Generalized-Star Cubeのトポロジー的特性及びそのブロードキャスティングアリゴリズム

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Topological Properties and Broadcasting Algorithms of the Generalized-Star Cube

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***Abstract*—In this research, another version of the star cube called the generalized-star cube, GSC(n, k, m), is presented as a three level interconnection topology. GSC(n, k, m) is a product graph of the (n, k)-star graph and the m-dimensional hypercube (m-cube). It can be constructed in one of two ways: to replace each node in an m-cube with an (n, k)-star graph, or to replace each node in an (n, k)-star graph with an m-cube. Because there are three parameters m, n, and k, the network size of GSC(n, k, m) can be changed more flexibly than the star graph, star-cube, and (n, k)-star graph. We first investigate the topological properties of the GSC(n, k, m), such as the node degree, diameter, average distance, and cost. Also, the regularity and node symmetry of the GSC(n, k, m) are derived. Then, we illustrate the broadcasting algorithms for both of the single-port and all-port models. To develop these algorithms, we use the spanning binomial tree, the neighbourhood broadcasting algorithm, and the minimum dominating set. The complexities of the broadcasting algorithms are also examined.**

I. Introduction

In recent years, study of parallel and distributed computing has been featured as one of the important research themes. Especially, there is increasing interest in large scale parallel computing. For such parallel computing systems, the wide variety of interconnection topologies were proposed. Among them, the hypercube [1] structure has been widely used because of its elegant topological properties and the ability to emulate a wide variety of other frequently used networks.

However, conventional hypercube network is not a good candidate for such large scale networks because hypercube has a major drawback. That is, the number of communication links for each node is a logarithmic function of the number of nodes in the network. To alleviate this drawback, several variations of the hypercube have been proposed in the literature. Cube-connected cycles [2] and reduced hypercube [3] focused on the reduction of the number of edges of the hypercube. Hierarchical cubic network [4] focused on reductions of the number of edges and the diameter of the hypercube. These topologies are the modification of the hypercube in one way or another with motivation to improve some of its properties.

In such circumstances, [5] and [6] pointed that many of these properties of the hypercube are in fact group theoretic properties possessed by a large class of networks called Cayley graphs. Some Cayley graphs not only possess all these properties but even offer a better degree and diameter than the hypercube. The star graph is an important class of Cayley graph and an attractive alternative to the hypercube in lower degree and shorter diameter [6].

However, star graph has also a major drawback such that the network size is restricted on the choice of the total number of nodes by $n!$. To mitigate the restriction of the significant gap between the two consecutive sizes of nodes $n!$ in the n-star graph, the incomplete star [7] and arrangement graphs [8] have been proposed. However, the incomplete star is a non-symmetric and irregular graph and the arrangement graphs have a problem of the very high node degree. These problems restrict the adoption of these topologies to the practical system design. To solve these problems, (n, k)-star graph [9] and star-cube [10] are proposed.

By generalizing the star graph with another parameter k, we can obtain the (n, k)-star graph. In this graph, two parameters n and k are used to control the number of nodes, thus making it convenient to design a network with a desirable size and a better degree/diameter trade-off than the star graph. The star-cube is a product graph based on the star graph and the hypercube and inherits all the attractive properties from both topologies. In this graph, two parameters of star graph n and hypercube m are used to control the network size. Therefore, its size grows in smaller steps than the star graph.

In [11], Daiki Arai and Yamin Li proposed a new interconnection network called the Generalized-Star Cube (GSC) with three parameters n, k, and m. A GSC(n, k, m) network consists of $2^m n!/(n-k)!$ nodes with a degree of $m+n-1$ and a diameter of $m+2k-1$ for $k \le \lfloor n/2 \rfloor$ and $m+k+\lfloor (n-1)/2 \rfloor$ for $k \ge \lfloor n/2 \rfloor + 1$. GSC$(n, k, m)$ is a product graph based on the (n, k)-star graph and m-dimensional hypercube. Using these three parameters, compared to star graph, star-cube, and (n, k)-star, the network size of GSC(n, k, m) can be changed flexibly.

For any interconnection network, we can classify it as either a single-port model or all-port model, depending on how a node communicates with its neighbors. In the single-port model, in one step, a node can send (receive) a message to (from) one and only one of its neighbor nodes. Meanwhile, in the all-port model, in one step, a node can send (receive) messages to (from) all of its neighbor nodes.

One of the simplest and most fundamental collective communication operations is one-to-all broadcasting algorithm. In one-to-all broadcast, a source node sends a message to all nodes. A similar problem which has been studied is the problem of neighbourhood broadcasting. It is an algorithm to

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send a message from a node to its all neighbors. It is clear for any interconnection network with N nodes, on a single-port model, that the problem of broadcasting has a trivial lower bound of $\mathbf{\Omega}(\log N)$ because in one step, the number of informed nodes can double at most. Similarly, the problem of neighbourhood broadcasting has a trivial lower bound of $\mathbf{\Omega}(\log n)$ where n is the degree of the source node. It is also clear for any interconnection network, on an all-port model, that a trivial lower bound for the problem of broadcasting is the diameter of the network and the neighbourhood broadcasting can be done in one step.

In this research, graph-theoretic properties of GSC(n,k,m) are addressed. Additionally, a shortest-path routing algorithm and a broadcasting algorithm for both of the single-port and all-port models for GSC(n,k,m) are established. The detailed shortest-path routing algorithm is formally given in [11]. For these routing algorithms, we separate them into hypercube part and (n,k)-star graph part and use existing optimal algorithms.In broadcasting algorithms at both of the single-port and all-port models, we use spanning binomial tree for hypercube part. On the other hand, in (n,k)-star graph part, on the single-port model, we use a neighbourhood broadcasting algorithm, and on the all-port model, we use minimum dominating set. As a result, we derived optimal algorithms for these three-type routing problems.

II. Generalized-Star Cube

The generalized-star cube, denoted by GSC(n,k,m), is a product graph of the (n,k)-star graph and m-cube. In a GSC(n,k,m), the node address of each vertex can be separated into two-part labels $\langle x_m x_{m-1}\ldots x_2 x_1\ ,\ y_1 y_2\ldots y_{k-1} y_k\rangle$, where the label of $x_m\ldots x_1$ signifies the m-cube part (cube-label) and $y_1\ldots y_k$ signifies the (n,k)-star graph part ((n,k)-star-label). Each node will be adjacent to two types of neighbors, namely the cube-neighbors and (n,k)-star-neighbors, respectively. The node addresses of cube-neighbors are represented as $\langle x_m\ldots\tilde{x}_i\ldots x_1\ ,\ y_1\ldots y_k\rangle$ for $1 \le i \le m$, where $\tilde{}$ means a bit inversion operation; and (n,k)-star-neighbors are represented as (1) $\langle x_m\ldots x_1\ ,\ y_j\ldots y_1\ldots y_k\rangle$ for $2 \le j \le k$, or (2) $\langle x_m\ldots x_1\ ,\ y'\ldots y_j\ldots y_k\rangle$ for $y' \in \{1,2,\ldots,n\} - \{y_j \mid 1 \le j \le k\}$. The edges of kind (1) are referred to as j-edges and (2) are referred to as 1-edges.

In the GSC(n,k,m), an (n,k)-star graph replaces each vertex of the m-cube or an m-cube replaces each vertex of the (n,k)-star graph. This means that there are $n!/(n-k)!$ m-cube subgraphs in the GSC(n,k,m), where the nodes of each m-cube are assigned with the same (n,k)-star-label. These subgraphs can be distinguished by their (n,k)-star-labels as shown in Fig. 1. Similarly, the GSC(n,k,m) can be considered as having 2^m (n,k)-star graphs, where the nodes of each (n,k)-star graph are assigned with the same cube-label. These sub-graphs can be distinguished by their cube-labels as shown in Fig. 2.

III. Properties of the Generalized-Star Cube

This section describes the topological properties of the proposed network GSC(n,k,m).

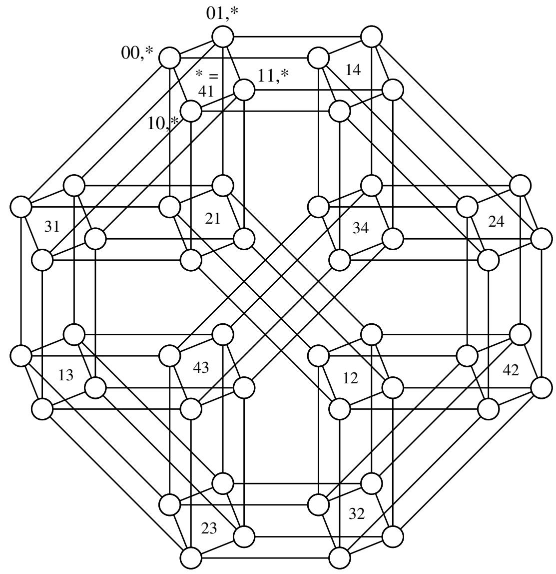

Fig. 1. GSC(4,2,2): A generalized star-connected-cube ((4,2)-star × 2-cube)

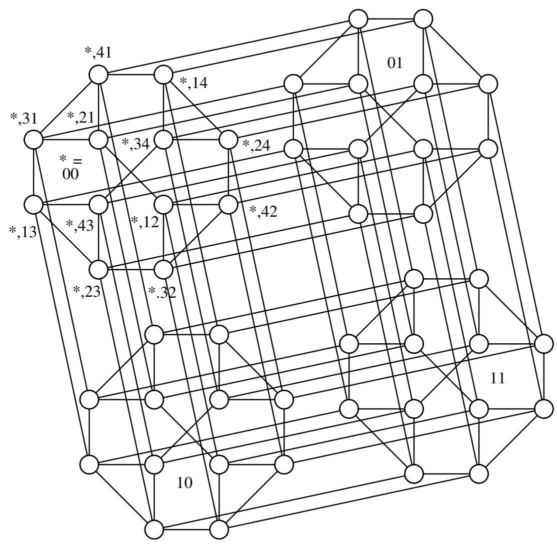

Fig. 2. GSC(4,2,2): A generalized cube-connected-star (2-cube × (4,2)-star)

A. The Basic Terminologies

Before illustrating the topological properties of the proposed GSC(n,k,m), the basic terminologies of the interconnection network are explained below. In this research, the interconnection network is thought of as an undirected graph. Therefore, the vertices correspond to the processors and the edges correspond to the bidirectional communication links.

Definition 1: The interconnection network is a finite graph $\boldsymbol{G} = \{\boldsymbol{V}, \boldsymbol{E}\}$, where $\boldsymbol{V}$ and $\boldsymbol{E}$ are a set of vertices (or nodes) and a set of edges (or links), respectively.

Definition 2: The degree of a vertex v in $\boldsymbol{G}$ is equal to the number of edges incident on v.

Definition 3: The diameter of a graph $\boldsymbol{G}$ denoted as D_G is defined to be $max\{d_G(u,v) \mid u,v \in \boldsymbol{V}\}$, where d_G is the distance between two nodes u and v.

Definition 4: A graph is called regular if all of its vertices have the same degree.

Definition 5: A graph $\boldsymbol{G}(\boldsymbol{V},\boldsymbol{E})$ is vertex symmetric if for any arbitrary pair of vertices, u and v, there exists an automorphism of the graph that maps u into v $(u, v \in \boldsymbol{V})$.

B. Topological Properties of GSC(n, k, m)

Theorem 1: The GSC(n, k, m) is a regular graph.

Proof: The m-cube and (n, k)-star graph are regular graphs. Then, GSC(n, k, m) is the product graph of them, so from Definition 4, the GSC(n, k, m) is a regular graph. ■

Theorem 2: The GSC(n, k, m) is vertex symmetric.

Proof: The m-cube and (n, k)-star graph are vertex symmetric. Then, GSC(n, k, m) is the product graph of them, so from Definition 5, the GSC(n, k, m) is vertex symmetric. However, GSC(n, k, m) could not be edge symmetric. For example, in Fig. 1, each 2-cube edge belongs to a cycle of length at least 4, but each edge of the $(4, 2)$-star graph may belong to a cycle of length at least 3. ■

The topological properties are summarized in TABLE I and the details are described in [11].

IV. Comparison on Degree and Diameter

The node degree and diameter are key properties of the interconnection networks. The node degree is the maximum number of the neighbors of a node in the whole network and the diameter is the value of maximum shortest distance of all pairs of the nodes. Node degree represents the port number of a switch module like an Infiniband. Generally, the more degree the network has, the higher hardware cost of the network requests. Diameter is used for estimating the maximum delay in transmitting a message from one processor to another and influences the message traffic density and the fault-tolerance. Thus a network with a lower node degree and a shorter diameter is desired. To evaluate such a network it is needed to compare these two properties simultaneously.

Fig. 3 and Fig. 4 show the comparison of the node degree and diameter against the total number of nodes of the generalized-star cube, respectively, with that of the hypercube, star graph, star-cube, and (n, k)-star graph. The proposed network can connect a more variety of the number of nodes than others. For example, when we want to make an about-100,000-node network, we need to connect at least 131,072, 362,880, 122,880, and 151,200 nodes with the hypercube, star graph, star-cube, and (n, k)-star graph, respectively. Contrastingly, the GSC(n, k, m) can connect 107,520 or 110,880 nodes.

The almost diameters of the GSC(n, k, m) fall in between the hypercube and (n, k)-star graph as shown in Fig. 4. But for degrees of the GSC(n, k, m), when the parameter k is much smaller than n, the node degree becomes higher than that of the hypercube as shown in Fig. 3. However, we do not have to consider such values of k, because as k approaches 1, the (n, k)-star graph of the GSC(n, k, m) is close to the complete graph of dimension n and such a network is unpractical for constructing a large-scale network. In Fig. 4, the diameter nonlinear variation of the GSC(n, k, m) is resulted from the domination of the cube-part and (n, k)-star-part. When the cube-part dominates the network, the diameter is close to the hypercube diameter. Similarly, when the (n, k)-star-part dominates the network, the diameter is close to the (n, k)-star graph diameter.

As observed from Fig. 3 and Fig. 4, the network size of the generalized-star cube changes in smaller steps. Thus we can choose more desirable network size than the hypercube, star graph, star-cube, and (n, k)-star graph.

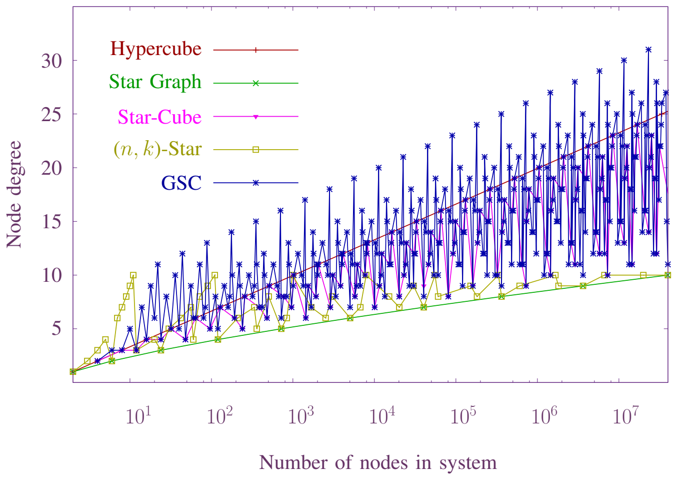

Fig. 3. Node degree

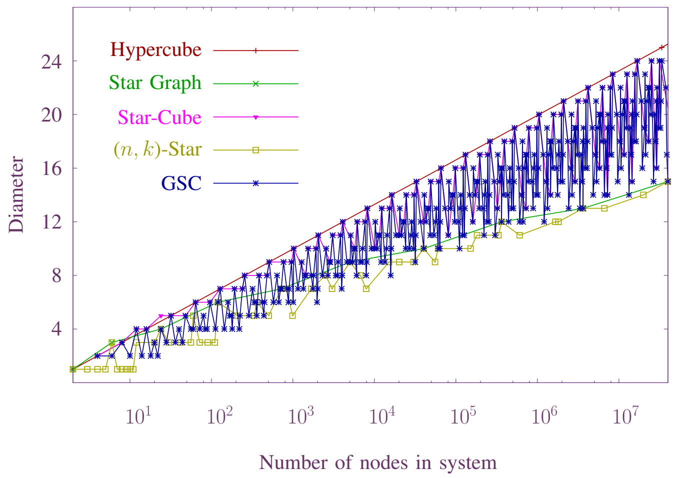

Fig. 4. Diameter

V. Broadcasting on the Single-Port Model

In this section, we develop the broadcasting algorithm for the single-port model. This algorithm is separated into two parts just like the shortest-path routing algorithm [11]: hypercube part and (n, k)-star graph part. First, we consider the broadcasting algorithm for the hypercube part. To implement the algorithm, we use the spanning binomial tree. By using this method, we can find optimal algorithms for hypercube part on both the single-port and all-port models. For the (n, k)-star graph part, we adopt an optimal neighbourhood broadcasting algorithm for developing an optimal broadcasting algorithm for GSC(n, k, m) on the single-port model.

TABLE I
COMPARISON ON THE TOPOLOGICAL PROPERTIES OF DIFFERENT NETWORKS

Parameters	HC(m)	n-Star	(n,k)-Star	SC(n,m)	GSC(n,k,m)
Nodes	2^m	$n!$	$\frac{n!}{(n-k)!}$	$2^m n!$	$2^m \frac{n!}{(n-k)!}$
Degree	m	$n-1$	$n-1$	$m+n-1$	$m+n-1$
Links	$m2^{m-1}$	$n!\frac{n-1}{2}$	$\frac{n!}{(n-k)!}\frac{n-1}{2}$	$2^{m-1}n!(m+n-1)$	$2^{m-1}\frac{n!}{(n-k)!}(m+n-1)$
Diameter	m	$\lfloor\frac{3(n-1)}{2}\rfloor$	$2k-1$ (if $1 \le k \le \lfloor\frac{n}{2}\rfloor$) $k+\lfloor\frac{n-1}{2}\rfloor$ (if $\lfloor\frac{n}{2}\rfloor+1 \le k \le n-1$)	$m+\lfloor\frac{3(n-1)}{2}\rfloor$	$m+2k-1$ (if $1 \le k \le \lfloor\frac{n}{2}\rfloor$) $m+k+\lfloor\frac{n-1}{2}\rfloor$ (if $\lfloor\frac{n}{2}\rfloor+1 \le k \le n-1$)
Average distance	$\frac{m}{2}$	$n-4+\frac{n}{2}$ $+\sum_{i=1}^{n}\frac{1}{i}$	$k-1+\sum_{i=1}^{k}\frac{1}{i}$ $-\frac{2(k-1)}{n}-\frac{k!(n-k)!}{n!}$	$\frac{m}{2}+n-4+\frac{n}{2}$ $+\sum_{i=1}^{n}\frac{1}{i}$	$\frac{m}{2}+k-1+\sum_{i=1}^{k}\frac{1}{i}$ $-\frac{2(k-1)}{n}-\frac{k!(n-k)!}{n!}$
Cost	m^2	$(n-1)\lfloor\frac{3(n-1)}{2}\rfloor$	$(n-1)(2k-1)$ (if $1 \le k \le \lfloor\frac{n}{2}\rfloor$) $(n-1)(k+\lfloor\frac{n-1}{2}\rfloor)$ (if $\lfloor\frac{n}{2}\rfloor+1 \le k \le n-1$)	$(m+n-1)(m+\lfloor\frac{3(n-1)}{2}\rfloor)$	$(m+n-1)(m+2k-1)$ (if $1 \le k \le \lfloor\frac{n}{2}\rfloor$) $(m+n-1)(m+k+\lfloor\frac{n-1}{2}\rfloor)$ (if $\lfloor\frac{n}{2}\rfloor+1 \le k \le n-1$)

A. Spanning Binomial Tree

We can use the spanning binomial tree communication scheme for broadcasting of hypercube, because hypercube's symmetric and binary recursive topology fit perfectly to this communication scheme. In general, a binomial tree is defined recursively as follows: (1) a binomial tree of order 0 is a single node; (2) a binomial tree of order m has a root node whose children are roots of binomial trees of orders $m-1, m-2, \ldots, 2, 1, 0$ (in this order); a binomial tree of order m has 2^m nodes, height m. Because the hypercube is both vertex and edge symmetric, we can place the root of a spanning binomial tree in any hypercube node and use the hypercube dimensions in any order [12]. The number of nodes that receive the message in step i is 2^{i-1} in the single-port model. In contrast, on the all-port model, the number is $\binom{m}{i}$ where m is the dimension of the hypercube. Therefore, in m step, $\sum_{i=1}^{m} 2^{i-1} = \sum_{i=1}^{m}\binom{m}{i} = 2^m - 1$. Hence, both schemes are transmission optimal ($\boldsymbol{O}(m)$).

B. Neighbourhood Broadcasting

The neighbourhood broadcasting problem, NBP for short, is a problem that a message of the source node is sent to all its neighbors in the single-port model. This problem for both star graph and (n,k)-star graph has been studied well and optimal algorithms were derived [13], [14], [15]. In [13], Fujita developed the algorithm by embedding binomial trees into the star graph. By contrast, in [14], [15], [16], more simple algorithm was developed with the cycle structures. In this research, we adopt this neighbourhood algorithm of the cycle structures. For some interconnection topologies with constant node degrees, the time required for neighbourhood broadcasting is constant. The lower bound of this NBP on a network with degree d is $\boldsymbol{\Omega}(\log d)$ [16]. For instance, the lower bound for NBP in $S_{n,k}$, an (n,k)-star graph, is $\boldsymbol{\Omega}(\log n)$ because the degree of $S_{n,k}$ is $n-1$.

For simplicity, we use the notation $i*$ to represent a node whose first symbol is i. Similarly, $*i$ represents a node whose last symbol is i. Let $S_{n-1,k-1}(i)$ be a subgraph where all the nodes are of the form $*i$, $1 \le i \le n$, then $S_{n-1,k-1}(i)$ is isomorphic to an $(n-1,k-1)$-star graph. This gives us one way to decompose an $S_{n,k}$ into n $S_{n-1,k-1}(i)$, for $1 \le i \le n$ [9], [17]. Unless otherwise stipulate, we will decompose the (n,k)-star graph at the last demension. Because the (n,k)-star graph is vertex symmetric, without loss of generality, we assume that the source node is $12\cdots k$. For this node, its i-edge neighbors are shown as:

$$21345\cdots k,\ 32145\cdots k,\ 42315\cdots k,\ \ldots,\ k234\cdots 1,$$

and its 1-edge neighbors are shown as:

$$(k+1)234\cdots k,\ (k+2)234\cdots k,\ \ldots,\ n234\cdots k.$$

The neighbourhood broadcating and broadcast algorithms for the two port models are based on the following observations on structual properties of the (n,k)-star graph. The proofs for these observations are fairly straightforward and can be found in [16]:

Observation 1: For any $r \neq 1$, $S_{r,1}$ is a clique K_r (a complete graph of size r).

Observation 2: In $S_{n,k}$, for any node u, u and all its 1-edge neighbors form a clique K_{n-k+1}.

Observation 3: For any i-edge neighbor $i*k = i23\cdots(i-1)1(i+1)\cdots k$ and 1-edge neighbor $j*k = j23\cdots(j-1)1(j+1)\cdots k$ of the node $12\cdots k$ (we assume that $i < j$ without loss of generality), they are on the same cycle of length 6. This cycle involves only i-edges. In fact, the above observation also holds true when $k+1 \le j \le n$:

Observation 4: For any i-edge neighbor $i*k = i23\cdots(i-1)1(i+1)\cdots k$ and 1-edge neighbor $j*k = j23\cdots k$ of the node $12\cdots k$, where $k+1 \le j \le n$, they are on the same cycle of length 6.

This cycle involves both i-edges and 1-edges.

Observation 5: Any two 6-cycles formed as in Observations 3 and 4 with distinct $2 \leq i_1, j_1, i_2, j_2 \leq n$ are disjoint except that they share the source node $12 \cdots k$.

Initially, only the source node has a message. In the first step, it sends a message to one of its neighbors through the direct link. In the second step, now two nodes have the message. One of them, the source node sends the message like the first step, whereas another one sends the message to a neighbor of the source node through a length-4 path that is part of a 6-cycle. However, the source node must wait until another node finishes forwarding the message. Now the number of neighbors including the source node are 4. Next, these four nodes send the message again in the same manner. Thus, three neighbours send the message to another three neighbours of the source node via disjoint paths of length-4 that are parts of three disjoint 6-cycles and the source node forwards directly. This algorithm ends when all neighbours of the source node receive the message.

The key idea of this algorithm is to design in such a way that: (1) a source node sends a message with direct links, (2) neighbors of the source node sends the message in parallel, and (3) if under four neighbors remain, the source node sends the message in three hops. Obviously, after each step, the number of neighbors with the message is doubled (but not done always in the last step). For example, in an $(8, 4)$-star graph, for the source node $\boldsymbol{s} = 1234$, the neighbourhood broadcasting is shown as follows:

- Step 1:
  $1234 \rightarrow 2134$
- Step 2:
  $1234 \rightarrow 3214$
  $2134 \rightarrow 4132 \rightarrow 1432 \rightarrow 2431 \rightarrow 4231$
- Step 3:
  $1234 \rightarrow 5234$
  $2134 \rightarrow 6134 \rightarrow 1634 \rightarrow 2634 \rightarrow 6234$
  $3214 \rightarrow 7214 \rightarrow 1274 \rightarrow 3274 \rightarrow 7234$
  $4231 \rightarrow 8231 \rightarrow 1238 \rightarrow 4238 \rightarrow 8234$

This running time for this algorithm is $\boldsymbol{O}(\log n)$ [16]. Because the lower bound is $\boldsymbol{\Omega}(\log n)$, this algorithm is optimal. Of course, when n is relatively small, it is better to simply forward a message from the source node to its $n-1$ neighbors in $n-1$ steps.

C. Broadcast Algoritm on the Single-Port Model

The broadcasting problem, BP for short, is a problem a message of the source node is sent to all the nodes in the network. For a single-port model, the BP has a lower bound of $\boldsymbol{\Omega}(\log N)$, where N is the total number of nodes in the network. Therefore, the lower bound for this broadcasting problem on single-port $S_{n,k}$ is $\boldsymbol{\Omega}(\log(n!/(n-k)!)) = \boldsymbol{\Omega}(k \log n)$. Several broadcast algorithms for (n, k)-star graph have been studied [18], [19], [16]. Among them, in [16], an optimal time algorithm is proposed by using the beighbourhood broadcasting.

The idea of this scheme can be described as follows. Since $S_{n,k}$ can be decomposed as n number of $S_{n-1,k-1}$, the source node will send message to one node in each of $S_{n-1,k-1}(i)$, where $1 \leq i \leq n$. Now every $S_{n-1,k-1}(i)$ has a node with the message, it recursively carries out the algorithm on each $S_{n-1,k-1}(i)$. Concretely, we assume that the source node is $\boldsymbol{e}_k = 123 \cdots k$ and wants to broadcast a message to all the other processors in $S_{n,k}$. In the first step, the source node forwards the message to its all neighbors using the neighbourhood broadcasting.

Now all i-neighbors of the source node $\boldsymbol{e}_k$ ($2 * k$, $3 * k$, ..., $(k-1) * k$ and $k * 1$) and all 1-neighbors ($(i+1) * k$, $(i+2) * k$, ..., $(n-1) * k$ and $n * k$) have the message. Then, these all neighbors (except $k * 1$) send the message through k-dimensional edges in one more time unit. Now n nodes ($*1$, $*2$, ..., $*n$) have the message and these nodes belong to every $S_{n-1,k-1}(i)$, $1 \leq i \leq n$. So we can recursively broadcast in each $S_{n-1,k-1}(i)$ in parallel.

This broadcasting algorithm has $\boldsymbol{O}(k \log n)$ time and is optimal in the view of the $\boldsymbol{\Omega}(k \log n)$ lower bound [16]. The key idea of this algorithm is to design in such a way that: (1) a source node sends a message to its all neighbors using the neibourhood broadcasting, (2) all neighbors (except $k * 1$) send the message through k-dimensional edges, (3) these nodes which received the messages in the previous step broadcast as new source nodes in their subgraphs, and (4) when subgraphs form clique, they simply perform a standard broadcasting algorithm.

Consequently, our broadcasting computational complexity of the GSC(n, k, m) on the single-port is $\boldsymbol{O}(m + k \log n)$. This broadcasting algorithm is optimal in the view of the $\boldsymbol{\Omega}(m + k \log n)(= \boldsymbol{\Omega}(\log 2^m + \log(n!/(n-k)!)))$ lower bound.

VI. Broadcasting on the All-Port Model

In this section, we develop the broadcasting algorithm for the all-port model. This algorithm is also separated into two part like the shortest-path routing algorithm and the broadcasting on the single-port model. First, we outline the minimum dominating set for (n, k)-star graph part. Then, we develop an optimal broadcasting algorithm on the all-port model using the minimum dominating set.

A. The Minimum Dominating Set of the (n, k)-Star Graph

Generally, in graph theory, a dominating set for a graph $\boldsymbol{G} = \{\boldsymbol{V}, \boldsymbol{E}\}$ is a subset $\boldsymbol{V'} \subseteq \boldsymbol{V}$ such that every vertex not in $\boldsymbol{V'}$ is adjacent to at least one member of $\boldsymbol{V'}$. The domination number is the number of vertices in $\boldsymbol{V'}$, and the minimum dominating set is a dominating set with the smallest domination number. The dominating set problem is to find a minimum dominating set $\boldsymbol{D_G}$ of a graph $\boldsymbol{G}$ with domination number $|\boldsymbol{D_G}|$.

Let $\boldsymbol{D}_{n,k}$ be a minimum dominating set of $\boldsymbol{S}_{n,k}$, then every vertex set $\boldsymbol{D}_{n,k} = \{i*\}$, for $1 \leq i \leq n$, and $|\boldsymbol{D}_{n,k}| = (n-1)!/(n-k)!$ [16]. For example, $S_{4,2}$ has four different minimum dominating sets depending on the value of i: (1) $\{12, 13, 14\}$ for $i = 1$, (2) $\{21, 23, 24\}$ for $i = 2$, (3) $\{31, 32, 34\}$ for $i = 3$, and (4) $\{41, 42, 43\}$ for $i = 4$.

$\boldsymbol{D}_{n,k}$ and its neighbors are all of nodes of $S_{n,k}$, therefore in the all-port model, we can send a message in one time unit by using the $\boldsymbol{D}_{n,k}$. We use this idea and the hierarchical structure of $S_{n,k}$ to develop a broadcasting algorithm for the all-port model in the next subsection.

B. Broadcasting Algorithm on the All-Port Model

When discussing the BP on interconnection networks of the all-port model, we need to consider the traffic, the total number of messages exchanged in addition to the time, the number of time steps required [20]. Hence, it is desirable to minimize both the time and traffic. By mitigating the traffic, we can reduce the message redundancy which is a problem that a node receives the same message many times. Broadcast algorithms for (n,k)-star on the all-port model has been studied [19], [16]. Among them, in [16], an optimal time algorithm is proposed based on the minimum dominating set.

Now we have the minimum dominating set $\boldsymbol{D}_{n,k}$ (all the nodes forming $i*$) from the previous subsection. Then, a simple broadcasting algorithm on the all-port model for $S_{n,k}$ can be designed by using $\boldsymbol{D}_{n,k}$ as follows: (1) we decompose current subgraph at the last dimension of the source node until forming a clique; (2) when a subgraph forms a clique, the source node sends the message along dimension 1; (3) all nodes with the message send along an upper current dimension; and (4) since the nodes that received the message in the previous step are the minimum dominating set in current dimension, each node in the dominating set sends its message along all dimensions except current dimension (if not finished, go back to step (3)). The optimal running time of this algorithm is proportional to the diameter of the network and $\boldsymbol{O}(k)$ [16]. Furthermore, there is no message redundancy.

Consequently, our broadcasting computational complexity of the GSC(n,k,m) on the all-port is $\boldsymbol{O}(m+k)$. Because this running time is proportional to the diameter of the GSC(n,k,m), thus it is optimal and there is no message redundancy.

VII. Conclusion

In this research, we proposed a new interconnection network, the generalized-star cube, described its topological properties, and gave a shortest-path routing algorithm and broadcast algorithms for both of the single-port and all-port models. The proposed generalized-star cube retains most of the properties of the hypercube and (n,k)-star graph. Compared to the hypercube, star graph, (n,k)-star graph, and star-cube, this network can change the network size in smaller steps and we can choose a more desirable network size.

In recent research, several product graphs have been proposed based on the star graph and cube-based derivatives [21], [22], [23], [24]. We can also derive new topologies by replacing the star graph of those product graphs with the (n,k)-star graph. Meanwhile, a lot of works concerning the generalized-star cube require further research. Some of them are: (1) to find disjoint-path in a generalized-star cube; (2) to develop fault-tolerant routing algorithms for the proposed network with faulty nodes; (3) to develop an efficient all-to-all broadcasting algorithm; and (4) to investigate the embedding of other frequently used topologies into this network.

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