Determination of a time-dependent free boundary in a two-dimensional parabolic problem

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Abstract

The retrieval of the timewise-dependent intensity of a free boundary and the temperature in a two-dimensional parabolic problem is, for the first time, numerically solved. The measurement, which is sufficient to provide a unique solution, consists of the mass/energy of the thermal system. A stability theorem is proved based on the Green function theory and Volterra's integral equations of the second kind. The resulting nonlinear minimization is numerically solved using the *lsqnonlin* MATLAB optimization routine. The results illustrate the reliability, in terms of accuracy and stability, of the time-dependent free surface reconstruction.

Keywords: Inverse problem; Free boundary; Two-dimensional parabolic equation.

1 Introduction

Free boundary problems for parabolic equations occur naturally in many branches of physics, engineering, chemistry, biology and other areas, see [5, 9, 12] to mention only a few. For instance, the simultaneous identification of transient coefficients and multiple unknown free boundaries were recently investigated by the authors in [15], whilst in [17], free boundary problems with nonlinear diffusion were considered. The numerical solution for inverse free boundary and Stefan problems, utilizing a meshless method, was obtained in [14, 20]. The heat/diffusion equation with an unknown timewise-dependent diffusivity or source, along with a free boundary was also investigated in [16] and [23], respectively. In [18], the authors investigated the determination of multiple time-dependent coefficients together with an unknown free boundary. Aggregation and nonlocal diffusive processes were discussed in [8]. However, only a few papers are concerned with time-dependent free boundary for parabolic equations in two-dimensions or more, [1, 2, 19]. These studies are theoretical and they are important because they establish sufficient conditions for the well-poseness of the inversely formulated problems. However, no numerical realization has been attempted and it is the goal of this paper to numerically recover the unknown time-dependent free boundary coefficient in a stable and accurate manner, along with the temperature from over-determination mass (average temperature) data.

2 Mathematical formulation

Consider the moving region $\Omega_T := \{(y_1, y_2, t)|\ 0 < y_1 < h,\ 0 < y_2 < g(t)\Psi(y_1),\ 0 < t < T\}$, with a free boundary of unknown intensity $g = g(t) > 0$, but with known space variation $\Psi(y_1) > 0$, we consider the two-dimensional parabolic equation

$$u_t = \Delta u + b_1(y_1, y_2, t)u_{y_1} + b_2(y_1, y_2, t)u_{y_2} + c(y_1, y_2, t)u + f(y_1, y_2, t), \ (y_1, y_2, t) \in \Omega_T, \quad (1)$$

for the unknown dependent variable $u(y_1, y_2, t)$, herein called temperature, given the known convection coefficients b_1 and b_2, reaction coefficient c and heat source f. At time $t = 0$ we have prescribed

$$u(y_1, y_2, 0) = \varphi(y_1, y_2), \quad 0 \leq y_1 \leq h, \quad 0 \leq y_2 \leq g(0)\Psi(y_1), \quad (2)$$

and the Dirichlet boundary conditions are

$$\begin{cases} u(0, y_2, t) = \mu_1(y_2, t), \quad 0 \leq y_2 \leq \Psi(0)g(t), \quad 0 \leq t \leq T, \\ u(h, y_2, t) = \mu_2(y_2, t), \quad 0 \leq y_2 \leq \Psi(h)g(t), \quad 0 \leq t \leq T, \\ u(y_1, 0, t) = \mu_3(y_1, t), \quad 0 \leq y_1 \leq h, \quad 0 \leq t \leq T, \\ u(y_1, \Psi(y_1)g(t), t) = \mu_4(y_1, t), \quad 0 \leq y_1 \leq h, \quad 0 \leq t \leq T. \end{cases} \quad (3)$$

The additional mass/energy specification is given by

$$\int_0^h dy_1 \int_0^{g(t)\Psi(y_1)} u(y_1, y_2, t)dy_2 = \mu_5(t), \quad 0 \leq t \leq T. \quad (4)$$

Of course, the additional condition (4) giving the mass/energy of the heat conducting system [6, 7] is measured in practice, and is needed in order to supply for the missing information represented by the unknown function $g(t)$. Alternative additional information to (4) such as an internal temperature measurement

$$u(y_1^0, y_2^0, t) = \mu_5^0(t), \quad 0 \leq t \leq T, \quad (5)$$

at a fixed interior point (y_1^0, y_2^0) with $0 < y_1^0 < h$, $0 < y_2^0 < g(t)\Psi(y_1)$, or a heat flux measurement

$$\frac{\partial u}{\partial n}(y_1^1, y_2^1, t) = q(y_1^1, y_2^1, t), \quad (6)$$

at a point (y_1^1, y_2^1, t) on the boundary $S_T := \{0\} \times \{(y_2, t)|\ 0 \leq y_2 \leq \Psi(0)g(t),\ t \in (0, T]\}$ $\cup\{h\} \times \{(y_2, t)|\ 0 \leq y_2 \leq \Psi(h)g(t), t \in (0, T]\}$ $\cup[0, h] \times (\{0\} \cup \{(y_1, y_2, t)|\ y_1 \in [0, h],\ y_2 = \Psi(y_1)g(t), t \in (0, T]\})$ can also be considered. In (6), n denotes the outward unit normal to the boundary S_T.

The unique local solvability of the inverse problem (10)–(13) hold, as proved in [19], and read as follows.

Theorem 1. *Assume that:*

(*A1*) $\varphi \in C([0, h] \times [0, \infty))$, $\mu_i \in C([0, \infty) \times [0, T])$, $i = 1, 2, \mu_j \in C([0, h] \times [0, T])$, $j = 3, 4$, $\mu_5 \in C^1[0, T]$, $b_k, c, f \in C([0, h] \times [0, \infty) \times [0, T])$, $k = 1, 2$;

(*A2*) $\varphi(y_1, y_2) \geq \varphi_0 > 0$, $(y_1, y_2) \in [0, h] \times [0, \infty)$, $\mu_i(y_2, t) \geq \mu_{i0} > 0$, $(y_2, t) \in [0, \infty) \times [0, T]$, $i = 1, 2$, $\mu_j(y_1, t) > 0, (y_1, t) \in [0, h] \times [0, T]$, $j = 3, 4$, $\mu_5(t) > 0$, $t \in [0, T]$, $f(y_1, y_2, t) \geq 0$, $(y_1, y_2, t) \in [0, h] \times [0, \infty) \times [0, T]$, $\Psi(y_1) > 0$, $y_1 \in [0, h]$;

(A3) $\mu_1 \in C^{2,1}([0, K_1\Psi(0)] \times [0,T])$, $\mu_2 \in C^{2,1}([0, K_1\Psi(h)] \times [0,T])$,
$\mu_i \in C^{2,1}([0,h] \times [0,T])$, $i = 3,4$, $b_1, b_2, c, f \in C^{1,0}(\overline{Q})$, *where*
$Q := \{(y_1, y_2, t) : 0 < y_1 < h, 0 < y_2 < K_1\Psi(y_1), 0 < t < T\}$,
$\varphi \in C^2(\overline{D_0})$, *where* $D_0 := \{(y_1, y_2) : 0 < y_1 < h, 0 < y_2 < g(0)\Psi(y_1)\}$,
$\Psi \in C^2[0,h]$, $\lim\limits_{y_1 \to 0} \Psi'(y_1) = +\infty$, $\lim\limits_{y_1 \to h} \Psi'(y_1) = -\infty$;

(A4) consistency conditions of order zero [21] between (2) and (3) hold;

(A5) $\varphi \in C([0,h] \times [0,\infty))$, $\mu_i \in C([0,\infty) \times [0,T])$, $i = 1,2$, $\mu_j \in C([0,h] \times [0,T])$, $j = 3,4$, $\mu_5 \in C^1[0,T]$, $b_k, c, f \in C^{1,0}([0,h] \times [0,\infty) \times [0,T])$, $k = 1,2$, $\Psi \in C^1[0,T]$.

If the assumptions $(A1)$–$(A3)$ *hold then there exists a number* $T_0 \in (0,T]$ *for which the problem (1)–(4) has a solution* $(g(t), u(y_1, y_2, t)) \in C^1[0,T_0] \times \left(C^{2,1}(\Omega_{T_0}) \cap C^{1,0}(\overline{\Omega_{T_0}})\right) =: A_{T_0}$ *with* $g(t) > 0$ *for* $t \in [0, T_0]$.

If $(A2)$, $(A4)$ *and* $(A5)$ *are satisfied the same existence result holds in the set* $(g(t), u(y_1, y_2, t)) \in C^1[0,T_0] \times (C^{2,1}(\Omega_{T_0}) \cap C\left(\overline{\Omega_{T_0}}\right))$ *with* $g(t) > 0$ *for* $t \in [0, T_0]$.

In $(A3)$, the positive constant K_1 represents an upper bound for $g(t)$ on the interval $t \in [0,T]$, which is obtained from (13) and the max-min principle for the function u, which under assumptions $(A1)$ and $(A2)$, yields, [19],

$$u(y_1, y_2, t) \geq M_0 > 0, \quad (y_1, y_2, t) \in \overline{\Omega_T} \tag{7}$$

for some positive constant M_0. Thus, we can take

$$K_1 = \max_{0 \leq t \leq T} |\mu_5(t)| / (M_0 h (\min_{0 \leq y_1 \leq h} \Psi(y_1))).$$

Note also that under assumptions $(A1)$ and $(A2)$, equation (4) applied at $t = 0$ yields the value of $g(0)$ as the unique positive solution of the nonlinear equation

$$\int_0^h dy_1 \int_0^{g(0)\Psi(y_1)} \varphi(y_1, y_2) dy_2 = \mu_5(0). \tag{8}$$

Theorem 2. *Assume that:*

(A6) $b_i, c, f \in C^{1,0}([0,h] \times [0,\infty) \times [0,T])$, $\mu_i \in C^{3,1}([0,\infty) \times [0,T])$, $i = 1,2$,
$0 < \Psi \in C^2[0,h]$, $\displaystyle\int_0^h \Psi(y_1)\mu_4(y_1, t) dy_1 \neq 0$ *for* $t \in [0,T]$;

(A7) $\varphi(y_1, y_2) \geq \varphi_0 > 0$, $(y_1, y_2) \in [0,h] \times [0,\infty)$.

Then, if $C^1[0,T] \ni \mu_5(t) \neq 0$, $t \in [0,T]$, *the inverse problem (1)–(4) cannot have more than one solution* $(g(t), u(y_1, y_2, t))$ *in the class* A_T, *with* $g(t) > 0$ *for* $t \in [0,T]$.

Remark that assumption $(A7)$ is needed to determine uniquely $g(0)$ from (8).

The following theorem establishes the stability of the inverse problem (1)–(4) under small perturbations in the measured additional mass/energy data (4).

Theorem 3. *Assume that (A6) and (A7) are satisfied. Let $\mu_5(t) \neq 0$ and $\mu_5^\epsilon(t)$ for $t \in [0,T]$, be two given data (4) in $C^1[0,T]$ satisfying*

$$||\mu_5 - \mu_5^\epsilon||_{C^1[0,T]} \leq \varepsilon \tag{9}$$

for some non-negative constant ε. Then, if it exists, the unique solution $(g(t), u(y_1, y_2, t)) \in A_T$ with $g(t) > 0$ for $t \in [0,T]$, of the inverse problem (1)–(4) under the perturbation $\mu_5^\epsilon(t)$ of $\mu_5(t)$ in (4), satisfying (9), is stable for small $\varepsilon > 0$.

Proof. First we perform the change of variables $z_1 = y_1$, $z_2 = y_2/g(t)$, to transform (1)–(4) into a new inverse problem for the unknown $g(t)$ and the new 'temperature' function $w(z_1, z_2, t) := u(z_1, z_2 g(t), t)$, as follows:

$$w_t = w_{z_1z_1} + \frac{1}{g^2(t)} w_{z_2z_2} + b_1(z_1, z_2g(t), t) w_{z_1} + \left(\frac{b_2(z_1, z_2g(t), t) + z_2 g'(t)}{g(t)} \right) w_{z_2}$$
$$+ c(z_1, z_2g(t), t) w + f(z_1, z_2 g(t), t), \quad (z_1, z_2, t) \in Q_T, \tag{10}$$

where $Q_T := \{(z_1, z_2, t) : \ 0 < z_1 < h, 0 < z_2 < \Psi(z_1), 0 < t < T\}$,

$$w(z_1, z_2, 0) = \varphi(z_1, z_2 g(0)), \quad (z_1, z_2) \in \overline{D}, \tag{11}$$

$$\begin{cases} w(0, z_2, t) = \mu_1(z_2 g(t), t), & (z_2, t) \in [0, \Psi(0)] \times [0, T], \\ w(h, z_2, t) = \mu_2(z_2 g(t), t), & (z_2, t) \in [0, \Psi(h)] \times [0, T], \\ w(z_1, 0, t) = \mu_3(z_1, t), & (z_1, t) \in [0, h] \times [0, T], \\ w(z_1, \Psi(z_1), t) = \mu_4(z_1, t), & (z_1, t) \in [0, h] \times [0, T], \end{cases} \tag{12}$$

$$g(t) \iint\limits_D w(z_1, z_2, t) dz_1 dz_2 = \mu_5(t), \quad t \in [0, T], \tag{13}$$

where $D := \{(z_1, z_2) : 0 < z_1 < h, 0 < z_2 < \Psi(x_1)\}$. Remark that now the domain $Q_T = D \times (0,T)$ is fixed, whilst the original domain Ω_T was moving (in time).

The problem (10)–(13) is equivalent to the problem (1)–(4) and, it possesses a unique solution $(g(t), w(z_1, z_2, t)) \in C^1[0,T] \times \left(C^{2,1}(Q_T) \cap C^{1,0}(\overline{Q_T})\right) =: \tilde{A_T}$ with $g(t) > 0$ for $t \in [0,T]$, if this solution exists. A similar problem, corresponding to the perturbed data $\mu_5^\epsilon(t)$ instead of $\mu_5(t)$ in (13), can be formulated for the solution $(g_\varepsilon(t), w_\varepsilon(z_1, z_2, t)) \in \tilde{A_T}$ with $g_\varepsilon(t) > 0$ for $t \in [0,T]$. Denote $G(t) := g(t) - g_\varepsilon(t)$ and $W(z_1, z_2, t) := w(z_1, z_2, t) - w_\varepsilon(z_1, z_2, t)$. Then,

$$W_t = W_{z_1z_1} + \frac{1}{g^2(t)} W_{z_2z_2} + b_1(z_1, z_2 g(t), t) W_{z_1}$$
$$+ \frac{(b_2(z_1, z_2g(t), t) + z_2 g'(t))}{g(t)} W_{z_2} + c(z_1, z_2 g(t), t) W + \left(\frac{1}{g^2(t)} - \frac{1}{g_\varepsilon^2(t)} \right) (w_\varepsilon)_{z_2z_2}$$
$$+ \left(\frac{g'(t)}{g(t)} - \frac{g'_\varepsilon(t)}{g_\varepsilon(t)} \right) z_2 (w_\varepsilon)_{z_2} + (b_1(z_1, z_2 g(t), t) - b_1(z_1, z_2 g_\varepsilon(t), t)) (w_\varepsilon)_{z_1}$$
$$+ \left(\frac{b_2(z_1, z_2 g(t), t)}{g(t)} - \frac{b_2(z_1, z_2 g_\varepsilon(t), t)}{g_\varepsilon(t)} \right) (w_\varepsilon)_{z_2} + (c(z_1, z_2 g(t), t)$$
$$- c(z_1, z_2 g_\varepsilon(t), t)) w_\varepsilon + f(z_1, z_2 g(t), t) - f(z_1, z_2 g_\varepsilon(t), t), \quad (z_1, z_2, t) \in Q_T, \tag{14}$$

$$W(z_1, z_2, 0) = 0, \quad (z_1, z_2) \in \overline{D}, \tag{15}$$

$$\begin{cases} W(0, z_2, t) = \mu_1(z_2 g(t), t) - \mu_1(z_2 g_\varepsilon(t), t), & (z_2, t) \in [0, \Psi(0)] \times [0, T], \\ W(h, z_2, t) = \mu_2(z_2 g(t), t) - \mu_2(z_2 g_\varepsilon(t), t), & (z_2, t) \in [0, \Psi(h)] \times [0, T], \\ W(z_1, 0, t) = W(z_1, \Psi(z_1), t) = 0, & (z_1, t) \in [0, h] \times [0, T], \end{cases} \tag{16}$$

$$\iint_D W(z_1, z_2, t) dz_1 dz_2 = \frac{\mu_5(t)}{g(t)} - \frac{\mu_5^\varepsilon(t)}{g_\varepsilon(t)}, \quad t \in [0, T]. \tag{17}$$

Let us rewrite equation (14) in a condensed form as

$$LW = F(z_1, z_2, t, G(t), G'(t)), \tag{18}$$

where

$$L := \frac{\partial}{\partial t} - \frac{\partial^2}{\partial z_1^2} - \frac{1}{g^2(t)} \frac{\partial^2}{\partial z_1^2} - b_1(z_1, z_2 g(t), t) \frac{\partial}{\partial z_1} - \frac{b_2(z_1, z_2 g(t), t)}{g(t)} \frac{\partial}{\partial z_2} \\ - c(z_1, z_2 g(t), t) I, \tag{19}$$

and I is the identity operator.

Following [19], let us define the satisfier function

$$\chi(z_1, z_2, t, G(t)) := \left(1 - \frac{z_1}{h}\right) (\mu_1(z_2 g(t), t) - \mu_1(z_2 g_\varepsilon(t), t)) + \frac{z_1}{h} (\mu_2(z_2 g(t), t) \\ - \mu_2(z_2 g_\varepsilon(t), t)) - \frac{z_2}{\Psi(z_1)} \left[\left(1 - \frac{z_1}{h}\right) (\mu_1(\Psi(z_1) g(t), t) - \mu_1(\Psi(z_1) g_\varepsilon(t), t)) \right. \\ \left. + \frac{z_1}{h} (\mu_2(\Psi(z_1) g(t), t) - \mu_2(\Psi(z_1) g_\varepsilon(t), t)) \right],$$

to homogenise the boundary conditions at $z_1 = 0$ and $z_1 = h$ in (16), for the new function

$$\tilde{W}(z_1, z_2, t) = W(z_1, z_2, t) - \chi(z_1, z_2, t, G(t)), \tag{20}$$

which satisfies

$$L\tilde{W} = F(z_1.z_2, t, G(t), G'(t)) - L\chi(z_1, z_2, t, G(t)), \quad (z_1, z_2, t) \in Q_T. \tag{21}$$

Solving the problem given by (21) with homogeneous initial and Dirichlet boundary conditions using the Green function $\mathcal{G}(z_1, z_2, t; \xi_1, \xi_2, \tau)$ we obtain

$$W(z_1, z_2, t) = \chi(z_1, z_2, t, G(t)) + \int_0^t \iint_D \mathcal{G}(z_1, z_2, t; \xi_1, \xi_2, \tau) \Big[F(\xi_1, \xi_2, \tau, G(\tau), G'(\tau)) \\ - L\chi(\xi_1, \xi_2, \tau, G(\tau)) \Big] d\xi_1 d\xi_2 d\tau, \quad (z_1, z_2, t) \in \overline{Q_T}. \tag{22}$$

Condition (17) yields

$$G(t) = -\frac{g(t) g_\varepsilon(t)}{\mu_5(t)} \iint_D W(z_1, z_2, t) dz_1 dz_2 + \frac{g(t)}{\mu_5(t)} (\mu_5(t) - \mu_5^\varepsilon(t)), \quad t \in [0, T]. \tag{23}$$

Differentiate (13) and use (10) to obtain, [19],

$$g'(t) = \frac{1}{\int_0^h \Psi(z_1)\mu_4(z_1,t)dz_1}\Bigg\{\mu_5'(t) - g(t)\Bigg[\int_0^{\Psi(0)} w_{z_1}(0,z_2,t)dz_2 - \int_0^{\Psi(h)} w_{z_1}(h,z_2,t)dz_2 + \mu_4(h,t) - \mu_4(0,t)$$
$$+ \iint_D \Big(b_1(z_1,z_2g(t),t)w_{z_1}(z_1,z_2,t) + \frac{b_2(z_1,z_2g(t),t)}{g(t)}w_{z_2}(z_1,z_2,t)$$
$$+c(z_1,z_2g(t),t)w(z_1,z_2,t) + f(z_1,z_2g(t),t)\Big)dz_1dz_2\Bigg]$$
$$-\frac{1}{g(t)}\int_0^h \Big(w_{z_2}(z_1,0,t) - w_{z_2}(z_1,\Psi(z_1),t)\Big)dz_1\Bigg\}, \quad t \in [0,T]. \tag{24}$$

Denoting $q(t) := G'(t) = g'(t) - g_\varepsilon'(t)$, from equation (24) written for $g'(t)$ and $g_\varepsilon'(t)$ and subtracted, we obtain

$$q(t) = \frac{1}{\int_0^h \Psi(z_1)\mu_4(z_1,t)dz_1}\Bigg\{\mu_5'(t) - (\mu_5^\varepsilon)'(t) + \frac{G(t)}{g(t)g_\varepsilon(t)}\int_0^h \Big((w_\varepsilon)_{z_2}(z_1,0,t)$$
$$-(w_\varepsilon)_{z_2}(z_1,\Psi(z_1),t)\Big)dz_1 - \frac{1}{g(t)}\int_0^h \Big(W_{z_2}(z_1,0,t) - W_{z_2}(z_1,\Psi(z_1),t)\Big)dz_1$$
$$-g(t)\Bigg[\int_0^{\Psi(0)} W_{z_1}(0,z_2,t)dz_2 - \int_0^{\Psi(h)} W_{z_1}(h,z_2,t)dz_2$$
$$+ \iint_D \Big[b_1(z_1,z_2g(t),t)W_{z_1}(z_1,z_2,t) + \frac{b_2(z_1,z_2g(t),t)}{g(t)}W_{z_2}(z_1,z_2,t)$$
$$+c(z_1,z_2g(t),t)W(z_1,z_2,t) + \Big(b_1(z_1,z_2g(t),t) - b_1(z_1,z_2g_\varepsilon(t),t)\Big)(w_\varepsilon)_{z_1}(z_1,z_2,t)$$
$$+\left(\frac{b_2(z_1,z_2g(t),t)}{g(t)} - \frac{b_2(z_1,z_2g_\varepsilon(t),t)}{g_\varepsilon(t)}\right)(w_\varepsilon)_{z_2}(z_1,z_2,t)$$
$$+\Big(c(z_1,z_2g(t),t) - c(z_1,z_2g_\varepsilon(t),t)\Big)w_\varepsilon(z_1,z_2,t) + f(z_1,z_2g(t),t)$$
$$-f(z_1,z_2g_\varepsilon(t),t)\Big]dz_1dz_2\Bigg] - G(t)\Bigg[\int_0^{\Psi(0)} (w_\varepsilon)_{z_1}(0,z_2,t)dz_2$$
$$-\int_0^{\Psi(h)} (w_\varepsilon)_{z_1}(h,z_2,t)dz_2 + \mu_4(h,t) - \mu_4(0,t)$$
$$+ \iint_D \Big[b_1(z_1,z_2g_\varepsilon(t),t)(w_\varepsilon)_{z_1}(z_1,z_2,t) + \frac{b_2(z_1,z_2g_\varepsilon(t),t)}{g_\varepsilon(t)}(w_\varepsilon)_{z_2}(z_1,z_2,t)$$
$$+c(z_1,z_2g_\varepsilon(t),t)w_\varepsilon(z_1,z_2,t) + f(z_1,z_2g_\varepsilon(t),t)\Big]dz_1dz_2\Bigg]\Bigg\}, \quad t \in [0,T]. \tag{25}$$

Thus, the problem (14)–(17) has been recast as the system of integral equations (23) and (25) for $G(t)$ and $q(t)$, where the function $W(z_1,z_2,t)$ and its first-order partial derivatives are obtainable from (22). For the differences of values of functions present in (25), we use the identity

$$Z(.,\alpha_1(t)) - Z(.,\alpha_2(t)) = (\alpha_1(t)-\alpha_2(t)) \int_0^1 \frac{\partial Z}{\partial s}(.,s)\Bigg|_{s=\alpha_2(t)+\theta(\alpha_1(t)-\alpha_2(t))} d\theta.$$

Remark that the dependence of $W(z_1, z_2, t)$ on $G(t)$ and $q(t)$ is linear. The right-hand side of the system of equations (23) and (25) is given by

$$RHS(t) = \left(\frac{g(t)}{\mu_5(t)} \left(\mu_5(t) - \mu_5^\varepsilon(t)\right), \frac{\mu_5'(t) - (\mu_5^\varepsilon)'(t)}{\int_0^h \Psi(z_1)\mu_4(z_1,t)dz_1} \right)^{\mathrm{T}}, \quad t \in [0,T]. \tag{26}$$

Assumptions of theorem ensure that the linear system (23) and (25) of two Volterra integral equations of the second kind with integrable kernels is well-posed and therefore, it possesses a unique solution $G(t) = g(t) - g_\varepsilon(t)$ and $q(t) = g'(t) - g_\varepsilon'(t)$ which tends to zero, as $\varepsilon \searrow 0$. This proof of the stability theorem is completed.

The stability Theorem 3 obviously implies, by taking $\varepsilon = 0$, the uniqueness Theorem 2. Theorem 3 ensures the stability in case the data (4) is smooth of class $C^1[0,T]$. However, the presence of the derivative $(\mu_5^\varepsilon)'(t)$ of the 'noisy' non-smooth function $\mu_5^\varepsilon(t)$, coming from measurement, in (26) highlights the practical ill-posedness of the inverse free surface problem under investigation.

3 Numerical discretization of the direct problem

For the numerical discretization in a fixed domain it is useful to employ further the change of variables $x_1 = y_1$, $x_2 = y_2/(g(t)\Psi(y_1))$, to transform (1)–(4) into the following inverse problem for the time-dependent free boundary intensity function $g(t)$ and the new 'temperature' $v(x_1, x_2, t) := u(x_1, x_2 g(t)\Psi(x_1), t)$:

$$\begin{aligned} v_t = v_{x_1x_1} + \frac{1}{g^2(t)\Psi^2(x_1)} v_{x_2x_2} + b_1(x_1, x_2 g(t)\Psi(x_1), t) v_{x_1} \\ + \left(\frac{b_2(x_1, x_2 g(t)\Psi(x_1), t) + x_2 g'(t)\Psi(x_1)}{g(t)\Psi(x_1)} \right) v_{x_2} + c(x_1, x_2 g(t)\Psi(x_1), t) v \\ + f(x_1, x_2 g(t)\Psi(x_1), t), \quad (x_1, x_2, t) \in \Omega \times (0,T), \end{aligned} \tag{27}$$

$$v(x_1, x_2, 0) = \varphi(x_1, x_2 g(0)\Psi(x_1)), \quad (x_1, x_2) \in \overline{\Omega}, \tag{28}$$

$$v(0, x_2, t) = \mu_1(x_2 g(t)\Psi(0), t),\ v(h, x_2, t) = \mu_2(x_2 g(t)\Psi(h), t),\ (x_2, t) \in [0,1] \times [0,T], \tag{29}$$

$$v(x_1, 0, t) = \mu_3(x_1, t),\ v(x_1, 1, t) = \mu_4(x_1, t),\ (x_1, t) \in [0,h] \times [0,T], \tag{30}$$

$$g(t) \iint_{\Omega} \Psi(x_1) v(x_1, x_2, t) dx_1 dx_2 = \mu_5(t), \quad t \in [0,T], \tag{31}$$

where $\Omega = (0,h) \times (0,1)$.

Next, we consider solving the direct problem (27)–(30), where the functions g, Ψ, b_1, b_2, c, f, φ and μ_i for $i = \overline{1,4}$, are known and the solution $v(x_1,x_2,t)$ is to be computed. We sub-divide $\Omega \times (0,T)$ into M_1, M_2 and N uniform intervals of lengths Δx_1, Δx_2 and Δt, where $\Delta x_1 = h/M_1$, $\Delta x_2 = 1/M_2$, and $\Delta t = T/N$, respectively. At the node (i,j,n), denote $v^n_{i,j} := v(x_{1i}, x_{2j}, t_n)$, where $x_{1i} = i\Delta x_1$, $x_{2j} = j\Delta x_2$, $t_n = n\Delta t$, $g_n := g(t_n)$, $\Psi_i := \Psi(x_{1i})$, $b^n_{1_{i,j}} := b_1(x_{1i}, x_{2j}g_n\Psi_i, t_n)$, $b^n_{2_{i,j}} := b_1(x_{1i}, x_{2j}g_n\Psi_i, t_n)$, $c^n_{i,j} := c(x_{1i}, x_{2j}g_n\Psi_i, t_n)$ and $f^n_{i,j} := f(x_{1i}, x_{2j}g_n\Psi_i, t_n)$ for $i = \overline{0,M_1}$, $j = \overline{0,M_2}$ and $n = \overline{0,N}$.

3.1 Alternating direction explicit (ADE) method

In this subsection, alternating direction explicit (ADE) method, [3,4,25], which is unconditionally stable, will be described for solving numerically the nonlinear the two-dimensional parabolic equation (27) with initial and boundary conditions (28)–(30).

Let $\tilde{v}^n_{i,j}$ and $\tilde{u}^n_{i,j}$ satisfy the following multilevel finite difference discretisations of equation (27):

$$\begin{aligned}
\frac{\tilde{v}^{n+1}_{i,j} - \tilde{v}^n_{i,j}}{\Delta t} &= \frac{\tilde{v}^n_{i+1,j} - \tilde{v}^n_{i,j} - \tilde{v}^{n+1}_{i,j} + \tilde{v}^{n+1}_{i-1,j}}{(\Delta x_1)^2} + \frac{1}{g_n^2\Psi_i^2}\Big(\frac{\tilde{v}^n_{i,j+1} - \tilde{v}^n_{i,j} - \tilde{v}^{n+1}_{i,j} + \tilde{v}^{n+1}_{i,j-1}}{(\Delta x_2)^2}\Big) \\
&+ b^n_{1_{i,j}}\Big(\frac{\tilde{v}^n_{i+1,j} - \tilde{v}^{n+1}_{i-1,j}}{2(\Delta x_1)}\Big) + \Big(\frac{b^n_{2_{i,j}} + x_{2j}g'_n\Psi_i}{g_n\Psi_i}\Big)\Big(\frac{\tilde{v}^n_{i,j+1} - \tilde{v}^{n+1}_{i,j-1}}{2(\Delta x_2)}\Big) + c^n_{i,j}\Big(\frac{\tilde{v}^{n+1}_{i,j} + \tilde{v}^n_{i,j}}{2}\Big) \\
&+ \frac{1}{2}\Big(f^{n+1}_{i,j} + f^n_{i,j}\Big), \quad i = \overline{1, M_1 - 1}, \quad j = \overline{1, M_2 - 1}, \quad n = \overline{0, N-1}, \qquad (32)
\end{aligned}$$

$$\begin{aligned}
\frac{\tilde{u}^{n+1}_{i,j} - \tilde{u}^n_{i,j}}{\Delta t} &= \frac{\tilde{u}^{n+1}_{i+1,j} - \tilde{u}^{n+1}_{i,j} - \tilde{u}^n_{i,j} + \tilde{u}^n_{i-1,j}}{(\Delta x_1)^2} + \frac{1}{g_n^2\Psi_i^2}\Big(\frac{\tilde{u}^{n+1}_{i,j+1} - \tilde{u}^{n+1}_{i,j} - \tilde{u}^n_{i,j} + \tilde{u}^n_{i,j-1}}{(\Delta x_2)^2}\Big) \\
&+ b^n_{1_{i,j}}\Big(\frac{\tilde{u}^{n+1}_{i+1,j} - \tilde{u}^n_{i-1,j}}{2(\Delta x_1)}\Big) + \Big(\frac{b^n_{2_{i,j}} + x_{2j}g'_n\Psi_i}{g_n\Psi_i}\Big)\Big(\frac{\tilde{u}^{n+1}_{i,j+1} - \tilde{u}^n_{i,j-1}}{2(\Delta x_2)}\Big) + c^n_{i,j}\Big(\frac{\tilde{u}^n_{i,j} + \tilde{u}^{n+1}_{i,j}}{2}\Big) \\
&+ \frac{1}{2}\Big(f^{n+1}_{i,j} + f^n_{i,j}\Big), \quad i = \overline{M_1 - 1, 1}, \quad j = \overline{M_2 - 1, 1}, \quad n = \overline{0, N-1}. \qquad (33)
\end{aligned}$$

Furthermore, let the $\tilde{v}^n_{i,j}$ and $\tilde{u}^n_{i,j}$ also satisfy the initial and boundary conditions (28)–(30), namely

$$\tilde{v}^0_{i,j} = \tilde{u}^0_{i,j} = \varphi(x_{1i}, x_{2j}), \quad i = \overline{0, M_1}, \quad j = \overline{0, M_2}, \qquad (34)$$

$$\tilde{v}^n_{0,j} = \tilde{u}^n_{0,j} = \mu_{11}(x_{2j}g_n\Psi(0), t_n), \quad \tilde{v}^n_{M_1,j} = \tilde{u}^n_{M_1,j} = \mu_{12}(x_{2j}g_n\Psi(h), t_n), \;\; j = \overline{0, M_2}, \;\; n = \overline{1, N}, \qquad (35)$$

$$\tilde{v}^n_{i,0} = \tilde{u}^n_{i,0} = \mu_{21}(x_{1i}, t_n), \quad \tilde{v}^n_{i,M_2} = \tilde{u}^n_{i,M_2} = \mu_{22}(x_{1i}, t_n), \quad i = \overline{0, M_1}, \quad n = \overline{1, N}. \qquad (36)$$

In expressions (32) and (33), the derivative of g is approximated, for simplicity, using forward finite differences as

$$g'_n := g'(t_n) \approx \frac{g(t_n) - g(t_{n-1})}{\Delta t} = \frac{g_n - g_{n-1}}{\Delta t}, \quad n = \overline{1, N}, \qquad (37)$$

though, more accurate, central finite differences may also be employed. Equations (32) and (33) are rearranged in order to obtain explicit expressions for $\tilde{v}_{i,j}^{n+1}$ and $\tilde{u}_{i,j}^{n+1}$. They, respectively, become

$$\begin{aligned}\tilde{v}_{i,j}^{n+1} = {} & A_{i,j}^{n}\tilde{v}_{i,j}^{n} + B_{i,j}^{n}(\tilde{v}_{i+1,j}^{n} + \tilde{v}_{i-1,j}^{n+1}) + C_{i,j}^{n}(\tilde{v}_{i,j+1}^{n} + \tilde{v}_{i,j-1}^{n+1}) \\ & + D_{i,j}^{n}(\tilde{v}_{i+1,j}^{n} - \tilde{v}_{i-1,j}^{n+1}) + E_{i,j}^{n}(\tilde{v}_{i,j+1}^{n} - \tilde{v}_{i,j-1}^{n+1}) + G_{i,j}^{*}, \\ & i = \overline{1, M_1 - 1}, \quad j = \overline{1, M_2 - 1}, \quad n = \overline{0, N-1},\end{aligned} \tag{38}$$

$$\begin{aligned}\tilde{u}_{i,j}^{n+1} = {} & A_{i,j}^{n}\tilde{u}_{i,j}^{n} + B_{i,j}^{n}(\tilde{u}_{i+1,j}^{n+1} + \tilde{u}_{i-1,j}^{n}) + C_{i,j}^{n}(\tilde{u}_{i,j+1}^{n+1} + \tilde{u}_{i,j-1}^{n}) \\ & + D_{i,j}^{n}(\tilde{u}_{i+1,j}^{n+1} - \tilde{u}_{i-1,j}^{n}) + E_{i,j}^{n}(\tilde{u}_{i,j+1}^{n+1} - \tilde{u}_{i,j-1}^{n}) + G_{i,j}^{*}, \\ & i = \overline{M_1 - 1, 1}, \quad j = \overline{M_2 - 1, 1}, \quad n = \overline{0, N-1},\end{aligned} \tag{39}$$

where

$$\begin{aligned} A_{i,j}^{n} &= \frac{1-\lambda_{i,j}^{n}}{1+\lambda_{i,j}^{n}}, \quad B_{i,j}^{n} = \frac{\Delta t}{(\Delta x_1)^2(1+\lambda_{i,j}^{n})}, \quad C_{i,j}^{n} = \frac{\Delta t}{g_n^2\Psi_i^2(\Delta x_2)^2(1+\lambda_{i,j}^{n})}, \\ D_{i,j}^{n} &= \frac{(\Delta t)b_{1_{i,j}}^{n}}{2\Delta x_1(1+\lambda_{i,j}^{n})}, \quad E_{i,j}^{n} = \frac{\Delta t}{2\Delta x_2}\left(\frac{b_{2_{i,j}}^{n} + x_{2j}g_n'\Psi_i}{g_n\Psi_i(1+\lambda_{i,j}^{n})}\right), \\ G_{i,j}^{*} &= \frac{\Delta t}{2(1+\lambda_{i,j}^{n})}\left(f_{i,j}^{n+1} + f_{i,j}^{n}\right), \quad \lambda_{i,j}^{n} = \Delta t\left(\frac{1}{(\Delta x_1)^2} + \frac{1}{g_n^2\Psi_i^2(\Delta x_2)^2} - \frac{c_{i,j}^{n}}{2}\right). \end{aligned} \tag{40}$$

From (38), (39) and (34)–(36) for $\tilde{v}$ and $\tilde{u}$, $\tilde{v}_{i,j}^{n+1}$ and $\tilde{u}_{i,j}^{n+1}$ can be explicitly computed, and the simple arithmetic mean approximation

$$v_{i,j}^{n+1} = \frac{\tilde{v}_{i,j}^{n+1} + \tilde{u}_{i,j}^{n+1}}{2} \tag{41}$$

finally yields the solution $v_{i,j}^{n+1}$.

The double integral in (31) is approximated using the trapezoidal rule [11, 13], as follows:

$$\begin{aligned} \int_0^h \int_0^1 \Psi(x_1)v(x_1,x_2,t)dx_2dx_1 = {} & \frac{1}{4M_1M_2}\Bigg[\Psi(0)v(0,0,t_n) + \Psi(h)v(h,0,t_n) \\ & + \Psi(0)v(0,1,t_n) + \Psi(h)v(h,1,t_n) + 2\sum_{i=1}^{M_1-1}\Psi(x_{1i})v(x_{1i},0,t_n) \\ & + 2\sum_{i=1}^{M_1-1}\Psi(x_{1i})v(x_{1i},1,t_n) + 2\sum_{j=1}^{M_2-1}\Psi(0)v(0,x_{2j},t_n) + 2\sum_{j=1}^{M_2-1}\Psi(h)v(h,x_{2j},t_n) \\ & + 4\sum_{j=1}^{M_2-1}\sum_{i=1}^{M_1-1}\Psi(x_{1i})v(x_{1i},x_{2j},t_n)\Bigg], \quad n = \overline{1, N}. \end{aligned} \tag{42}$$

4 Inverse numerical solution

For the inverse problem solution of (27)–(31) we minimize

$$F(g) := \left\| g(t)\int_0^h \int_0^1 \Psi(x_1)v(x_1,x_2,t;g)dx_2dx_1 - \mu_5(t)\right\|_{L^2[0,T]}^2 \tag{43}$$

or, in discretizations form,

$$F(\underline{g}) = \sum_{n=1}^{N} \left[g_n \int_0^h \int_0^1 \Psi(x_1) v(x_1, x_2, t_n; \underline{g}) dx_2 dx_1 - \mu_5(t_n) \right]^2, \tag{44}$$

where $\underline{g} = (g_n)_{n=\overline{1,N}}$ and $v(x_1, x_2, t; g)$ solves (27)–(30) for given g. This minimization is accomplished using the *lsqnonlin* MATLAB toolbox routine, [10, 22].

5 Results and discussion

We define the root mean square errors (rmse) as

$$rmse(g) = \left[\frac{T}{N} \sum_{n=1}^{N} \left(g_n - g^{Exact}(t_n) \right)^2 \right]^{1/2}. \tag{45}$$

We take $h = T = 1$, for simplicity. The lower and upper bounds for the coefficient $g(t)$ are taken as 10^{-9} and 10^2, respectively. The initial guesses for $g(t)$ is taken as the value of $g(0)$, which is obtainable from (31).

The inverse problem given by equations (27)–(31) is solved subject to the noisy data (31)

$$\mu_5^{\epsilon}(t_n) = \mu_5(t_n) + \epsilon_n, \quad n = \overline{1, N}, \tag{46}$$

where $\underline{\epsilon} = (\epsilon_n)_{n=\overline{1,N_t}} := normrnd(0, \sigma, N)$, $\sigma = p \times \max_{t \in [0,1]} |\mu_5(t)|$ and p is the percentage of noise.

Let us solve the inverse problem (1)–(4) with the input data:

$$\begin{aligned}
\Psi(y_1) = 1, \quad b_1(y_1, y_2, t) = \frac{1}{2}(y_1 + y_2 + t), \quad b_2(y_1, y_2, t) = \frac{1}{2}(y_1 + y_2 + t), \\
c(y_1, y_2, t) = \frac{1}{2}(y_1 + y_2 + t), \quad \varphi(y_1, y_2) = 3 - (-1 + 2y_1)^2 - (-1 + y_2)^2, \\
\mu_1(y_2, t) = 2 + t - (-1 + y_2)^2, \quad \mu_2(y_2, t) = 2 + t - (-1 + y_2)^2, \\
\mu_3(y_1, t) = 2 + t - (-1 + 2y_1)^2, \quad \mu_4(y_1, t) = \frac{1}{9}(14 + 13t - t^2 + 36y_1 - 36y_1^2) \\
-(-1 + 2y_1)^2, \quad f(y_1, y_2, t) = 11 + 2(-1 + 2y_1)(t + y_1 + y_2) \\
+(-1 + y_2)(t + y_1 + y_2) - \frac{1}{2}(t + y_1 + y_2)(1 + t + 4y_1 - 4y_1^2 + 2y_2 - y_2^2),
\end{aligned} \tag{47}$$

$$\mu_5(t) = \frac{1}{81}(1 + t)(53 + 34t - t^2). \tag{48}$$

Conditions of Theorem 2 hold and hence, the solution's uniqueness is guaranteed. In fact, the exact solution of (27)–(30) is

$$v(x_1, x_2, t) = u(x_1, x_2 g(t) \Psi(x_1), t) = 3 + t - (1 - 2x_1)^2 - \frac{1}{9}(-3 + x_2 + t x_2)^2, \tag{49}$$

and

$$g(t) = \frac{1}{3}(1 + t). \tag{50}$$

Also,

$$u(y_1, y_2, t) = 3 + t - (-1 + 2y_1)^2 - (-1 + y_2)^2. \quad (51)$$

First, we assess the accuracy of the direct problem given by (1)–(3) (or (27)–(30)) with the input data (47) when $g(t)$ is known and given by (50). Table 1 illustrates that the analytical and numerical solutions for the data (4) (or (31)), which analytically is given by (48), obtained with various numbers of space grids $M_1 = M_2 \in \{5, 10, 20\}$ and with various numbers of time steps $N \in \{20, 40, 80\}$ are in excellent agreement.

Table 1: The numerical and exact (48) solutions for $\mu_5(t)$, with various $M_1 = M_2 \in \{5, 10, 20\}$ and $N \in \{20, 40, 80\}$, for the direct problem.

$M_1 = M_2$	N	$t = 0.1$	$t = 0.2$	$t = 0.3$	...	$t = 0.9$	$rmse(\mu_5)$
	20	0.7554	0.8741	1.0006	...	1.9232	0.0148
5	40	0.7556	0.8741	1.0007	...	1.9233	0.0147
	80	0.7556	0.8742	1.0007	...	1.9234	0.0146
	20	0.7629	0.8823	1.0096	...	1.9369	0.0039
10	40	0.7631	0.8825	1.0098	...	1.9372	0.0038
	80	0.7632	0.8825	1.0098	...	1.9373	0.0037
	20	0.7648	0.8841	1.0116	...	1.9398	0.0015
20	40	0.7649	0.8845	1.0120	...	1.9405	0.0011
	80	0.7651	0.8846	1.0121	...	1.9408	0.0009
exact		0.7658	0.8853	1.0129	...	1.9420	0

Next, we investigate the inverse problem. We fix $M_1 = M_2 = 10$ and $N = 40$, which was found sufficiently dense to ensure that any finer mesh (such as $M_1 = M_2 = 20$ and $N = 80$) did not influence the stability and accuracy of the numerical solution.

Figure 1 illustrates the absolute error between the exact solution (49) and the numerical solutions for $v(x_1, x_2, t)$. It can be observed that the accuracy of the numerical solution improves, as the noise level p decreases.

The results for $g(t)$ are illustrated in Figure 2. The $rmse(g)$ obtained values (45) are $0.0014, 0.0127, 0.0250$ and 0.0373 for $p \in \{0, 1, 2, 3\}\%$ noise, respectively. As expected, for noise free data, i.e. $p = 0$, the unique solution (50), which is guaranteed from Theorem 2, is retrieved very accurately. As noise p is included in the data (46), Figure 2 illustrates that the numerical recoveries are reasonably accurate but start to build up oscillations as the amount of noise p increases. To restore stability we penalise the least-squares function (43) by adding a first-order smoothing term $\lambda||g'(t)||^2_{L^2[0,T]}$ to it since the theory provides $g \in C^1[0, T]$, where $\lambda > 0$ is the Tikhonov's regularization parameter to be selected. Then, in discretised form this first-order Tikhonov functional recasts as

$$F_\lambda(\underline{g}) = F(\underline{g}) + \lambda \sum_{n=1}^{N} \left(\frac{g_n - g_{n-1}}{\Delta t} \right)^2 . \quad (52)$$

For $p = 5\%$ noise, Figure 3 illustrates the analytical solution (50) and the numerical solutions obtained by minimizing the objective functional (52) for various regularization parameters. The $rmse(g)$ values are 0.0618, 0.0440, 0.0292 and 0.0322 for $\lambda \in \{0, 10^{-3}, 10^{-2}, 10^{-1}\}$, respectively. It can be noted that the numerical unregularized solution obtained with $\lambda = 0$ manifests instability, however, inclusion of regularization with $\lambda = 10^{-2}$ to 10^{-1} provides a stable solution which is consistent in accuracy with the $p = 5\%$ noise contaminating the input data (46). The last remaining thing to do is to provide some reasoning on how to choose the regularization parameter $\lambda > 0$ in the functional (52). One possible argument for this choice is given by the L-curve shown in Figure 4. For several parameters λ for the obtained minimizer $\underline{g}_\lambda$ of (52), we plot the derivative norm $||\underline{g}'_\lambda|| = \sqrt{\sum_{n=1}^{N} \left(\frac{g_n - g_{n-1}}{\Delta t}\right)^2}$ versus the residual norm $\sqrt{F(\underline{g}_\lambda)}$. The 'corner' of the obtained L-curve around $\lambda = 10^{-2}$, illustrated in Figure 4, is taken as a good selection for λ compromising/balancing the fit of measured data (residual comparable to the amount of noise included) with the stability of solution (bounded derivative solution norm).

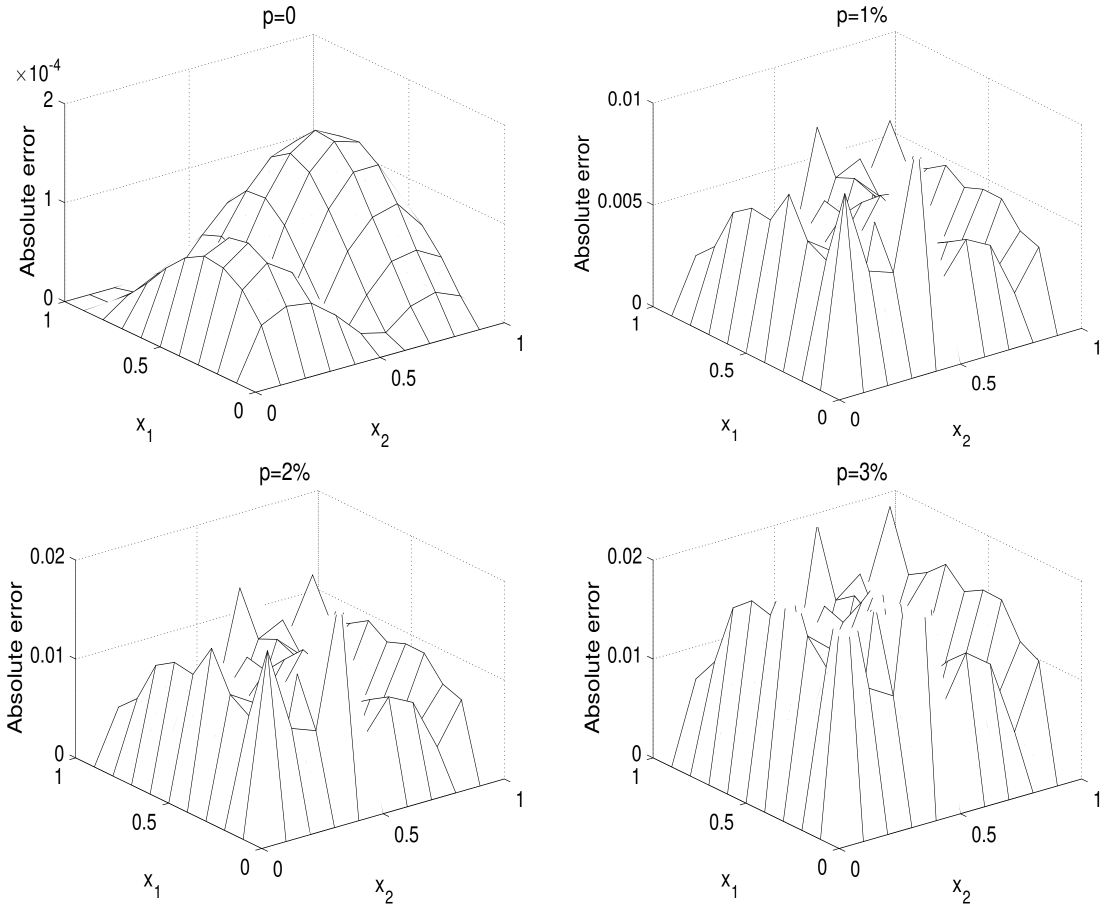

Figure 1: The absolute error between the numerical and analytical (49) solutions for the transformed temperature $v(x_1, x_2, 1)$, for $p \in \{0, 1, 2, 3\}\%$ noise, without regularization.

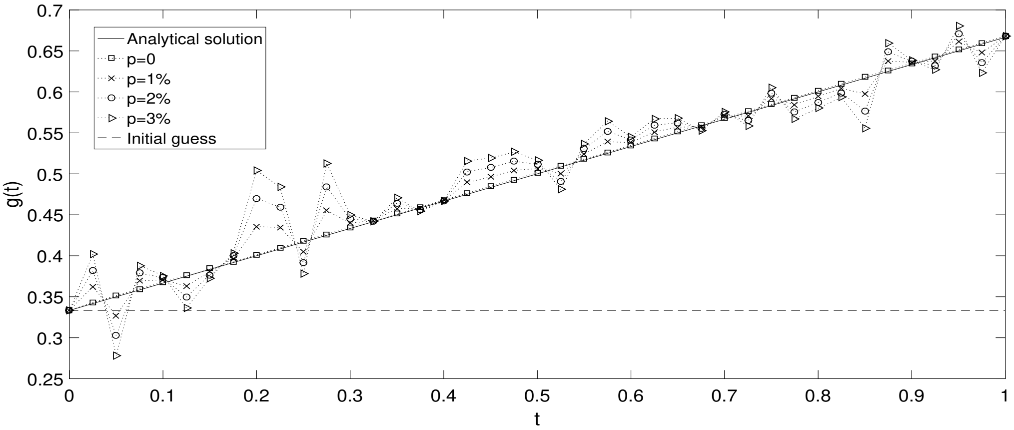

Figure 2: The numerical and analytical (50) solutions for the intensity $g(t)$ of the free boundary, for $p \in \{0, 1, 2, 3\}\%$ noise, without regularization.

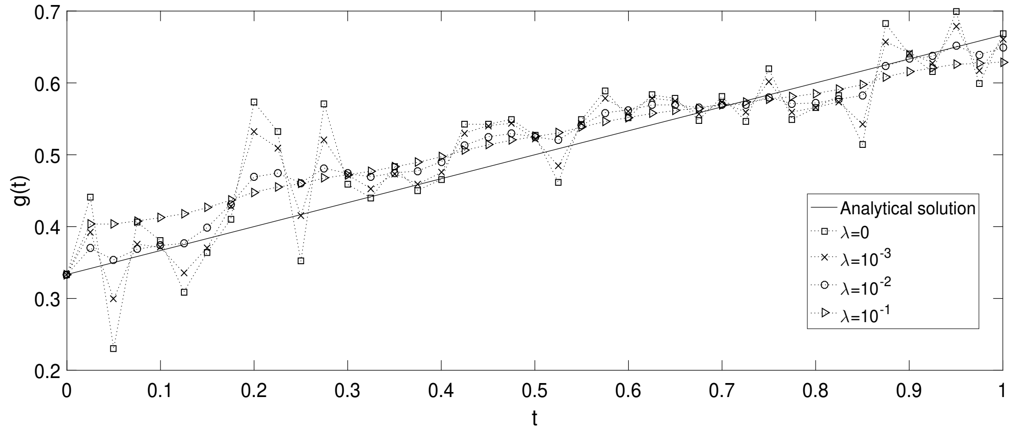

Figure 3: The numerical and analytical (50) solutions for the intensity $g(t)$ of the free boundary, for $p = 5\%$ noise, with various $\lambda \in \{0, 10^{-3}, 10^{-2}, 10^{-1}\}$.

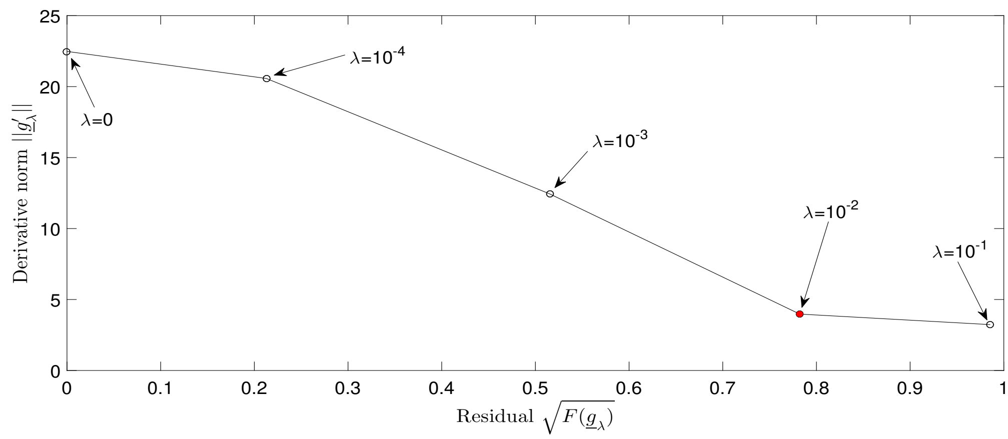

Figure 4: The derivative norm $||\underline{g}'_\lambda||$ versus the residual norm $\sqrt{F(\underline{g}_\lambda)}$ for the L-curve, for $p = 5\%$ noise.

6 Conclusion

The retrieval of the transient intensity of a free boundary and the temperature in a two-dimensional parabolic problem from mass (energy) measurement has been studied. This nonlinear inverse problem has been shown to be (locally) well-posed and a stability theorem has been proven. The free boundary formulation has been changed to a fixed domain, and the direct solver based on a ADE-FDM has been utilized. The inverse solution has been obtained based on the *lsqnonlin* MATLAB optimisation procedure for minimizing the least-squares function further penalised with first-order regularization for noisy data. Numerical illustrations show that accurate and stable solutions have been attained. Extension to three-dimensions is in principle straightforward. Future work will consider recovering the intensity $g(t)$ of the free boundary together with the minor coefficient c (depending on the time t only) form the mass specification (4) and an additional Stefan-type condition [24].

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