} - 1 \\
&= \frac{1}{2}(r+1) - \frac{3}{2} \\
&= \frac{|M^t|}{2} - \frac{|M^t|}{2} + \frac{1}{2}(r+1) - \frac{3}{2} \\
&= |B| - \frac{|M^t|}{2}\left(\frac{|M^t| - (r+1)}{|M^t|}\right) - \frac{3}{2} \\
&= |B| - |B|\epsilon - \frac{3}{2} \\
&= (1-\epsilon)|B| - O(1),
\end{aligned}
$$

where $\epsilon = \left(\frac{|M^t|-(r+1)}{|M^t|}\right)$, and where we have made use of the fact that $|B| = \frac{|M^t|}{2}$ since the only half edges adjacent to M^t connect it to sets with exactly half as many rays. $\square$

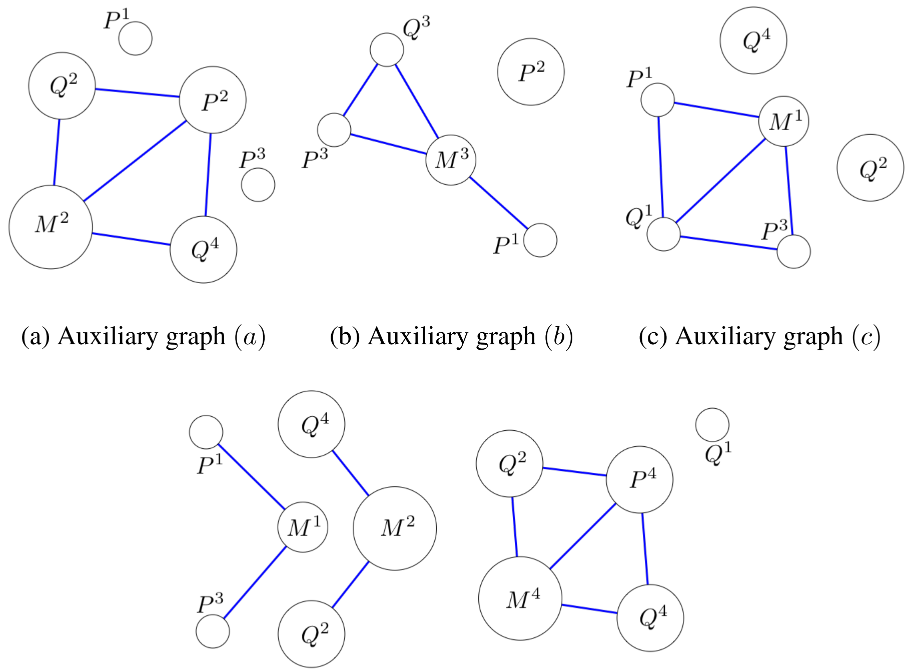
For a vertex cover C of G, this ϵ may be thought of as the fraction of M^t included in C due to this particular half edge. What this means is that $|M^t \cap C| \geq \epsilon|M^t| - 1$.

Moreover, applying the lemma to an M^t adjacent to two half edges yields both an ϵ_1 and an ϵ_2, marking fractions of M^t included in the cover C from each side of the ordered list of rays in M^t (there are only two such orderings of the rays because the order comes from viewing the anchors of rays clockwise or counterclockwise around the unit circle). In other words, when M^t is adjacent to two half edges, $|M^t \cap C| \geq (\epsilon_1 + \epsilon_2)|M^t| - 2$.

Lemma 2.19 is the final tool we need to prove Theorem 2.15.

Proof of Theorem 2.15. The five subgraphs in of G^h_α whose complements correspond to vertex covers of G^c with total weight below $\frac{3}{4}n - O(1)$. Were a vertex cover of G to have weight smaller than $\frac{3}{4}n - O(1)$, it must extend one of these subgraphs' complements by adding enough vertices of the subgraph itself; we therefore show that each such extension leading to a minimum vertex cover must still have total weight at least $\frac{3}{4}n - O(1) = 6m - O(1)$.

These subgraphs are labeled $S_{(a)}, S_{(b)}, S_{(c)}, S_{(d)}$, and $S_{(e)}$, where each $S_{(\cdot)} = (V_{(\cdot)}, E_{(\cdot)})$, and we address extending in each case separately. However, arguments for all cases are structured similarly: first we note the total weight of the complement of $S_{(\cdot)}$ (i.e. the rays already known to be in the cover of G), and then we use Lemma 2.19 to bound below the number of rays that must be added from $S_{(\cdot)}$ to make a vertex cover of G, based on the edges appearing in $S_{(\cdot)}$.



(a) Auxiliary graph (a) (b) Auxiliary graph (b) (c) Auxiliary graph (c)

(d) Auxiliary graph (d) (e) Auxiliary graph (e)

Figure 2.10: The subgraphs of G^c_α whose vertices are maximal independent sets in G^c_α, and that correspond to subgraphs of G that need to be covered. Note that edges shown are from G^h_α, since these are vertexes from independent sets in G^c_α

$S_{(a)}$: $V_\alpha \setminus V_{(a)}$ is a vertex cover of G^c_α with weight $|V_\alpha \setminus V_{(a)}| = |P^4| + |Q^1| + |Q^3| + |M^1| + |M^3| + |M^4| = m\,[(1-\alpha) + \alpha + \alpha + 2\alpha + 2\alpha + 2(1-\alpha)] = (3+3\alpha)m - O(1)$.

Any vertex cover of G with no rays in M^2 must include all rays in Q^2, P^2, and Q^4 except for up to 3, since each of these sets is the neighborhood of some ray of M^2 (except at most one ray each). In this case, the weight of the extended vertex cover is $(3+3\alpha)m + |Q^2| + |P^2| + |Q^4| = (3+3\alpha)m + 3(1-\alpha)m - O(1) = 6m - O(1)$.

If instead C is a vertex cover of G with at least one ray in M^2, then as in Figure 2.8, those rays in M^2 anchored further clockwise have more neighbors in Q^2 and Q^4, and simultaneously have *fewer* neighbors in P^2. So the ordering of M^2 in the half edge with P^2 is opposite the ordering in the half edges with Q^2 and Q^4.

By Lemma 2.19, then, there are $0 \leq \epsilon_1, \epsilon_2 \leq 1$ so that $|M^2 \cap C| \geq (\epsilon_1 + \epsilon_2)|M^2| - 2$ and $|\,(P^2 \cup Q^2 \cup Q^4) \cap C| \geq (1 - \epsilon_1)|P^2| + (1 - \epsilon_2)(|Q^2| + |Q^4|)$.

Therefore, the number of additional rays required to get a vertex cover of G is at least

$$\begin{aligned}
&|(M^2 \cup P^2 \cup Q^2 \cup Q^4) \cap C| \\
&\quad \geq (\epsilon_1 + \epsilon_2)|M^2| - 2 + (1 - \epsilon_1)|P^2| + (1 - \epsilon_2)(|Q^2| + |Q^4|) \\
&\quad \geq (\epsilon_1 + \epsilon_2) \cdot 2(1 - \alpha)m + (1 - \epsilon_1) \cdot (1 - \alpha)m + (1 - \epsilon_2) \cdot 2(1 - \alpha)m - O(1) \\
&\quad \geq (3 + \epsilon_1)(1 - \alpha)m - O(1) \\
&\quad \geq (3 - 3\alpha)m - O(1),
\end{aligned}$$

and so the size of a vertex cover extending $V_\alpha \setminus V_{(a)}$ is at least $(3 + 3\alpha)m + (3 - 3\alpha)m - O(1) = 6m - O(1)$.

Note that we did not need to address rays from (Q^2, P^2) or from (Q^4, P^2) directly.

$S_{(b)}$: The complement of $S_{(b)}$ is a vertex cover of G^c_α satisfying:

$$\begin{aligned}
|V_\alpha \setminus V_{(b)}| &= |P^4| + |Q^1| + |Q^2| + |Q^4| + |M^1| + |M^2| + |M^4| \\
&= (7 - 4\alpha)m - O(1),
\end{aligned}$$

Note that if $\alpha \leq 1/4$, then $|V_\alpha \setminus V_{(b)}| \geq 6m - O(1)$ and this case is done.

A vertex cover of G with no rays from M^3 must include all rays in P^1, P^3, and Q^3, which together have weight $3\alpha m \geq (4\alpha - 1)m$ since $\alpha \leq 1$. In this case, the extended vertex cover has weight at least $6m - O(1)$.

So suppose C is a vertex cover of G taking at least one ray of M^3. Rays of M^3 anchored more counterclockwise have more neighbors in Q^3 and fewer in P^1 and

P^3. By Lemma 2.19, there are $0 \leq \epsilon_1, \epsilon_2 \leq 1$ so that $|M^3 \cap C| \geq (\epsilon_1 + \epsilon_2)|M^3|$ while $|(Q^3 \cup P^1 \cup P^3) \cap C| \geq (1 - \epsilon_1)|Q^3| + (1 - \epsilon_2)(|P^1| + |P^3|)$.

$$
\begin{aligned}
|(M^3 \cup P^3 \cup Q^3 \cup P^1) \cap C| \\
&\geq (\epsilon_2 + \epsilon_1)|M^3| + (1 - \epsilon_2)(|P^1| + |P^3|) + (1 - \epsilon_1)|Q^3| \\
&= [(\epsilon_2 + \epsilon_1)(2\alpha) + (1 - \epsilon_2)(\alpha + \alpha) + (1 - \epsilon_1)\alpha]m - O(1) \\
&= (3 + \epsilon_1)\alpha m - O(1) \\
&\geq (4\alpha - 1)m - O(1),
\end{aligned}
$$

which again means the extended vertex cover has weight at least $(7 - 4\alpha)m + (4\alpha - 1)m - O(1) = 6m - O(1)$.

$S_{(c)}$: $|V_\alpha \setminus V_{(c)}| = |P^2| + |P^4| + |Q^1| + |M^2| + |M^3| + |M^4| = [(1 - \alpha) + (1 - \alpha) + \alpha + 2(1 - \alpha) + 2\alpha + 2(1 - \alpha)]m = (6 - 3\alpha)m - O(1)$, so the goal is to show any vertex cover C of G extending the complement of $S_{(c)}$ takes more than $3\alpha m - O(1)$ vertices from $S_{(c)}$, because $(6 - 3\alpha)m - 3\alpha m - O(1) = 6m - O(1)$.

A vertex cover of G with no rays from M^1 must take all of P^1, P^3, and Q^1, totaling at least $3\alpha m - O(1)$ additional rays.

If instead (again applying Lemma 2.19) at least $(\epsilon_1 + \epsilon_2)|M^1| - 2$ rays are taken from M^1, then at least $(1 - \epsilon_1)|Q^1| + (1 - \epsilon_2)(|P^1| + |P^3|)$ rays are taken from Q^1, P^1, and P^3, giving that

$$
\begin{aligned}
|(M^1 \cup P^1 \cup P^3 \cup Q^1) \cap C| \\
&\geq (\epsilon_2 + \epsilon_1)|M^1| + (1 - \epsilon_2)(|P^1| + |P^3|) + (1 - \epsilon_1)|Q^1| \\
&= [(\epsilon_2 + \epsilon_1)(2\alpha) + (1 - \epsilon_2)(\alpha + \alpha) + (1 - \epsilon_1)(\alpha)]m - O(1)
\end{aligned}
$$

$$= (3+\epsilon_1)\alpha m - O(1)$$
$$\geq 3\alpha m - O(1),$$

and so the extended cover contains at least $(6-3\alpha)m + 3\alpha m - O(1) = 6m - O(1)$ rays.

$S_{(d)}$: $|V_\alpha \setminus V_{(d)}| = |P^2| + |P^4| + |Q^1| + |Q^3| + |M^3| + |M^4| = 4m - O(1)$.

Since $S_{(d)}$ has two connected components with any edges (one component with M^1 and one with M^2), we can address them independently.

A cover with no rays from M^1 must include all rays in both P^1 and P^3, totaling $2\alpha m - O(1)$. If instead it takes $\epsilon|M^1| - 1$ rays from M^1 as in Lemma 2.19, then the contribution from this connected component is $\epsilon|M^1| + (1-\epsilon)(|P^1| + |P^3|) = 2\epsilon\alpha m + (1-\epsilon)(\alpha+\alpha)m - O(1) = 2\alpha m - O(1)$. Note that the same ϵ applies because neighborhood size for edges from M^1 into P^1 and into P^3 increase together. Hence, vertex covers must take at least $2\alpha m - O(1)$ rays from this component.

Similarly, a cover with no rays from M^2 must include both Q^2 and Q^4, totaling $2(1-\alpha)m - O(1)$. Instead taking $\epsilon|M^2| - 1$ from M^2 means the contribution from this component is $\epsilon|M^2| + (1-\epsilon)(|Q^2| + |Q^4|) = 2(1-\alpha)m - O(1)$, and the contribution to any vertex cover from this component is at least $2(1-\alpha)m - O(1)$.

The two components together thus contribute at least $2m - O(1)$ rays to any vertex cover of G, and thus any cover extending $V_\alpha \setminus V_{(d)}$ contains at least $4m + 2m - O(1) = 6m - O(1)$ rays.

$S_{(e)}$: $|V_\alpha \setminus V_{(e)}| = |P^1| + |P^2| + |P^3| + |Q^3| + |M^1| + |M^2| + |M^3| = (3+4\alpha)m - O(1)$.

A vertex cover of G with nothing from M^4 takes all of P^4, Q^2, and Q^4, which have a combined weight of $(3-3\alpha)m \geq (3-4\alpha)m - O(1)$.

Taking instead $(\epsilon_1 + \epsilon_2)|M^4|$ rays from M^4 in the cover C of G, we have that $|(M^4 \cup P^4 \cup Q^2 \cup Q^4) \cap C| \geq (\epsilon_1 + \epsilon_2)|M^4| + (1 - \epsilon_1)(|Q^2| + |Q^4|) + (1 - \epsilon_2)|P^4| = [(3 + \epsilon_2) - \epsilon_2 \alpha]m$, which is at least $(3 - 4\alpha)m$ since $0 \leq \epsilon_2 \leq 1$.

But $(3 + 4\alpha)m + (3 - 4\alpha)m - O(1) \geq 6m - O(1)$.

Finally, because every vertex cover of G extends a vertex cover of G^c_α, and as we have shown, every extension of such a cover to a cover of G contains at least $6m - O(1)$ elements of G, it follows that every vertex cover of G has at least $6m - O(1) = \frac{3}{4}n - O(1)$ vertices in it. This, together with the results of Theorem 2.14, yields $R_2(n) = \frac{3}{4}n - O(1)$. $\square$

2.4 Acknowledgements

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CHAPTER 3: EXPOSING IN HIGHER DIMENSIONS

3.1 d-dimensional Exposure and Joint Exposure

The notion of protecting points from exposure with lines in the plane has a natural analog in d dimensions: protecting points in $\mathbb{R}^d$ with affine hyperplanes, which we will hereafter simply call hyperplanes.

Although $\mathbb{R}^d$ also contains lines, for $d > 2$ no finite number of these can protect a point from an intruder. Indeed, flats of dimension less than $d - 1$ do not partition the rest of $\mathbb{R}^d$ into multiple cells. In this sense, hyperplane "sensors" are the most natural analog of line sensors in higher dimensions.

Consider a set H of n hyperplanes in $\mathbb{R}^d$ and a set $P \subseteq \mathbb{R}^d$ of k points. H partitions $\mathbb{R}^d$ into convex sets, called the *t-faces* of the hyperplane arrangement $\boldsymbol{\mathcal{A}}(H)$, such that for two points p and q in the same t-face and any $A \in H$, p is above (respectively below, or on) A if and only if q is. We refer to the d-faces of $\boldsymbol{\mathcal{A}}(H)$ as its ***cells***, or ***d-cells***.

We will say H and P are, together, in ***general position*** if no point $p \in P$ is contained in any hyperplane $A \in H$. Unless otherwise specified, all point sets and hyperplane arrangements are taken to be in general position.

As before, a point $p \in P$ is ***protected*** if the d-cell containing it is bounded, and ***exposed*** otherwise; an exposing set $X \subseteq H$ is one such that $H \setminus X$ exposes P, and a ***witness set*** is a set W of k rays with each point of P as an anchor.

The lemma below is the d-dimensional analog of Lemma 2.1, connecting witness sets to the exposing sets that they witness; it has essentially the same proof as well, since unbounded d-cells also contain at least one unbounded ray in their boundary, and since d-cells are also convex.

Lemma 3.1. *If $H \setminus X$ exposes P, then there is a witness set W with $E(W) \subseteq X$.*

Now, the generalized version of Question 2 is straightforward:

Question 4. *Given k designated target locations and allowing the placement of n hyperplane barriers in $\mathbb{R}^d$, what number of those barriers is always sufficient and sometimes necessary to remove so that all target locations are exposed?*

To describe the answer to this question, we define functions analogously to their two-dimensional counterparts:

Definition 3.2. *For a finite set $P \subseteq \mathbb{R}^d$ of points and finite set H of hyperplanes in $\mathbb{R}^d$, $h^d(P, H)$ is the size of the smallest exposing set $X \subseteq H$. In addition,*

$$\begin{aligned} h^d(P, n) &= \max\{h^d(P, H) : H \textit{ is a set of } n \textit{ hyperplanes in } \mathbb{R}^d\} \\ h^d(k, H) &= \max\{h^d(P, H) : P \subseteq \mathbb{R}^d \textit{ and } |P| = k\} \\ h^d_k(n) &= \max\{h^d(P, n) : P \subseteq \mathbb{R}^d \textit{ and } |P| = k\} \\ &= \max\{h^d(k, H) : H \textit{ a set of } n \textit{ hyperplanes in } \mathbb{R}^d\} \end{aligned}$$

For points in convex position, we again use a tilde to recognize this assumption: $\widetilde{h}_k^d(n)$ is the smallest number so that every set of n d-dimensional hyperplanes protecting a set of k points in convex position in $\mathbb{R}^d$ has an exposing set of size at most $\widetilde{h}_k^d(n)$.

3.2 Lower Bounds on $h_k^d(n)$

Our approach to examining the functions $h_k^d(n)$ is again to find nontrivial upper and lower bounds. In two dimensions, we were able to find matching bounds and thus the exact value of the functions $\lambda_k(n) = h_k^2(n)$ for some values of k. In higher dimensions, such precision has been more elusive.

However, the fact that each finite dimensional Euclidean space is a subspace of the next allows us to carry lower bounds upward to higher dimensions.

Theorem 3.3 (Lower Bound Dimension Raising)**.** *If $k \le t < d$, then $h_k^d(n) \le h_k^t(n)$.*

Proof. Let $P \subseteq \mathbb{R}^d$ be a set of k points and let H be an arrangement of hyperplanes in general position in $\mathbb{R}^d$. Let Π be any t-dimensional flat of $\mathbb{R}^d$ containing P (always possible since $k \le t$). For each $\pi \in H$, if $\pi \cap \Pi \neq \emptyset$ then let π' be any subflat of Π, with dimension $t-1$, containing $\Pi \cap \pi$ but not containing a point of P.

This choice of π' is always possible: if $\Pi \cap \pi$ itself has dimension less than $t-1$ it can be extended to any $(t-1)$-dimensional subspace Φ containing $\Pi \cap \pi$ and translating it along some vector in Π so that it still misses all of P.

Π is a copy of $\mathbb{R}^t$ containing P and sitting inside of $\mathbb{R}^d$, and the set $H' = \{\pi' : \pi \in H, \pi \cap \Pi \neq \emptyset\}$ is a collection of $(t-1)$-dimensional flat inside of Π. These are hyperplanes in Π and by construction are in general position with respect to P.

If $X' \subseteq H'$ is an exposing set for P in the context of Π, then by definition there is a witness set W' whose rays are all contained in Π, and such that $E_{H'}(W') = X'$ is of minimum size. Let $X = \{\pi \in H : \pi' \in X'\}$, which can be thought of as the set of hyperplanes π in $\mathbb{R}^d$ whose corresponding hyperplanes π' in Π are represented in the exposing set X'.

For any $w \in W'$, if $\pi' \in S' \setminus X'$, then $w \cap \Pi \cap \pi \subseteq w \cap \pi' = \emptyset$. But $w \cap \pi = w \cap \Pi \cap \pi$ since $w \in \Pi$, and thus $w \cap \pi \subseteq w \cap \pi' = \emptyset$. Thus, $E_S(W') \subseteq X$; in other words, W' witnesses that X is an exposing set for H. Moreover, since X' is of minimum size, every hyperplane in X' intersects some ray of W'.

But $|X| = |X'|$, and so we have found an exposing set, X', for P and H whose size is no larger than $h_k^t(n)$. $\square$

Corollary 3.4. *For $d > 2$,*

$$h_1^d(n) \leq \left\lfloor \frac{1}{2}(n-1) \right\rfloor$$

$$h_2^d(n) \leq \left\lfloor \frac{2}{3}(n-1) \right\rfloor$$

$$h_3^d(n) \leq \left\lfloor \frac{1}{4}(3n-2) \right\rfloor = \frac{3}{4}n - O(1).$$

Now, the argument we used to prove our lower bound on $\lambda_k(n)$ for all k can be extended to apply in d dimensions, and to support this generalization we introduce the following lemma.

Lemma 3.5. *There are arbitrarily large sets of vectors V in $\mathbb{R}^d$ such that every subset $S \subseteq V$, with $|S| \leq d$, is linearly independent.*

Proof. Recall that the Vandermonde matrix of size n,

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

is invertible, and that such a matrix is defined for any n distinct real values x_n.

Fix $n > 0$. Choose n distinct nonzero real values $x_1, x_2, \ldots, x_n$, and let $\mathbf{v_i}$ be $\left\{1, x_i, x_i^2, \ldots, x_i^d\right\}$ for $1 \leq i \leq n$. Any subset of $\{\mathbf{v_i}\}$ provides the rows of a size d Vandermonde matrix, and are thus linearly independent. $\square$

Now that we have shown such sets of vectors exist, we will say a set of vectors is a ***d-independent set*** if it contains at least d vectors and every subset of size d is linearly independent.

Theorem 3.6. *In any dimension d, and for any k points in $\mathbb{R}^d$, we have that*

$$h_k^d(n) \geq \frac{k}{k+1}n - dk.$$

Proof. Fix any set P of k points in $\mathbb{R}^d$.

Suppose $n = m(k+1)$ for some $m \in \mathbb{N}$. Let V be any d-independent set of n vectors in $\mathbb{R}^d$ so that the normal of any hyperplane containing more than one point of P is not a multiple of any $\mathbf{v_i} \in V$.

For each $1 \leq i \leq m$ let X_i be any set of $k+1$ hyperplanes that interlace the points in P and which all have normal $\mathbf{v_i}$. (This interlacing is always possible by the choice of V).

For convenience, we will refer to the direction of $\mathbf{v_i}$ as “right”, and its opposite as “left” when considering a fixed i, and we will refer to the linear order imposed by this vector on the hyperplanes of X_i as well as on P. See Figure 3.1 for an example in three dimensions.

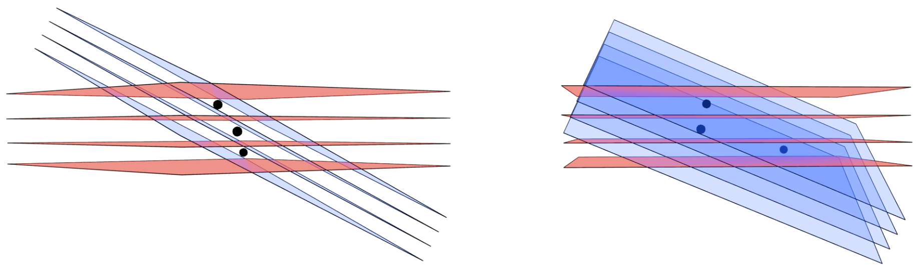

Figure 3.1: Two sets X_i of 4 parallel planes for a set of 3 points in three dimensions, viewed from two angles.

Fix any set W of k rays each anchored at a different point of P, and fix i.

Let k'_i be the number of rays in W orthogonal to $\mathbf{v_i}$, let ℓ_i be the rightmost of the rays in W that points to the left, and let r_i be the leftmost of the rays in W that points to the right. ℓ_i intersects all hyperplanes of X_i to its left and r_i intersects all hyperplanes in X_i to its right.

If ℓ_i is to the right of r_i, all hyperplanes of X_i are intersected by ℓ_i and r_i together.

Suppose instead ℓ_i is left of r_i. Then all rays of W anchored between ℓ_i and r_i (with respect to $\mathbf{v_i}$) must be orthogonal to $\mathbf{v_i}$, since otherwise the offending ray would replace ℓ_i or r_i by definition.

If the anchors of ℓ_i and r_i are adjacent in the order imposed by $\mathbf{v_i}$ on P, one hyperplane of X_i lies between them missed by W. Otherwise, for each ray in W between them we find another hyperplane of X_i that is not intersected by any ray in W and there are at most k'_i such rays anchored between ℓ_i and r_i.. So, all but at most $k'_i + 1$ hyperplanes in X_i are intersected by a ray in W, for a total of $k+1-(k_i'+1) = k-k_i'$ hyperplanes in X_i witnessed by W.

Because this argument applies for all i, the total number of hyperplanes witnessed by some ray in W is at least $\sum_i (k-k_i') = mk - \sum_i k_i'$.

Recall that k_i' is the number of rays in W orthogonal to $v_i \in V$. Let W and V be the partite sets of a bipartite graph which has an edge for every pair $w \in W, v \in V$ satisfying $w \perp v$. Then $\sum_i k_i'$ is the number of edges in this graph.

Now, since every size d subset of V spans $\mathbb{R}^d$, each ray in W is orthogonal to at most $d-1$ vectors in V. If n_w is the number of vectors in V orthogonal to $w \in W$, then the number of edges in the auxiliary graph is also $\sum_{w\in W} n_w$ this gives that $\displaystyle\sum_i k_i' = \sum_{w\in W} n_w \leq (d-1)k$, and therefore at least $mk-(d-1)k = \frac{k}{k+1}n-(d-1)k$ hyperplanes are witnessed by W.

If instead $n = m(k+1)+b$ for some $b < k+1$, then choose V as above but with $m+1$ vectors, and choose m sets of $k+1$ hyperplanes for each of the first m vectors. For the additional vector $\mathbf{v}_{m+1}$, choose a set of b hyperplanes with normal parallel to $\mathbf{v}_{m+1}$ so that each pair of hyperplanes is separated by at least one point of P.

Then all but $1+k_i'$ hyperplanes in each X_i intersect some ray of W except when $i = m+1$; all but at most $b+k_{m+1}'$ hyperplanes of X_{m+1} intersect some ray of W.

The number of hyperplanes witnessed in X_i for $i < m+1$ is then $|X_i|-(1+k_i')$, and the number witnessed in X_{m+1} is $|X_{m+1}|-(b+k_{m+1}')$.

The total number of hyperplanes witnessed by W is thus at least

$$\begin{aligned}
\sum_i |X_i| - \sum_{i<m+1} (1+k_i') - (b+k_{m+1}') \\
&= n - (b+k_{m+1}') - \sum_{i<m+1} (1+k_i') \\
&= m(k+1) + b - (b+k_{m+1}') - \left(m + \sum_{i<m+1} k_i'\right) \\
&= mk + m + b - m - b - k_{m+1}' - \sum_{i<m+1} k_i' \\
&= mk - \sum_i k_i' \\
&\geq mk - (d-1)k \\
&= \frac{k}{k+1}(m(k+1) + b - b) - (d-1)k \\
&= \frac{k}{k+1}(n-b) - (d-1)k \\
&\geq \frac{k}{k+1}n - \frac{k}{k+1}b - (d-1)k \\
&= \frac{k}{k+1}n - dk.
\end{aligned}$$

$\square$

3.3 Upper Bounds on $h_k^d(n)$

Our approach to upper bounds for $h_k^d(n)$ is similar to our approach for upper bounds for $\lambda_k(n)$; we describe a canonical set of rays from which we select a witness set whose size yields the bounds we desire.

Lemma 3.7. *Let $P \subseteq \mathbb{R}^d$ with $P = \{p_1, \ldots, p_k\}$ and let H be a set of n hyperplanes in $\mathbb{R}^d$ in general position with respect to P. If H_I is the set of hyperplanes separating at least one pair of points in P, then there is an exposing set $X \subseteq H$ of size at most $|H_I| + \frac{1}{2}(n - |H_I| - 1)$.*

Proof. Remove H_I, and choose any line through p_1 parallel to some hyperplane $\pi \in H$. The two opposite rays anchored at p_1 along this line together intersect at most $n - |H_I|$ hyperplanes in $H \setminus H_I$, and thus one of these rays, r, intersects at most half of them. Translating r to be anchored at other p_i does not change the subset of hyperplanes in H it intersects since no remaining hyperplane passes between two points of P. Thus, the set of these translations of r constitutes a witness set witnessing an exposing set of at most half of the remaining hyperplanes other than π, or $\frac{1}{2}(n - |H_I| - 1)$ of them. Together with H_I itself, these hyperplanes form an exposing set of size $|H_I| + \frac{1}{2}(n - |H_I| - 1)$. $\square$

We begin with a bound in 3 dimensions for clarity, in part because these are planes we can visualize and draw.

Theorem 3.8. $h_4^3(n) \leq \frac{7}{8}n - O(1)$.

Proof. Let $P = \{p_1, p_2, p_3, p_4\} \subseteq \mathbb{R}^d$ be any four points in $\mathbb{R}^3$.

For $\{i, j\} \subset [4]$, let r_{ij} be the ray anchored at p_j, along the line between p_i and p_j, pointing away from p_i.

For each of $i \in [4]$, let P_i be the set $\{r_{ij} : j \in [4], j \neq i\}$. So, P_i is the set of rays pointing away from p_i, anchored at other points of P.

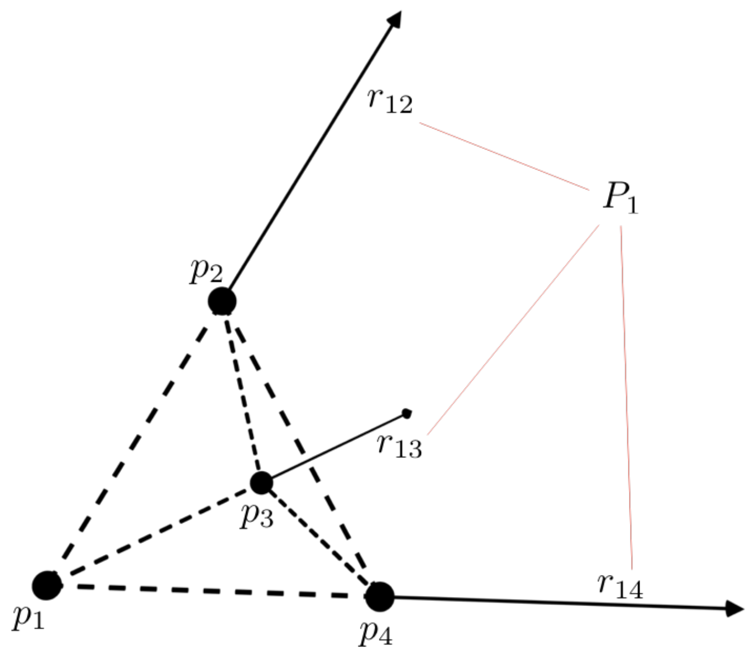

Figure 3.2: The ray set $P_1 = \{r_{12}, r_{13}, r_{14}\}$

Claim. *Any plane π intersecting* $\mathrm{conv}(P)$*, but not P, hits rays in at most two of the ray sets P_i.*

Proof of claim: Fix $p_i \in P$. If p_j is on the same side of π as p_j and no closer to π than p_i, then r_{ij} does not intersect π (it points away from π). If p_j is on the other side of π from p_i, then r_{ij} still does not intersect π.

Therefore, if p_a and p_b are the closest elements of P to π on each side, then π misses all rays in both P_a and P_b.

See Figure 3.3 for a visualization with the points of P arranged in a tetrahedron.

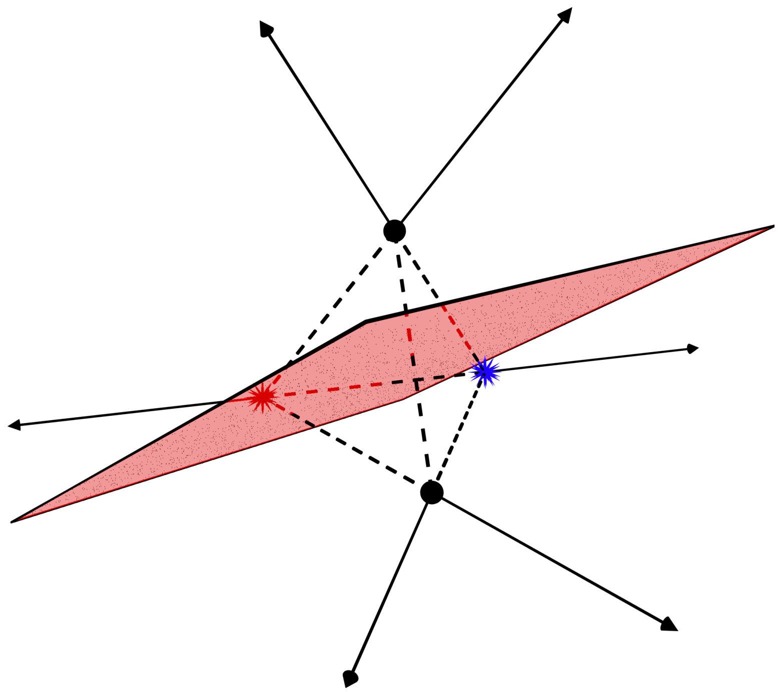

Figure 3.3: A plane π, the closest points to it on either side represented by stars (on the left, behind the plane and on the right, in front of it), and the rays pointing away from these points which also point away from π.

A plane thus may both intersect $\mathrm{conv}(P)$ and hit rays from exactly one, exactly two, or none of the ray sets P_i.

Let H be any collection of n planes that do not contain any points of P, and write $H = H_I \cup H_E$, where H_I are those planes intersecting $\mathrm{conv}(P)$ ("internal planes") and H_E are the rest ("external planes").

For a set $S \subseteq [k] = [4]$, let H_S be the sets of internal planes which hit rays from P_i whenever $i \in S$, and miss P_i whenever $i \in S$. Note that for $i \neq j$, this definition implies that $P_i \cap P_j = \emptyset$.

Then, by the Claim above, H_I is the union of all H_S with $1 \leq |S| \leq 2$, i.e. $H_I = \left(\bigcup_{i \in [4]} H_{\{i\}}\right) \cup \left(\bigcup_{\substack{i,j \in [4] \\ i \neq j}} H_{\{i,j\}}\right)$, and we have that $\left|\bigcup_{i \in [4]} H_{\{i\}}\right| = \alpha|H_I|$ while $\left|\bigcup_{\substack{i,j \in [4] \\ i \neq j}} H_{\{i,j\}}\right| = (1-\alpha)|H_i|$ for some $0 \leq \alpha \leq 1$.

Next, for each pair of indices $\{x, y\} \subset [4]$, define the set of hyperplanes:

$$X_{xy} = X_{yx} = \bigcup_{\substack{S \subset [4] \\ x \in S \text{ or } y \in S \\ 1 \leq |S| \leq 2}} H_S.$$

Note that each $H_E \cup X_{xy}$ is an exposing set for P, because none of the hyperplanes that remain, after removing these, intersect any of the rays in P_x or P_y. The exposure of p_i with $i \neq x$ is witnessed by $r_{xi} \in P_x$, and the exposure of p_x is witnessed by $r_{yx} \in P_y$.

Consider the size of X_{xy} for fixed $\{x, y\} \subset [4]$. Because the H_S are all disjoint, this size can be rewritten as the sum of their sizes:

$$|X_{xy}| = \sum_{\substack{S \subset [4] \\ x \in S \text{ or } y \in S \\ 1 \leq |S| \leq 2}} |H_S|.$$

Using this, we consider the sum of all of these sizes:

$$\sum_{\{x,y\}\subset[4]} |X_{xy}| = \sum_{\{x,y\}\subset[4]} \sum_{\substack{S\subset[4] \\ x\in S \text{ or } y\in S \\ 1\leq|S|\leq 2}} |H_S|.$$

If $S = \{i\} \subseteq [4]$, then $H_S \subseteq X_{ix}$ for $x \neq i$, and so each H_S with $|S| = 1$ appears three times in the summation.

If instead $S = \{i,j\} \subseteq [4]$, then $H_S \subset X_{ij}$, which occurs once in the summation, and each of $H_S \subseteq X_{it}$ or $H_S \subseteq X_{jt}$ for $t \notin \{i,j\}$ occurs 2 times in the summation, once for each of the 2 other values of t not equal to i or j. Such a set H_S therefore appears $1 + 2(2) = 5$ times in the summation.

With this information, we can rewrite the summation over all X_{xy} as follows:

$$\begin{aligned}
\sum_{\{x,y\}\subset[4]} |X_{xy}| &= \sum_{\{x,y\}\subset[4]} \sum_{\substack{S\subset[4] \\ x\in S \text{ or } y\in S \\ 1\leq|S|\leq 2}} |H_S| \\
&= \left(\sum_{i\in[4]} 3|H_{\{i\}}|\right) + \left(\sum_{\{i,j\}\subset[4]} 5|H_{\{i,j\}}|\right) \\
&= 3\alpha|H_I| + 5(1-\alpha)|H_I|.
\end{aligned}$$

This sum is maximized when $\alpha = 0$, which implies that

$$\sum_{\{x,y\}\subset[4]} |X_{xy}| \leq 5|H_I|.$$

There are 6 such hyperplane sets X_{xy}, so there must be some pair $\{x', y'\} \subset [4]$ for which $|X_{x'y'}| \leq \frac{5}{6}|H_I|$, which means that the exposing set $H_E \cup X_{x'y'}$ contains at most $n - |H_I| + \frac{5}{6}|H_I| = n - \frac{1}{6}|H_I|$ hyperplanes. Call this exposing set X.

Recall that, by Lemma 3.7, there is also an exposing set consisting of all but one hyperplane of H_I and about half of H_E, with size at most $|H_I| + \frac{1}{2}(n - |H_I| - 1)$. Call this alternative exposing set X'.

$|X|$ and $|X'|$ are linear functions of $|H_I|$, and $|H_I|$ takes integer values between 0 and n, with each value easily realizable. $|X|$ decreases and $|X'|$ increases as $|H_I|$ increases, and therefore the largest value of $\min\{|X|, |X'|\}$ is obtained when $|X| = |X'|$. This occurs when $|H_I| = \frac{3}{4}(n+1)$, for which the value of $|X|$ is $\frac{7}{8}n - \frac{1}{8}$.

By definition, $h^3(P, H) \leq \min\{|X|, |X'|\}$, as it is the minimum size among the sizes of all exposing sets for P and H, and thus $h^3(P, H) \leq \frac{7}{8}n - \frac{1}{8}$.

Then, from the definition of $h_k^d(n)$, we have that

$$\begin{aligned} h_4^3(n) &= \max_{|H|=n, |P|=4} \{h^3(P, H)\} \\ &\leq \max_{|H|=n} \left\{ \frac{7}{8} \cdot n - \frac{1}{8} \right\} \\ &= \frac{7}{8} \cdot n - \frac{1}{8}. \qquad \square \end{aligned}$$

We can generalize this approach for any k points in $\mathbb{R}^d$, since the structure of the argument doesn't depend on the dimension. We begin with the following lemma.

Lemma 3.9. *Let* $P = \{p_i\}_{i=1}^k$ *be any set of points in* $\mathbb{R}^d$*. Let* P_i *be the set of rays, anchored at* $\{p_j\}_{j \neq i}$*, each along the line between* p_i *and* p_j*, pointing away from* p_i*. Then any hyperplane intersecting* $\operatorname{conv}(P)$*, but not P, intersects rays from at most* $k - 2$ *of the* P_i*.*

Proof. Fix a vertex p_i. If p_j is on the same side of π as p_j and no closer to π than p_i, then r_{ij} does not intersect π. If p_j is on the other side of π from p_i, then r_{ij} still does not intersect π because it points away from p_i and thus away from π.

Therefore, if p_a and p_b are the closest elements of P to π on each side, then π misses all rays in both P_a and P_b. This proves that there are always at least two ray sets that π misses, and that rays from at most $k - 2$ sets are hit. $\square$

Theorem 3.10. $h_k^d(n) \leq \dfrac{k^2 - k + 2}{k^2 - k + 4} \cdot n - O(1)$.

Proof. Fix $k \in \mathbb{N}$. Let $P = \{p_i\}_{i=1}^k$ be any set of k points in $\mathbb{R}^d$. Let H be any set of n hyperplanes in $\mathbb{R}^d$ not containing any of the p_i. Let $H = H_I \cup H_E$, where H_I is the set of "interior" hyperplanes of H (passing between two points of P) and H_E is the rest of H.

For each $\{i, j\} \subseteq [k]$, let r_{ij} be the ray anchored at p_j and pointing away from p_i along the line between p_i and p_j.

For each $i \in [k]$, let $P_i := \{r_{ij} : j \in [k], j \neq i\}$. So, P_i is the set of these rays pointing away from p_i.

If $S \subseteq [k]$, let $H_S \subseteq H_I$ be the subset of interior hyperplanes that hit some ray in P_i for each $i \in S$, and hit no rays in P_j for each $j \notin S$. Note that for distinct $S, S' \subseteq [k]$, $H_S \cap H_{S'} = \emptyset$. Also note that $H_S = \emptyset$ for $|S| > k - 2$, by Lemma 3.9.

For each $s \in [k]$, let $0 \leq \alpha_s \leq 1$ be the fraction of H_I made up of hyperplanes in some H_S with $|S| = s$, i.e.

$$\sum_{\substack{S \subseteq [k] \\ |S| = s}} |H_S| = \alpha_s |H_I|$$

and note that $\sum_{s=1}^{k-2} \alpha_s = 1$.

Next, for any pair of indices $\{x, y\} \subset [k]$, define the set of hyperplanes

$$X_{xy} = X_{yx} = \bigcup_{\substack{S \subseteq [k] \\ x \in S \text{ or } y \in S \\ |S| \leq k-2}} H_S.$$

This set is such that $H_E \cup X_{xy}$ is an exposing set, because none of the hyperplanes in H_I that remain after removing all of H_E and X_{xy} intersect any rays in P_x or in P_y. The exposure of p_i with $i \neq x$ is witnessed by $r_{xi} \in P_x$, and the exposure of p_x is witnessed by $r_{yx} \in P_y$.

Now, consider the size of X_{xy} for fixed $\{x, y\} \subset [k]$. Because the H_S are all disjoint, this size can be rewritten as the sum of their sizes:

$$|X_{xy}| = \sum_{\substack{S \subset [k] \\ x \in S \text{ or } y \in S \\ |S| \leq k-2}} |H_S| = \sum_{s=1}^{k-2} \sum_{\substack{S \subset [k] \\ x \in S \text{ or } y \in S \\ |S| = s}} |H_S|.$$

Next, consider the sum of all of these sizes,

$$\sum_{\{x,y\} \subset [k]} |X_{xy}|. \tag{3.1}$$

Suppose $S = \{i\} \subset [k]$. Then $H_S \subseteq X_{ix}$ for any $x \in [k]$ with $x \neq i$, and there are $k-1$ distinct choices of x, so each H_S with $|S| = 1$ appears $k-1$ times in the summation in Equation 3.1.

If $S = \{i, j\} \subset [k]$ instead, then $H_S \subset X_{ij}$, which occurs once in the summation; $H_S \subset X_{it}$ and $H_S \subset X_{jt}$ for $t \notin S$, each of which occurs $k-2$ times (once for each t different from i and j). Such a set H_S therefore appears $1 + 2(k-2) = 2k-3$ times in the summation.

In general, if $S = \{i_1, \ldots, i_s\} \subset [k]$ for $1 \leq s \leq k-2$, then $H_S \subset X_{i_\ell i_r}$ for $1 \leq \ell < r \leq s$ which occurs $\binom{s}{2}$ times in the summation (once for each pair of indices in S); $H_S \subset X_{i_\ell t}$ for $1 \leq \ell \leq s$ and $t \notin S$, which occurs $s(k-s)$ times in the summation (counting all ℓ, t pairs). Thus, such an H_S occurs $\binom{s}{2} + s(k-s)$ times in the summation.

With this information, since all the H_S are disjoint and appear at most once per X_{xy}, we can rewrite the summation over all X_{xy} as follows:

$$\begin{aligned}
\sum_{\{x,y\}\subset[k]} |X_{xy}| &= \sum_{\{x,y\}\subset[k]} \sum_{s=1}^{k-2} \sum_{\substack{S\subset[k] \\ x\in S \text{ or } y\in S \\ |S|=s}} |H_S| \\
&= \left(\sum_{s=1}^{k-2} \sum_{\substack{S\subseteq[k] \\ |S|=s}} \left(\binom{s}{2} + s(k-s) \right) |H_S| \right) \\
&= \left(\sum_{s=1}^{k-2} \left(\binom{s}{2} + s(k-s) \right) \sum_{\substack{S\subseteq[k] \\ |S|=s}} |H_S| \right) \\
&= \left(\sum_{s=1}^{k-2} \left(\binom{s}{2} + s(k-s) \right) \alpha_s |H_I| \right) \\
&= \left(\sum_{s=1}^{k-2} \left(\binom{s}{2} + s(k-s) \right) \alpha_s \right) |H_I|.
\end{aligned}$$

We would like an upper bound that doesn't depend on the values of the α_s, and since H could be any arrangement of hyperplanes in general position whatsoever, we look for the largest value the above sum could take, ranging over possible values of the α_s. The sum is thus maximized when $\left(\sum_{s=1}^{k-2}(\binom{s}{2} + s(k-s))\alpha_s\right)$ is maximized. Each α_s is nonnegative, and $\sum_s \alpha_s = 1$, so this function is largest when all the α_s are zero except for the one whose coefficient is largest.

The coefficient of α_s in the sum is $f(s) = \binom{s}{2} + s(k-s) = ks - \frac{s}{2} - \frac{s^2}{2}$, for which $f'(s) = k - \frac{1}{2} - s$. For $1 \leq s \leq k-2$, this means $f(s)$ is largest for $s = k-2$. Therefore, the sum is maximized when $\alpha_{k-2} = 1$ and $\alpha_i = 0$ for $i \neq k-2$.

In concrete terms, this corresponds to an arrangement for which all interior hyperplanes intersect rays from $k-2$ different sets of witness rays P_i.

This yields the inequality

$$\sum_{\{x,y\}\subset[k]} |X_{xy}| \leq \left(\binom{k-2}{2} + (k-2)(k-(k-2))\right)|H_I| \leq \left(\frac{k^2-k-2}{2}\right)|H_I|.$$

There are $\binom{k}{2}$ such sets X_{xy}, so there must be some pair $\{x', y'\} \subset [k]$ for which $|X_{x'y'}| \leq \frac{k^2-k-2}{2\binom{k}{2}}|H_I|$. So, the exposing set $H_E \cup X_{x'y'}$ contains at most $n - |H_I| + \frac{k^2-k-2}{2\binom{k}{2}}|H_I| = n - \frac{1}{\binom{k}{2}}|H_I|$ hyperplanes. Call this exposing set X.

Recall that, due to Lemma 3.7, there is also an exposing set consisting of all but one hyperplane of H_I and about half of H_E, with size at most $|H_I| + \frac{1}{2}(n - |H_I| - 1)$. Call this exposing set X'.

We may think of $|X|$ and $|X'|$ as linear functions of $|H_I|$, and $|H_I|$ takes integer values between 0 and n, each of which is easily realizable.

$|X|$ decreases and $|X'|$ increases as $|H_I|$ increases, and therefore the largest value of $\min\{|X|, |X'|\}$ is obtained when $|X| = |X'|$. For fixed k, this occurs when $|H_I| = \frac{k(k-1)(n+1)}{k^2-k+4}$, for which the value of $|X|$ is $\frac{k^2-k+2}{k^2-k+4}n - \frac{2}{k^2-k+4}$.

By definition, $h^d(P, H) \leq \min\{|X|, |X'|\}$, as it is just the minimum of the sizes of all exposing sets for P and H, and thus $h^d(P, H) \leq \frac{k^2-k+2}{k^2-k+4}n - \frac{2}{k^2-k+4}$.

But then, working from the definition of $h_k^d(n)$, we have that

$$\begin{aligned} h_k^d(n) &= \max_{|P|=k,|H|=n} \{h^d(P,H)\} \\ &\leq \max_{|H|=n} \left\{ \frac{k^2-k+2}{k^2-k+4} \cdot n - \frac{2}{k^2-k+4} \right\} \\ &= \frac{k^2-k+2}{k^2-k+4} \cdot n - \frac{2}{k^2-k+4}. \qquad \square \end{aligned}$$

The general upper and lower bounds that we found for $h_k^d(n)$ can both be expressed as fractions of n that tend to 1 as k tends to infinity. The lower bound can be expressed as

$$\frac{k}{k+1}n - O(1) = n - \frac{1}{k+1}n - O(1) = \left(1 - O\left(\frac{1}{k}\right)\right) n - O(1),$$

and the upper bound can be expressed as

$$\frac{k^2-k+2}{k^2-k+4}n - O(1) = n - \frac{2}{k^2-k+4}n - O(1) = \left(1 - O\left(\frac{1}{k^2}\right)\right) n - O(1).$$

We note, however, that the factor in the upper bound tends to 1 much more quickly. It would be interesting to resolve this difference.

3.4 Point Exposing Algorithms and Related Problems

There are a number of problems whose settings are very similar to the problem of exposing points. The first is a notion of data depth in statistics called the "arrangement depth" [10], or "hyperplane depth" [1], of a point p with respect to a set of hyperplanes H, defined as the minimum number of hyperplanes in H crossed by a ray anchored at p.

The definition of the d-dimensional hyperplane depth $hdepth(p, H)$ initially suggests that the depth is the same as $h^d(\{p\}, H)$, but the depth takes objects to intersect at infinity, so that a point between two lines is "deeper" if those lines are parallel. This means that $h(\{p\}, H) \leq hdepth(p, H)$, and in fact the two values are distinct in many cases with parallel lines. The two values *are* the same in the cases considered by [10], however, because those authors take general position of hyperplanes to forbid parallel hyperplanes.

$hdepth(p, H)$ is the geometric dual of the Regression Depth [11], a distribution-free measure of the fit quality of a linear regression. The dual line $\mathcal{D}(p)$ is a regression fit to the set of data points $\mathcal{D}[H]$. A ray w anchored at p corresponds in the dual to a transformation, analogous to a homotopy, between the dual of p and some vertical line, where $\mathcal{D}(p)$ is one boundary of the bowtie $\mathcal{D}[w]$ and the vertical line is the other boundary. If w witnesses no lines, then the data points are all outside of the bowtie, and consequently to points that the transformation does not have to "pass by" on the way from the regression fit $\mathcal{D}(p)$ to the non-fit vertical line.

If p is between two parallel lines $\ell, \ell' \in H$, then the duals of these lines are data points with the same x-coordinate with the regression $\mathcal{D}(p)$ passing between them. There is no vertical line attainable by this homotopy-like transformation from $\mathcal{D}(p)$ that does not pass through one or both of the data points $\mathcal{D}(\ell)$ or $\mathcal{D}(\ell')$, and yet a ray anchored at p and parallel to ℓ does not intersect ℓ or ℓ'. Hence the assumption that parallel objects intersect at infinity.

Because we are not tied to the dual relationship with statistics, we are not interested in, and do not make, this assumption. This assumption, along with the concern with the statistical setting, is likely the primary reason that our extremal problems have not been explored by those authors or others in their field, despite being a meaningful generalization of the setting; the simultaneous interdependent non-fit status of a set of multiple distinct linear regressions with respect to a common set of data does have particularly obvious applications.

Another problem, called the Densest Hemisphere problem [7], asks for a hemisphere of a d-dimensional sphere which contains the largest number of some finite set of points on the sphere. While on its face less directly related to exposing points, it turns out that solving the Densest Hemisphere problem yields the value of $h^d(\{p\}, H)$, and conversely any Densest hemisphere instance can be solved by computing some $h^d(\{p\}, H)$.

Formally, we use the following version of the problem, which is more conveniently described in terms of vectors:

Definition 3.11 (Densest Hemisphere Problem [7]). *Let K be a set of points in $\mathbb{R}^d$. Find $x \in \mathbb{R}^d$ with $|x| > 0$ so that $|\{P \in K : x \cdot P \geq 0\}|$ is maximized.*

Theorem 3.12. *$h^d(\{p\}, H)$ may be computed by solving a particular instance of the Densest Closed Hemisphere problem. Conversely, any instance of Densest Closed Hemisphere may be solved by finding a witness set witnessing the value of $h^d(\{p\}, H)$ where p is the origin and H is a particular set of hyperplanes in $\mathbb{R}^d$.*

Proof. Let $p \in \mathbb{R}^d$ and let H be a set of n hyperplanes in $\mathbb{R}^d$, for which we would like to compute $h^d(\{p\}, H)$.

First, without loss of generality we may assume that p is the origin, and otherwise translate p to the origin and translate all hyperplanes in H by the same vector. For $\pi \in H$, orthogonally project the origin onto π and reflect this projection about the origin. Let q_π be the point obtained by this process. Then let $K = \{q_\pi : \pi \in H\}$.

Note that a point $q_\pi \in K$, viewed as a vector, is simply a negative multiple of the normal vector of π.

The solution to the Densest Hemisphere Problem with input K is a vector x such that $|\{q_\pi \in K : x \cdot q_\pi \geq 0\}|$ is maximized, and thus $|\{q_\pi \in K : x \cdot q_\pi < 0\}| = n - |\{q_\pi \in K : x \cdot q_\pi \geq 0\}|$ is minimized. But, by construction, $x \cdot q_\pi < 0$ if and only if x points toward the hyperplane π, and thus any ray in the direction of x, anchored at the origin, intersects a minimum number of hyperplanes in H.

Therefore, x is a witness to the fact that $h^d(\{p\}, H) = n - |\{q_\pi \in K : x \cdot q_\pi \geq 0\}|$.

On the other hand, suppose K' is the set of input points to an instance of Densest Closed Hemisphere. Then for a point q, if π_q is the hyperplane containing q and with normal vector q, let $H' = \{\pi_q : -q \in K'\}$. Intuitively, H' is a set of hyperplanes tangent to the sphere containing the antipodes of the input points.

Let w be a ray witnessing that $X \subseteq H'$ is a smallest exposing set for the origin, and $-w$ its opposite ray also anchored at the origin. Then any point x on $-w$ is such that $x \cdot q < 0$ for all $q \in X$, and moreover $|X| = |\{q \in K' : x \cdot q < 0\}|$ is of minimum size for such a set. This means that, for the same x, $|\{q \in K' : x \cdot q \geq 0\}|$ is maximized, and thus x is a solution to the original Densest Hemisphere Problem with input K'. $\square$

A major contribution of [7] is that the Densest Hemisphere problem is NP-complete, if the dimension is not fixed in advance. This immediately leads to the following corollaries.

Corollary 3.13. *Finding an exposing set that witnesses the value of* $h^d(\{p\}, H)$ *is NP-complete if* d *is not fixed.*

Corollary 3.14. *Finding an exposing set witnessing the value of* $h^d(P, H)$ *for any* P *is also NP-complete if* d *is not fixed.*

CHAPTER 4: CONCLUSION

Starting in chapter 1, we have made considerable progress toward the classification of ray-barrier graphs: they are tripartite, perfect graphs with a strong neighborhood rigidity property that implies that few bipartite graphs are barrier graphs.

In proving the neighborhood rigidity of barrier graphs, we provided a result that is an interesting geometric result in its own right: that a set of rays can be stabbed by another ray in polynomially many ways, as opposed to the naive expectation of exponentially many ways. We also provided realizations for several natural classes of bipartite and non-bipartite graphs.

A number of open questions about barrier graphs remain. The primary, overarching question is whether there is a completely graph-theoretic classification of them. In this direction, we would like an algorithm that could compute a ray sensor network that corresponds to a given graph if that graph is a barrier graph, and otherwise reports failure.

In Section 1.5 we showed that finding sensor networks realizing bipartite graphs of resilience (minimum vertex cover size) 2 and 3 is straightforward; thus, all bipartite graphs of resilience at most 3 are barrier graphs. On the other hand, the neighborhood rigidity property indirectly implies that bipartite graphs with resilience larger than 16 can never be barrier graphs. We asked for the minimum resilience $r_\star$ such that there exists a bipartite graph of resilience $r_\star$ that is not realizable as a barrier graph; we have shown that $4 \leq r_\star \leq 16$.

In chapter 2 we introduced two extremal problems regarding Exposing and Jointly Exposing points in Euclidean space protected by networks of linearly shaped sensors: lines, rays, and segments in two dimensions, and hyperplanes in higher dimensions.

The first extremal problem seeks the value of $h_k^d(n)$, which is the number of d-dimensional hyperplanes whose removal from a set of n is sometimes necessary and always sufficient to expose a set of k points. We provided a general lower bound for the value of $h_k^d(n)$, with a way to construct an n-sensor network in d dimensions protecting k points that has a large minimum exposing set of about $\left(1 - O\left(\frac{1}{k}\right)\right) n$ sensors.

We provided matching upper bounds for $1 \leq k \leq 4$ in the two dimensional case, with the assumption for $k = 4$ that the points are in convex position. Our more general upper bound in d dimensions, which does not match the lower bound, says that an exposing set of $\left(1 - O\left(\frac{1}{k^2}\right)\right) n$ sensors always exists. This result is the strongest we have, even in two dimensions. Future work should aim to tighten these bounds.

Our second extremal problem asks about the joint exposure of points for rays in the plane, rather than lines (hyperplanes in 2 dimensions); the problem seeks the value of $R_k(n)$, the number of rays in the plane whose removal from a set of n is sometimes necessary and always sufficient to jointly expose a set of k points. We determined the asymptotic value of $R_2(n)$ to be $\frac{3}{4}n$ using an upper and lower bound that meet.

In the same way that we generalized the first extremal problem to higher dimensions, it would be interesting to generalize joint exposure to higher dimensions, but for essentially the same reasons as in two dimensions, joint exposure is not interesting for hyperplane sensors, as placing all sensors between any pair of input points forces a single attacker to pass through all of them.

The first step in generalizing is then to choose a sensor geometry, and there are several natural choices. One choice is to use half-planes, which is in a sense what a ray is in two dimensions; take a $(d-1)$-dimensional flat and cut it in half by a $(d-2)$-dimensional flat. Another choice is to use a sensor that is the intersection of d such half-spaces at a common point. There are a number of other choices between the two of these that could also raise interesting questions.

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