Exploring students adaptive use of domain specific knowledge

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Adaptive use of meaningful knowledge is widely adopted as key learning objective in the changing society. This paper presents the results of a teaching experiment in the domain of partitive division. It is designed to explore how grade-3 students do adapt personal knowledge to the variation in task conditions. Under the first condition groups of four and six students explore the process of distributing 52 carts between four/six persons. They can use 52 unifix cubes to model the process directly. The second condition requires that they mentally anticipate the results of sharing the same quantity of carts between respectively two and three children. The study shows that the variation in conditions combined with classroom climate challenge a great part of the students to use adaptively "pieces of knowledge" acquired in different areas of reasoning in equal group situations.

Keywords: multiplicative thinking, adaptive expertise.

Introduction

By the turn of the century, "Mathematical proficiency" is proposed as basis for a large consensual agreement about the goal of mathematics instructions in the changing society (Kilpatrick, Swafford, & Findell, 2001). Said globally, primary school students should develop and organize domains meaning full knowledge in such a way that it should be used adaptively to tackle and solve new or less familiar problems.

According to Hatano & Inagaki (1986) it is "domains-specific knowledge" that develops along the accumulation of learning experiences and what adapts to constraints of new situation are, according to Vergnaud (2009), the "schemes for reasoning". Considering the crucial role of developing these schemes for reasoning we explore in our research (i) what grade-3 students know about the quantitative and numerical relationships involved by combining, sharing, and segmenting in equal groups setting, and (ii) how they do adapt meaningful personal knowledge to well-chosen variations of task-conditions.

Theoretical framework

Our framework connects four aspects of students' expertise involved in the exploration of the process of partitioning in grade 3 needed to analyze the relationship between the conditions of the tasks and the personal way of tackling and solving these tasks.

Schemes of reasoning

Combining *n* equal groups to obtain an intended quantity of "things" (e.g. 5 bags with 6 cookies each) constitutes children earlier encounter with an application for multiplication (Greer, 1992). Children extract from their activities the notion of number as composite unit and the invariant relationship between the number of groups and the number of units per group that constitute the conceptual base of the arising scheme "multiplicative double counting" (Tzur, Johnson, McClitock, Kenny, Xin, Si, Woordward, Hord & Jin, 2013).

According to Freudenthal´s (2002) phenomenological analysis of division as mental act, division arises in three ways, as (i) continually taking away (by repeated subtractions), (ii) distributing in equal parts (distributing cyclically the same share to several persons), and (iii) inverting a multiplication. The former and second one corresponds to the difference made between quotative (measurement; ratio) division and partitive division, and between the two associated schemes of reasoning – segmenting and partitioning (e.g. Tzur et all, 2013; Thompson & Saldanha, 2003). This third variant consists in figuring out the effect of both segmenting (e.g. 52 cookies into bags of 6 cookies each) and partitioning (e.g. distributing 52 cookies between 6 children) by constructing an appropriate arithmetical sequence of repeated addition (or multiples). In this case, the remainder represents the four simple steps left to reach 52 after 8 steps of 6 on the (mental) number line.

Research results show that sharing one-by-one is rarely used and that student prefer to build-up the quantity (inverting multiplication by repeated additions) instead of subtracting repetitively (e.g. Heirdsfield, Cooper. Mulligan & Calvin, 1999). Taking in advantage of this tendency and adopting the idea that students must observe variation in key variables to constitute deep understanding (Lo, 2012), we focus students' activity on exploring the critical differences among partitioning problems (variation in total number of objects, number of persons/parts, number of objects in each part; remainder) in relation to the invariant multiplicative structure ($a = q \times d (+ r)$) of any partition. We conjecture that this relationship should function as tie between partitioning and combining, on the one hand (Greer, 2012) and between partitioning and segmenting, on the other hand (Thompson & Saldanha, 2003; Downton, 2008).

Mathematical principles and numerical relations

Envisioning the learning process we conjecture that, reaching the highest level of comprehension, students should (i) formulate the numerical equivalence of $a \times b = c$ and $c \div a = b$ and (ii) use it explicitly to derive unknown quotients from memorized correspondent products (e.g. $100 \div 25$ via $4 \times 25 = 100$) and to tackle and solve partitioning problem using appropriate patterns of multiples. Students should use different expressions to symbolize the same quantitative relationship (e.g. $13 \times 4 = 52$; $52 \div 4 = 13$; $4 \times 13 = 52$; $52 = 13 \times 4$) knowing that each number and consequently each operation can be composed and decomposed on different ways (Gray & Tall, 1994; Tall, 2013). Last but not least, understanding the product (mn) as being in multiple reciprocal relationships to n and to m, they should derive a lot of quotients from familiar numerical relations (e.g. $60 \div 15$ via $15 = \frac{1}{4}$ of 60) (Thompson & Saldanha, 2003). We expected differences in reasoning, computing and symbolization in function of the progression through the well-documented sequence of multiplication procedures: from counting all strategies, through sequences of repeated addition and doubling procedures, to using patterns in numbers and operations, and finally, to deriving unknown products from surrounding memorize facts (e.g. Verschaffel, Greer & de Corte, 2007).

Strategic skills

Adopting Threlfall´s (2002; 2009) conception of flexible mental calculation as "interaction between noticing and knowledge", we conjecture that an appropriate variation of task conditions should motivate and foster students to adapt the above domain-specific knowledge to the constraints of situations (Vergnaud, 2009; Hatano & Inagaki, 1986). In this perspective, specific tasks should give students the opportunity to develop particular strategic skills: (i) relating the numbers of problems to other familiar situations, (ii) composing and decomposing numbers multiplicatively, (iii) using patterns of multiples, (iii) transforming multiplication and division.

Classroom culture

It is well known that classroom climate motivates students to reflect about how they should tackle the situation taking advantage of what is met before. This factor is included in the following three conditions proposed by Hatano and Inagaki (1986) for promoting adaptive expertise: i) variability inherent to the task environment, ii) variability permitted in the individual´s procedural application, iii) variability of explanation permitted by the culture.

Methodology

This article reports part of a research project that follows a design research methodology, specifically a teaching experience (Gravemeijer & Cobb, 2006) that has the objective to understand how students can develop the ability to tackle and solve problems, adapting personal knowledge to new situations' constrains.

The team project developed two teaching experiences: one focused in addition/subtraction and the other one focused on multiplication/division. Each teaching experience includes a set of tasks that was designed and reformulated using a three-step cyclic process: (1) design tasks, (2) analyse what children noticed in the numbers and how they use their knowledge about numbers and operations to solve the task presented in the class or along clinical interviews and (3) reformulate the previous task.

We present part of the teaching experiment on multiplication/division that involved a third grade class (students age 8-9) with 20 students. The underlying "conjectural hypothetical theory" (Gravemeijer & Cobb, 2006) for this experiment concerns a possible learning process between the construction of the products of the multiplication tables and related quotients (start point) and elementary forms of reasoning proportionally (end point).

The first three tasks of this teaching experiment that involved a total of nine tasks, intend to help students to see the multiplicative structure of equal group situations as "some numbers of composed units". This paper focuses the second task– What is sharing?. The objective is to observe how these students adapt personal knowledge to the variation in task conditions. The invariant condition is the number of objects distributed (52 stickers). The variant conditions are: i) possibility to use (part one), or not (part three) concrete material; ii) the number of persons/parts and ii) with or without remainder.

The teacher of the class analysed and discussed with the researchers all the underlying justifications for the tasks, classroom organization and proposed focus for discussion with students.

Data was collected through video recordings of the classroom work, researchers' field notes and students' written answers. According to the task design, the teacher organized the class in two groups of 4 and two of 6. In the first part of the task students could model the process of distributing 52 stickers[1] using 52 unifix cubes and register their shares in a given table.

The objective of the second part of the task is to register and connect the numbers of the distribution on a given diagram (Figure 1, two left images) and to symbolize the structure of the distribution using the expression a = qd + r (distributions with rest) or with a = qd (distributions without rest).

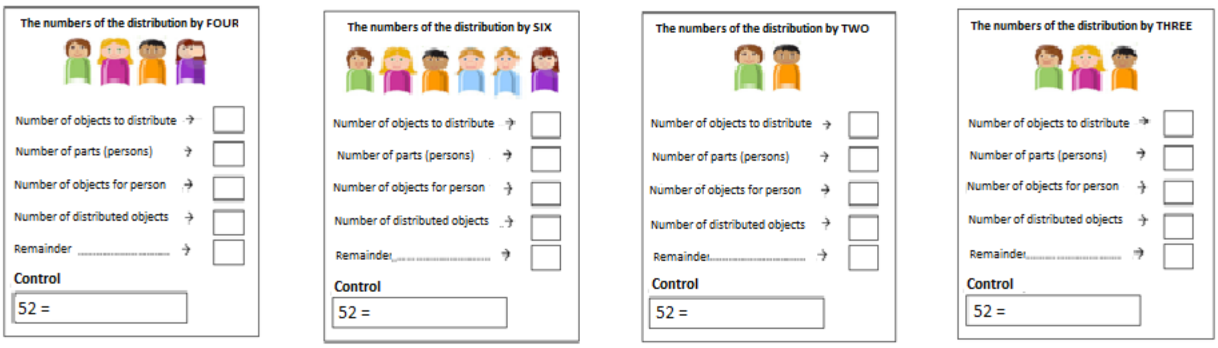

Figure 1: Distribution numbers

In the third part ((Figure 1, two right images) students must envision the result of partitioning connecting the numbers of the new partition with those of first one (deriving using proportional relationship).

We formulated the following conjectures:

(i) as students know that they can approach and solve the task in their own way and that the work will be discussed in a final phase, we expect them to use what they already know, adapting it to the task conditions (Hatano, 2003);

(ii) as they have 52 objects, we expect students to distribute them in two different ways: one by one or two by two (an intuitive way of distributing);

(iii) the second part of the task allows students to discern the meaning of the different numbers of the distribution and to connect them in an appropriate way that represents the underlying structure (multiplicative structure of partitioning);

(iii) as the number of people is half of the number of people in the part 1 of the task, we expect that most students will deduct the part they each receive using the double / half ratio (idea of proportional relations).

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[1] In Portugal children often have collections of Panini stickers.

The categories for analysing data were constructed from the theoretical framework and focused on students' resolution processes focusing: (i) ways of modelling and representing the situation and (ii) relations used when reasoning, representing and calculating (how they relate numbers and operations, properties of operations and numerical relations used).

Results

Distributing in equal parts

Confirming our conjecture some groups cyclically distribute the same amount of objects (one or two) to each element of the group. However, some groups did not model the situation as expected and followed a personal way of acting, reasoning and representing.

One of the groups with 4 students organized the cubes in 13 bars with 4 cubes each (see picture). They explained:

Student 1: We only need the first round.

Teacher: Why?

Student 2: We formed groups of 4 and we counted.

(…)

Student 1: We formed groups of 4 and we counted them. We have 13.

Teacher: And what does this mean?

Student 2: Each one with 13.

Teacher: Each one has ...

Student 3: 13 stickers.

Teacher: Why did you form groups of 4?

Student 1: Because we had to divide 52 by 4.

The other group with 4 students used the relation "to divide by 4 is the same as half of the half". One student explained "First of all the number of rounds is 13. It was a quarter of 52. The number of objects distributed in the 13 (points to the thirteenth round) is 52".

As expected, understanding what happens when some cubes remain, originated some hesitations and discussion within the two groups with six students. For instance, one of them understood that there were 4 cards left but still continued to pose other possibilities for the number of cards that each could receive:

Students: We have 4 left.

Teacher: Ok. We have 4 left. And can we have another round?

Student 1: No. We had to rip the cart.

Teacher: We do not usually rip cards, do we?

Student 2: And if we give 7 cards to each of us?

Teacher: And if we give 7 cards to each of us? What happens?

Student 3: And if we give 6 cards?

Teacher: And if we give 7 cards to each of us? What happens?

Student 2: There are more left.

The register of the distribution of the 52 objects in the table originated some mistakes. For instance, some of them register the sequence of multiples of 4 instead of the numbers of objects distributed in each round.

Analysed data shows that the table presented in the task was not adequate to register the reasoning used by the groups that modelled the distribution via “one fourth is half of the half” or that related distributing one by one to the final distribution of 8 to each one:

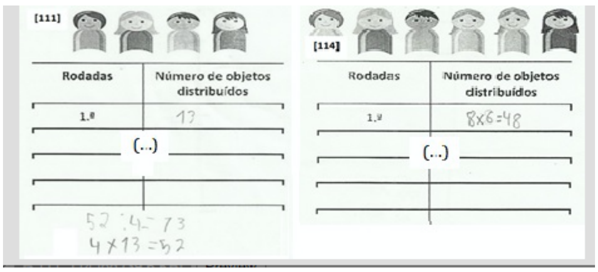


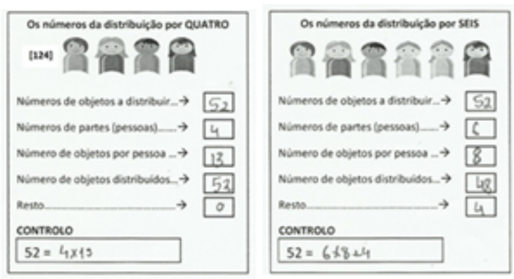
Figure 2: Adapted registration

Relating the numbers of the distribution to control the outcome

It was expected that students, in the extension of the modelling, would recognize and register correctly the meaning of the numbers of both distribution on the dispensed diagram.

Relating the number of groups to the numbers of objects per group and what remains, they would understand that the outcomes of both “division” can be controlled by symbolizing the multiplicative structure of the distribution with the adequate sequence of repetitive additive and the corresponding decomposition of 52 into a = qd (+r).

The video recording, registration of the meaning (Figure 3) and representations of the distribution (Figure 3 and Figure 4) on the individual worksheets show that, in open reflection and discussion under direction of the teacher, all the students succeed to control the outcomes as expected.



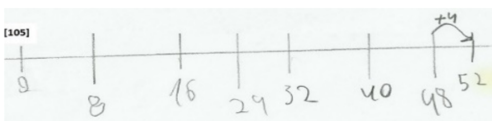


Figure 3: Meaning of the numbers **Figure 4: Multiplicative structure**

Deriving distributing by 2 and 3 from distributing by 4 and 6

The results confirm our conjecture: students do not use cubes and deduce the part that receives each one using the rule that if we have half of the persons each one receives twice as many objects. All the students apply the rule by comparing the two distributions without rest: 52 ÷ 2 with 52 ÷ 4, halving the number of person goes with doubling the number of objects per person.

However, some groups apply this relationship to 52 ÷ 6 without considering the 4 remaining objects and give the incorrect answer of 16 objects per person.

Discussion

Data analyse confirms the conjecture that varying conditions of the task (possibility to use or not concrete material; different number of persons/parts; division with or without remainder) stimulates the adaptive use of the knowledge and procedures that the students already have, which favours the possibility of adapting the acquired knowledge and procedures to the numbers involved in the task. This complex situation of sharing gives students the opportunity to explore ways of thinking that allow them to take advantage of what they already know.

Data analysis suggests three critical aspects of the learning process in this domain that we propose for further investigation and discussion:

- *Envisioning the process and the result of dividing instead of directly modelling*. The fact that some groups do not distribute objects cyclically suggests that modelling with objects does not make sense because students already have an idea of the inverse relationship between combining and dividing that allows a more abstract approach. Our new conjecture is to give only 4 (or 6) cubes to envision and represent numerically the process of sharing 52 by 4 (or 6) persons.

- *Exploring remainder patterns using the inverse relation.* Several divisions without rest and with rest raise the question "what explains this difference?". Using the knowledge of the inverse relationship between combining and sharing/segmenting students could investigate 'from where the remains come' (divisibility).

- *Understanding the ambiguity of symbolization with numerical expressions*. Segmenting/distributing can be represented with different numerical expressions. Giving students the opportunity to think multiplicatively in the context of division to explore numerical patterns can promote the ability to compose and decompose numbers using operations to represent them (12 is 3 × 4, 12 is a quarter of 48, 4 is 48 to divide by 12).

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