Hindawi Publishing Corporation
Journal of Control Science and Engineering
Volume 2008, Article ID 523749, 11 pages
doi:10.1155/2008/523749

Research Article

Stability of Discrete Systems Controlled in the Presence of Intermittent Sensor Faults

Rui Vilela Dionísio[1, 2] and João M. Lemos[3, 2]

[1] *Laboratório de Instrumentação e Medida, Escola Superior de Tecnologia de Setúbal, Instituto Politécnico de Setúbal, Campus do IPS, Estefanilha, 2910-761 Setúbal, Portugal*
[2] *Instituto de Engenharia de Sistemas e Computadores-Investigação e Desenvolvimento em Lisboa, Rua Alves Redol 9, 1000-029 Lisboa, Portugal*
[3] *Instituto Superior Técnico, Universidade Técnica de Lisboa, Avenida Rovisco Pais, 1049-001 Lisboa, Portugal*

Correspondence should be addressed to Rui Vilela Dionísio, rvilela@est.ips.pt

Received 10 April 2007; Accepted 7 October 2007

Recommended by Jakob Stoustrup

This paper presents sufficient conditions for stability of unstable discrete time invariant models, stabilized by state feedback, when interrupted observations due to intermittent sensor faults occur. It is shown that the closed-loop system with feedback through a reconstructed signal, when, at least, one of the sensors is unavailable, remains stable, provided that the intervals of unavailability satisfy a certain time bound, even in the presence of state vanishing perturbations. The result is first proved for linear systems and then extended to a class of Hammerstein systems.

1. INTRODUCTION

In recent years, the mass advent of digital communication networks and systems has boosted the integration of teleoperation in feedback control systems. Applications like unmanned vehicles [1] or internet-based real time control [2] provide significant examples raising, in turn, new problems.

This paper deals with one of such problems, if the communication channel through which feedback information passes is not completely reliable, sensors' measurements may not be available to the controller during some intervals of time. In such a situation, one has to couple the controller with a block, hereafter called supervisor, which is able to discriminate between intervals of signal availability (availability time T_{a_i}) and unavailability (unavailability time $T_{u_{i+1}}$), and to generate an estimate of the plant's state during this $T_{u_{i+1}}$ intervals. Methods for detection and estimation for abruptly changing systems [3] can be applied in the problem considered here. For that purpose an algorithm based on Bayesian decision could be implemented, for example.

Somehow related with the problem of temporary sensor unavailability presented in this paper are the problem of data packet dropout, and the problem of network-induced delay, in networked control systems [4, 5].

Moreover, the approach suggested in this paper can be compared with different techniques based, for example, on the idea of the unknown input observer, as suggested in [6]. On the other hand, it is obvious to exploit Kalman filters and fuzzy logic for sensor fusion, applied to autonomous underwater vehicle systems, as described in [7]. It was, also, shown in [8, 9] that the design of fault-tolerant observers can be successfully applied to the control of rail traction drives. Finally, the stability analysis for a real application example in the presence of intermittent faults is described in [10].

Biomedical applications provide, as well, examples in that the sensor used for feedback is intermittently unavailable. In [11] the artifacts in the neuromuscular blockade level measurement in patients subject to general anaesthesia are modeled as sensor faults. The occurrence of these faults is detected with a Bayesian algorithm and, during the periods of unavailability of the signal, the feedback controller is fed with an estimate generated by a model.

It is shown, throughout the paper, that with the above described scheme, the controlled open-loop unstable plant will be stable (in some sense, to be defined later) if the time

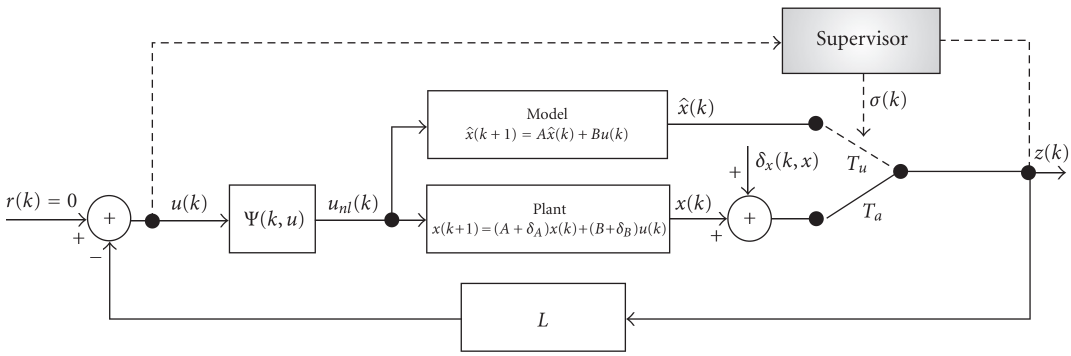

FIGURE 1: Block diagram of a discrete feedback system with nonlinear actuator and interrupted observations supervisor.

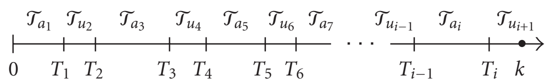

FIGURE 2: Operation time line with availability intervals alternating with unavailability intervals.

interval, during which at least one of the sensors measure is unavailable, is somehow "small", and that the Euclidean norm of the state $x(k)$, at the end of each $T_{u_{i+1}}$ interval, is a monotonic descent sequence. Moreover, if the plant state is perturbed by a class of vanishing perturbations $\delta_x(k,x)$, similar stability results are derived.

The contributions of the paper consist in providing sufficient conditions for stability of feedback controlled open-loop unstable systems with intermittent sensors faults. Linear as well as nonlinear systems are considered.

This paper is organized in four sections and two appendices. After this introduction, Section 2 makes a detailed system description referring the functionality of the supervisor in terms of detection and estimation of the state, and the way the feedback system with linear as well as with nonlinear actuators behaves when intermittent sensors faults occur. Section 3 presents two theorems with sufficient conditions, one for uniform stability of the system with linear and nonlinear actuators, and respective corollaries, also with sufficient conditions, and the other for uniform exponential stability and descent monotonicity of the Euclidean norm of the state $x(k)$ at the end of each $T_{u_{i+1}}$ interval. Moreover, Section 3 presents two other theorems, again with sufficient conditions, that prove that the system with linear and nonlinear actuators, subject to a vanishing perturbation, is asymptotically stable. In Section 4 conclusions are drawn. Appendix A.1 gives a full proof of Theorem 1 and Corollaries 1 and 2, and Appendix A.2 gives a full proof of Theorem 2 and Corollaries 3 and 4.

2. SYSTEM DESCRIPTION

The system depicted in Figure 1 is composed of two subsystems: (i) the supervisor responsible for detection of sensors' measures interruptions, and for switching state feedback from plant to model and from model to plant; (ii) the plant and the model rendered stable through state feedback.

An example of supervisor based on Bayesian inference is provided in [12, 13]. It is beyond the scope of this paper to detail this block.

The supervisor decides whether the state $x(k)$ is being correctly measured by the sensors or not and commands the switch signal $\sigma(k)$. During the time intervals in which the sensors do not provide a reliable measure of the actual plant state (it is admitted that the state coincides with the output) one possibility is to replace it by an estimate $\hat{x}(k)$ obtained from a plant model. This yields a loss of performance with respect to the ideal situation in which the sensors are always available, and may pose stability problems if the plant is open-loop unstable.

In order to understand the system functioning, consider the time line of operation, depicted in Figure 2, divided in alternate intervals where all sensors operate correctly (T_{a_j}, with $j = 1,3,5,7,\dots,i$), and where, at least, one of them fails (T_{u_j}, with $j = 2,4,6,\dots,i-1,i+1$) being replaced by the model estimate. Note that the index j does not represent discrete instants of time, but is rather used to enumerate both the availability, T_{a_j}, and the unavailability, T_{u_j}, intervals. These intervals are identified in script font in the upper part of the time line of Figure 2. The time instants corresponding to the beginning of each interval, wether it is an availability or an unavailability interval, are represented in the lower part of the time line of Figure 2. Let k_0 denote the beginning of one such intervals. It is assumed that the first interval always corresponds to an availability interval, and that the intervals are open at their end. Furthermore, the time analysis always finishes in an unavailability interval at time k. Therefore, in a complete time sequence there are $(i+1)$ intervals, where $(i+1)$ is an even number.

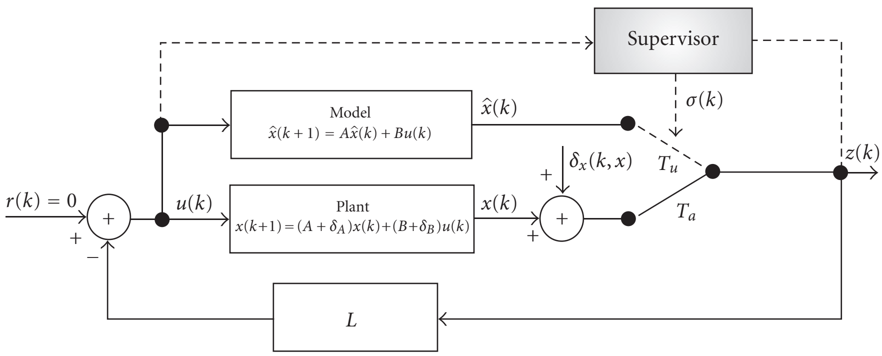

FIGURE 3: Block diagram of a discrete feedback system with linear actuator, and interrupted observations supervisor.

The model initial sate $\hat{x}$ is made equal to the last available observation of the state x when an interrupted observation occurs ($\hat{x}(k_0) = x(k_0) = x(k_0 - 1)$), since the sate x is no longer available.

3. STABILITY RESULTS

Three distinct situations regarding system's stability are considered. In the first case, the nonlinear function $\psi(u(k))$ does not exist ($\psi(u(k)) = I$; see Figure 3). Moreover, the perturbation function $\delta_x(k,x)$ is also considered not to exist ($\delta_x(k,x) = 0$, see Figure 3). The second case is referred to the feedback system with nonlinear actuator function $\psi(u(k))$ but, also, without the perturbation function $\delta_x(k,x)$; see Figure 1. The third case considers the existence of the perturbation function $\delta_x(k,x)$ in both feedback systems, with linear actuator, and with nonlinear actuator.

In all the situations the reference signal, $r(k)$, is considered to be zero, for all $k \geq 0$ (regulation problem).

Throughout the text, matrices norms are the ones induced by the Euclidean norm of vectors, being given by their largest singular value ($\|A\| = \sigma_{\max}[A] = \sigma_A \geq 0$).

3.1. System with linear input

Consider Figure 3 with $\delta_x(k,x) = 0$, and $r(k) = 0$, for all $k \geq 0$. The plant and the model depicted are described in the state-space form by (1) and (2), respectively

$$x(k+1) = (A+\delta_A)x(k) + (B+\delta_B)u(k), \quad (1)$$
$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) \quad (2)$$

with x and $\hat{x} \in \mathbb{R}^n$, accessible for direct measurement, $u \in \mathbb{R}^p$, A, B, δ_A, and δ_B are of appropriate dimensions, and (A,B) is controllable. Moreover, δ_A and δ_B represent modeling uncertainties. It is assumed that the plant is time invariant, and open-loop unstable. The state feedback of signal $z(k)$, yielded by the sensor

$$u(k) = -Lz(k), \quad (3)$$

is implemented by L, a matrix of feedback gains assumed to stabilize the model. Furthermore, $z(k) = x(k)$ during availability intervals, when all sensors are working properly, and $z(k) = \hat{x}(k)$ during unavailability intervals, when measuring interruptions take place.

During availability intervals the plant state equation is

$$x(k+1) = [(A+\delta_A) - (B+\delta_B)L]x(k) \quad (4)$$

and during unavailability intervals the plant state equation is

$$x(k+1) = (A+\delta_A)x(k) - (B+\delta_B)L\hat{x}(k). \quad (5)$$

Define the plant closed-loop dynamics matrix as

$$A_{\delta_{\text{CL}}} := (A+\delta_A) - (B+\delta_B)L = A_\delta - B_\delta L, \quad (6)$$

the model closed-loop dynamics matrix as

$$A_{\text{CL}} := A - BL, \quad (7)$$

the plant open and closed-loop transition matrices

$$\Phi_\delta(k,k_0) := A_\delta^{k-k_0}, \qquad \Phi_{\delta_{\text{CL}}}(k,k_0) := A_{\delta_{\text{CL}}}^{k-k_0}, \quad (8)$$

and the model open and closed-loop transition matrices

$$\Phi(k,k_0) := A^{k-k_0}, \qquad \Phi_{\text{CL}}(k,k_0) := A_{\text{CL}}^{k-k_0}. \quad (9)$$

Theorem 1. *Consider the closed-loop system of Figure 3 with the unstable model in open-loop (bounded by $\|\Phi(k,k_0)\| \leq \alpha\beta^{k-k_0}$ with $\alpha \geq 1$ and $\beta > 1$, finite constants) rendered stable in closed-loop ($\|\Phi_{CL}(k,k_0)\| \leq \gamma\lambda^{k-k_0}$ with $\gamma \geq 1$ a finite constant and $0 \leq \lambda < 1$), through proper design of L. Consider, also, that $\sigma_B := \|B\|$, $\sigma_L := \|L\|$, and that model uncertainties are bounded $\|\delta_A\| \leq \sigma_{\delta_A}$, and $\|\delta_B\| \leq \sigma_{\delta_B}$. The system with initial condition $x(0) = x_0$ is globally uniformly stable provided that the total unavailability time T_u, up to discrete time k inside the unavailability interval $T_{u_{i+1}}$, satisfies the bound*

$$T_u < \frac{\log M_1}{\log(\beta + \alpha\cdot\sigma_{\delta_A})} - \frac{(i+1)}{2} \cdot \frac{\log[(1 + (\sigma_B + \sigma_{\delta_B})\sigma_L\cdot\gamma/(\beta + \alpha\cdot\sigma_{\delta_A} - \lambda))\alpha\gamma]}{\log(\beta + \alpha\cdot\sigma_{\delta_A})} - T_a\frac{\log(\lambda + \gamma\Sigma)}{\log(\beta + \alpha\cdot\sigma_{\delta_A})} \quad (10)$$

with $M_1 \geq 1$, a finite constant, and $\Sigma := \sigma_{\delta_A} + \sigma_{\delta_B} \cdot \sigma_L$ is such that verifies

$$0 \leq \Sigma < \frac{1-\lambda}{\gamma}, \tag{11}$$

T_a is the total availability time.

A result derived from the previous theorem is stated on the following corollary.

Corollary 1. *Under the assumptions of Theorem 1, $\|x(T_j)\|$, for $j = 0, 2, 4, \ldots, i-1$, (the state norm at the beginning of each availability interval) is a monotonic descent sequence provided that the unavailability interval $T_{u_{i-1}}$ satisfies*

$$T_{u_{i-1}} < \frac{\log\left[(1 + (\sigma_B + \sigma_{\delta_B})\sigma_L \cdot \gamma/(\beta + \alpha \cdot \sigma_{\delta_A} - \lambda))\alpha\gamma\right]}{\log(\beta + \alpha \cdot \sigma_{\delta_A})} - T_{a_{i-2}} \frac{\log(\lambda + \gamma\Sigma)}{\log(\beta + \alpha \cdot \sigma_{\delta_A})} \tag{12}$$

and $\Sigma := \sigma_{\delta_A} + \sigma_{\delta_B} \cdot \sigma_L$ is such that verifies

$$0 \leq \Sigma < \frac{1-\lambda}{\gamma}, \tag{13}$$

$T_{a_{i-2}}$ is the availability time previous to $T_{u_{i-1}}$.

Concerning global uniform exponential stability, consider the next corollary.

Corollary 2. *Under the assumptions of Theorem 1, the system with initial condition $x(0) = x_0$ is globally uniformly exponentially stable provided that the total unavailability time T_u, up to discrete time k inside the unavailability interval $T_{u_{i+1}}$, satisfies*

$$T_u < \frac{\log M_2}{\log((\beta + \alpha \cdot \sigma_{\delta_A})/N_2)} - \frac{(i+1)}{2} \cdot \frac{\log\left[(1 + (\sigma_B + \sigma_{\delta_B})\sigma_L \cdot \gamma/(\beta + \alpha \cdot \sigma_{\delta_A} - \lambda))\alpha\gamma\right]}{\log((\beta + \alpha \cdot \sigma_{\delta_A})/N_2)} - T_a \frac{\log((\lambda + \gamma\Sigma)/N_2)}{\log((\beta + \alpha \cdot \sigma_{\delta_A})/N_2)} \tag{14}$$

with $M_2 \geq 1$, a finite constant, and $\Sigma := \sigma_{\delta_A} + \sigma_{\delta_B} \cdot \sigma_L$ is such that verifies

$$0 \leq \Sigma < \frac{1-\lambda}{\gamma} \tag{15}$$

and $0 \leq N_2 < 1$ is a constant constrained to

$$N_2 > \lambda + \gamma\Sigma, \tag{16}$$

T_a is the total availability time.

A proof of the theorem and of the corollaries is presented in Appendix A.1.

Remark 1. The constraint $\Sigma < (1-\lambda)/\gamma$ is imposed to assure that the plant closed-loop transition matrix is such that $\|\Phi_{\delta_{CL}}(k, k_0)\| \leq \gamma(\lambda + \gamma\Sigma)^{k-k_0}$ with $0 \geq (\lambda + \gamma\Sigma) < 1$ (see the proof in Appendix A.1).

Remark 2. Notice that since $(\beta + \alpha \cdot \sigma_{\delta_A}) > 1$ and $0 \leq (\lambda + \gamma\Sigma) < 1$, then the bound on T_u has a monotonous crescent linear relation with T_a in the result from Theorem 1, and $T_{u_{i-1}}$ also has a monotonous crescent linear relation with $T_{a_{i-2}}$ in the result from Corollary 1.

Remark 3. The constant N_2 (in Corollary 2) represents an upper bound on the rate of exponential decay of the overall system. If $N_2 < \lambda + \gamma\Sigma$, then the result of Corollary 2 would indicate a negative solution for T_u, which, clearly, is not possible, since $T_u \in [0, \infty[$. Being $N_2 > \lambda + \gamma\Sigma$, then the bound on T_u has also a monotonous crescent linear relation with T_a, as mentioned in the previous remark.

Remark 4. Concerning Theorem 1 and Corollary 2, constants M_1 and M_2 represent an offset term for the upper bound function on the evolution of $\|x(k)\|$. The bigger these constants are, the more conservative is the referred upper bound on uniform stability and uniform exponential stability, respectively.

Remark 5. Theorem 1 and Corollaries 1 and 2 present only conservative sufficient stability conditions for the system of Figure 3.

3.2. System with nonlinear input

Consider Figure 1 with $\delta_x(k, x) = 0$, and $r(k) = 0$, for all $k \geq 0$. The plant and the model depicted are described in the state-space form by (17) and (18), respectively,

$$x(k+1) = (A + \delta_A)x(k) + (B + \delta_B)u_{\text{nl}}(k), \tag{17}$$

$$\hat{x}(k+1) = A\hat{x}(k) + Bu_{\text{nl}}(k), \tag{18}$$

with x and $\hat{x} \in \mathbb{R}^n$, accessible for direct measurement, u and $u_{\text{nl}} \in \mathbb{R}^p$, A, B, δ_A, and δ_B are of appropriate dimensions, and (A, B) is controllable. Moreover, δ_A and δ_B represent modeling uncertainties. It is assumed that the plant is time invariant, and open-loop unstable.

The vector u_{nl} represents the nonlinear input to both the plant and the model, $u_{\text{nl}}(k) = \psi(k, u)$. A memoryless nonlinearity, $\psi : [0, \infty[\times \mathbb{R}^p \to \mathbb{R}^p$, is said to satisfy a sector condition globally [14] if

$$\left[\psi(k, u) - K_{\min} u(k)\right]^T \left[\psi(k, u) - K_{\max} u(k)\right] \leq 0 \tag{19}$$

for all $t \geq 0$, for all $u \in \mathbb{R}^p$, for some real matrices $K_{\min}$ and $K_{\max}$, where $K = K_{\max} - K_{\min}$ is a positive definite symmetric matrix. The nonlinearity $\psi(k, u)$ is said to belong to a sector $[K_{\min}, K_{\max}]$.

Proposition 1. *Consider $K_{\min} = -(\gamma_2/2)I$ and $K_{\max} = (\gamma_2/2)I$ with γ_2 a finite positive constant. The nonlinearity $\psi(k, u)$ can be decomposed in a linear component and a nonlinear component, [15]*

$$\psi_s(k, u) = \psi(k, u) - K_{\min} u(k), \tag{20}$$

where $\psi_s(k, u)$ represents the nonlinear component and verifies the sector condition

$$\psi_s^T(k, u)\left[\psi_s(k, u) - Ku(k)\right] \leq 0. \tag{21}$$

Proof. This result is straightforward using (20) in (19), and considering matrix K definition. □

Proposition 2. *For the defined matrices $K_{\min}$ and $K_{\max}$, the memoryless sector nonlinearity $\psi(k,u)$ is bounded by $\|\psi(k,u)\| \le (\gamma_2/2)\|u(k)\|$, for all $t \ge 0$, for all $u \in \mathbb{R}^p$.*

Proof. Replacing $K_{\min}$ and $K_{\max}$ by their respective values in (19), and since $\psi^T(k,u)u(k)$ is a scalar, yields

$$\|\psi(k,u)\|^2 - \left(\frac{\gamma_2}{2}\right)^2 \|u(k)\|^2 \le 0. \tag{22}$$

By definition $\|\psi(k,u)\| \ge 0$, $\|u(k)\| \ge 0$, and $\gamma_2/2 > 0$, which implies

$$\|\psi(k,u)\| \le \frac{\gamma_2}{2}\|u(k)\|. \tag{23}$$

□

In order to find a bound on $\psi_s(k,u)$ starting from (20), using (23), and $K_{\min}$ definition, it follows that

$$\|\psi_s(k,u)\| \le \gamma_2\|u(k)\|. \tag{24}$$

The state feedback of signal $z(k)$, yielded by the sensor

$$u(k) = -Lz(k), \tag{25}$$

is implemented by L, a matrix of feedback gains assumed to stabilize the model. Furthermore, $z(k) = x(k)$ during availability intervals, when all sensors are working properly, and $z(k) = \hat{x}(k)$ during unavailability intervals, when measuring interruptions take place.

During availability intervals, the plant state equation is

$$\begin{aligned} x(k+1) &= [(A+\delta_A) - (B+\delta_B)K_{\min}L]x(k) \\ &\quad + (B+\delta_B)\psi_s(-Lx(k)) \end{aligned} \tag{26}$$

and during unavailability intervals, the plant state equation is

$$\begin{aligned} x(k+1) &= (A+\delta_A)x(k) + (B+\delta_B) \\ &\quad \cdot (\psi_s(-L\hat{x}(k)) - K_{\min}L\hat{x}(k)). \end{aligned} \tag{27}$$

Define the plant closed-loop dynamics matrix as

$$\overline{A}_{\delta_{\mathrm{CL}}} := (A+\delta_A) - (B+\delta_B)K_{\min}L = A_\delta - B_\delta K_{\min}L, \tag{28}$$

the model closed-loop dynamics matrix as

$$\overline{A}_{\mathrm{CL}} := A - BK_{\min}L, \tag{29}$$

the plant open and closed-loop transition matrices as

$$\begin{aligned} \Phi_\delta(k,k_0) &:= (A+\delta_A)^{k-k_0} = A_\delta^{k-k_0}, \\ \overline{\Phi}_{\delta_{\mathrm{CL}}}(k,k_0) &:= \overline{A}_{\delta_{\mathrm{CL}}}^{k-k_0}, \end{aligned} \tag{30}$$

the model open and closed-loop transition matrices as

$$\begin{aligned} \Phi(k,k_0) &:= A^{k-k_0}, \\ \overline{\Phi}_{\mathrm{CL}}(k,k_0) &:= \overline{A}_{\mathrm{CL}}^{k-k_0}, \end{aligned} \tag{31}$$

and matrix $P = K_{\min}L$, considered to stabilize the model in closed loop.

Theorem 2. *Consider the closed-loop system of Figure 1 where the model is unstable in open-loop (bounded by $\|\Phi(k,k_0)\| \le \alpha\beta^{k-k_0}$, with $\alpha \ge 1$, and $\beta > 1$, finite constants), rendered stable in closed-loop ($\|\overline{\Phi}_{CL}(k,k_0)\| \le \gamma\lambda^{k-k_0}$, with $\gamma \ge 1$ a finite constant, and $0 \le \lambda < 1$), through proper design of P. The nonlinearity $\psi_s(k,u)$ satisfies $\|\psi_s(k,u)\| \le \gamma_2\|u(k)\|$ for all $t \ge 0$, for all $u \in \mathbb{R}^p$. Consider, also, that $\sigma_B := \|B\|$, $\sigma_L := \|L\|$, $\sigma_P := \|P\|$ and that model uncertainties are bounded $\|\delta_A\| \le \sigma_{\delta_A}$ and $\|\delta_B\| \le \sigma_{\delta_B}$. The system with initial condition $x(0) = x_0$ is globally uniformly stable provided that the total unavailability time T_u, up to discrete time k inside the unavailability interval $T_{u_{i+1}}$, satisfies the bound*

$$\begin{aligned} T_u &< \frac{\log M_1}{\log(\beta+\alpha\cdot\sigma_{\delta_A})} - \frac{(i+1)}{2} \\ &\quad \cdot \frac{\log\left[\left(1+\dfrac{(\sigma_B+\sigma_{\delta_B})(\sigma_P+\gamma_2\cdot\sigma_L)\gamma}{(\beta+\alpha\cdot\sigma_{\delta_A})-(\lambda+\gamma\sigma_{\delta_B}\cdot\gamma_2\cdot\sigma_L)}\right)\alpha\gamma\right]}{\log(\beta+\alpha\cdot\sigma_{\delta_A})} \\ &\quad - T_a\frac{\log[(\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L]}{\log(\beta+\alpha\cdot\sigma_{\delta_A})} \end{aligned} \tag{32}$$

with $M_1 \ge 1$, a finite constant, and $\Sigma := \sigma_{\delta_A} + \sigma_{\delta_B}\cdot\sigma_P$ is such that verifies

$$0 \le \Sigma < \frac{1-\lambda}{\gamma} \tag{33}$$

and γ_2 is the less of the following two inequalities:

$$\begin{aligned} \gamma_2 &< \frac{1-(\lambda+\gamma\Sigma)}{\gamma(\sigma_B+\sigma_{\delta_B})\sigma_L}, \\ \gamma_2 &< \frac{\beta+\alpha\cdot\sigma_{\delta_A}-\lambda}{\gamma\cdot\sigma_{\delta_B}\cdot\sigma_L}, \end{aligned} \tag{34}$$

T_a is the total availability time.

As in the previous subsection the following two corollaries are derived.

Corollary 3. *Under the assumptions of Theorem 1, $\|x(T_j)\|$, for $j = 0,2,4,\dots,i-1$, (the state norm at the beginning of each availability interval) is a monotonic descent sequence provided that the unavailability interval $T_{u_{i-1}}$ satisfies*

$$\begin{aligned} T_{u_{i-1}} &< -\frac{\log\left[\left(1+\dfrac{(\sigma_B+\sigma_{\delta_B})(\sigma_P+\gamma_2\cdot\sigma_L)\gamma}{(\beta+\alpha\cdot\sigma_{\delta_A})-(\lambda+\gamma\sigma_{\delta_B}\cdot\gamma_2\cdot\sigma_L)}\right)\alpha\gamma\right]}{\log(\beta+\alpha\cdot\sigma_{\delta_A})} \\ &\quad - T_{a_{i-2}}\frac{\log[(\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L]}{\log(\beta+\alpha\cdot\sigma_{\delta_A})} \end{aligned} \tag{35}$$

and $\Sigma := \sigma_{\delta_A} + \sigma_{\delta_B}\cdot\sigma_L$ is such that verifies

$$0 \le \Sigma < \frac{1-\lambda}{\gamma} \tag{36}$$

and γ_2 is the less of the following two inequalities:

$$\gamma_2 < \frac{1-(\lambda+\gamma\Sigma)}{\gamma(\sigma_B+\sigma_{\delta_B})\sigma_L},\qquad \gamma_2 < \frac{\beta+\alpha\cdot\sigma_{\delta_A}-\lambda}{\gamma\cdot\sigma_{\delta_B}\cdot\sigma_L}, \tag{37}$$

$T_{a_{i-2}}$ is the availability time previous to $T_{u_{i-1}}$.

Corollary 4. *Under the assumptions of Theorem 2, the system with initial condition $x(0)=x_0$ is globally uniformly exponentially stable provided that the total unavailability time T_u, up to discrete time k inside the unavailability interval $T_{u_{i+1}}$, satisfies*

$$T_u < \frac{\log M_2}{\log((\beta+\alpha\cdot\sigma_{\delta_A})/N_2)} - \frac{(i+1)}{2} \cdot \frac{\log\left[\left(1+\dfrac{(\sigma_B+\sigma_{\delta_B})(\sigma_P+\gamma_2\cdot\sigma_L)\gamma}{(\beta+\alpha\cdot\sigma_{\delta_A})-(\lambda+\gamma\sigma_{\delta_B}\cdot\gamma_2\cdot\sigma_L)}\right)\alpha\gamma\right]}{\log((\beta+\alpha\cdot\sigma_{\delta_A})/N_2)} - T_a\frac{\log(((\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L)/N_2)}{\log((\beta+\alpha\cdot\sigma_{\delta_A})/N_2)} \tag{38}$$

with $M_1 \geq 1$, a finite constant, and $\Sigma := \sigma_{\delta_A}+\sigma_{\delta_B}\cdot\sigma_P$ is such that verifies

$$0 \leq \Sigma < \frac{1-\lambda}{\gamma} \tag{39}$$

and γ_2 is the less of the following two inequalities:

$$\gamma_2 < \frac{1-(\lambda+\gamma\Sigma)}{\gamma(\sigma_B+\sigma_{\delta_B})\sigma_L},\qquad \gamma_2 < \frac{\beta+\alpha\cdot\sigma_{\delta_A}-\lambda}{\gamma\cdot\sigma_{\delta_B}\cdot\sigma_L}, \tag{40}$$

and $0 \leq N_2 < 1$ is a constant constrained to

$$N_2 > (\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L, \tag{41}$$

T_a is the total availability time.

A proof of the theorem and of the corollaries is presented in Appendix A.2.

Remark 6. Notice that since $(\beta+\alpha\cdot\sigma_{\delta_A}) > 1$ then it must be $(\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L < 1$, which leads to (34), (37), and (40), so that the bound on T_u has a monotonous crescent linear relation with T_a in the result from Theorem 2, and $T_{u_{i-1}}$ also has a monotonous crescent linear relation with $T_{a_{i-2}}$ in the result from Corollary 3.

Remark 7. The constant N_2 (in Corollary 4) represents an upper bound on the rate of exponential decay of the overall system. If $N_2 < (\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L$, then the result of Corollary 4 would indicate a negative solution for T_u, which, clearly, is not possible, since $T_u \in [0,\infty[$. Being $N_2 > (\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L$, then the bound on T_u has also a monotonous crescent linear relation with T_a, as mentioned in the previous remark.

Remark 8. Concerning Theorem 2 and Corollary 4, constants M_1 and M_2 represent, once again, an offset term for the upper bound function on the evolution of $\|x(k)\|$. The bigger these constants are, the more conservative is the referred upper bound on uniform stability and uniform exponential stability, respectively.

Remark 9. Theorem 2 and Corollaries 3 and 4 present only conservative sufficient stability conditions for the system of Figure 1.

3.3. Perturbed system with linear and nonlinear inputs

Consider that both systems depicted in Figures 1 and 3, suffer the influence of perturbation $\delta_x(k,x)$, where $\delta_x : [0,\infty[\times D \to \mathbb{R}^n$ is piecewise continuous in k and locally Lipschitz in x on $[0,\infty[\times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x=0$. Also, $\|\delta_x(k,x)\| \leq \epsilon\|x(k)\|$ for all $k \geq 0$, for all $x \in D$, and ϵ is a nonnegative constant, meaning that the perturbation satisfies a linear growth bound, therefore, considering a vanishing perturbation, [14].

During availability intervals T_{a_j}, for $j = 1,3,5,\ldots,i$, both systems can be represented by the autonomous equation

$$x(k+1) = F(k,x), \tag{42}$$

where $F(k,x)$, for the system depicted in Figure 3, is

$$F(k,x) = A_{\delta_{CL}}x(k), \quad k \in T_{a_j}, \tag{43}$$

and for the system depicted in Figure 1, $F(k,x)$ is

$$F(k,x) = \overline{A}_{\delta_{CL}}x(k) + (B+\delta_B).\psi_s(-Lx(k)), \quad k \in T_{a_j}. \tag{44}$$

Clearly, $F(0) = 0$ in both situations (from (19) and matrices' $K_{\min}$ and $K_{\max}$ definition in Proposition 1, the sector memoryless nonlinearity verifies $\psi_s(0) = 0$). Recalling the state equations (5) and (27) during unavailability intervals T_{u_j}, for $j = 2,4,6,\ldots,i+1$, and the fact that the initial model state $\hat{x}$ is made equal to the last available observation of the state x when an interrupted observation occurs, $(\hat{x}(k_0) = x(k_0) = x(k_0-1))$, it is clearly understood that if the state becomes zero during an availability interval, then it will remain zero for all time instants belonging to any unavailability interval that may occur. The function's $F(k,x)$ branch related with the unavailability interval is not of obvious writing in terms only of $x(k)$. It has an easier writing in terms of $x(k)$ and of $\hat{x}(k)$. Nevertheless, since these two states are related at the switching time between availability and unavailability intervals (as recalled above), it can be understood that during an unavailability interval, $F(k,x)$ exists.

It is important to stress out that an unavailability interval cannot occur without having previously existed an availability interval. Bearing this in mind, it is possible to state that $F(0) = 0$, for all $k \geq 0$, (including availability and unavailability intervals).

Also, linear and nonlinear systems were proved to be globally uniformly exponentially stable, under the conditions of Corollaries 2 and 4, respectively, therefore, both $F(k,x)$ are Lipschitz not only near the origin, but in $\mathbb{R}^n$, and verify $\|F(x_1) - F(x_2)\| \le L_v\|x_1 - x_2\|$.

Combining the results from Corollaries 2 and 4 with the above comments, and with the result presented in [16], is reproduced in the next theorem.

Theorem 3. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfy a Lipschitz condition in a neighborhood of the origin, with $F(0) = 0$. If the origin is an exponentially stable fixed point of $x(k+1) = F(x(k))$, it is an asymptotically stable fixed point of the perturbed system $\overline{x}(k+1) = F(\overline{x}(k)) + \delta_x(k,\overline{x})$.*

This leads to the next two theorems.

Theorem 4. *The nonperturbed system from Figure 3, $x(k+1) = F(k,x)$, verifying Corollary 2 and Theorem 3 sufficient conditions, has a globally asymptotically stable fixed point of the perturbed system $\overline{x}(k+1) = F(\overline{x}(k)) + \delta_x(k,\overline{x})$ in the origin, and $\delta_x : [0,\infty[\times D \to \mathbb{R}^n$ is piecewise continuous in k and locally Lipschitz in x on $[0,\infty[\times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$. Also, $\|\delta_x(k,x)\| \le \epsilon\|x(k)\|$ for all $k \ge 0$, for all $x \in D$ with ϵ a nonnegative constant satisfies a linear growth bound.*

Theorem 5. *It is the same redaction of Theorem 4, but considering the system from Figure 1.*

Remark 10. These results are global since both $F(k,x)$ are Lipschitz continuous in $\mathbb{R}^n$, and the original systems are uniformly exponentially stable, [16].

4. CONCLUSIONS

The paper presents and proves sufficient conditions that allow a discrete time analysis of sensor unavailability (interrupted observations) intervals, bounding these intervals in order to state that the unstable open-loop plant represented in Figure 1, when controlled in closed-loop, is globally uniformly exponentially stable. These results are proved under the existence of modeling uncertainties and if plant state vanishing perturbations occur, then global asymptotical stability is achieved for the perturbed system. The results were proved for either systems with linear actuators, or with memoryless sector nonlinear actuators.

It is interesting to note that in a related work [4], a similar conservative theoretical result regarding uniform exponential stability is reported, showing that longer intervals of unavailability can be reached in practice and that these theoretical results might be too conservative for practical purposes.

APPENDIX

Throughout the appendix, the matrices norms are the ones induced by the Euclidean norm of vectors, being given by their largest singular values.

Consider the discrete time line represented in Figure 2. The intervals where the sensors yield correct measures are designated as T_{a_j}, with $j = 1,3,5,7,\dots,i$, and the intervals where the observations are interrupted are designated as T_{u_j}, with $j = 2,4,6,\dots,i-1,i+1$. Let the discrete time instant k_0 denote the beginning of a generic interval.

Since it will be often used in the following proofs, a Gronwall-Bellman type of inequality for sequences is presented [17].

Lemma 1. *Suppose the scalar sequences $v(k)$ and $\phi(k)$ are such that $v(k) \ge 0$ for $k \ge k_0$, and*

$$\phi(k) \le \begin{cases} \Psi, & k = k_0, \\ \Psi + \eta \sum_{j=k_0}^{k-1} v(j)\phi(j), & k \ge k_0 + 1, \end{cases} \tag{A.1}$$

where Ψ and η are constants with $\eta \ge 0$. Then

$$\phi(k) \le \Psi \prod_{j=k_0}^{k-1} [1 + \eta v(j)]. \tag{A.2}$$

Consider, also, the sum of the $(k - k_0)$ terms of a geometric progression with ratio r,

$$\sum_{j=k_0}^{k-1} r^j = \frac{r^{k_0} - r^k}{1 - r}. \tag{A.3}$$

If $|r| < 1$, then, as $k \to \infty$, (A.3) becomes

$$\sum_{j=k_0}^{k-1} r^j = \frac{r^{k_0}}{1 - r}. \tag{A.4}$$

A.1. Stability proofs for system with linear input

Proof of Theorem 1. Consider the system depicted in Figure 3. During availability time intervals T_{a_j}, with $j = 1,3,5,7,\dots,i$, it is $z(k) = x(k)$, and the plant state $x(k)$ evolves according to

$$x(k) = \Phi_{\delta_{CL}}(k,k_0)x(k_0), \quad k \ge k_0 + 1. \tag{A.5}$$

On the other hand, during unavailability time intervals T_{u_j}, with $j = 2,4,6,\dots,i-1,i+1$, it is $z(k) = \hat{x}(k)$, the model state $\hat{x}(k)$ evolves according to

$$\hat{x}(k) = \Phi_{CL}(k,k_0)\hat{x}(k_0), \quad k \ge k_0 + 1, \tag{A.6}$$

and the plant state $x(k)$ evolves according to

$$x(k) = \Phi_\delta(k,k_0)x(k_0) - \sum_{j=k_0}^{k-1} \Phi_\delta(k,j+1)(B + \delta_B)L\hat{x}(j), \quad k \ge k_0 + 1. \tag{A.7}$$

Replacing (A.6) in (A.7), and knowing that the model initial sate $\hat{x}$ is made equal to the last available observation of the state x when an interrupted observation occurs ($\hat{x}(k_0) = x(k_0) = x(k_0 - 1)$), the plant state $x(k)$ evolution is

$$x(k) = \Phi_\delta(k, k_0)x(k_0) - \sum_{j=k_0}^{k-1} \Phi_\delta(k, j+1)(B + \delta_B)L\Phi_{CL}(j, k_0)x(k_0), \quad k \geq k_0 + 1. \tag{A.8}$$

It is assumed that the model in closed-loop is stable and bounded by $\|\Phi_{CL}(k, k_0)\| \leq \gamma\lambda^{k-k_0}$, $k \geq k_0$, with $0 \leq \lambda < 1$ and $\gamma \geq 1$, and that the model is unstable in open-loop, but bounded by $\|\Phi(k, k_0)\| \leq \alpha\beta^{k-k_0}$, $k \geq k_0$, with $\beta > 1$ and $\alpha \geq 1$.

For bounded model uncertainties $\|\delta_A\| \leq \sigma_{\delta_A}$, and considering the bound on $\|\Phi(k, k_0)\|$, with $\beta > 1$ (this corresponds to assume an unfavorable situation), it can be proved through the use of Lemma 1, if δ_A is seen as a perturbation in the system $x(k+1) = (A + \delta_A)x(k)$, [17], that $\|\Phi_\delta(k, k_0)\| \leq \alpha(\beta + \alpha\cdot\sigma_{\delta_A})^{k-k_0}$, with $(\beta + \alpha\cdot\sigma_{\delta_A}) > 1$. This means, as expected, that if the model dynamics are open-loop unstable, then there will be a δ_A such that the plant dynamics will be open-loop unstable (the use of a continuity argumentation could also explain such assertion). A similar proof can be given for the stability of the plant in closed-loop since the model is stable in closed-loop ($\|\Phi_{CL}(k, k_0)\| \leq \gamma\lambda^{k-k_0}$, with $0 \leq \lambda < 1$). Again, recurring to Lemma 1, and considering that $(\delta_A - \delta_B L)$ is seen as a perturbation in the system $x(k+1) = [(A + \delta_A) - (B + \delta_B)L]x(k)$, it can be proved that $\|\Phi_{\delta_{CL}}(k, k_0)\| \leq \gamma(\lambda + \gamma\Sigma)^{k-k_0}$, $k \geq k_0$, with $0 \leq \Sigma < (1 - \lambda)/\gamma$, and $\Sigma := \sigma_{\delta_A} + \sigma_{\delta_B}\cdot\sigma_L$.

Upper bounds for (A.5) during availability time intervals, and for (A.8) during unavailability time intervals, are obtained, respectively

$$||x(k)|| = ||\Phi_{\delta_{CL}}(k, k_0)x(k_0)||, \tag{A.9}$$

$$||x(k)|| = \left\| \Phi_\delta(k, k_0)x(k_0) - \sum_{j=k_0}^{k-1} \Phi_\delta(k, j+1)(B + \delta_B)L\Phi_{CL}(j, k_0)x(k_0) \right\|. \tag{A.10}$$

Starting from (A.9), yields

$$||x(k)|| \leq \gamma(\lambda + \gamma\Sigma)^{k-k_0}||x(k_0)|| \tag{A.11}$$

and for (A.10), recalling that $\|B\| := \sigma_B$, $\|\delta_B\| \leq \sigma_{\delta_B}$, and $\|L\| := \sigma_L$,

$$||x(k)|| \leq \left(\alpha(\beta + \alpha\cdot\sigma_{\delta_A})^{k-k_0} + \sum_{j=k_0}^{k-1} \alpha(\beta + \alpha\cdot\sigma_{\delta_A})^{k-j-1}(\sigma_B + \sigma_{\delta_B})\sigma_L\gamma\lambda^{j-k_0} \right) \cdot ||x(k_0)||. \tag{A.12}$$

Since $0 \leq \lambda/(\beta + \alpha\cdot\sigma_{\delta_A}) < 1$, and considering (A.4), after some calculations

$$||x(k)|| \leq \left[1 + (\sigma_B + \sigma_{\delta_B}) \frac{\sigma_L\cdot\gamma}{\beta + \alpha\cdot\sigma_{\delta_A} - \lambda} \right] . \alpha(\beta + \alpha\cdot\sigma_{\delta_A})^{k-k_0}\cdot||x(k_0)||. \tag{A.13}$$

The complete state evolution from time instant $k = 0$, up to the final time instant at $k \in T_{u_{i+1}}$, is given by the alternate product of (A.5) by (A.8), where $\hat{x}(k_0) = x(k_0) = x(k_0 - 1)$ is considered. Applying results (A.11) and (A.13) to this product originates

$$\begin{aligned} \|x(k)\| \leq{}& \left[1 + (\sigma_B + \sigma_{\delta_B}) \frac{\sigma_L\cdot\gamma}{\beta + \alpha\cdot\sigma_{\delta_A} - \lambda} \right] \alpha(\beta + \alpha\cdot\sigma_{\delta_A})^{k-T_i} \\ &\cdot\gamma(\lambda + \gamma\Sigma)^{(T_i - 1) - T_{i-1}} \\ &\cdots \left[1 + (\sigma_B + \sigma_{\delta_B}) \frac{\sigma_L\cdot\gamma}{\beta + \alpha\cdot\sigma_{\delta_A} - \lambda} \right] \alpha(\beta + \alpha\cdot\sigma_{\delta_A})^{T_2 - T_1} \\ &\cdot\gamma(\lambda + \gamma\Sigma)^{T_1 - 1}\cdot||x_0|| \\ ={}& c_1^{(i+1)/2}\cdot(\beta + \alpha\cdot\sigma_{\delta_A})^{T_u}\cdot(\lambda + \gamma\Sigma)^{T_a}\cdot||x_0||, \end{aligned} \tag{A.14}$$

where T_u and T_a represent the entire duration of all unavailability and availability time intervals, respectively, and

$$c_1 := \left[1 + (\sigma_B + \sigma_{\delta_B}) \frac{\sigma_L\cdot\gamma}{\beta + \alpha\cdot\sigma_{\delta_A} - \lambda} \right] \alpha\gamma. \tag{A.15}$$

In order for the system to be uniformly stable, it must verify $\|x(k)\| \leq M_1\|x(k_0)\|$, $k \geq k_0$, with $M_1 \geq 1$. Therefore, from (A.14)

$$c_1^{(i+1)/2}\cdot(\beta + \alpha\cdot\sigma_{\delta_A})^{T_u}\cdot(\lambda + \gamma\Sigma)^{T_a} \leq M_1 \Longrightarrow T_u \leq \frac{\log M_1 - ((i+1)/2)\log c_1 - T_a \log(\lambda + \gamma\Sigma)}{\log(\beta + \alpha\cdot\sigma_{\delta_A})}. \tag{A.16}$$

Replacing (A.15) in (A.16) gives the desired result from Theorem 1 subject to the constraint $\Sigma < (1 - \lambda)/\gamma$, and the result holds globally since it is valid for any $\|x(k_0)\|$. □

Proof of Corollary 1. Consider the Euclidean norm of $x(k)$ at discrete times $k = T_{i-1}$, and $k = T_{i-3}$, at the end of the unavailability intervals $T_{u_{i-1}}$, and $T_{u_{i-3}}$, respectively. In order for $\|x(T_j)\|$, for $j = 0, 2, 4, 6, \ldots, i-1$, to be a monotonic descent sequence, it should verify

$$\frac{||x(T_{i-1})||}{||x(T_{i-3})||} < 1 \tag{A.17}$$

equivalently, from the first two lines of (A.14), and considering (A.15)

$$c_1(\beta + \alpha\cdot\sigma_{\delta_A})^{T_{i-1} - T_{i-2}}\cdot(\lambda + \gamma\Sigma)^{(T_{i-2} - 1) - T_{i-3}} < 1 \tag{A.18}$$

or, since $T_{i-1} - T_{i-2} = T_{u_{i-1}}$, and $(T_{i-2} - 1) - T_{i-3} = T_{a_{i-2}}$

$$T_{u_{i-1}} < \frac{-\log c_1 - T_{a_{i-2}} \log(\lambda + \gamma\Sigma)}{\log(\beta + \alpha\cdot\sigma_{\delta_A})}. \tag{A.19}$$

Replacing (A.15) in (A.19) gives the desired result from Corollary 1 subject to the constraint $\Sigma < (1-\lambda)/\gamma$, and the result holds globally since it is valid for any $\|x(k_0)\|$. □

Proof of Corollary 2. In order for the system to be uniformly exponentially stable, it must verify $\|x(k)\| \leq M_2 N_2^{k-k_0}\|x(k_0)\|$, $k \geq k_0$, with $M_2 \geq 1$, and $0 \leq N_2 < 1$. Therefore, from (A.14) and considering $k_0 = 0$, and $(k - k_0) = T_u + T_a$,

$$c_1^{(i+1)/2}\cdot(\beta+\alpha\cdot\sigma_{\delta_A})^{T_u}\cdot(\lambda+\gamma\Sigma)^{T_a} \leq M_2 N_2^{T_u+T_a}$$
$$\Longrightarrow T_u \leq \frac{\log M_2 - ((i+1)/2)\log c_1 - T_a \log((\lambda+\gamma\Sigma)/N_2)}{\log((\beta+\alpha\cdot\sigma_{\delta_A})/N_2)}. \tag{A.20}$$

Replacing (A.15) in (A.20) gives the desired result from Corollary 2 subject to the constraint $\Sigma < (1-\lambda)/\gamma$, and the result holds globally since it is valid for any $\|x(k_0)\|$. □

A.2. *Stability proofs for system with nonlinear input*

Proof of Theorem 2. Consider the system depicted in Figure 1. During availability time intervals T_{a_j}, with $j = 1, 3, 5, 7, \ldots, i$, it is $z(k) = x(k)$, and the plant state $x(k)$ evolves according to

$$x(k) = \overline{\Phi}_{\delta_{\mathrm{CL}}}(k,k_0)x(k_0) + \sum_{j=k_0}^{k-1}\overline{\Phi}_{\delta_{\mathrm{CL}}}(k,j+1)(B+\delta_B) \cdot\psi_s(-Lx(j)), \quad k \geq k_0+1. \tag{A.21}$$

On the other hand, during unavailability time intervals T_{u_j}, with $j = 2, 4, 6, \ldots, i-1, i+1$, it is $z(k) = \hat{x}(k)$, the model initial sate $\hat{x}$ is made equal to the last available observation of the state x when an interrupted observation occurs ($\hat{x}(k_0) = x(k_0) = x(k_0-1)$), the model state $\hat{x}(k)$ evolves according to

$$\hat{x}(k) = \overline{\Phi}_{\mathrm{CL}}(k,k_0)x(k_0) + \sum_{j=k_0}^{k-1}\overline{\Phi}_{\mathrm{CL}}(k,j+1)B\psi_s(-L\hat{x}(j)), \quad k \geq k_0+1 \tag{A.22}$$

and the plant state $x(k)$ evolves according to

$$x(k) = \Phi_\delta(k,k_0)x(k_0) + \sum_{j=k_0}^{k-1}\Phi_\delta(k,j+1)(B+\delta_B) \cdot[\psi_s(-L\hat{x}(j)) - P\hat{x}(j)], \quad k \geq k_0+1. \tag{A.23}$$

It is assumed that the model in closed-loop is stable and bounded by $\|\overline{\Phi}_{\mathrm{CL}}(k,k_0)\| \leq \gamma\lambda^{k-k_0}$, $k \geq k_0$, with $0 \leq \lambda < 1$ and $\gamma \geq 1$, and that the model is unstable in open-loop, but bounded by $\|\Phi(k,k_0)\| \leq \alpha\beta^{k-k_0}$, $k \geq k_0$, with $\beta > 1$ and $\alpha \geq 1$.

For bounded model uncertainties $\|\delta_A\| \leq \sigma_{\delta_A}$, and considering the bound on $\|\Phi(k,k_0)\|$ with $\beta > 1$ (this corresponds to assume an unfavorable situation), it was proved in Appendix A.1 that $\|\Phi_\delta(k,k_0)\| \leq \alpha(\beta+\alpha\cdot\sigma_{\delta_A})^{k-k_0}$ with $(\beta+\alpha\cdot\sigma_{\delta_A}) > 1$. A similar proof can be given for the stability of the plant in closed-loop since the model is stable in closed-loop ($\|\overline{\Phi}_{\mathrm{CL}}(k,k_0)\| \leq \gamma\lambda^{k-k_0}$ with $0 \leq \lambda < 1$). Recurring to Lemma 1, and considering that $(\delta_A - \delta_B P)$ is seen as a perturbation in the system $x(k+1) = [(A+\delta_A)-(B+\delta_B)P]x(k)$, it can be proved that $\|\overline{\Phi}_{\delta_{\mathrm{CL}}}(k,k_0)\| \leq \gamma(\lambda+\gamma\Sigma)^{k-k_0}$, $k \geq k_0$ with $0 \leq \Sigma < (1-\lambda)/\gamma$, and $\Sigma := \sigma_{\delta_A} + \sigma_{\delta_B}\cdot\sigma_P$, [17].

Upper bounds for (A.21) during availability time intervals, and for (A.23) during unavailability time intervals, are obtained, respectively

$$\|x(k)\| = \left\| \overline{\Phi}_{\delta_{\mathrm{CL}}}(k,k_0)x(k_0) + \sum_{j=k_0}^{k-1}\overline{\Phi}_{\delta_{\mathrm{CL}}}(k,j+1)(B+\delta_B)\psi_s(-Lx(j)) \right\|,$$
$$\|x(k)\| = \left\| \Phi_\delta(k,k_0)x(k_0) + \sum_{j=k_0}^{k-1}\Phi_\delta(k,j+1)(B+\delta_B) \cdot[\psi_s(-L\hat{x}(j)) - P\hat{x}(j)] \right\|. \tag{A.24}$$

From (A.24), recalling that $\|B\| := \sigma_B$, $\|\delta_B\| \leq \sigma_{\delta_B}$, $\|L\| := \sigma_L$ and considering (24) yield, respectively

$$\|x(k)\| \leq \gamma(\lambda+\gamma\Sigma)^{k-k_0}\cdot\|x(k_0)\| + \sum_{j=k_0}^{k-1}\gamma(\lambda+\gamma\Sigma)^{k-j-1}\cdot(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L\cdot\|x(j)\|, \tag{A.25}$$

$$\|x(k)\| \leq \alpha(\beta+\alpha\cdot\sigma_{\delta_A})^{k-k_0}\cdot\|x(k_0)\| + \sum_{j=k_0}^{k-1}\alpha(\beta+\alpha\cdot\sigma_{\delta_A})^{k-j-1} \cdot(\sigma_B+\sigma_{\delta_B})(\sigma_P+\gamma_2\cdot\sigma_L)\cdot\|\hat{x}(j)\|. \tag{A.26}$$

Applying Lemma 1 to (A.25) gives

$$\|x(k)\| \leq \gamma[(\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L]^{k-k_0}\cdot\|x(k_0)\|. \tag{A.27}$$

An upper bound for (A.22) is obtained from

$$\|\hat{x}(k)\| \leq \gamma\lambda^{k-k_0}\cdot\|x(k_0)\| + \sum_{j=k_0}^{k-1}\gamma\lambda^{k-j-1}\sigma_B\cdot\gamma_2\cdot\sigma_L\cdot\|\hat{x}(j)\|. \tag{A.28}$$

Applying Lemma 1 and recalling that $\hat{x}(k_0) = x(k_0) = x(k_0-1)$ yield

$$\|\hat{x}(k)\| \leq \gamma(\lambda+\gamma\cdot\sigma_B\cdot\gamma_2\cdot\sigma_L)^{k-k_0}\cdot\|x(k_0)\|. \tag{A.29}$$

Using (A.29) in (A.26),

$$\|x(k)\| \leq \Big[\alpha(\beta+\alpha\cdot\sigma_{\delta_A})^{k-k_0} + \alpha(\sigma_B+\sigma_{\delta_B})(\sigma_P+\gamma_2\cdot\sigma_L)\gamma \cdot (\beta+\alpha\cdot\sigma_{\delta_A})^{k-1}\cdot(\lambda+\gamma\cdot\sigma_B\cdot\gamma_2\cdot\sigma_L)^{-k_0} \cdot \sum_{j=k_0}^{k-1}\left(\frac{\lambda+\gamma\cdot\sigma_B\cdot\gamma_2\cdot\sigma_L}{\beta+\alpha\cdot\sigma_{\delta_A}}\right)^{j}\Big]\cdot\|x(k_0)\|. \tag{A.30}$$

Making use of (A.3) in (A.30) gives

$$\|x(k)\| \leq \Big[\alpha(\beta+\alpha\cdot\sigma_{\delta_A})^{k-k_0} + \alpha(\sigma_B+\sigma_{\delta_B})(\sigma_P+\gamma_2\cdot\sigma_L)\gamma \cdot \frac{(\beta+\alpha\cdot\sigma_{\delta_A})^{k-k_0} - (\lambda+\gamma\cdot\sigma_B\cdot\gamma_2\cdot\sigma_L)^{k-k_0}}{(\beta+\alpha\cdot\sigma_{\delta_A}) - (\lambda+\gamma\cdot\sigma_B\cdot\gamma_2\cdot\sigma_L)}\Big] \cdot\|x(k_0)\|. \tag{A.31}$$

Providing $\gamma_2 < (\beta+\alpha\cdot\sigma_{\delta_A}-\lambda)/(\gamma\cdot\sigma_B\cdot\sigma_L)$, and considering (A.4), after some calculations, (A.31) yields

$$\|x(k)\| \leq \left[1+\frac{(\sigma_B+\sigma_{\delta_B})\cdot(\sigma_P+\gamma_2\cdot\sigma_L)\gamma}{(\beta+\alpha\cdot\sigma_{\delta_A}) - (\lambda+\gamma\cdot\sigma_B\cdot\gamma_2\cdot\sigma_L)}\right] \cdot \alpha(\beta+\alpha\cdot\sigma_{\delta_A})^{k-k_0}\cdot\|x(k_0)\|. \tag{A.32}$$

From this point on, the demonstration follows closely the one of Theorem 1 (see Appendix A.1), and applying results (A.27) and (A.32) originates

$$\|x(k)\| \leq c_2^{(i+1)/2}\cdot(\beta+\alpha\cdot\sigma_{\delta_A})^{T_u} \cdot\left[(\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L\right]^{T_a}\cdot\|x_0\|, \tag{A.33}$$

where T_u and T_a represent the entire duration of all unavailability and availability time intervals, respectively, and

$$c_2 := \left[1+\frac{(\sigma_B+\sigma_{\delta_B})\cdot(\sigma_P+\gamma_2\cdot\sigma_L)\gamma}{(\beta+\alpha\cdot\sigma_{\delta_A}) - (\lambda+\gamma\cdot\sigma_B\cdot\gamma_2\cdot\sigma_L)}\right]\alpha\gamma. \tag{A.34}$$

In order for the system to be uniformly stable, it must verify $\|x(k)\| \leq M_1\|x(k_0)\|$, $k \geq k_0$ with $M_1 \geq 1$. Therefore, from (A.33)

$$c_2^{(i+1)/2}\cdot(\beta+\alpha\cdot\sigma_{\delta_A})^{T_u}\cdot\left[(\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L\right]^{T_a} \leq M_1 \Longrightarrow T_u \leq \frac{\log M_1 - ((i+1)/2)\log c_2}{\log(\beta+\alpha\cdot\sigma_{\delta_A})} - \frac{T_a\log\left[(\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L\right]}{\log(\beta+\alpha\cdot\sigma_{\delta_A})}. \tag{A.35}$$

Replacing (A.34) in (A.35) gives the desired result from Theorem 2 subject to the constraints $\Sigma < (1-\lambda)/\gamma$, and $\gamma_2 < (\beta+\alpha\cdot\sigma_{\delta_A}-\lambda)/(\gamma\cdot\sigma_B\cdot\sigma_L)$. The result holds globally since it is valid for any $\|x(k_0)\|$. □

Proof of Corollary 3. Consider the Euclidean norm of $x(k)$ at discrete times $k = T_{i-1}$, and $k = T_{i-3}$, at the end of the unavailability intervals $T_{u_{i-1}}$, and $T_{u_{i-3}}$, respectively. In order for $\|x(T_j)\|$, for $j = 0, 2, 4, 6, \ldots, i-1$, to be a monotonic descent sequence, it should verify

$$\frac{\|x(T_{i-1})\|}{\|x(T_{i-3})\|} < 1. \tag{A.36}$$

This proof is outlined in the very same way as Corollary 1 proof, therefore, the following equation yields naturally after Theorem 2 proof calculations:

$$T_{u_{i-1}} < \frac{-\log c_2 - T_{a_{i-2}}\log\left[(\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L\right]}{\log(\beta+\alpha\cdot\sigma_{\delta_A})}. \tag{A.37}$$

Replacing (A.34) in (A.37) gives the desired result from Corollary 3 subject to the constraints $\Sigma < (1-\lambda)/\gamma$, and $\gamma_2 < (\beta+\alpha\cdot\sigma_{\delta_A}-\lambda)/(\gamma\cdot\sigma_B\cdot\sigma_L)$. The result holds globally since it is valid for any $\|x(k_0)\|$. □

Proof of Corollary 4. In order for the system to be uniformly exponentially stable, it must verify $\|x(k)\| \leq M_2N_2^{k-k_0}\|x(k_0)\|$, $k \geq k_0$, with $M_2 \geq 1$, and $0 \leq N_2 < 1$. Therefore, from (A.33), and considering $k_0 = 0$, and $(k-k_0) = T_u + T_a$,

$$c_2^{(i+1)/2}\cdot(\beta+\alpha\cdot\sigma_{\delta_A})^{T_u}\cdot\left[(\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L\right]^{T_a} \leq M_2N_2^{T_u+T_a} \Longrightarrow T_u \leq \frac{\log M_2 - ((i+1)/2)\log c_2}{\log((\beta+\alpha\cdot\sigma_{\delta_A})/N_2)} - \frac{T_a\log\left(\dfrac{\left[(\lambda+\gamma\Sigma)+\gamma(\sigma_B+\sigma_{\delta_B})\gamma_2\cdot\sigma_L\right]}{N_2}\right)}{\log((\beta+\alpha\cdot\sigma_{\delta_A})/N_2)}. \tag{A.38}$$

Replacing (A.34) in (A.38) gives the desired result from Corollary 4 subject to the constraints $\Sigma < (1-\lambda)/\gamma$ and $\gamma_2 < (\beta+\alpha\cdot\sigma_{\delta_A}-\lambda)/(\gamma\cdot\sigma_B\cdot\sigma_L)$. The result holds globally since it is valid for any $\|x(k_0)\|$. □

ACKNOWLEDGMENT

This work was produced in the framework of the project IDEA—Integrated Design for Automated Anaesthesia, PTDC/EEA-ACR/69288/2006.

REFERENCES

[1] E. Halberg, I. Kaminer, and A. Pascoal, "Development of a flight test system for unmanned air vehicle," *IEEE Control Systems*, vol. 19, no. 1, pp. 55–65, 1999.

[2] J. W. Overstreet and A. Tzes, "An Internet-based real-time control engineering laboratory," *IEEE Control Systems*, vol. 19, no. 5, pp. 19–34, 1999.

[3] J. Tugnait, "Detection and estimation for abruptly changing systems," *Automatica*, vol. 18, pp. 607–615, 1982.

[4] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of networked control systems," *IEEE Control Systems*, vol. 21, no. 1, pp. 84–99, 2001.

[5] T. Estrada, H. Lin, and P. J. Antsaklis, "Model-based control with intermittent feedback," in *Proceedings of the 14th Mediterranean Conference on Control and Automation (MED '06)*, pp. 1–6, Ancona, Italy, June 2006.

[6] H. Yang and M. Saif, "Observer design and fault diagnosis for state-retarded dynamical systems," *Automatica*, vol. 34, no. 2, pp. 217–227, 1998.

[7] D. Loebis, R. Sutton, J. Chudley, and W. Naeem, "Adaptive tuning of a Kalman filter via fuzzy logic for an intelligent AUV navigation system," *Control Engineering Practice*, vol. 12, no. 12, pp. 1531–1539, 2004.

[8] S. M. Bennett, R. J. Patton, and S. Daley, "Sensor fault-tolerant control of a rail traction drive," *Control Engineering Practice*, vol. 7, no. 2, pp. 217–225, 1999.

[9] C. J. Lopez-Toribio and R. J. Patton, "Takagi-Sugeno fuzzy fault-tolerant control for a non-linear system," in *Proceedings of the 38th IEEE Conference on Decision and Control (CDC '99)*, vol. 5, pp. 4368–4373, Phoenix, Ariz, USA, December 1999.

[10] O. R. Gonzalez, W. S. Gray, A. Tejada, and S. Patilkulkarni, "Stability analysis of upset recovery methods for electromagnetic interference," in *Proceedings of the 40th IEEE Conference on Decision and Control*, vol. 5, pp. 4134–4139, Orlando, Fla, USA, December 2001.

[11] J. M. Lemos, H. Magalhães, T. Mendonça, and R. V. Dionísio, "Control of neuromuscular blockade in the presence of sensor faults," *IEEE Transactions on Biomedical Engineering*, vol. 52, no. 11, pp. 1902–1911, 2005.

[12] J. M. Lemos, H. Magalhães, R. V. Dionísio, and T. Mendonça, "Control of physiological variables in the presence of interrupted feedback measurements," in *Proceedings of the 16th International Conference on Systems Engineering (ICSE '03)*, vol. 2, pp. 433–438, Coventry, UK, September 2003.

[13] R. V. Dionísio and J. Lemos, "Stability results for discrete systems controlled in the presence of interrupted observations," in *Proceedings of the 11th IEEE Mediterranean Conference on Control and Automation (MED '03)*, Rhodes, Greece, June 2003.

[14] H. K. Khalil, *Nonlinear Systems*, Prentice Hall, Upper Saddle River, NJ, USA, 2nd edition, 1996.

[15] W. M. Haddad and V. Kapila, "Static output feedback controllers for continuous-time and discrete-time systems with input non-linearities," *European Journal of Control*, vol. 4, pp. 22–31, 1998.

[16] P. O. M. Scokaert, J. B. Rawlings, and E. S. Meadows, "Discrete-time stability with perturbations: application to model predictive control," *Automatica*, vol. 33, no. 3, pp. 463–470, 1997.

[17] W. J. Rugh, *Linear System Theory*, Prentice-Hall, Upper Saddle River, NJ, USA, 2nd edition, 1996.