A Remark on Gentzen's Notes

著者	ANDOU Yuuki
出版者	法政大学文学部
journal or publication title	Bulletin of Faculty of Letters, Hosei University
volume	76
page range	61-65
year	2018-03-13
URL	http://hdl.handle.net/10114/13768

A Remark on Gentzen's Notes

ANDOU Yuuki*

1 Introduction

Some bundles of handwritten notes of Gerhard Gentzen (1909-1945) have been found and published [3, 4, 5] more than fifty years after his death. In this paper, we remark on Gentzen's proof of normalization theorem for intuitionistic natural deduction, which is included in his notes above.

Natural deduction is one of the deduction systems that Gentzen formalized in [2] (an English translation is in [8]). As noticed in the synopsis of his paper, through the investigation on the system, he obtained the conception of the *Hauptsatz* (fundamental theorem) the intuitive meaning of which is we can remove all the detours from arbitrary given derivations. It holds for both on intuitionistic logic and on classical logic, but in the classical case the system natural deduction is not suitable for the representation of the theorem, so Gentzen also provided in [2] another deduction system called sequent calculus and proved the theorem by using the latter. This course of consideration is expressed in [2], not showing a proof of the fundamental theorem (or the normalization theorem) for intuitionistic logic in the style of natural deduction. He proved the fundamental theorem (or the cut-elimination theorem) in a unified form for the two logics by the sequent calculus.

Therefore, we have an interest in Gentzen's *original proof* of the normalization theorem for intuitionistic natural deduction. Historically, after Gentzen, Prawitz [6] proved the normalization theorems on natural deduction for several logics including intuitionistic logic and restricted (in the sense of fragment on logical symbols) classical logic. Furthermore, after Prawitz, the normalization theorem for classical logic without restriction (i.e. with disjunction and existential quantifier) have been proved by Stålmarck [7], Andou [1], and others.

In the next section, we first observe the similarity of Gentzen's proof and Prawitz' one, focusing on the permutation conversion, and after that, give an example of their reduction procedure. Moreover, in the appendix, we state the instance of our reduction defined in [1] corresponding the example above.

*Department of Philosophy, Hosei University, Tokyo 102-8160, Japan.
E-mail: norakuro@hosei.ac.jp

2 Normalization for intuitionistic natural deduction

In this section, we compare two proofs, namely Gentzen's [3] and Prawitz' [6], of normalization theorem on natural deduction for first order intuitionistic logic including disjunction and existential quantifier as logical connectives.

We turn our attention to permutation conversions. In the case that the fragment the intuitionistic system treats does not have either disjunction or existential quantifier, we can define the normalization procedure quite simply, because the redexes stand on between introduction rules and elimination rules directly. But otherwise, we have to deal with stretched redexes that are represented as the sequences of same formulas.

To settle this problem, Gentzen, in his notes, as the description in Plato's translation [3], had introduced the notion *hillock* and defined in his reduction procedure a preparatory step concerning the diminution of the length of the hillock. Hillock is essentially the same notion with *maximum segment* defined by Prawitz [6]. The former is a sequence of formula occurrences in a deduction, starting from the conclusion of an introduction or intuitionistic absurdity rule, being a minor premiss of an elimination rule for disjunction or existential quantifier followed by the conclusion of the inference rule except for the last one, and terminating at the major premiss of an elimination rule or at the end formula of the deduction. (To be precise, maximum segments correspond with *inner hillocks*, that are ones not terminating at the end formula of the deduction).

Then, the *hillock theorem* in the notation of Gentzen, that states we can remove all inner hillocks in an arbitrary given deduction, is equivalent to Prawitz' normalization theorem. Moreover, there are many correspondences between two proofs. Not only the essential conversions for each logical connectives, but also the permutation conversions concerning the elimination rules for disjunction and for existential quantifier are exactly the same ones respectively. For choosing the first target to be reduced in normalization, they gave the same order to the sets of inner hillocks or maximum segments under a restricted condition. The order is expressed by the terminology *higher* in the case of Gentzen, that can be roughly defined as *above* or *right standing* if the minor premisses are to be placed at the right hand side of the major premiss in each elimination rule.

On the details, there is a little technical difference on the use of the induction for proving the theorem. In Gentzen's proof, permutation (i.e. structural) conversions are preparatory steps for an essential (i.e. logical) conversion, and the induction value decreases at the time when the essential conversion has been done. On the other hand, in Prawitz' proof, permutation and essential conversions are treated simultaneously in the sense of induction, and both conversions diminish the induction value.

In the following, we show an example of a part of Gentzen's reduction that consists of permutation conversions as preparatory steps and one essential conversion.

Let $\mathcal{P}$ be a deduction of the form:

$$\begin{array}{c}\dfrac{\dfrac{\Gamma \quad \dfrac{\Delta \quad \dfrac{\Delta' \quad \dfrac{\Theta}{C}\,I_1}{C}\,I_2}{C}\,I_3 \quad \dfrac{\Delta'' \quad \dfrac{\Theta'}{C}\,I_4}{C}\,I_5}{C}\,I_6 \quad \dfrac{\Lambda \quad \dfrac{\Sigma}{D}\,I_7}{D}\,I_8 \quad \dfrac{\Lambda' \quad \dfrac{\Sigma'}{D}\,I_9}{D}\,I_{10}}{D}\,I_{11} \quad \begin{array}{c}\Xi\\ E\end{array}}{E}\,I_{12} \\ \Pi \end{array},$$

where the conditions

(a) I_1, I_4, I_7, and I_9 are introduction rules for existential quantifier,

(b) I_6 and I_{11} are elimination rules for disjunction,

(c) C and D have the greatest grade (i.e. the number of occurrence of logical symbols), say m, in $\mathcal{P}$,

(d) E has the grade less than m,

(e) Θ, Θ', Σ, Σ', and Ξ do not contain any hillocks of grade m, and

(f) Π does not contain any hillocks of grade m whose representants, i.e. uppermost formulas, are higher than the conclusion of I_{12}

are satisfied. Then, the sequences

l_1: four occurrences of C from the conclusion of I_1 to the conclusion of I_6,

l_2: three occurrences of C from the conclusion of I_4 to the conclusion of I_6,

l_3: three occurrences of D from the conclusion of I_7 to the conclusion of I_{11}, and

l_4: three occurrences of D from the conclusion of I_9 to the conclusion of I_{11}

are all inner hillocks of grade m. Since the greatest grade of hillocks in $\mathcal{P}$ is m and also the condition (d), (e) and (f) hold, the inner hillocks l_3 and l_4 are two of main hillocks in $\mathcal{P}$. We choose l_3 as the main hillock to reduce here. After renaming variables appropriately, we first diminish the length of l_3, that is, a permutation conversion (in present termination) between I_{11} and I_{12} is applied. So we have the derivation, say $\mathcal{P}_1$, below.

$$\begin{array}{c}\dfrac{\dfrac{\Gamma \quad \dfrac{\Delta \quad \dfrac{\Delta' \quad \dfrac{\Theta}{C}}{C}}{C} \quad \dfrac{\Delta'' \quad \dfrac{\Theta'}{C}}{C}}{C} \quad \dfrac{\dfrac{\Lambda \quad \dfrac{\Sigma}{D}}{D} \quad \begin{array}{c}\Xi\\ E\end{array}}{E} \quad \dfrac{\dfrac{\Lambda' \quad \dfrac{\Sigma'}{D}}{D} \quad \begin{array}{c}\Xi\\ E\end{array}}{E}}{E} \\ \Pi \end{array}$$

Consecutively, we apply permutation conversions until the length of l_3 becomes 1. Then we have the following derivation, say $\mathcal{P}_2$,

$$\dfrac{\dfrac{\Gamma \quad \dfrac{\Delta \quad \dfrac{\Delta' \quad \dfrac{\Theta}{C}}{C}}{C} \quad \dfrac{\Delta'' \quad \dfrac{\Theta'}{C}}{C}}{C} \quad \dfrac{\Lambda \quad \dfrac{\dfrac{\Sigma}{D} \quad \begin{array}{c}\Xi\\ E\end{array}}{E}\,I_0}{E} \quad \dfrac{\dfrac{\Lambda' \quad \dfrac{\Sigma'}{D}}{D} \quad \begin{array}{c}\Xi\\ E\end{array}}{E}}{\begin{array}{c}E\\ \Pi\end{array}}$$

where I_0, corresponding with I_{12}, is an elimination rule for existential quantifier. Now we can remove the inner hillock the origin of which is l_3. The application of essential conversion for the major premiss of I_0 leads the following derivation, say $\mathcal{P}_3$,

$$\dfrac{\dfrac{\Gamma \quad \dfrac{\Delta \quad \dfrac{\Delta' \quad \dfrac{\Theta}{C}}{C}}{C} \quad \dfrac{\Delta'' \quad \dfrac{\Theta'}{C}}{C}}{C} \quad \dfrac{\Lambda \quad \begin{array}{c}[\tilde{\Sigma}]\\ \Xi\\ E\end{array}}{E} \quad \dfrac{\dfrac{\Lambda' \quad \dfrac{\Sigma'}{D}}{D} \quad \begin{array}{c}\Xi\\ E\end{array}}{E}}{\begin{array}{c}E\\ \Pi\end{array}}$$

where $[\tilde{\Sigma}]$ represents the substitution of Σ with appropriate renaming of variables respectively for the assumptions discharged at I_0.

As above, by Gentzen's method, $\mathcal{P}$ is reduced to $\mathcal{P}_3$ by applying the reduction procedure for a main hillock l_3. Through this reduction, the greatest grade of inner hillocks is unchanged, but the number of the inner hillocks of the greatest grade is decreased by 1.

On the other hand, in Prawitz' proof, $\mathcal{P}$ is reduced to $\mathcal{P}_1$ if the reduction procedure is applied for the maximum segment l_3, and the reduction value is decreased from $\langle m, 13 + L\rangle$ to $\langle m, 11 + L\rangle$, where L is the sum of the length of all maximum segments of degree m in Γ, Δ's, Λ's or Π.

3 Appendix

In Andou's proof [1] of normalization theorem for classical natural deduction with full logical connectives (i.e. including disjunction and existential quantifier), the notion segment is expanded to classical logic, and the reduction procedure is defined in unified form both for intuitionistic and classical case. According to the difference of the order of choosing a maximum segment (or formula) to be reduced, if we apply the reduction procedure described in [1] for the above mentioned deduction $\mathcal{P}$, we can have the following deduction, say $\mathcal{P}_4$, because the maximal length 4 of the segments terminating at the maximum formula C of I_{11} is greater than the maximal length 3 of the segments terminating at the maximum formula D of I_{12}.

$$\dfrac{\dfrac{\Gamma \quad \dfrac{\dfrac{\Delta \quad \dfrac{\Delta' \quad \dfrac{\Theta}{C}}{C}}{C} \quad \dfrac{\Lambda \quad \dfrac{\Sigma}{D}}{D} \quad \dfrac{\Lambda' \quad \dfrac{\Sigma'}{D}}{D}}{D} \quad \dfrac{\dfrac{\Delta'' \quad \dfrac{\Theta'}{C}}{C} \quad \dfrac{\Lambda \quad \dfrac{\Sigma}{D}}{D} \quad \dfrac{\Lambda' \quad \dfrac{\Sigma'}{D}}{D}}{D}}{D} \quad \dfrac{\Xi}{E}}{\begin{array}{c}E\\ \Pi\end{array}}$$

In this case, the reduction value is decreased form $\langle\langle g,4\rangle, 1+n, 2+i\rangle$ to $\langle\langle g,4\rangle, 1+n, 1+i\rangle$, where n is the number of the maximum formulas of degree $\langle g,4\rangle$ in Γ, Δ's, Ξ, or Π, and i is the number of inferences below the conclusion of I_{12}.

References

[1] Y. Andou, A Normalization-Procedure for the First Order Classical Natural Deduction with Full Logical Symbols, *Tsukuba Journal of Mathematics* 19 (1995) 153-162.

[2] G. Gentzen, Untersuchungen über das logische Schliessen, *Mathematische Zeitschrift* 39 (1935) 176-210, 405-431.

[3] J. von Plato, Gentzen's Proof of Normalization for Natural Deduction, *The Bulletin of Symbolic Logic* 14 (2008) 240-257.

[4] J. von Plato, Gentzen's Logic, in *Handbook of the History of Logic Volume 5 -Logic from Russell to Church* (edited by D. M. Gabbay and J. Woods), North-Holland, Amsterdam (2009) 667-721.

[5] J. von Plato, *Saved from the Cellar - Gerhard Gentzen's Shorthand Notes on Logic and Foundations of Mathematics*, Springer, Switzerland (2017).

[6] D. Prawitz, *Natural Deduction - A Proof Theoretical Study*, Almqvist & Wiksell, Stockholm (1965).

[7] G. Stålmarck, Normalization theorems for full first order classical natural deduction, *Journal of Symbolic Logic* 56 (1991) 129-149.

[8] M. E. Szabo (editor), *The Collected Papers of Gerhard Gentzen*, North-Holland, Amsterdam (1969).