Comment on "Quantum Solution to the Arrow-of-Time Dilemma"

In [1], it is claimed that entropy decreases can occur, but that any such decrease necessarily coincides with an erasure of memory. It is argued that this resolves the directionality of the arrow of time, since we can only ever have records of entropy increasing events.

Specifically, a memory of an event E is defined [1] as a physical system A that has a nonzero *classical* mutual information with a system C that bears the consequences of event E. For the sake of discussion, we adopt this formulation. A purifying environment R may be assumed, and it is clear that if the entropy of A and C is to decrease, with no entropy change in the reservoir, $\Delta S(\rho_R) = 0$, then the quantum mutual information between A and C must decrease.

Given this setting, the principal claim in [1] is that "any decrease in entropy of a system that is correlated with an observer entails a memory erasure of said observer" (*). In a classical setting, (*) is true by definition of the entropies—classical events involving the reduction of local entropies trivially coincide with a reduction of memory records, since these are defined as the classical correlations between the memory system A and system C. However, the extension of the argument to all quantum mechanical states, where there is a much richer correlation structure, is nontrivial. Consequently, in this fuller setting, the claim is really that entropy-decreasing events always coincide with a reduction in classical mutual information in spite of the freedoms of quantum mechanics.

This result only follows if a reduction in the quantum mutual information $I_q(A{:}C)$ implies a reduction in the classical mutual information $I_c(A{:}C)$ {[1] only proves that $I_q(A{:}C) \geq I_c(A{:}C)$}. Since $I_c(A{:}C)$ is identified with the memory recorded in A of the effects on C, it would follow that a reduction in entropy for C always involves an erasure of memory of the event for A.

First and foremost, we point out that demanding an entropy-decreasing event E is very different from demanding an event in which all correlations are eliminated. In the central proof of [1], the author mistakenly assumes the elimination of all correlations, when in fact his actual focus is that of entropy-decreasing events. Clearly given a complete elimination of correlations, memory erasure occurs *by assumption*, and so for this highly atypical (measure zero) case, the argument is tautological.

For the correct case in question (the set of entropy-decreasing events), we can show that the central claim (*) of [1] is false by presenting a simple counterexample.

The specific example that we consider is a 3 qubit W state, $|\Phi_{ACR}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$, and the event E that we consider is the action of a CNOT gate controlled on C with R being the target. Before the event, the reduced state $\rho_{AC,i} = \frac{1}{3}|00\rangle\langle 00| + \frac{2}{3}|\psi^+\rangle\langle\psi^+|$, while after the CNOT event the reduced state is $\rho_{AC,f} = \frac{1}{3}|0\rangle\langle 0| \otimes |1\rangle\langle 1| + \frac{2}{3}|+\rangle\langle +| \otimes |0\rangle\langle 0|$. It is straightforward to see that the entropy of the reservoir R or system A does not change, $\Delta S_R = \Delta S_A = 0$; however, $\Delta S_C = -0.3683$ and so the quantum mutual information changes from $I_q(A{:}C; i) = 0.9183$ to $I_q(A{:}C; f) = 0.5500$.

The calculation of the classical mutual information is a bit involved. Numerics show that for both $\rho_{AC,i}$ and $\rho_{AC,f}$, the optimal measurements are actually projective. It can then readily be derived that for the case of $\rho_{AC,i}$, the optimal measurements are projections in the $|\pm\rangle$ basis, for which the classical mutual information is $I_c(A{:}C; i) = 0.3499$. For the final state $\rho_{AC,f}$, the optimal projective measurements for the classical mutual information are straightforward to deduce. The optimal measurement on C is clearly to measure in the $|0\rangle, |1\rangle$ basis; on A, we simply maximize the discrimination of $|0\rangle$ and $|+\rangle$ with priors of $1/3$ and $2/3$, respectively, which is done by the projective measurement onto $\Pi_\pm = (I \pm \cos\theta Z \mp \sin\theta X)/2$ with $\theta = \arctan(\frac{\sqrt{5}-1}{2})$. One then finds $I_c(A{:}C; f) = 0.3683$, which is higher than the initial classical correlations, despite the reduction of local entropy for C.

In light of this example, one might think that $I_q(A{:}C)$ is the correct measure to use instead of $I_c(A{:}C)$; however, since $\Delta S_R = \Delta S_{AC} = 0$, we have that $I_q(A{:}C) = S_A + S_C$ up to an additive constant. The quantum mutual information is thus an elementary rewriting of the total local entropy, and so considering its reduction contributes nothing new to the problem.

In a sense, we see that instead of quantum mechanics resolving the fact that we have no classical memory records of entropy-decreasing events, it actually makes the issue worse.

In some ongoing work [2], we have been investigating the relationship between quantum entanglement and the thermodynamic arrow of time.

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Received 9 September 2009; published 9 April 2010
DOI: 10.1103/PhysRevLett.104.148901
PACS numbers: 03.65.Ta, 03.65.Ud, 03.67.−a, 05.70.Ce

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