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Developing flexible-adaptive reasoning and computing: Pedro's understanding of the task "Prawn skewers"

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Abstract

The project 'Numerical thinking and flexible calculation: critical issues' aims to study students' conceptual knowledge associated with the understanding of the different levels of learning numbers and operations. We follow the idea proposed by several authors that *flexibility* refers to the ability to manipulate numbers as mathematical objects which can be decomposed and recomposed in multiple ways using different symbolisms for the same objet (Gravemeijer, 2004; Gray &Tall, 1994;). The project plan is based on a qualitative and interpretative methodology (Denzin & Lincoln, 2005) with a design research approach (Gravemeijer & Cobb, 2006). This article focus the preparation of a teaching experience centered in the flexible learning of multiplication. It describes the analysis of a clinical interview where Pedro (9 years) solves the task 'Prawn skewers'. It illustrates how we identify and describe Pedro's conceptual knowledge associated with the different levels of understanding of numbers and multiplication/division and analyzes if and how this knowledge facilitates adaptive thinking and flexible calculation.

Key words: Multiplicative reasoning; Flexible calculation; Mathematical tasks

Introduction

Acquiring proficiency **with whole numbers, fractions and decimals is one of the main goals in the reform curriculum** (Kilpatrick, Swafford, & Findell, 2001). In this perspective, researchers focus on what Hatano and Inagaki (1986) called "adaptive expertise". In Hatano´s (2003) words, the general question from this perspective is how students can be taught so that they learn to use what they have learned and invent effective procedures to solve new problems. Specifically, we have to identify: i) first the conceptual knowledge underlying procedures ("operating knowledge", Thompson &

Saldanha, 2003), and ii) second, how students do construct it. A lot of researchers as Baroody (2003) and Thompson and Saldanha (2003) assume that adaptive expertise depends on conceptual understanding (well-connected knowledge) and its integration with procedural knowledge.

In this paper we focus on the understanding of multiplication and the flexible and adaptive use of "products" as one aspect of 'multiplicative reasoning' that children have to develop along primary education. We illustrate how the research team of the project *Numerical thinking and flexible calculation: critical issues* tries to identify and describe students' conceptual knowledge associated with the different levels of understanding of numbers and multiplication/division and analyze if and how this knowledge facilitates adaptive thinking and flexible calculation.

We present the theoretical justification and the design principles that we use to develop each mathematical task. We also present how one nine year student – Pedro – interprets the task and how we analyze and interpret his way of thinking.

Theoretical framework

General idea

Authors like Freudenthal (1991) and Sfard (1991) characterize the process of constructing mathematics as a series of transitions in which mathematical processes are transformed in objects, which in turn became part of new processes. From this point of view they stress the importance of organizing the learning of mathematics from a conceptual development perspective.

We use this stand point as a model to organize the development of flexible and adaptive reasoning and computing, in the domain of addition-subtraction and multiplication-division with integers and fractions. This means that we use the learning process of humanity to model individual learning in mathematics education (Freudenthal, 1991; Sfard, 1994;Tall, 2004).

We follow the idea proposed by several authors that flexibility refers to the ability to manipulate numbers as mathematical objects which can be decomposed and recomposed in different ways using properties of the arithmetic operations and different symbolisms for the same object (Baroody, 2003; Gravemeijer, 2004; Gray & Tall, 1994; Sfard, 1991; Verschaffel, Greer & De Corte, 2007). This means that, for the comprehension of `multiplication´, children have to understand that `4 × 5´ both represent the process of multiplying (the procedure `four multiplied by five´), and the number 'twenty' (the concept of 'product'), and that this number is associated with a lot of other numbers (objects) in a great network of multiplicative relations, for instance 5 × 4, 2,5 × 8, 2 × 10, 40 × ½, etc.

On one side, we consider this dual nature of mathematical conception and the hierarchical nature of the development of multiplicative reasoning. On the other side, we consider what Gray and Tall (1994) called 'proceptual reasoning' (the way of thinking arithmetically), based on the intelligent use of the ambiguity of mathematical symbolism and connections between concepts and procedures, as prerequisite to become flexible and inventive expert in the sense of Hatano (1982, 2003).

Core elements of the conceptual analysis of multiplicative reasoning

We globally present the theoretical framework we constructed to develop the instructional sequence in our teaching experiment called *Flexible and adaptive reasoning and computing*. Following the idea of conceptual analysis of Thompson and Saldanha (2003), we organized a chart of what Sfard (1991) called the operational (as process) and structural (as objects) conceptions/understanding of ´multiplication´. This means that we identified and connected the core elements of conceiving/understanding problems `multiplicatively´ in situations related with `measuring´, `multiplication´, `division´, `fractions´ and ´proportionality´, regardless of the nature of the quantities (discrete and continuous) and the numbers (integers, decimals, fractions) (Figure 1).

In this paper we focus on the understanding of ´multiplication´ with whole numbers and discrete quantities. We take into account five aspects of multiplication referred in the theoretical literature:

i. Multiplication of whole numbers is the "systematic creation of units of units". What means 'quantifying something made of identical copies of some quantity' and/or 'envisioning the result of having multiplied' – 'anticipating a multiplicity', mentally, and before calculating (Thompson & Saldanha, 2003).

ii. This involves the use of the expression '… × ….' with two complementary meanings: as the symbolic notation of the process of multiplication, and as representation of a number or fraction of some quantity (Gray & Tall,1994; Sfard, 1991; Thompson & Saldanha, 2003).

iii. 'Times' is used to compare two quantities of objects or measures (multiplicative comparison) (Freudenthal, 2002; Vergnaud, 1983, 1988): The '3' in '3 rolls' is the number that is 3 times as large as one roll and the cost of 3 rolls is 3 times as large as the cost of one roll. Analyzing this type of situations children abstract the idea that the product increases as the increasing factor.

iv. Envisioning multiplicities and comparing the quantity structures, children discover the corresponding proportional relations involved in the product. We distinguish two clusters of mathematical relations:
   - given the product, to maintain the relation between the quantities, increasing one factor implies decreasing the other. 2 × 10 is as much as 4 × 5 and 6 × 4 is as much as 12 × 2;
   - given a product as 4 × 5 = 20, if 4 is 1/5 of 20 then5 is ¼ of 20 and reciprocally.

v. These relations conceptually connect multiplication with division (Freudenthal, 1981) and open the possibility to conceive the isomorphism of the two division structures (portioning/distributive division and segmenting/ratio division) (Freudenthal, 1981; Thompson & Saldanha,2003).

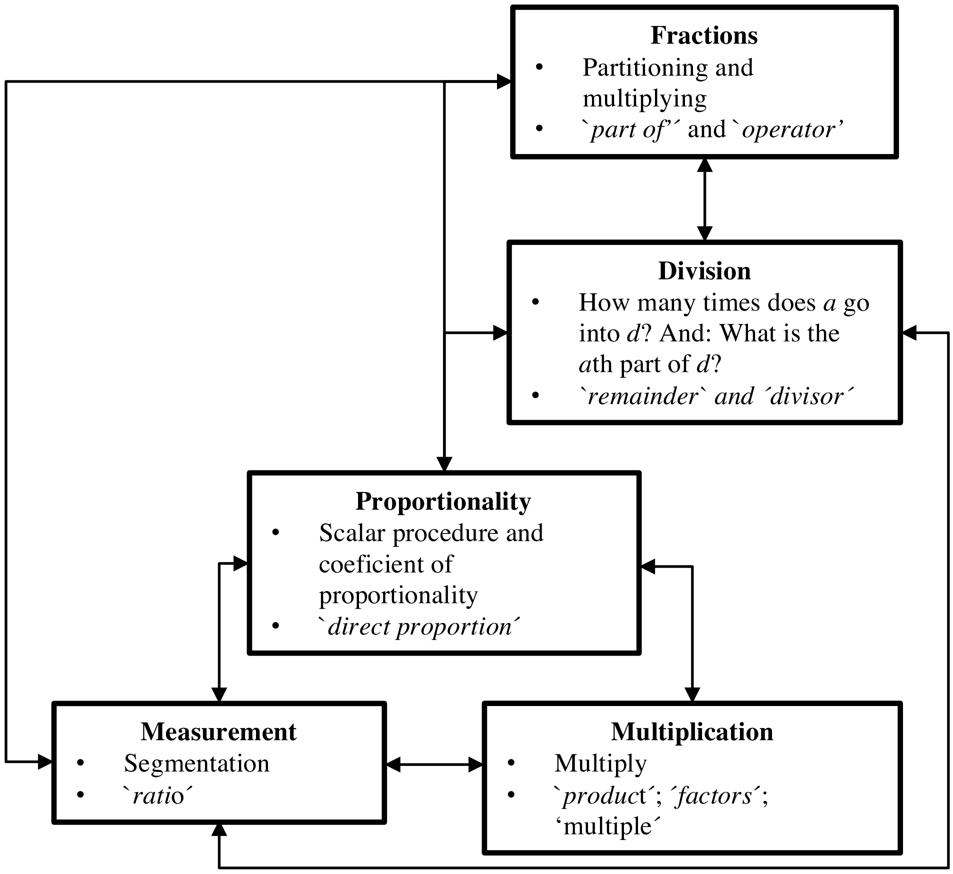

Figure 1. Core elements of the conceptual analysis of multiplicative reasoning

Methodology

The project plan is based on a qualitative and interpretative methodology (Denzin & Lincoln, 2005) with a design research approach (Gravemeijer & Cobb, 2006). The preparation of teaching experiences is a crucial aspect of the project plan. To prepare teaching experiences we design and reformulate mathematical tasks using a three steps cyclic process: (1) design tasks thought as adequate to develop students' flexible and adaptive mental calculation, (2) analyze what children noticed in the numbers and how they use their knowledge about numbers and operations to solve the task presented along clinical interviews and (3) reformulate the previous task:

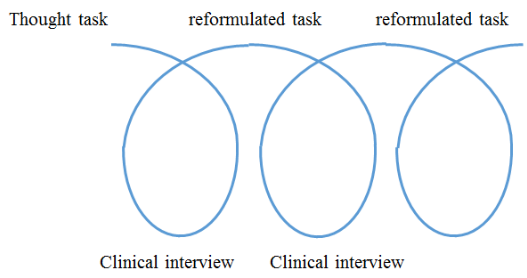

Figure 2 - The cyclic process of designing and reformulating tasks

The tasks developed by the project team are problems that can be solved using different manipulations of the context and different numerical relations and proprieties, according with what students see in the numbers of the task.

During clinical interviews, conducted by one of the researchers of the project, children are encouraged to freely explore the task. They also are asked to explain their approach to the task, justifying their reasoning and computation process. The researcher also makes clear that they can change their initial approach and adopt another one, considered more suitable. All clinical interviews were audio taped and transcript. Data is analyzed and interpreted keeping the objective of developing adaptive reasoning.

This article focuses on the analysis of a clinical interview where Pedro (9 years) solves the task 'Prawn skewers'. It illustrates how we try to identify and describe Pedro's conceptual knowledge associated with the different levels of understanding of numbers and multiplication/division and analyze if and how this knowledge facilitates adaptive thinking and flexible calculation.

Task design

We use the example of the task `Prawn skewer´ to present the general principle of the task design and the core characteristics of the common characteristics of experimental tasks.

For Vasco's birthday lunch he prepares prawn skewers. He hesitates between using three or five prawns in each skewer.

1. Can you explain what Vasco is thinking?
- How would you explain it to one of your colleagues?
- Which type of skewers would you prepare? Why?
2. Vasco is counting the prawns that his mother bought:

... 52, 54, 56, 58, 60, 61!
- Think about your choice. Imagine the number of skewers you can do with this number of prawns. How many, more or less? More than 5? More than 10? More than 20? ...
- How would you find the exact number of skewers?

Figure 3 - The tasks 'Prawn skewer'

General design principle. According to Freudenthal (1973), the students should be given the opportunity to experience a process similar to the process by which a given piece of mathematics was invented. It means, that we must imagine a route that would allow the students to connect the elements of their understanding of multiplication as mapped in figure 1 and to discover which concept and/or procedure allow them to shift from one point of view to the other in order to use efficiently well-known relation of facts to find what they don't know.
To design the tasks we take into account three different aspects of students' activity: orientation, mathematical discourse topics and levels of reasoning and calculating.

Orientation. The idea of the task *Prawn skewers* is to introduce multiplication in situations that stimulate children to "envision something in a particular way- to think of copies (including parts of copies) of some amount" (Thompson & Saldanha, 2003, p. 24).

In this perspective, contexts, given numbers and relationships, illustrations and / or models available to students, focus their attention on the relationship between the quantities in question and /or the structure of these relationships (order structure). This is different from conventional tasks that engage in calculating a product or quotient - the process of multiplying / dividing.

As Thompson and Saldanha (2003), we consider, in the perspective of flexible and adaptive reasoning and computation required at the end of primary education (12 years), that "non-calculational ways to think of products will be important in comprehending situations in which multiplicative calculations might be useful. The comprehension will enable students to decide on appropriate actions" (p. 24-25).

Mathematical discourse topics. In the context of a birthday, students will describe and compare two ways of making prawn skewers and justify their own preference - choose 3 skewers prawns or 5 skewers prawns.

From an arithmetic point of view they have to build, compare and segment multiplicatively imagined quantities of identical objects (skewers with 3 or 5 prawns), using memorized facts and their concept/understanding of ´product', 'proportionality', properties of multiplication and equivalence between multiplication and division.

In the terminology of Vergnaud (1983, 1988) the proposed situation belongs to the class of isomorphic problems of measurement. This is a situation that can be analyzed in terms of simple ratio of the measure of two quantities, in this case the number of prawns and the number of skewers.

Levels of reasoning and calculating. In this context and with this numbers students can think and calculate in different levels of conceptualization and / or understanding and use of multiplicative relationships and familiarity with calculation procedures (multiplicative and / or proportional).

The literature (for instance Vergnaud, 1983, 1988) identifies three levels:

- additive reasoning through counting all the represented objects (repetitive addition);

- multiplicative reasoning starting from the representation of the structure of accumulation described with the product.

- proportional reasoning focusing on the relations between the factors and the product.

Analyzing Pedro's multiplicative reasoning

We selected 13 episodes that are relevant to understand Pedro's conceptual knowledge associated with the different levels of understanding of numbers and multiplication/division. We analyze these episodes in the light of the conceptual framework of multiplicative reasoning that we presented earlier.

Episode 1

Pedro: It is like this, he will make three, five skewers with three prawns, is it not? **Researcher**: See if he is going to prepare three skewers ... **Pedro**: No, three skewers with five prawns.	*Pedro imagines the quantity in two different ways (five "threes" and three "fives") and verbalizes this correctly.* *Pedro uses the product as a symbol of a quantity and does not mention the total number of prawns.*

Episode 2

Researcher: If you were Vasco what would you do? **Pedro**: I would prepare … prepare 5 skewers with … I don't know!	*Temporary confusion (he doesn't associates 5 with the number of prawns, he associates it with the number of skewers)* *From the measure point of view Pedro confuses the number of groups with the number of unities in each group (measure).*

Episode 3

Researcher: Five skewers? **Pedro**: No, three skewers, for instance, with 5 skewers. **Researcher**: You would prepare skewers with more prawns, with 5? **Pedro:** Yes. And to prepare 3 skewers I would use 5 prawns in each one. I had 3 skewers and I would do 5 times 3 that is 15. So, there were 15 prawns, 5 in each skewer, adding it all I would have 15.	*Pedro corrects his own thinking.* *Verbalization of the structure 3×5.* *Pedro speaks of a total quantity of prawns and associates 5×3 to describe the structure that gives 15.* *Pedro repeats a verbalization, but with another expression.*

Episode 4

Pedro: [reads "*52, 54, 56, 58, 60, 61. Think about your choice. Imagine the number of skewers you can do with this number of*	*Does he continue or not the table of 5? Is*

prawns. How many, more or less? More than 5?] More than 5. (…) **Researcher**: Why? **Pedro:** Because … hum … 13 times 5 are 65.	*this a known fact?*

Episode 5

Researcher: And no more than 10? **Pedro**: Hum, hum! **Researcher**: And it is more than 20? **Pedro**: Probably. Yes! It is more than 20. **Researcher**: Are you sure? **Pedro**: Yes. **Researcher**: Why? **Pedro**: Because 20 times 1 is 20, 20 times 2 is 40, 20 times 3 is 60, until 5 prawns in each skewer. As until now I have 60, this gives me 80, and this is more than 61.	*Confusion related with the number 20 that seems to be associated to a pattern with the multiples of 20 that Pedro knows.* *Another inconsistence between the image of the structure and the number that symbolizes the product. What Pedro verbalizes does not match with the structure: he thinks 2×20 and he says 20×2.*

Episode 6

Pedro: Than, if each skewer has only 5 he only can invite 12 friends. **Researcher**: How do you know it is 12? Explain it to me. **Pedro**: Because if we take here 13, we do 13 times 5 that is 65. If I have 12 times 5 I have 60. [writes 12×5 = 60 + 1 = 61]	*Connection between 13×5 and 12×5 and the understanding of remainder.*

Episode 7

Researcher: How do you know so quickly that 12 times 5 is 60? Explain it to me.	*Pedro justifies that 1 prawn corresponds to the remainder.*

Pedro: Because 10 times 5 is 50, so 11 times 5 is 55 and 12 times 5 is 60! **Researcher**: Good! **Pedro**: And here more 1 it gives 61. So there is one prawn left. **Researcher**: There is one prawn left? **Pedro**: Yes because he only can invite 12 friends. I've already done 13x5 that gives 65, so I have more than the prawns I had.	

Episode 8

Researcher: And if we wanted to invite more friends, what could he do? **Pedro:** Probably he would take less skewers. **Researcher:** The number of prawns in each skewer … **Pedro:** Yes, or he could ask the mother to buy more. **Researcher:** But imagine that she couldn't do it. **Pedro:** Than he can take fewer prawns in each skewer.	Pedro is able to express a proportional reasoning: more friends, less prawns per skewer.

Episode 9

Pedro: He could do … I'll do like this, as 30, still is less, 60 is 20 times 3, it is 60. **Researcher:** Write it here. [he writes $20\times3 = 60 + 1 = 61$] **Pedro:** I have to have 20 friends. He can invite 20 friends. **Researcher:** Ah, if he wants to invite 20 friends …, write it there. **Pedro:** 20 times 3 is 60 60 plus 1 is 61.	*Pedro associates 60 to 3×20 or to 20×3?* *He repeats the justification for the remainder.*

Episode 10

Researcher: If he wants to invite more friends he can invite up to how many? **Pedro:** Up to 60, oh, up to 20 friends. **Researcher:** 20 friends. **Pedro:** Because the number of prawns decreases and this gives more friends.	*Confirmation of the understanding of proportional relationships between the number of prawns and the number of skewers.*

Episode 11

Pedro: I do not know if you can do it but if I diminish increasingly the number of prawns, I could invite more friends! **Researcher:** Can you explain this? **Pedro:** Because, for instance, 30×2 is 60, I invite 30 friends; 1×60 is 60, I invite 60 friends. **Researcher:** How many prawns did each friend eat? **Pedro:** 1.	*Example of isomorphic structures: 30x2; 20x3; 1x60 and its interpretation in the context that is being discussed.*

Concluding remarks

We propose the following conclusion, using the core aspects of understanding presented in the beginning of the paper. We focus on two questions. First, what tells us Pedro´s activity during this session about his understanding of multiplication? Second, what could we say about his ability of `adaptive expert´?

Understanding of multiplication

Multiplication of whole numbers. Pedro's description of the way of making skewers and the structure of each skewer shows his understanding of multiplication as the 'systematic creation of units of units' (Thompson & Saldanha, 2003). He correctly describes the two multiplicative structures in question: $n \times 3$ and $n \times 5$.

'Times'. Pedro used spontaneously the notion that the product increases as the increasing factor. He uses the expression $n \times 3$ and $n \times 5$ as symbol for the number of prawns. He connects 5 × 3 with 15 and associates at the same time this number with 3 × 5.

We don't know how Pedro finds the product 13 × 5. It could be through repetitive addition or through the continuation of the table of 5, from 10 × 5 to 13 × 5.

Pedro describes with understanding the difference between 12 × 5 that gives 60 and 13 × 5 that gives 65, noticing that the second product is greater that the number of prawns available (see division).

Proportional relations between factors and product. Pedro describes with 'a kind of common sense´ the proportional relation between the number of prawns and the number of skewers. He understands that increasing the number of units implies decreasing the number of groups.

Relation with division. Pedro uses multiplicative structure to find the number of skewers that he can prepare with 61 prawns. He compares multiplicatively 12 × 5 and 13 × 5, interprets this with the available numbers of prawns, understands that some quantities cannot be exactly divided and symbolizes this arithmetically, using the notion of remainder: 12 × 5 = 60 + 1 = 61.

Local temporary confusions. One temporary/local confusion focus the attention on the transition from thinking in terms of the relation between the number of units and the number of groups and reasoning in terms of the relation between the product and its two factors.

Pedro confuses 5 skewers with 3 prawns and 3 skewers with 5 prawns.
It seems that Pedro´s familiarity with the pattern of multiples of 20 (40, 60, …) inhibits him to estimate the number of skewers.

Adaptive expertise

Pedro seems to be relatively familiar with these type of multiplicative structures. Hatano (2003) associates expertise with the use of what is learned to invent effective procedures for solving new problems. We identify five elements that could allow Pedro to learn to do this:

1. the use of product (3 × 5; 13 × 5) and of expressions as `12 × 5 + 1´ as symbols for a quantity;
2. the use of memorized facts as 3 × 5 = 15 and 3 × 20 = 60;
3. reasoning on the base of properties of multiplication, for instance commutative (3 x 5 = 5 × 3) and distributive (13 × 5 = 12 × 5 + 1 × 5);
4. the solution of the ratio division (How many times does 5 go into 61) through seeing division as the inversion of multiplication ;
5. The association of the remainder with the pattern of multiples of *n*.

This combination of operational and structural knowledge about `multiplication´ (Sfard, 1991) connected with the use of the ambiguity of expression as 3 × 5 and 13 × 5 (Gray & Tall, 1994) seems to give Pedro the prerequisites to think and operate in a flexible and inventive way in the field of multiplication and division problems.

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