A CLOSED LOOP CONTROL SCHEME FOR A CLASS OF PIECEWISE AFFINE SYSTEMS [1]

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Abstract: This paper presents a robustly stable closed loop control scheme for Piecewise Affine (PWA) systems. The control scheme is composed by a robust mode hybrid controller that stabilizes a PWA system subject to additive input disturbances while ensuring that all the uncertainty associated with the continuous state is contained in a single discrete mode of the system. The continuous state uncertainty is described by a polytope and is determined by a set-valued hybrid observer using polytopic arithmetics. Closed loop stability is ensured by employing a Model Predictive Control strategy. A very simple demonstrative example is also presented.

Keywords: Hybrid Systems, Model Predictive Control, Closed Loop Stability, Robust Mode Hybrid Control, Set-valued Observer.

1. INTRODUCTION

The present work focuses on the closed loop control problem for stochastic hybrid systems. In recent years the industry and research community have shown an increasing interest on hybrid systems due to their capability of describing the interaction between dynamical and logical components (Antsaklis, 2000). This interaction can be found in many real world systems, embedded control systems and in the control of many complex industrial systems via the combination of classical continuous control laws with supervisory switching logic.

The class of hybrid systems considered in this paper is Piecewise Affine (PWA) systems. These are basically composed by a set of affine dynamics and a discrete mode that defines the active dynamics. In (Heemels *et al.*, 2001) PWA systems are proven to be equivalent, under some mild assumptions, to many other classes of hybrid systems, and so, the proposed techniques can be interchanged among all the referred classes.

Closed loop control schemes have already been used to control hybrid systems, for instance the widely spread Model Predictive Control (MPC) (Mayne *et al.*, 2000). However, the state is always assumed to be exactly known and no observers or estimators are needed to reconstruct it. Some of the MPC schemes were even proven to be robustly stable in the presence of small disturbances (Jalali and Nadimi, 2006; Bemporad and Morari, 1999). In this paper, the robust MPC of (Botto *et al.*, 2005) will be used, and a set-valued state observer (Shamma and Tu, 1995) will reconstruct the state uncertainty set from the measured outputs. The stability is ensured for the unknown but bounded input disturbances and measurement noise explicitly considered in the stochastic PWA model.

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The closed loop stability arises from the robust stability of the MPC scheme in the presence of uncertainty and, from the fact that the information gathered by the set-valued observer must reduce the uncertainty.

The remainder of this paper is organized as follows: Section 2 describes the stochastic PWA model that will be considered. Sections 3 and 4 describe the robust mode hybrid controller and the set-valued hybrid observer, respectively. In section 5, the receding horizon strategy presented and the stability of the entire control loop is analyzed. A numerical example is presented in section 6 to further clarify the introduced concepts. Finally, some conclusions are drawn in section 7.

2. SYSTEM MODEL

The proposed control scheme is developed for PWA systems which were introduced in (Sontag, 1981). The following stochastic PWA model will be considered:

$$x(k+1) = A_{i(k)}x(k)+B_{i(k)}u(k)+f_{i(k)}+L_{i(k)}w(k) \quad (1a)$$

$$y(k) = C_{i(k)}x(k)+D_{i(k)}u(k)+g_{i(k)}+v(k) \quad (1b)$$

$$\text{iff } \begin{bmatrix} x(k) \\ u(k) \\ w(k) \end{bmatrix} \in \Omega_{i(k)} \quad (1c)$$

where $k \in \mathbb{N}_0$ is the discrete time, $x(k) \in \mathbb{X} \subset \mathbb{R}^{n_x}$ is the continuous state, $u(k) \in \mathbb{U} \subset \mathbb{R}^{n_u}$ is the input, $y(k) \in \mathbb{R}^{n_y}$ is the output and, $i(k) \in \mathcal{I} = \{1, \ldots, s\}$ is the discrete mode where s the total number of discrete modes. The input disturbance $w(k) \in \mathbb{W}_i \subset \mathbb{R}^{n_w}$ and measurement noise $v(k) \in \mathbb{V}_i \subset \mathbb{R}^{n_v}$ are modelled as random variables taking values in the respective polytopes which contain the origin and may depend on the discrete mode i. These polytopes are defined by sets of inequalities, as follows:

$$H_{\mathbb{W}_{i(k)}}\, w(k) \leq h_{\mathbb{W}_{i(k)}} \quad , \quad \forall_{k\in\mathbb{N}_0} \quad (2)$$

$$H_{\mathbb{V}_{i(k)}}\, v(k) \leq h_{\mathbb{V}_{i(k)}} \quad , \quad \forall_{k\in\mathbb{N}_0} \quad (3)$$

The discrete mode i is a piecewise constant function of the state, input and input disturbance of the system whose value is defined by the polytopic regions Ω_i:

$$\Omega_i \triangleq \left\{ \begin{bmatrix} x \\ u \\ w \end{bmatrix} \in \mathbb{R}^{n_x+n_u+n_w} : S_i\, x + R_i\, u + Q_i\, w \leq T_i \right\} \quad (4)$$

The matrices and vectors A_i, B_i, f_i, L_i, C_i, D_i, g_i, S_i, R_i, Q_i, T_i depend on the discrete mode i and have appropriate dimensions.

3. THE ROBUST OPTIMAL CONTROL PROBLEM

As an MPC strategy will be adopted, this section defines the robust optimal control problem that must be solved at each sampling time to determine the control input.

3.1 *The Open-Loop Min-Max Optimal Control Problem*

Consider the PWA system subject to bounded additive exogenous disturbances defined in (1). The finite horizon min-max optimal control problem for the disturbed PWA system under operational constraints is defined as follows (Kerrigan and Maciejowski, 2003).

Problem 1. The PWA formulation
Given an initial convex set of continuous states $\mathcal{X}_k \subseteq \mathbb{X}$ at time k and a final time $k+N$, find (if it exists) the control sequence $\mathbf{u}(k) \equiv \mathbf{u}_k^{k+N-1} \equiv (u'(k|k), u'(k+1|k), \ldots, u'(k+N-1|k))'$ which (*i*) transfers the state from $\mathcal{X}_k$ to a given final set $\mathbb{X}_f \subseteq \mathbb{X}$, which contains a final (target) nominal equilibrium state x_f, and (*ii*) minimizes the performance index $V_N\,(x_k,\mathbf{u})$ given by:

$$\max_{\substack{\mathbf{w}\in\mathbb{W}_{i(k|k)}\times\ldots\times\mathbb{W}_{i(k+N-1|k)} \\ x_k\in\mathcal{X}_k}} J_N\,(x_k,\mathbf{u},\mathbf{w}) \quad (5)$$

where $J_N\,(x_k,\mathbf{u},\mathbf{w})$ is defined as:

$$\sum_{t=k}^{k+N-1} \min_{a(t|k)\in\mathbb{X}_f} \|x(t|k)-a(t|k)\|_{P_x,l} + \|u(t|k)-u_f\|_{P_u,l} \quad (6)$$

subject to:

$$x(k|k) = x_k \quad (7a)$$

$$x(t+1|k) = A_i x(t|k)+B_i u(t|k)+f_i+L_i w(t|k)$$
$$\text{for } \begin{bmatrix} x(t|k) \\ u(t|k) \\ w(t|k) \end{bmatrix} \in \Omega_i \quad (7b)$$

$$\begin{bmatrix} x(t|k) \\ u(t|k) \\ w(t|k) \end{bmatrix} \in \mathbb{C},\ \forall\, w(t|k)\in\mathbb{W}_{i(t|k)},\ t=k,\ldots,k+N-1$$
$$\text{with } \mathbb{C} \triangleq \left\{ \begin{bmatrix} x \\ u \\ w \end{bmatrix} : S_{\mathbb{C}}x+R_{\mathbb{C}}u+Q_{\mathbb{C}}w \leq T_{\mathbb{C}} \right\} \quad (7c)$$

$$x(N|k) \in \mathbb{X}_f\ , \quad \forall\, w(.|k) \in \mathbb{W}_{i(.|k)}, \quad (7d)$$

where $x(t|k)$ represents the state trajectory, $\|x\|_{P_x,l}$ and $\|u\|_{P_u,l}$ the l-norm of vector x and u weighted with matrices P_x and P_u, respectively, with P_x and P_u being full column rank matrices, and u_f is the steady-sate equilibrium input when $x(t|k) = x_f$ and disturbances are not present.

Problem 1 minimizes the worst-case performance cost (5)–(6) and robustly guarantees constraints (7), at all time steps k. It penalizes the distance from the given final state-set $\mathbb{X}_f$, while the state at the end of the horizon is not penalized. This structure and properties of the stage cost, terminal cost, and terminal state-set are important to achieve closed-loop robust asymptotic stability of the MPC controlled system. Problem 1 has *infinite* dimension since it requires the solution of an optimization problem for every admissible state $x_k \in \mathcal{X}_k$ and disturbance sequence $\mathbf{w}(k) \equiv \mathbf{w}_k^{k+N-1} \equiv (w'(k|k), w'(k+1|k), \ldots, w'(k+N-1|k))'$. It must be transformed into a *finite* dimension problem, using the methodologies presented in (Boyd and Vandenberghe, 2004) to solve robust optimization problems.

3.1.1. *Min-Max as a Finite-Dimensional Problem* The robust mode hybrid control strategy described in (Silva *et al.*, 2004) ensures that the mode of the system is "certain" regardless of the disturbances over a fixed horizon. As a consequence, for each possible mode trajectory $\mathbf{i}(k) \equiv \mathbf{i}_k^{k+N-1} \equiv (i'(k|k), i'(k+1|k), \ldots, i'(k+N-1|k))'$ the system behaves as a linear (affine) system, though time-variant, and so convex state-sets are generated given that both the initial state and the disturbance sequences take values in convex sets. Besides, since the stage cost (5) of Problem 1 is a convex function, the technique presented in (Scokaert and Mayne, 1998) to convert a min-max problem into an equivalent convex program based on the linearity of the dynamic model and convexity of the stage cost and uncertainties can be adopted here. In view of this, consider the following infinite-dimensional min-max optimization problem, where $\mathbb{U}^N$, $\mathcal{X}_k$ and $\mathbb{W}_{\mathbf{i}} \equiv \mathbb{W}_{i(k|k)} \times \ldots \times \mathbb{W}_{i(k+N-1|k)}$ are convex polytopes, and function $L(.,.,.)$ is convex:

$$\min_{\mathbf{u}\in\mathbb{U}^N} \max_{x_k\in\mathcal{X}_k, \mathbf{w}\in\mathbb{W}_{\mathbf{i}}} L\left(x_k, \mathbf{u}, \mathbf{w}\right) \tag{8}$$

Consider also that $\Upsilon_{\mathcal{X}_k}$ is the set of all vertices $\nu_{\mathcal{X}_k}$ of $\mathcal{X}_k$ and $\Upsilon_{\mathbb{W}_{\mathbf{i}}}$ is the set of all vertices $\nu_{\mathbb{W}_{\mathbf{i}}}$ of $\mathbb{W}_{\mathbf{i}}$. As $L(.,.,.)$ is assumed convex relatively to x_k and $\mathbf{w}$, the above infinite-dimensional optimization problem is equivalent to the following finite-dimensional one:

$$\min_{\mathbf{u}\in\mathbb{U}^N} \max_{x_k\in\Upsilon_{\mathcal{X}_k}, \mathbf{w}\in\Upsilon_{\mathbb{W}_{\mathbf{i}}}} L\left(x_k, \mathbf{u}, \mathbf{w}\right) \tag{9}$$

The previous step was obtained by knowing that the maximum of a convex function L over a convex set $\mathcal{X}_k$ ($\mathbb{W}_{\mathbf{i}}$) is at one of the vertices of $\mathcal{X}_k$ ($\mathbb{W}_{\mathbf{i}}$) (see e.g. (Boyd and Vandenberghe, 2004)). In turn, the optimization problem (9) is also equivalent to the convex program:

$$\min_{\mathbf{u},\gamma} \left\{\gamma \,\middle|\, \mathbf{u}\in\mathbb{U}^N, L\left(x_k, \mathbf{u}, \mathbf{w}\right) \le \gamma, \forall x_k \in \Upsilon_{\mathcal{X}_k}, \mathbf{w}\in\Upsilon_{\mathbb{W}_{\mathbf{i}}}\right\} \tag{10}$$

Based on the previous technique, the min-max Problem 1 is now converted into an equivalent finite-dimensional minimizing one, however restricted by the robust mode condition that imposes the same mode trajectory $\mathbf{i}$ for all admissible $x_k \in \mathcal{X}_k$ and $\mathbf{w} \in \mathbb{W}_{\mathbf{i}}$ (or equivalently $x_k \in \Upsilon_{\mathcal{X}_k}$ and $\mathbf{w} \in \Upsilon_{\mathbb{W}_{\mathbf{i}}}$). Consider the system dynamics, operational constraints, and the robust mode condition represented within the PWA framework. The robust mode min-max optimal control problem equivalent to Problem 1 is defined as follows:

$$J_N\left(\mathcal{X}_k\right) \triangleq \min_{\mathbf{u},\mathbf{a},\gamma} \gamma \;,\; \forall x_k \in \Upsilon_{\mathcal{X}_k}, \mathbf{w}\in\Upsilon_{\mathbb{W}_{\mathbf{i}}} \tag{11}$$

4. SET-VALUED HYBRID OBSERVER

The hybrid controller presented in the previous section requires, at each time instant, the knowledge of a polytope that contains all possible values for the continuous state, the continuous state uncertainty set $\mathcal{X}_k$. So, the hybrid observer must determine, at every time instant, the polytope of uncertainty from the collected measurements and applied inputs. Usually, an hybrid observer would also provide estimates for the continuous state and for the discrete mode of the system, however, in this special setup, the continuous state estimate is not required and the discrete mode is imposed by the hybrid controller through the robust mode condition, and so it does not need to be estimated.

At a given time instant k the observer must determine the uncertainty polytope $\mathcal{X}(k+1|k)$ from the imposed discrete modes $i(k)$ and $i(k+1)$, the applied input $u(k)$ and measured output $y(k)$ and the polytope of the previous time instant $\mathcal{X}(k|k-1)$. The uncertainty polytope is propagated to time instant $k+1$ since it must be used in the determination the input $u(k+1)$ and, according to the system model (1) the most recent measurement is $y(k)$ while $y(k+1)$ will only be available after $u(k+1)$ has been applied.

The uncertainty polytope $\mathcal{X}(k+1|k)$ requires the use of polytopic arithmetic (Veres, 2002) and can be computed in 3 steps:

Measurement Collection The information of the collected measurement, $y(k)$, is used to refine the uncertainty polytope $\mathcal{X}(k|k-1)$:

$$\begin{aligned}\mathcal{X}(k|k) = \mathcal{X}(k|k-1) \bigcap \\ \bigcap \left[\bar{\mathcal{X}}_{i(k)} \overrightarrow{+} \left(y(k) - D_{i(k)}u(k) - g_{i(k)}\right)\right]\end{aligned} \tag{12}$$

where $\bar{\mathcal{X}}_{i(k)}$ is the admissible state polytope for zero output of discrete mode $i(k)$, the sign $\overrightarrow{+}$ represents the shift of a polytope by a vector and $\bigcap$ is the intersection of polytopes. $\bar{\mathcal{X}}_i$ are given by:

$$\bar{\mathcal{X}}_i \triangleq \left\{x \in \mathbb{R}^{n_x} : -H_{\mathbb{V}_i} C_i x \le h_{\mathbb{V}_i}\right\} \tag{13}$$

State Prediction The system dynamics (1a) are used to predict $\mathcal{X}(k+1|k)$ from $\mathcal{X}(k|k)$. First each vertex $\nu_{\mathcal{X}(k|k)}$ of $\mathcal{X}(k|k)$ is propagated to the next time instant according to:

$$\nu_{\tilde{\mathcal{X}}(k+1|k)} = A_{i(k)}\nu_{\mathcal{X}(k|k)} + B_{i(k)}u(k) + f_{i(k)} \tag{14}$$

then, the set of admissible input disturbances are also propagated using:

$$\nu_{\tilde{\mathcal{W}}_{i(k)}} = W_{i(k)}\nu_{\mathbb{W}_{i(k)}} \tag{15}$$

Finally, these two polytopes must be added using the Minkwosky sum $\bar{+}$:

$$\mathcal{X}(k+1|k) = \tilde{\mathcal{X}}(k+1|k) \,\bar{+}\, \tilde{\mathcal{W}}_{i(k)} \tag{16}$$

Constraint Verification The constraints (1c) of the discrete mode $i(k+1)$ are verified to ensure that all the uncertainty $\mathcal{X}(k+1)$ is contained in a single discrete mode:

$$\mathcal{X}(k+1|k) \bigcap \Omega_{i(k+1)} = \mathcal{X}(k+1|k) \tag{17}$$

This step may be ignored since it only serves as a verification of the controller constraints.

5. THE RECEDING HORIZON CONTROL STRATEGY

The set-valued observer described in section 4 computes the uncertainty polytope $\mathcal{X}(k|k-1)$ associated with the continuous state of the system, which is then used to determine the solution of the open-loop min-max robust mode optimal control Problem 1 giving the optimal input sequence $\mathbf{u}(k)$ at time instant k. Based on this solution, a Receding Horizon Control (RHC) strategy can be implemented such that state-feedback is obtained. Therefore, consider the following Model Predictive Control algorithm.

Algorithm 1. **Model Predictive Control Algorithm**.

(1) At time k, collect the measurement $y(k)$.
(2) Determine the uncertainty polytope $\mathcal{X}(k+1|k)$, and set $\mathcal{X}_{k+1} = \mathcal{X}(k+1|k)$.
(3) Solve Problem 1, and obtain the optimal input sequence $\mathbf{u}^*(k+1)$.
(4) When sampling time $k+1$ is reached, apply the first component of $\mathbf{u}^*(k+1)$ to the hybrid system, i.e. apply $u(k+1) = u^*(k+1|k+1)$.
(5) Set $k=k+1$ and go to 1.

5.1 *Robust Stability*

As the PWA system is subject to persistent disturbances the system must be steered to a target state-set. So, convergence to a final equilibrium state-set (or a desired reference trajectory tube) must be studied. Next, some important definitions to establish closed-loop robust stability are presented.

Definition 1. The pair (x_f, u_f) is said to be a *nominal equilibrium pair* of a PWA system (1) if the equilibrium state $x_f \in \mathbb{R}^{n_x}$ and equilibrium input $u_f \in \mathbb{R}^{n_u}$ satisfy:

$$x_f = A_i x_f + B_i u_f + f_i + L_i 0$$
$$\begin{bmatrix} x_f \\ u_f \\ 0 \end{bmatrix} \in \Omega_i$$

for some $i \in \mathcal{I}$.

Notice that an equilibrium pair can be computed by solving a mixed integer program. Consider also the following definitions, which can be found in e.g. (Kerrigan and Mayne, 2002) and (Blanchini, 1999).

Definition 2. A set $\mathbb{X}_f$ is *robustly stable* iff, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x(0), \mathbb{X}_f) \leq \delta$ implies $d(x(k), \mathbb{X}_f) \leq \epsilon$, $\forall k \geq 0$ and all admissible disturbance sequences (where $d(z, \mathcal{Z}) \triangleq \min_{y \in \mathcal{Z}} \|z - y\|$, such that $\mathcal{Z} \subset \mathbb{R}^n$ and $\|.\|$ denotes any norm).

Definition 3. The set $\mathbb{X}_f$ is *robustly asymptotically (finite-time) attractive* with domain of attraction $\mathbb{X}$ iff for all $x(0) \in \mathbb{X}$, $d(x(k), \mathbb{X}_f) \to 0$ as $k \to \infty$ (there exists a time M such that $x(k) \in \mathbb{X}_f$, $\forall k \geq M$) for all admissible disturbance sequences.

Definition 4. The set $\mathbb{X}_f$ is *robustly asymptotically (finite-time) stable* with domain of attraction $\mathbb{X}$ iff it is robustly stable and robustly asymptotically (finite-time) attractive with domain of attraction $\mathbb{X}$.

Definition 5. The set $\mathbb{X}_f$ is *robustly positively invariant* for the system $x(k+1) = F(x(k), w(k))$ iff $\forall x(0) \in \mathbb{X}_f$ and $\forall w(k) \in \mathbb{W}$, the system behavior is such that $x(k) \in \mathbb{X}_f$, $\forall k \in \mathbb{N}$.

Definition 6. The set $\mathbb{X}_f$ is *robustly controlled invariant* for the system $x(k+1) = F(x(k), u(k), w(k))$ iff there exists a feedback control law $u(k) = \kappa_f(x(k))$ such that $\mathbb{X}_f$ is a robust positively invariant set for the closed-loop system $x(k+1) = F(x(k), \kappa_f(x(k)), w(k))$ and $u(k) \in \mathbb{U}$, $\forall x(k) \in \mathbb{X}_f$.

In order to prove robust stability of the PWA when the RHC strategy is applied, consider the following set of assumptions regarding the stage cost $L(.)$, the terminal cost $P(.)$, and the terminal state constraint $\mathbb{X}_f$.

Assumptions 1.

a) $L(x, u)$ is a convex function over $\mathbb{X} \times \mathbb{U}$ and there exists a $c > 0$ such that $L(x, u) \geq c\,(d(x, \mathbb{X}_f))$, $\forall (x, u) \in (\mathbb{X} \backslash \mathbb{X}_f) \times \mathbb{U}$.
b) The stage cost $L(x, u) = 0$ if $x \in \mathbb{X}_f$ and $u = u_f$.
c) The terminal cost $P(x) = 0$, $\forall x \in \mathbb{R}^{n_x}$.
d) The terminal state constraint $\mathbb{X}_f \subseteq \mathbb{X}$ is a compact convex polyhedron containing the final nominal state x_f in its interior.
e) If the nominal equilibrium pair (x_f, u_f) is such that $\begin{bmatrix} x_f \\ u_f \\ 0 \end{bmatrix} \in \Omega_i$ then $\begin{bmatrix} x \\ u_f \\ w \end{bmatrix} \in \Omega_i$, $\forall x \in \mathbb{X}_f$, $\forall w \in \mathbb{W}_i$.
f) The terminal state constraint $\mathbb{X}_f$ is robustly controlled invariant for $u = u_f \in \mathbb{U}$ and $\forall w \in \mathbb{W}_i$.

Based on the previous set of assumptions, the following theorem is presented.

Theorem 1.
Consider that Assumptions 1 hold for Problem 1, and that $\mathbb{X}_N$ is a non-empty set defined by all initial states x_k such that Problem 1 is feasible at time k for $\mathcal{X}_k = x_k$. Then $\mathbb{X}_f$ is robustly asymptotically stable, with domain of attraction $\mathbb{X}_N$, for the closed-loop system when the MPC Algorithm 1 is applied.

The proof follows from considering standard Lyapunov arguments.

The set-valued observer is not explicitly referred in this stability analysis because it will only reduce the state uncertainty polytope at each time instant. If $\mathbb{X}_f$ is robustly asymptotically stable for this setting without the consideration of the set-valued observer then, it will remain robustly asymptotically stable when the observer is introduced.

6. NUMERICAL EXAMPLE

Consider the following stochastic PWA system:

$$x(k+1) = \begin{cases} \begin{bmatrix} 0.25 & -0.40 \\ 0.40 & 0.25 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(k) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \qquad \text{iff } \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \leq 0 \qquad \text{(mode 1)} \\ \begin{bmatrix} 0.25 & -0.50 \\ 0.50 & 0.25 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(k) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ \qquad \text{iff } \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) > 0 \qquad \text{(mode 2)} \end{cases}$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + v(k)$$

$$x(k) \in \mathbb{X} \triangleq \begin{bmatrix} -2 \, , \, 2 \end{bmatrix} \times \begin{bmatrix} -2 \, , \, 2 \end{bmatrix}$$
$$u(k) \in \mathbb{U} \triangleq \begin{bmatrix} -1 \, , \, 1 \end{bmatrix}$$
$$w(k) \in \mathbb{W} \triangleq \begin{bmatrix} -0.1 \, , \, 0.1 \end{bmatrix} \text{, Uniformly distributed}$$
$$v(k) \in \mathbb{V} \triangleq \begin{bmatrix} -0.1 \, , \, 0.1 \end{bmatrix} \text{, Uniformly distributed}$$

The uncertainty associated with the initial state is defined as:

$$x(0) \in \mathcal{X}(0) \triangleq \begin{bmatrix} 1.3 \, , \, 1.5 \end{bmatrix} \times \begin{bmatrix} 1.5 \, , \, 1.7 \end{bmatrix} \qquad (18)$$

The objective of the control scheme is to move the state and the corresponding uncertainty polytope into the final set:

$$\mathbb{X}_f \triangleq \begin{bmatrix} 0.1 \, , \, 0.9 \end{bmatrix} \times \begin{bmatrix} -0.5 \, , \, -1.3 \end{bmatrix} \qquad (19)$$

The control horizon is chosen as $N = 3$, the weights of the controller are $P_x = I_{n_x}$, $P_u = 10 I_{n_u}$ and the norm l is the 1-norm. The solution of the initial optimal control problem is represented in figure 1, while the MPC solution is shown in figure 2.

To study the influence of the estimation on the proposed closed-loop scheme, the previous MPC solution will be recomputed considering the following characteristics of the measurement noise:

$$v(k) \in \mathbb{V} \triangleq \begin{bmatrix} -0.01 \, , \, 0.01 \end{bmatrix} \text{, Uniformly distributed} \qquad (20)$$

The new MPC solution is represented in figure 3.

Comparing figures 2 and 3, it is clear that the state uncertainty is reduced in the last case, having a direct impact on the controller performance. This influence reflects itself in the cost function values of the complete trajectory, which are presented in figure 4.

As can be seen in figure 4 there is a sensible reduction in the cost function of the trajectory defined by the MPC when the measurement noise is reduced, showing the benefits of introducing a state estimator in the control loop. For very wide intervals of the measurement noise these benefits are reduced to the point where the estimator becomes irrelevant.

A quantitative measure of the cost functional reduction can not be determined *a priori* since it depends on the actual input disturbances and measurement noises

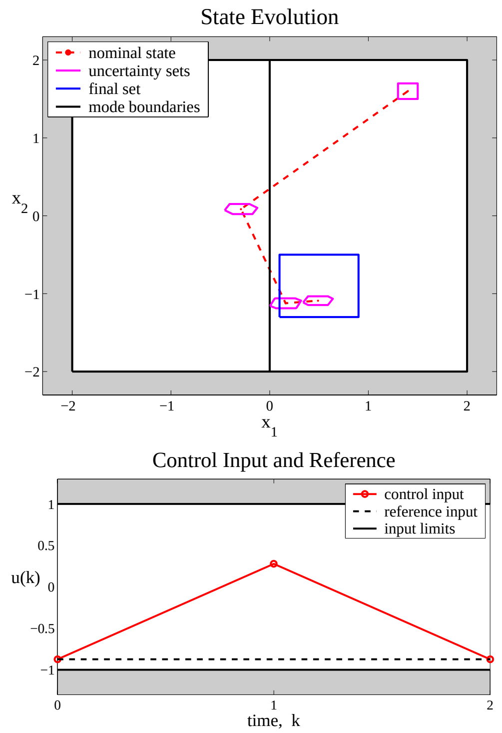

Fig. 1. Solution of the initial optimal control problem.

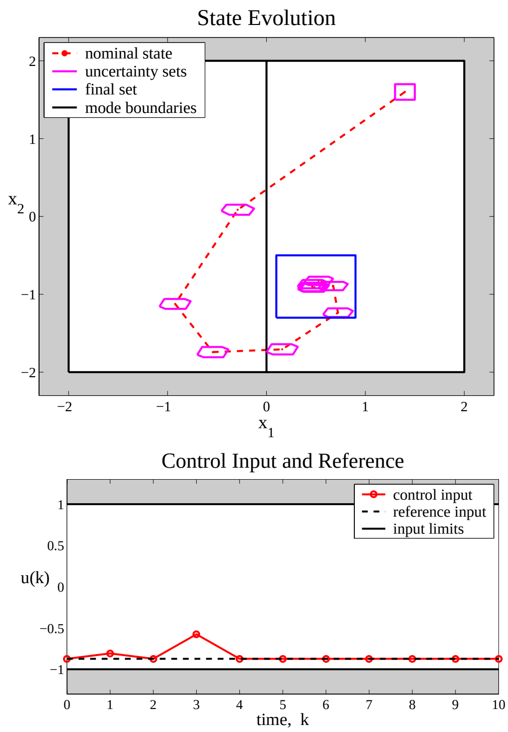

Fig. 2. Solution of the MPC problem with, $\mathbb{V} \triangleq \begin{bmatrix} -0.1 \, , \, 0.1 \end{bmatrix}$.

that acted on the system. Further, if the actual input disturbances are the ones predicted in the worst case scenario, no reduction of the cost functional occurs.

7. CONCLUSIONS

This paper presented the first closed-loop control scheme for stochastic hybrid systems. The state esti-

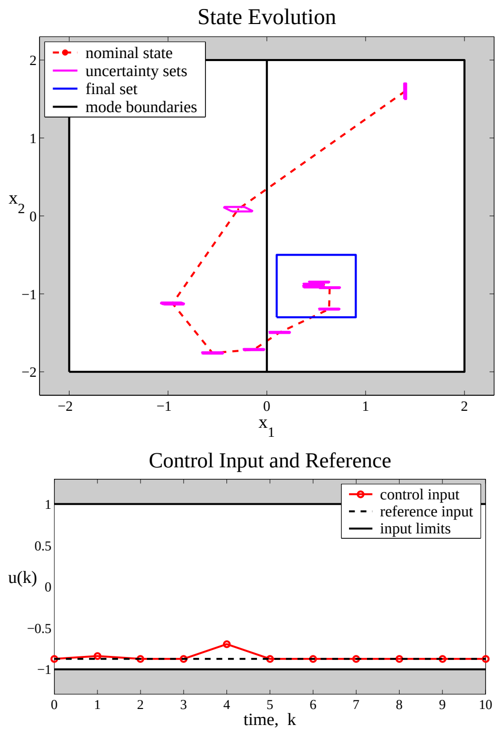

Fig. 3. Solution of the MPC problem with, $\mathbb{V} \triangleq \left[-0.01\ ,\ 0.01\right]$.

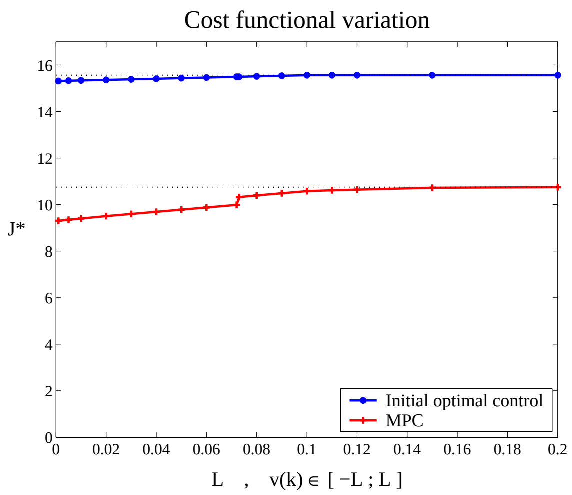

Fig. 4. Optimal cost function values of the initial optimal control and MPC for different measurement noise intervals.

mator reduces, at each time instant, the uncertainty in the continuous state, leading to an improvement of the controller performance. This improvement depends on the actual input disturbance and measurement noise instances, and may be zero in a worst case scenario.

Other MPC methodologies for hybrid systems will be considered in future research aiming at more significant performance improvements.

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