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On the Optimal Control of a Class of Non-Newtonian Fluids

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Abstract We consider optimal control problems of systems governed by stationary, incompressible generalized Navier-Stokes equations with shear dependent viscosity in a two-dimensional or three-dimensional domain. We study a general class of viscosity functions including shear-thinning and shear-thickening behavior. We prove an existence result for such class of optimal control problems.

Keywords Optimal control · electro-rheological fluids · shear-thinning · shear-thickening.

Mathematics Subject Classification (2000) 49K20 · 76D55 · 76A05

1 Introduction

This paper is devoted to the proof of the existence of steady solutions for a distributed optimal control problem of a viscous and incompressible fluid. The control and state variables are constrained to satisfy a generalized Navier-Stokes system of equations with shear dependent viscosity which switches from shear-thinning to shear-thickening behavior. More specifically, we deal with the following generalization of the stationary Navier-Stokes system

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$$\begin{cases} -div\ (S(D\mathbf{y})) + \mathbf{y}\cdot\nabla\mathbf{y} + \nabla p = \mathbf{u} & \text{in } \Omega \\ div\,\mathbf{y} = 0 & \text{in } \Omega \\ \mathbf{y} = 0 & \text{on } \partial\Omega\ , \end{cases} \tag{1}$$

where S is the extra stress tensor given by

$$S(\eta) = (1 + |\eta|)^{\alpha(x)-2}\eta$$

and $\alpha(x)$ is a positive bounded continuous function. The vector $\mathbf{y}$ denotes the velocity field, p is the pressure, $D\mathbf{y} = \frac{1}{2}\big(\nabla\mathbf{y} + (\nabla\mathbf{y})^T\big)$ is the symmetric part of the velocity gradient, $\mathbf{u}$ is the given body force and Ω is an open bounded subset of $I\!R^n$ $(n = 2$ or $n = 3)$.

System (1) can be used to model steady incompressible electro-rheological fluids. It is based on the assumption that electro-rheological materials, composed by suspensions of particles in a fluid, can be considered as a homogenized single continuum media. The corresponding viscosity has the property of switching between shear-thinning and shear-thickening behavior under the application of a magnetic field. This model is described and analyzed in [13], [14], [15] or [7]. More recently, in [5], the authors proved the existence and uniqueness of a $C^{1,\gamma}(\bar{\Omega}) \cap \mathbf{W}^{2,2}(\Omega)$ solution under smallness data conditions for system 1. This regularity result was the motivation for the analysis of the associated distributed optimal control problem that we describe in what follows.

Let us look for the control $\mathbf{u}$ and the corresponding $\mathbf{y}_u$ solution of (1) such that the pair $(\mathbf{u}, \mathbf{y}_u)$ solves

$$(P_\alpha)\begin{cases} \text{Minimize } J(\mathbf{u}, \mathbf{y}_u) \\ \text{subject to (1)} \end{cases} \tag{2}$$

where the functional $J : \mathbf{L}^2(\Omega) \times \mathbf{W}_0^{1,2}(\Omega) \to I\!R$ is given by

$$J(\mathbf{u}, \mathbf{y}) = \frac{1}{2}\int_\Omega |\mathbf{y}_u - \mathbf{y}_d|^2\,dx + \frac{\nu}{2}\int_\Omega |\mathbf{u}|^2\,dx. \tag{3}$$

and $\mathbf{y}_d$ denotes a fixed element of $\mathbf{L}^2(\Omega)$.

Such type of optimal control problems has been a subject of intensive research in the past decades. For non-Newtonian fluid equations we mention the results in [1], [3], [4], [8], [9], [12] and [16] where the authors used several techniques to deal properly with the shear-thinning and shear-thickening viscosity laws, defined both in 2D and 3D domains. For the existence of solution, such techniques consist in exploring correctly the properties of the tensor S in order to establish compactness results necessary for the application of the direct method of the Calculus of Variations. Our purpose here is to show that, based on the regularity results obtained in [5], it is possible to easily extend those techniques to the case of electro-rheological fluids modeled by (1). Treating the optimality conditions associated to problem (P_α) is also an important, yet delicate, issue. We will therefore study this problem in a forthcoming work.

In section 2 we introduce the notation that we are going to use, and recall some useful results. In section 3 we characterize the tensor S including its continuity, coercivity and monotonicity properties. Finally, in section 4 we prove the main existence result of this paper.

2 Notation and classical results

We denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support in Ω, $\mathcal{D}'(\Omega)$ denotes its dual (the space of distributions). The standard Sobolev spaces are represented by $\mathbf{W}^{k,\alpha}(\Omega)$ ($k \in I\!N$ and $1 < \alpha < \infty$), and their norms by $\|\cdot\|_{k,p}$. We set $\mathbf{W}^{0,\alpha}(\Omega) \equiv \mathbf{L}^{\alpha}(\Omega)$ and $\|\cdot\|_{\alpha} \equiv \|\cdot\|_{L^\alpha}$. The dual space of $\mathbf{W}_0^{1,\alpha}(\Omega)$ is denoted by $\mathbf{W}^{-1,\alpha'}(\Omega)$ and its norm by $\|\cdot\|_{-1,\alpha'}$. We consider the space of divergence free functions defined by

$$\mathcal{V} = \big\{\psi \in \mathcal{D}(\Omega) \mid \nabla \cdot \psi = 0\big\},$$

to eliminate the pressure in the weak formulation. The space $\mathbf{V}_\alpha$ is the closure of $\mathcal{V}$ with respect to the gradient norm, i.e.

$$\mathbf{V}_\alpha = \left\{\psi \in \mathbf{W}_0^{1,\alpha}(\Omega) \mid \nabla \cdot \psi = 0\right\}.$$

The space of Hölder continuous functions is a Banach space defined as

$$C^{m,\gamma}(\bar{\Omega}) \equiv \{\mathbf{y} \in C^m(\bar{\Omega}) : \|\mathbf{y}\|_{C^{m,\gamma}(\bar{\Omega})} < \infty\}$$

where

$$\|\mathbf{y}\|_{C^{m,\gamma}(\bar{\Omega})} \equiv \sum_{|\alpha|=0}^{m} \|D^\alpha \mathbf{y}\|_\infty + [\mathbf{y}]_{C^{m,\gamma}(\bar{\Omega})}, \tag{4}$$

and the semi-norm is denoted by

$$[\mathbf{y}]_{C^{m,\gamma}}(\bar{\Omega}) \equiv \sum_{|\alpha|=m} \sup_{\{x_1,x_2 \in \bar{\Omega}, x_1 \neq x_2\}} \frac{|D^\alpha \mathbf{y}(x_1) - D^\alpha \mathbf{y}(x_2)|}{|x_1 - x_2|^\gamma} < +\infty,$$

for m a nonnegative integer, $0 < \gamma < 1$ and where

$$D^\alpha \mathbf{y} \equiv \frac{\partial^{|\alpha|}\mathbf{y}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}},$$

with $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_i \in I\!N_0$ and $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

Now we recall two fundamental classical inequalities.

Lemma 1 (Poincaré's inequality) *Let* $\mathbf{y} \in \mathbf{W}_0^{1,\alpha}(\Omega)$ *with* $1 \leq \alpha < +\infty$. *There exists a constant* C_1 *depending on* α *and* Ω *such that*

$$\|\mathbf{y}\|_\alpha \leq C_1(\alpha, \Omega)\|\nabla \mathbf{y}\|_\alpha.$$

Proof. Found in [2].

Lemma 2 (Korn's inequality) *Let* $\mathbf{y} \in \mathbf{W}_0^{1,\alpha}(\Omega)$ *with* $1 < \alpha < +\infty$. *There exists a constant* C_2 *depending on* Ω *such that*

$$C_2(\Omega)\|\mathbf{y}\|_{1,\alpha} \leq \|D\mathbf{y}\|_\alpha.$$

Proof. Found in [11].

Finally two simple, yet very useful, properties of the convective term.

Lemma 3 *Let us consider* $\mathbf{u}$ *in* $\mathbf{V}_2$ *and* $\mathbf{v}$, $\mathbf{w}$ *in* $\mathbf{W}_0^{1,2}(\Omega)$. *Then*

$$(\mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{w}) = -(\mathbf{u}\cdot\nabla\mathbf{w}, \mathbf{v}) \quad \textit{and} \quad (\mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{v}) = 0. \tag{5}$$

3 Properties of the extra tensor S

Denoting by $I\!R^{n\times n}_{sym}$ the set of all symmetric $n \times n$ matrices, we assume that tensor $S : I\!R^{n\times n}_{sym} \longrightarrow I\!R^{n\times n}_{sym}$ has a potential, i.e., there exists a function $\Phi \in C^2(I\!R^+_0, I\!R^+_0)$ with $\Phi(0) = 0$ such that

$$S_{ij}(\eta) = \frac{\partial\Phi(|\eta|^2)}{\partial\eta_{ij}} = 2\Phi'(|\eta|^2)\,\eta_{ij}, \qquad S(0) = 0$$

for all $\eta \in I\!R^{n\times n}_{sym}$. An example of such tensor is the one we are going to work with, namely

$$S(\eta) = (1 + |\eta|)^{\alpha(x)-2}\eta,$$

where $\alpha(x)$ is a continuous function in $\bar{\Omega}$ such that

$$\alpha(x) : \bar{\Omega} \to (1, +\infty)$$

and

$$1 < \alpha_0 \leq \alpha(x) \leq \alpha_\infty < +\infty \tag{6}$$

$$\min \alpha(x) = \alpha_0,$$

$$\max \alpha(x) \leq \alpha_\infty, \quad \text{for} \quad \alpha_\infty > 2.$$

The case $1 < \alpha(x) < 2$ corresponds to shear-thinning viscosity fluids, while $\alpha(x) > 2$ corresponds to shear-thickening fluids. The case $\alpha(x) = 2$ corresponds to a Newtonian fluid. For such function $\alpha(x)$, it can be proved that tensor S satisfies standard properties.

Proposition 1 *Consider $\alpha(x) \in (1, \infty)$, C_3 and C_4 positive constants. Then the following inequalities hold*

A1 -

$$\left| \frac{\partial S_{k\ell}(\eta)}{\partial \eta_{ij}} \right| \leq C_3 \left(1 + |\eta|\right)^{\alpha(x)-2}.$$

A2 -

$$S'(\eta) : \zeta : \zeta = \sum_{ijk\ell} \tfrac{\partial S_{k\ell}(\eta)}{\partial \eta_{ij}} \zeta_{k\ell} \zeta_{ij} \geq C_4 \left(1 + |\eta|\right)^{\alpha(x)-2} |\zeta|^2$$

for all $\eta, \zeta \in I\!R^{n\times n}_{sym}$ and $i, j, k, \ell = 1, \cdots, d$.

Proof. In fact,

$$\begin{aligned} \left| \frac{\partial S_{kl}}{\partial \eta_{ij}} \right| &= \left| (\alpha(x) - 2) \left(1 + |\eta|\right)^{\alpha(x)-3} \frac{\eta_{ij}}{|\eta|} \eta_{kl} + \left(1 + |\eta|\right)^{\alpha(x)-2} \delta_{ik} \delta_{jl} \right| \\ &\leq |\alpha(x) - 2| \left(1 + |\eta|\right)^{\alpha(x)-3} \frac{|\eta_{ij} \eta_{kl}|}{|\eta|} + \left(1 + |\eta|\right)^{\alpha(x)-2} |\delta_{ik} \delta_{jl}|. \end{aligned} \quad (7)$$

Taking into account that

$$\delta_{ik} \delta_{jl} = \begin{cases} 1 \text{ if } i = k, j = l \\ 0 \quad \text{otherwise} \end{cases}$$

and

$$|\eta_{ij} \eta_{kl}| \leq |\eta|^2,$$

we can write

$$\begin{aligned} (7) &\leq |\alpha(x) - 2| \left(1 + |\eta|\right)^{\alpha(x)-3} |\eta| + \left(1 + |\eta|\right)^{\alpha(x)-2} \\ &\leq |\alpha(x) - 2| \left(1 + |\eta|\right)^{\alpha(x)-3} \left(1 + |\eta|\right) + \left(1 + |\eta|\right)^{\alpha(x)-2} \\ &= |\alpha(x) - 2| \left(1 + |\eta|\right)^{\alpha(x)-2} + \left(1 + |\eta|\right)^{\alpha(x)-2} \\ &= \left(|\alpha(x) - 2| + 1\right) \left(1 + |\eta|\right)^{\alpha(x)-2}. \end{aligned} \quad (8)$$

If $\alpha(x) - 2 \geq 0$, we have

$$\begin{aligned}(8) &= (\alpha(x)-1)\,(1+|\eta|)^{\alpha(x)-2} \\ &\le (\alpha_\infty - 1)\,(1+|\eta|)^{\alpha(x)-2}.\end{aligned}$$

Otherwise, if $\alpha(x)-2<0$, we have

$$\begin{aligned}(8) &= (3-\alpha(x))\,(1+|\eta|)^{\alpha(x)-2} \\ &\le (3-\alpha_0)\,(1+|\eta|)^{\alpha(x)-2},\end{aligned}$$

and therefore we have

$$\left|\frac{\partial S_{k\ell}(\eta)}{\partial \eta_{ij}}\right| \le \begin{cases} (3-\alpha_0)\,(1+|\eta|)^{\alpha(x)-2} & \text{if } \alpha(x)-2<0 \\ (\alpha_\infty-1)\,(1+|\eta|)^{\alpha(x)-2} & \text{if } \alpha(x)-2\ge 0. \end{cases}$$

This proves inequality **A1**.

In order to obtain **A2**, we write

$$\begin{aligned} S'(\eta):\zeta:\zeta &= \sum_{ijk\ell} \tfrac{\partial S_{k\ell}(\eta)}{\partial \eta_{ij}} \zeta_{k\ell}\zeta_{ij} \\ &= \sum_{ijk\ell}\left[(\alpha(x)-2)\,(1+|\eta|)^{\alpha(x)-3}\,\frac{\eta_{ij}\eta_{kl}}{|\eta|} + (1+|\eta|)^{\alpha(x)-2}\,\delta_{ik}\delta_{jl}\right]\zeta_{ij}\zeta_{k\ell} \\ &= (\alpha(x)-2)\frac{(1+|\eta|)^{\alpha(x)-3}}{|\eta|}\sum_{ijk\ell}\eta_{ij}\eta_{kl}\zeta_{ij}\zeta_{k\ell} + (1+|\eta|)^{\alpha(x)-2}\sum_{ijk\ell}\delta_{ik}\delta_{jl}\zeta_{ij}\zeta_{k\ell}. \end{aligned} \tag{9}$$

Considering that

$$\sum_{ijk\ell}\eta_{ij}\eta_{kl}\zeta_{ij}\zeta_{k\ell} = \sum_{ij}\eta_{ij}\zeta_{ij}\sum_{kl}\eta_{kl}\zeta_{kl} = |\eta:\zeta|^2$$

and

$$\sum_{ijk\ell}\delta_{ik}\delta_{jl}\zeta_{ij}\zeta_{k\ell} = \sum_{ij}\zeta_{ij}\zeta_{ij} = |\zeta|^2,$$

expression (9) is equal to

$$(\alpha(x)-2)\frac{(1+|\eta|)^{\alpha(x)-3}}{|\eta|}|\eta:\zeta|^2 + (1+|\eta|)^{\alpha(x)-2}\,|\zeta|^2. \tag{10}$$

Taking into account that $\alpha(x)-2<0$, $\alpha_0 \le \alpha(x)$ and $|\eta : \zeta|^2 \le |\eta|^2|\zeta|^2$, it follows

$$
\begin{aligned}
(10) &\ge (\alpha(x)-2)\frac{(1+|\eta|)^{\alpha(x)-3}}{|\eta|}|\eta|^2|\zeta|^2 + (1+|\eta|)^{\alpha(x)-2}\,|\zeta|^2 \\
&= \Big((\alpha(x)-2)(1+|\eta|)^{\alpha(x)-3}|\eta| + (1+|\eta|)^{\alpha(x)-2}\Big)\,|\zeta|^2 \\
&\ge \Big((\alpha(x)-2)(1+|\eta|)^{\alpha(x)-3}(1+|\eta|) + (1+|\eta|)^{\alpha(x)-2}\Big)\,|\zeta|^2 \\
&= \Big((\alpha(x)-2)(1+|\eta|)^{\alpha(x)-2} + (1+|\eta|)^{\alpha(x)-2}\Big)\,|\zeta|^2 \\
&= (\alpha(x)-1)(1+|\eta|)^{\alpha(x)-2}|\zeta|^2 \\
&\ge (\alpha_0-1)(1+|\eta|)^{\alpha(x)-2}|\zeta|^2
\end{aligned}
$$

Instead, $\alpha(x)-2\ge 0$ gives

$$
\begin{aligned}
(10) &\ge (1+|\eta|)^{\alpha(x)-3}\left((\alpha(x)-2)\frac{|\eta:\zeta|^2}{|\eta|} + (1+|\eta|)|\zeta|^2\right) \\
&\ge (1+|\eta|)^{\alpha(x)-3}\,(1+|\eta|)|\zeta|^2 \\
&= (1+|\eta|)^{\alpha(x)-2}\,|\zeta|^2.
\end{aligned}
$$

Then we have

$$
S'(\eta):\zeta:\zeta \ge \begin{cases} (\alpha_0-1)\,(1+|\eta|)^{\alpha(x)-2}\,|\zeta|^2 & \text{if } \alpha(x)-2<0 \\ (1+|\eta|)^{\alpha(x)-2}\,|\zeta|^2 & \text{if } \alpha(x)-2\ge 0. \end{cases}
$$

This proves inequality **A2**.

Assumptions **A1-A2** imply the following standard properties for S (see [11]).

Proposition 2 *In the same conditions we have*

1. *Continuity*
$$|S(\eta)| \le (1+|\eta|)^{\alpha(x)-2}|\eta|, \tag{11}$$
2. *Coercivity*
$$S(\eta):\eta \ge \begin{cases} \nu(1+|\eta|)^{\alpha(x)-2}|\eta|^2, & \textit{if } \alpha(x)-2<0 \\ |\eta|^2, & \textit{if } \alpha(x)-2\ge 0 \end{cases} \tag{12}$$
3. *Monotonicity*
$$(S(\eta)-S(\zeta)):(\eta-\zeta) \ge \nu(1+|\eta|+|\zeta|)^{\alpha(x)-2}|\eta-\zeta|^2. \tag{13}$$

Proof. Continuity is trivially derived from

$$|S(\eta)| = \left|(1+|\eta|)^{\alpha(x)-2}\eta\right| = \left|(1+|\eta|)^{\alpha(x)-2}\right| |\eta| = (1+|\eta|)^{\alpha(x)-2}|\eta|.$$

Coercivity is equivalent to monotonocity taking $S(\zeta) = \zeta = 0_M$. Therefore it is enough to prove monotonocity. Taking into account that

$$\begin{aligned} S_{ij}(\eta) - S_{ij}(\zeta) &= \int_0^1 \frac{\partial}{\partial t} S_{ij}(t\eta + (1-t)\zeta)\, dt \\ &= \int_0^1 \sum_{kl} \frac{\partial S_{ij}(t\eta + (1-t)\zeta)}{\partial D_{kl}} (\eta-\zeta)_{kl} \end{aligned}$$

we can write

$$\begin{aligned} (S(\eta) - S(\zeta) : (\eta - \zeta) &= \int_0^1 \sum_{ij} \sum_{kl} \frac{\partial S_{ij}(t\eta + (1-t)\zeta)}{\partial D_{kl}} (\eta-\zeta)_{kl} : (\eta-\zeta)_{ij}\, dt \\ &= \int_0^1 S'(t\eta + (1-t)\zeta)) : (\eta-\zeta) : (\eta-\zeta)\, dt \qquad (14) \end{aligned}$$

Using **A2** and considering $\alpha(x) - 2 < 0$ we have

$$(14) \geq \int_0^1 (\alpha_0 - 1)\left(1 + |t\eta + (1-t)\zeta|\right)^{\alpha(x)-2} |\eta - \zeta|^2\, dt \qquad (15)$$

Since $t \in [0,1]$ then

$$1 + |t\eta + (1-t)\zeta| \leq 1 + |\eta + \zeta| \leq 1 + |\eta| + |\zeta| \qquad (16)$$

and therefore, we can write

$$(15) \geq \int_0^1 (\alpha_0 - 1)\left((1 + |\eta| + |\zeta|)\right)^{\alpha(x)-2} |\eta - \zeta|^2\, dt$$

Using **A2** and considering $\alpha(x) - 2 \geq 0$ we show that

$$(14) \geq \int_0^1 (1+|t\eta+(1-t)\zeta|)^{\alpha(x)-2}\,|\eta-\zeta|^2\,dt$$
$$\geq \int_0^1 1^{\alpha(x)-2}|\eta-\zeta|^2\,dt$$
$$\geq |\eta-\zeta|^2. \tag{17}$$

Then we conclude

$$(S(\eta)-S(\zeta)):(\eta-\zeta) \geq \begin{cases} (\alpha_0-1)(1+|\eta|+|\zeta|)^{\alpha(x)-2}|\eta-\zeta|^2 & \text{if } \alpha(x)-2<0 \\ |\eta-\zeta|^2 & \text{if } \alpha(x)-2\geq 0. \end{cases}$$

4 Main Result

Definition 1 Assume that $\mathbf{u}\in\mathbf{L}^2(\Omega)$. A function $\mathbf{y}$ is a $C^{1,\gamma}$-solution of (1) if $\mathbf{y}\in C^{1,\gamma}(\bar{\Omega})$, for $\gamma\in(0,1)$, $div\,\mathbf{y}=0$, $\mathbf{y}|_{\partial\Omega}=0$ and it satisfies the following equality

$$(S(D\mathbf{y}),D\varphi)+(\mathbf{y}\cdot\nabla\mathbf{y},\varphi)=(\mathbf{u},\varphi), \quad \text{for all } \varphi\in\mathbf{V}_2, \tag{18}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$.

Next proposition, due to [5], presents an existence and uniqueness result of a C^{1,γ_0} solution of system (1) with certain conditions imposed to $\mathbf{u}$, but with no aditional conditions on the exponent α.

Proposition 3 *We assume that $\mathbf{u}\in\mathbf{L}^q(\Omega)$, for some $q>n$. Let us consider Ω a C^{1,γ_0} domain, and $\alpha\in C^{0,\gamma_0}(\bar{\Omega})$, with $\gamma_0=1-\frac{n}{q}$. Then, for any $\gamma<\gamma_0$, there exist positive constants C_5 and C_6, depending on $\|\alpha\|_{C^{0,\gamma}(\bar{\Omega})}$, n, q and Ω such that, if $\|\mathbf{u}\|_q<C_5$, there exists a $C^{1,\gamma}$ solution $(\mathbf{y},p)$ of problem (1) verifying*

$$\|\mathbf{y}\|_{C^{1,\gamma}(\bar{\Omega})}+\|p\|_{C^{0,\gamma}(\bar{\Omega})}\leq C_6\|\mathbf{u}\|_q. \tag{19}$$

Furthermore, there exists a constant C_7 depending on α_0, $\|\alpha\|_{C^{0,\gamma}(\bar{\Omega})}$, n, q and Ω such that, if $\|\mathbf{u}\|_q\leq C_7$, the solution is unique.

Proposition 4 *Assume that properties* **A1** *and* **A2** *are fulfilled. Considering $\mathbf{y}\in C^{1,\gamma}(\bar{\Omega})$ we have*

$$\|D\mathbf{y}\|_2\leq C_8\|\mathbf{u}\|_2, \tag{20}$$

where $\mathbf{y}$ is the associated state to $\mathbf{u}$.

Proof. Taking $\varphi = \mathbf{y}$ in (18) and recalling the convective term properties, we have

$$(S(D\mathbf{y}), D\mathbf{y}) = (\mathbf{u}, \mathbf{y}). \tag{21}$$

Since $\mathbf{y} \in C^{1,\gamma}(\bar{\Omega})$ then $\mathbf{y}$ belongs to $C^1(\bar{\Omega})$ which means that $\mathbf{y}$ and $D\mathbf{y}$ are bounded functions in $\bar{\Omega}$ and consequently belong to $\mathbf{L}^\alpha(\Omega)$, for any $\alpha > 1$. In particular we consider $\mathbf{y} \in \mathbf{L}^2(\Omega)$. Then, by using Hölder's inequality and the Poincaré and Korn inequalities there exists a constant C_8 such that

$$|(\mathbf{u}, \mathbf{y})| \leq \|\mathbf{u}\|_2 \|\mathbf{y}\|_2 \leq C_8 \|\mathbf{u}\|_2 \|D\mathbf{y}\|_2.$$

On the other hand, by coercivity we write

$$\|D\mathbf{y}\|_2^2 \leq (S(D\mathbf{y}), D\mathbf{y}).$$

Putting together both inequalities with (21) we prove the pretended result.

Once we have the guarantee of existence of a solution of problem (1) provided by Proposition 3 and an estimative of energy for $D\mathbf{y}$ given by proposition 4, we can now formulate and prove the following existence result for the control problem (P_α).

Theorem 1 (Main Result) *Assume that* **A1**-**A2** *are fulfilled. Then* (P_α) *admits at least one solution.*

To prove this theorem we need to establish some important results.

Proposition 5 *Assume that* $(\mathbf{u}_k)_{k>0}$ *converges to* $\mathbf{u}$ *weakly in* $\mathbf{L}^2(\Omega)$. *Let* $\mathbf{y}_k$ *be the associated state to* $\mathbf{u}_k$. *Then there exists* $\mathbf{y} \in \mathbf{W}_0^{1,2}(\Omega)$ *and* $\tilde{S} \in \mathbf{L}^2(\Omega)$ *such that the following convergences are verified*

$$(\mathbf{y}_k)_k \rightharpoonup \mathbf{y} \quad in \quad \mathbf{W}_0^{1,2}(\Omega) \tag{22}$$

$$(D\mathbf{y}_k)_k \rightharpoonup D\mathbf{y} \quad in \quad \mathbf{L}^2(\Omega) \tag{23}$$

$$(S(D\mathbf{y}_k))_k \rightharpoonup \tilde{S} \quad in \quad \mathbf{L}^2(\Omega). \tag{24}$$

Proof. The convergence of $(\mathbf{u}_k)_{k>0}$ to $\mathbf{u}$ in the weak topology of $\mathbf{L}^2(\Omega)$ implies that $(\mathbf{u}_k)_{k>0}$ is bounded, i.e, there exists a positive constant M such that

$$\|\mathbf{u}_k\|_2 \leq M, \quad \text{for } k > k_0. \tag{25}$$

Due to (20) and (25), it follows that

$$\|D\mathbf{y}_k\|_2 \leq C_8 M.$$

By Korn's inequality $\mathbf{y}_k$ is then bounded in $\mathbf{W}_0^{1,2}(\Omega)$ and thus there is a subsequence still indexed in k that weakly converges to a certain $\mathbf{y}$ in $\mathbf{W}_0^{1,2}(\Omega)$. Moreover, by using a Sobolev's compact injection, $\mathbf{y}_k$ converges strongly (then also weakly) to $\mathbf{y}$ in $\mathbf{L}^2(\Omega)$. It is straightforward to conclude (23).
Finally, the previous estimate, together with (11) implies

$$\begin{aligned}\|S(D\mathbf{y}_k)\|_2^2 &\le \int_\Omega (1+|D\mathbf{y}_k|)^{(\alpha(x)-2)2}|D\mathbf{y}_k|^2\,dx\\ &\le \int_\Omega (1+|D\mathbf{y}_k|)^{(\alpha(x)-2)2}(1+|D\mathbf{y}_k|)^2\,dx\\ &= \int_\Omega (1+|D\mathbf{y}_k|)^{2(\alpha(x)-1)}\,dx\\ &\le C_8\int_\Omega (1+|D\mathbf{y}_k|^{2(\alpha(x)-1)})\,dx\\ &\le C_8\left(|\Omega|+\int_\Omega |D\mathbf{y}_k|^{2(\alpha_\infty-1)}\,dx\right)\\ &= C_8\left(|\Omega|+\|D\mathbf{y}_k\|_{2(\alpha_\infty-1)}^{2(\alpha_\infty-1)}\right).\end{aligned}$$

This last expression is bounded once $D\mathbf{y}_k \in C(\bar{\Omega})$ and consequently the sequence $(S(D\mathbf{y}_k))_k$ is bounded in $\mathbf{L}^2(\Omega)$ and we finish the proof by establishing the existence of a subsequence, still indexed by k, and $\tilde{S} \in \mathbf{L}^2(\Omega)$ such that $(S(D\mathbf{y}_k))_{k>0}$ weakly converges to $\tilde{S} \in \mathbf{L}^2(\Omega)$.

Proposition 6 *Assume that (23), (22) and (24) are verified. Then the weak limit of $(\mathbf{y}_k)_k$, $\mathbf{y}$, is the solution of (18) corresponding to $\mathbf{u} \in \mathbf{L}^2(\Omega)$.*

Proof. Let us consider

$$(S(D\mathbf{y}_k)-S(\mathbf{y}),D\varphi)+(\mathbf{y}_k\cdot\nabla\mathbf{y}_k-\mathbf{y}\cdot\nabla\mathbf{y},\varphi)=(\mathbf{u}_k-\mathbf{u},\varphi), \tag{26}$$

for all $\varphi \in \mathbf{V}_2$. Taking into account the convective term properties and the regularity results assumed on $\mathbf{y}$, we have

$$\begin{aligned}|(\mathbf{y}_k\cdot\nabla\mathbf{y}_k-\mathbf{y}\cdot\nabla\mathbf{y},\varphi)&=|((\mathbf{y}_k-\mathbf{y})\cdot\nabla\mathbf{y}_k,\varphi)+(\mathbf{y}\cdot\nabla(\mathbf{y}_k-\mathbf{y}),\varphi)|\\ &=|((\mathbf{y}_k-\mathbf{y})\cdot\nabla\mathbf{y}_k,\varphi)-(\mathbf{y}\cdot\nabla\varphi,(\mathbf{y}_k-\mathbf{y}))|\\ &\le|((\mathbf{y}_k-\mathbf{y})\cdot\nabla\mathbf{y}_k,\varphi)|+|(\mathbf{y}\cdot\nabla\varphi,(\mathbf{y}_k-\mathbf{y}))|\\ \le C_E^2\left(\|\nabla\mathbf{y}_k\|_2\|\varphi\|_4+\|\mathbf{y}\|_4\|\nabla\varphi\|_2\right)\|\mathbf{y}_k-\mathbf{y}\|_4 &\to 0, \quad \text{when} \quad k\to+\infty.\end{aligned}$$

This result is a consequence of the compact injection of $\mathbf{W}_0^{1,2}(\Omega)$ into $\mathbf{L}^4(\Omega)$ which provide a strong convergence in $\mathbf{L}^4(\Omega)$ due to (22). Note that C_E corresponds to the embedding constant.

Hence, passing to the limit in

$$(S(D\mathbf{y}_k), D\varphi) + (\mathbf{y}_k \cdot \nabla \mathbf{y}_k, \varphi) = (\mathbf{u}_k, \varphi), \quad \text{for all} \quad \varphi \in \mathbf{V}_2,$$

we obtain

$$(\tilde{S}, D\varphi) + (\mathbf{y} \cdot \nabla \mathbf{y}, \varphi) = (\mathbf{u}, \varphi), \quad \text{for all} \quad \varphi \in \mathbf{V}_2, \tag{27}$$

In particular, taking $\varphi = \mathbf{y}$ and considering (5) we may write

$$(\tilde{S}, D\mathbf{y}) = (\tilde{S}, D\mathbf{y}) + (\mathbf{y} \cdot \nabla \mathbf{y}, \mathbf{y}) = (\mathbf{u}, \mathbf{y}). \tag{28}$$

On the other hand, the monotonocity assumption (13) gives

$$(S(D\mathbf{y}_k) - S(D\varphi), D(\mathbf{y}_k) - D\varphi) \geq 0, \quad \text{for all} \quad \varphi \in \mathbf{V}_2. \tag{29}$$

Since,

$$(S(D\mathbf{y}_k), D\mathbf{y}_k) = (\mathbf{u}_k, \mathbf{y}_k),$$

replacing the first member in (29), we obtain

$$(\mathbf{u}_k, \mathbf{y}_k) - (S(D\mathbf{y}_k), D\varphi) - (S(D\varphi), D\mathbf{y}_k - D\varphi) \geq 0, \quad \text{for all} \quad \varphi \in \mathbf{V}_2.$$

Passing to the limit it follows

$$(\mathbf{u}, \mathbf{y}) - (\tilde{S}, D\varphi) - (\tau(D\varphi), D\mathbf{y} - D\varphi) \geq 0, \quad \text{for all} \quad \varphi \in \mathbf{V}_2.$$

This inequality together with (28), implies that

$$(\tilde{S} - S(D\varphi), D\mathbf{y} - D\varphi) \geq 0, \quad \text{for all} \quad \varphi \in \mathbf{V}_2$$

Taking $\varphi = \mathbf{y} - \lambda \mathbf{v}$ (see [10]), which is possible considering any $\mathbf{v} \in \mathbf{V}_2$ and $\lambda > 0$, we have

$$(\tilde{S} - S(D(\mathbf{y} - \lambda \mathbf{v})), D\mathbf{y} - D(\mathbf{y} - \lambda \mathbf{v})) \geq 0, \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}_2 \tag{30}$$

which is equivalent to

$$\lambda(\tilde{S} - S(D(\mathbf{y} - \lambda \mathbf{v})), D\mathbf{v})) \geq 0, \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}_2 \tag{31}$$

and then, since $\lambda > 0$, it comes

$$(\tilde{S} - S(D(\mathbf{y} - \lambda \mathbf{v})), D\mathbf{v})) \geq 0, \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}_2 \tag{32}$$

Passing to the limit when $\lambda \to 0$ and considering the continuity of S we obtain

$$(\tilde{S} - S(D(\mathbf{y})), D\mathbf{v})) \geq 0, \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}_2. \tag{33}$$

This implies that

$$\tilde{S} = S(D(\mathbf{y}))$$

and then

$$(S(D\mathbf{y}), D\varphi) + (\mathbf{y} \cdot \nabla \mathbf{y}, \varphi) = (\mathbf{u}, \varphi), \quad \text{for all} \quad \varphi \in \mathbf{V}_2.$$

Hence, $\mathbf{y} \equiv \mathbf{y}_u$, i.e, $\mathbf{y}$ is the solution associated to $\mathbf{u}$.

Proposition 7 *Assume that* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are satisfied. Then* $(\mathbf{y}_k)_k$ *strongly converges to* $\mathbf{y}_u$ *in* $\mathbf{W}_0^{1,2}(\Omega)$.

Proof. Setting $\varphi = y_k - y_u$ in (18) and taking (13) we obtain

$$(S(D\mathbf{y}_k) - S(D\mathbf{y}_u), D(\mathbf{y}_k - \mathbf{y}_u)) \geq \|D(\mathbf{y}_k - \mathbf{y}_u)\|_2^2 \tag{34}$$

Therefore, using (5) and classical embedding results, we obtain

$$\begin{aligned}
\|D(\mathbf{y}_k - \mathbf{y}_u)\|_2^2 &\leq (S(D\mathbf{y}_k) - S(D\mathbf{y}_u), D(\mathbf{y}_k - \mathbf{y}_u)) \\
&= ((\mathbf{u}_k - \mathbf{u}, \mathbf{y}_k - \mathbf{y}_u) - (\mathbf{y}_k \cdot \nabla \mathbf{y}_k - \mathbf{y}_u \cdot \nabla \mathbf{y}_u, \mathbf{y}_k - \mathbf{y}_u)) \\
&= ((\mathbf{u}_k - \mathbf{u}, \mathbf{y}_k - \mathbf{y}_u) - ((\mathbf{y}_k - \mathbf{y}_u) \cdot \nabla \mathbf{y}_u, \mathbf{y}_k - \mathbf{y}_u)) \\
&\leq \big((\mathbf{u}_k - \mathbf{u}, \mathbf{y}_k - \mathbf{y}_u) - \|\mathbf{y}_k - \mathbf{y}_u\|_4^2 \|\nabla \mathbf{y}_u\|_2\big) \to 0,
\end{aligned}$$

since the limit of the last expression, when k goes to infinity, is also zero. And therefore, by Korn's inequality,

$$\|\mathbf{y}_k - \mathbf{y}\|_{1,2} \to 0.$$

Now we can prove our main result.

Proof of Theorem 1. Let $(\mathbf{u}_k)_k$ be a minimizing sequence in $\mathbf{L}^2(\Omega)$ and $(\mathbf{y}_k)_k$ the sequence of associated states. Considering the properties of the functional J defined by (3), we obtain

$$\frac{\nu}{2} \|\mathbf{u}_k\|_2^2 \leq J(\mathbf{u}_k, \mathbf{y}_k) \leq J(0, \mathbf{y}_0), \quad \text{for} \quad k > k_0$$

implying that $(\mathbf{u}_k)_k$ is bounded in $\mathbf{L}^2(\Omega)$. From Proposition 7, we deduce that (y_k) converges strongly to y_u. J is a sum of quadratic terms and is convex. On the other hand, if

$$(\mathbf{v}_k, \mathbf{z}_k) \to (\mathbf{v}, \mathbf{z}) \quad \text{in} \quad \mathbf{L}^2(\Omega) \times \mathbf{W}_0^{1,2}(\Omega)$$

this implies

$$J(\mathbf{v}_k, \mathbf{z}_k) \to J(\mathbf{v}, \mathbf{z}) \quad \text{in} \quad I\!R$$

and then the functional J is also a continuous function. In fact, once we have

$$
\begin{aligned}
&|J(\mathbf{v}_k,\mathbf{z}_k)-J(\mathbf{v},\mathbf{z})| = \left|\|\mathbf{z}_k-\mathbf{y}_d\|_2^2+\|\mathbf{v}_k\|_2^2-\|\mathbf{z}-\mathbf{y}_d\|_2^2-\|\mathbf{v}\|_2^2\right| \\
\leq &\left|\left(\|(\mathbf{z}_k-\mathbf{y}_d)-(\mathbf{z}-\mathbf{y}_d)\|_2+\|(\mathbf{z}-\mathbf{y}_d)\|_2\right)^2+\left(\|\mathbf{v}_k-\mathbf{v}\|_2+\|\mathbf{v}\|_2\right)^2-\|\mathbf{z}-\mathbf{y}_d\|_2^2-\|\mathbf{v}\|_2^2\right| \\
\leq &\left|\left(\|(\mathbf{z}_k-\mathbf{z}\|_2+\|(\mathbf{z}-\mathbf{y}_d)\|_2\right)^2+\left(\|\mathbf{v}_k-\mathbf{v}\|_2+\|\mathbf{v}\|_2\right)^2-\|\mathbf{z}-\mathbf{y}_d\|_2^2-\|\mathbf{v}\|_2^2\right|.
\end{aligned}
$$

Since $\mathbf{z}_k \to \mathbf{z}$ strongly also in $\mathbf{L}^2(\Omega)$, the last expression converges to zero when $k \to \infty$ and therefore, J is a semicontinuous function (see [2]). We may now apply the direct method of the Calculus of Variations (see e.g. [6])

$$
\inf_k J \leq J(\mathbf{u},\mathbf{y}_u) \leq \liminf_k J(\mathbf{u}_k,\mathbf{y}_k) \leq \inf_k J
$$

to conclude that $(\mathbf{u},\mathbf{y}_u)$ is in fact a minimizer and therefore a solution of the control problem (P_α).

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References

1. Arada, N.: Optimal control of shear-thinning fluids, SIAM Journal on Control and Optimization, 50, pp. 2515-2542 (2012)
2. Brézis, H.: Análisis funcional. Teoría y aplicaciones, Alianza Editorial, Madrid (1984)
3. Casas, E, Fernández, L.A.: Boundary control of quasilinear elliptic equations, Rapport de Recherche 782, INRIA (1988)
4. Casas, E, Fernández, L.A.: Distributed control of systems governed by a general class of quasilinear elliptic equations, J. Differential Equations 35, pp. 20-47 (1993)
5. Crispo, F., Grisanti, C.R.: On the $C^{1,\gamma}(\bar{\Omega}) \cap W^{2,2}(\Omega)$ regularity for a class of electro-rheological fluids, J. Math. Anal. and App., 356, pp. 119-132 (2009)
6. Dacorogna, B.: Introduction au Calcul des Variations, Press Polytechiques et Universitaires Romandes, (1992)
7. Diening, L., Růžička, M.: Non-Newtonian fluids and function spaces, Nonlinear Anal. Func. Spaces App 8,95-143, (2007)
8. Guerra, T.: Distributed control for shear-thinning non-Newtonian fluids, Journal of Mathematical Fluid Mechanics, Volume 14, 771-789, Issue 4 (2012)
9. Gunzburger, M., Trenchea, C.: Analysis of an optimal control problem for the three-dimensional coupled modified Navier-Stokes and Maxwell equations, J. Math. Anal. Appl. 333, pp. 295-310 (2007)
10. Lions, J.L.: Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Gauthier-Villars, Paris (1969)
11. Málek, J., Nečas, J. , Rokyta, J., Růžička, M.: Weak and Measure-valued Solutions to Evolutionary PDEs, Applied Mathematics and Mathematical Computation, 13, Chapman and Hall, London (1996)
12. Slawig, T.: Distributed control for a class of non-Newtonian fluids, J. Differential Equations, 219, pp. 116-143 (2005)
13. Rajagopal, K.R., Růžička, M.: Mathematical modeling of electrorheological fluids materials, Contin Mech Thermodyn, 13, pp. 59-78 (2001)
14. Růžička, M.: Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Math., vol 1748, Springer Verlag, Berlin, Heidelberg, New York (2000)

15. Růžička, M.: Modeling, Mathematical and Numerical Analysis of Electrorheological Fluids, Applications of Mathematics, Volume 49, Issue 6, pp 565-609, (2004)
16. Wachsmuth, D., Roubíček, T.: Optimal control of incompressible non-Newtonian fluids, Z. Anal. Anwend 29, pp. 351-376 (2010)