

Inflation, Endogenous Growth, Transaction Costs, and Varying Discount Rates

著者	MIYAZAKI Kenji
出版者	Institute of Comparative Economic Studies, Hosei University
journal or publication title	比較経済研究所ワーキングペーパー
volume	153
page range	1-33
year	2010-02-15
URL	http://hdl.handle.net/10114/6297

逐次的効用関数・曖昧さ・時間不整合性を考慮した国際マクロ分析 シリーズ No.7

Inflation, Endogenous Growth, Transaction Costs, and Varying Discount Rates

Kenji Miyazaki

Inflation, Endogenous Growth, Transaction Costs, and Varying Discount Rates*

Kenji Miyazaki[†]

Faculty of Economics, Hosei University

February 8, 2010

Abstract

This study explores the relation between inflation and economic growth using the transaction costs model with a socially determined discount rate and a linear production technology. Even when the labor decision is inelastic, this study demonstrates that inflation affects the endogenous growth with nonconstant time preferences. In particular, if the degree of impatience increases in the economy-wide average ratio of total assets (the sum of capital and money) to consumption, then zero rate of money supply could achieve maximized endogenous growth; the relationship between inflation and economic growth is hump-shaped. The numerical examples confirm such a hump-shaped relationship, but the impact of inflation on economic growth is quantitatively small.

Keywords: monetary policy; impatience; hump-shaped

JEL classifications: E5; O42

*I would like to thank Nobusumi Sagara and Yoichi Gokan for their helpful comments, and Hitoshi Utsunomiya for excellent research assistance. All remaining errors are my own. I am grateful for a grand-in-aid from the Ministry of Education and Science, the Government of Japan.

[†]Correspondence to: Kenji Miyazaki, Faculty of Economics, Hosei University, 4342 Aihara, Machida, Tokyo, Japan, 194-0298; e-mail: miya_ken@hosei.ac.jp; tel: +81-42-783-2591; fax: +81-42-783-2611

1 Introduction

Beginning with Tobin (1965) and Sidrauski (1967), the relationship between inflation and economic growth has been one of the central issues in monetary economic theory. Wang and Yip (1992) considered the money-in-the-utility-function (MIUF) approach, the cash-in-advance (CIA) approach, and the transactions-costs (TC) approach, and demonstrated that when labor demand is inelastic, the three approaches leads to superneutrality. When labor demand is elastic, inflation generally lowers the level of capital stocks in the long run. To derive this conclusion, they considered exogenous growth models and examined the relationships between inflation and the level of capital stocks rather than the rate of economic growth.

Several authors have investigated a correlation between inflation and the long-run growth rate in an endogenous growth framework. In the CIA approach, Gomme (1993) introduced a trade-off between labor and consumption, and Jones and Manuelli (1995) considered nominal rigidities. In the TC approach, DeGregorio (1993) and Jha et al. (2002) used the transaction technology in which the transaction-cost function increases with consumption and decreasing in money in DeGregorio and additionally increasing in real output in Jha et al. The studies cited above in general supported the negative effect of an increasing money supply on economic growth.

These implications cannot, however, explain the empirical evidence reported by Gomme (1993) and Bullard and Keating (1995) that there may exist a positive link between inflation and the growth rate in low-inflation countries. For a possible explanation, Fukuda (1994) and Itaya and Mino (2003) considered a monetary version of the Benhabib and Farmer (1994) model. Fukuda (1996) and Itaya and Mino (2003) adopted the CIA approach and the TC approach, respectively. Both results found that with sufficient large labor externality, a

positive relationship emerges. However, their models cannot explain the hump-shaped relationship between inflation and economic growth within one country with the same economic environment.¹

We explore the relationship between inflation and economic growth using the transaction costs model with a socially determined discount rate and a linear production technology. Even when the labor decision is inelastic, this study demonstrates that inflation affects the endogenous growth with nonconstant time preferences. In particular, if the degree of impatience increases in the economy-wide average ratio of general assets to consumption, then the relationship between inflation and economic growth could be hump-shaped, which supports empirical evidences. In the case of total assets (the sum of capital and money), zero monetary policy achieves maximized economic growth.

The intuition is as follows. The higher the rates of money supply the lower are real balances and the higher are the transaction costs, and this discourages consumption or increases the ratio of capital to consumption. At the same time, higher cost of money holdings decreases the money supply or the ratio of money to consumption. When an economic agent is less patient as the ratio of assets to consumption increases, the agent becomes either less patient if the capital effect is stronger or more patient if the money effect is stronger. When the cost of holding money is sufficiently high, the capital effect is dominant, lowering economic growth.

We establish the existence and uniqueness of a balanced growth path (BGP) and demonstrate that such a BGP is locally stable. Furthermore, we conduct several numerical exercises and compare ours with the model with the discount rates determined internally by the individual. Our numerical exercises confirm

¹One explanation for this hump-shaped relationship within the same technology and preferences was given by Dutta and Kapur (1998), who considered a three-period overlapping model with preference shocks, cash-in-advance constrains, irreversible investment, and exogenous technology.

that the hump-shaped relationship between inflation and the rate of economic growth is established with a set of plausible parameters, but show that the impact of inflation on economic growth is small quantitatively. Our comparison proves that even if the discount rate depends not on aggregate but on individual consumption, then the results of the comparative statics on the BGP are observationally equivalent in both models.

This study is not the first attempt to consider the relationship between money and growth with varying discounting rates. Kam (2005) considered the MIUF model with the discount rates depending on economy-wide total assets and found that when discount rates increase in total assets, then a positive relationship between inflation and capital stocks, a Tobin effect, emerges. Chen et al. (2008) used the MIUF and TC models with the degree of impatience is internally determined by individuals. Chen et al. demonstrated that increasing impatience in money (resp. consumption) leads to a Tobin effect in the MIUF (resp. TC) model. However, both studies lay in an exogenous growth framework and did not generate a hump-shaped relationship.

This study is also not the first attempt to use an endogenous growth model with varying discount rate. Palivos et al. (1997) and Meng (2006) investigated the relationship among the BGP and the functional forms of the felicity and the discount rate in the framework of linear technology. Both studies found that under the BGP and the discount rate function invariant to the BGP, the discount rate should be constant or a homogeneous degree of zero, and that the elasticity of marginal felicity must be constant. We demonstrate that their results hold in our monetary economy.

Our studies makes three contributions. First, this model gives a solution to the puzzle posed by Gomme (1993) and Bullard and Keating (1995), and sheds new light on monetary policy. Second, we consider the monetary model with

varying discount rate in an endogenous growth framework, whereas Palivos et al. (1997) and Meng (2006) focused their attention on a real economy. Third, we find that in some types of models, the comparative statics on the BGP has the same results whether consumption in the discount rate is determined externally and internally.

Our paper is organized as follows. Section 2 describes a model economy with the externally determined discount rates, and Section 3 establishes the existence and uniqueness of the BGP. Section 4 conducts comparative statics analysis, and Section 5 discusses the local stability. Section 6 performs several numerical exercises, and Section 7 compares our model to the model with discount rates determined internally by the individual. Section 8 concludes.

2 The Model with the Externally Determined Discount Rates

We construct an endogenous growth model augmented with money or real balances. The representative agent is infinitely lived and endowed with an initial capital $k_0 > 0$, an initial nominal money stock $M_0 > 0$, and a normalized initial price level $P_0 = 1$. The production technology is linear in capital, whereas the labor supply is inelastic. The population of the economy stays constant.

The representative agent has the following lifetime utility:

$$U = \int_0^{\infty} u(c_t) e^{-\Delta_t} dt, \quad (1)$$

where c_t is consumption at t , u represents the felicity, Δ_t represents the cumulative discount rate at period t .

The felicity function u is continuous, increasing, and concave. The cumula-

tive discount rate function Δ_t is determined by

$$\dot{\Delta} = \rho(C, X), \quad (2)$$

where ρ represents the discount rate function, C_t is the economy-wide average level of consumption at t , and X_t is an economy-wide average level of assets at t . The discounting rate for each agent is not necessarily constant, but it is taken as exogenous. The society determines the time preference, and individuals accept such a social norm to maximize their preference. We assume $\Delta_0 = 0$ and $0 < \rho < \infty$ for all C and X . We will impose more restrictions on u and ρ after we present two propositions in Section 2.

Equation (2) indicates that not only aggregate consumption, but also assets affect the degree of impatience. We consider two types of assets: productive and nonproductive. Economy-wide productive assets, denoted by K , contribute directly to production at a macroeconomic level, whereas non-productive assets, denoted by M , facilitate transactions. In what follows, we respectively call K and M capital and money, and the sum of capital and money $K+M$ are referred to as total assets. As the variable X in (2), we can consider the economy-wide general assets:

$$X = \alpha_1 K + \alpha_2 M,$$

where $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$. When $\alpha_1 = 1$ and $\alpha_2 = 1$, then X represents total assets.

Individual money or real balances, denoted by m , is introduced into the model by considering the costs with individual transactions. The amount of transaction costs increases with consumption, but decreases with real money balances. Such transaction costs technology is represented by $T(c, m)$. For the existence of a balanced growth path, we assume $T = s(m/c)c$, $s \geq 0$, $s' < 0$, $s'' >$

0 for all $m/c > 0$, $\lim_{m/c \rightarrow \infty} s'(m/c) = 0$, and $\lim_{m/c \rightarrow 0} s'(m/c) = -\infty$. This assumption assures $T_c > 0$, $T_m < 0$, $T_{cc} > 0$, $T_{mm} > 0$, and $T_{cc}T_{mm} - T_{cm}^2 = 0$. We call $s(m/c)$ the transaction-costs function. An example is:

$$s(m/c) = s_0(m/c)^{-\eta}$$

where $s_0 > 0$ and $\eta > 0$.

The budget constraint is

$$[1 + s(m/c)]c + \dot{a} = Ak + \nu - \pi m, \quad (3)$$

where $a = m + k$ is individual total assets, k is individual capital, $\pi = \dot{P}/P$ is expected rates of inflation, ν is lump-sum transfers from the government, and A is a constant parameter representing a linear technology of production function.²

Given C and X , the economic agent chooses c , k , and m to maximize (1) subject to (2), (3), and the boundary conditions $k_0 > 0$, $M_0 > 0$, $P_0 = 1$, and $\lim \lambda(k + m)e^{-\Delta} = 0$. Since the utility function is bounded, the optimization problem is well-defined. When λ is the costate variable of (3), Pontryagin's maximum principle yields

$$\lambda = u'/(1 + s - s'm/c) \quad (4)$$

$$\lambda(A + \pi + s') = 0 \quad (5)$$

$$\dot{\lambda} = \lambda(\rho - A). \quad (6)$$

The government behaves in a (monetary-theoretically) conventional way. It

²Since the individual level of total assets is denoted by a , the economy-wide average level of total assets should have been A . However, the notation A has been conventionally used to represent a constant technology of production function and we follow this convention. Thus, as the average level of total assets, we use $K + M$.

prints nominal money at a constant rate μ and runs a balanced budget by transferring seigniorage revenues to the consumers in a lump-sum manner: $\nu_t = \mu m_t$.

In equilibrium, the money and the goods markets are clear:

$$\dot{m} = (\mu - \pi)m \quad (7)$$

$$\dot{k} = Ak - (1 + s)c. \quad (8)$$

The aggregate consistency condition requires $c = C$, $k = K$, and $m = M$. A monetary equilibrium is a set of path $\{c_t, k_t, m_t, \pi_t\}_{t \in [0, \infty)}$ that maximizes (1) subject to (2) and (3) for given initial conditions, in which the government behavior condition, the market equilibrium conditions, and the aggregate consistency condition hold. We can obtain the following dynamic system of c , m , and k under a monetary equilibrium:

$$\theta \frac{\dot{c}}{c} = A - \rho + \frac{s'' \cdot m/c^2}{1 + s - s' \cdot m/c} \dot{m}, \quad (9)$$

$$\frac{\dot{m}}{m} = \mu + A + s', \quad (10)$$

and (8) with the boundary conditions, where

$$\theta = -c \frac{u''}{u'} + \frac{s'' \cdot m^2/c^2}{1 + s - s' \cdot m/c}, \quad (11)$$

Note that the rate of inflation π is not included in the above dynamic system, but is determined by $\pi = \mu - \dot{m}/m = -A - s'$.

3 A Balanced Growth Path

In this section, we establish the existence and uniqueness of a balanced growth path (BGP). A non-degenerate balanced-growth-path (BGP) monetary equilibrium is a set of monetary equilibrium paths $\{c_t, k_t, m_t, \pi_t\}_{t \in [0, \infty)}$ such that the quantity variables c , k , and m grow at a constant rate $\gamma > 0$. On the BGP,

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{m}}{m} = \gamma$$

and a constant $\pi = \mu - \gamma$. Note that the rate of nominal interests is $A + \pi = A + \mu - \gamma = -s'$ from (10). For a positive nominal interest rates, the government should set the growth of money to be $\mu \geq \gamma - A$ on the BGP.

As Meng (2006) has shown, we can easily prove the following proposition:

Proposition 1: If a BGP exists and the discount rate function $\rho(C, X)$ is invariant to the BGP, then (i) $\rho(C, X)$ must be constant or homogeneous of degree zero in C and X , and (ii) the elasticity of marginal felicity must be constant.

Proof: The proof is essentially the same as that of Lemma 4.1 of Meng (2006), and omitted.

In what follows, we assume $u(c) = c^{1-\sigma}/(1-\sigma)$ for $\sigma > 1$, and $\rho(C, X) = \rho(X/C)$. Note that the objective function (1) is well-defined in this utility case because $\lim_{c \rightarrow \infty} c^{1-\sigma}/(1-\sigma) < 0$ for $\sigma > 1$.

The discounting rate $\rho(X/C)$ depends on the ratio of economy-wide assets to consumption. When $\rho' = 0$, the time preference rate is constant. When $\rho' > 0$, the degree of impatience decreases with the economy-wide average level of consumption, but it increases with the economy-wide average level of capital, money, or total assets, given all other variables fixed. In particular, $\rho'(X/C) > 0$ indicates that as the society gets wealthier or consumes less, people become more

impatient or less willing to defer consumption. Meng (2006) has examined the ratio of consumption to income in a real economy. The ratio is the same as the inverse ratio of capital to consumption in our framework because the production function is Ak .

Let $Z_1 = k/c$ and $Z_2 = m/c$. On the BGP, (8), (9), and (10) are expressed as

$$\sigma\gamma = A - \rho(\xi), \quad (12)$$

$$\gamma = \mu + A + s'(Z_2), \quad (13)$$

$$\gamma = A - \{1 + s(Z_2)\}/Z_1, \quad (14)$$

where $\xi = \alpha_1 Z_1 + \alpha_2 Z_2$. Note that as long as the government sets the growth rate of the money supply to be $\mu \geq (1 - \sigma)A/\sigma$, the nominal interest rate is always positive. In fact, $A + \mu - \gamma = \mu + (\sigma - 1)A/\sigma + \rho/\sigma > 0$ for all $\mu \geq (1 - \sigma)A/\sigma$.

With additional assumptions regarding ρ , we can prove that there exists a unique BGP.

Proposition 2: Assume $\rho(\xi) > 0$ for all $\xi = \alpha_1 Z_1 + \alpha_2 Z_2 > 0$, and $\sup \rho(\xi) \equiv \bar{\rho} < A$. Then there exists a non-degenerating BGP for any $\mu \geq (1 - \sigma)A/\sigma$ with $\sigma > 1$. When ρ is constant, or an increasing function only of Z_2 ($\rho'(\xi) \geq 0$ and $\alpha_1 = 0$), then the BGP is unique. Even in the case with $\alpha_1 > 0$, there exists a unique BGP when $\rho'(\xi) \geq 0$ and $(1 + s)s'' - s'(\mu + s') > 0$ for all $Z_2 > 0$.

Proof: See the Appendix.

In what follows, we assume that $\rho(\xi) > 0$, $\rho'(\xi) \geq 0$, and $\sup \rho(\xi) \equiv \bar{\rho} < A$. When $\alpha_1 > 0$, we need an additional assumption $(1 + s)s'' - s'(\mu + s') > 0$ for a unique BGP. This assumption is satisfied when $\mu \geq 0$ and $s(Z_2) = s_0(Z_2)^{-\eta}$

where $s_0 > 0$ and $\eta > 0$.³ This functional specification is used in Section 6, where the uniqueness of BGP is numerically confirmed in the case with $\mu < 0$.

4 Comparative Statics

In this section, we investigate the effect of monetary policy on the BGP. Our model allows the government to control the growth rate of money. We conduct a comparative static analysis on the BGP. We examine the effect of monetary expansion on economic growth, the ratio of capital to consumption, and the ratio of money to consumption. Then, we demonstrate that there exists an optimal monetary policy in some case. We finally discuss the relationship between the economic growth and the rate of inflation.

We perform the comparative statics with respect to γ , Z_1 , and Z_2 in response to the money growth rate μ . The total differentiation of (12), (13), and (14) leads to the comparative-static results with respect to γ , Z_1 , and Z_2 in response to the growth rate of money:⁴

$$\begin{bmatrix} d\gamma/d\mu \\ dZ_1/d\mu \\ dZ_2/d\mu \end{bmatrix} = \frac{1}{\Omega} \begin{bmatrix} -\rho' \{ \alpha_1 s'(Z_2)/Z_1 + \alpha_2 (1 + s(Z_2))/Z_1^2 \} \\ \sigma s'(Z_2)/Z_1 - \alpha_2 \rho' \\ \alpha_1 \rho' + \sigma (1 + s(Z_2))/Z_1^2 \end{bmatrix}$$

where

$$\Omega = -\sigma s''(1 + s)/Z_1^2 - \alpha_1 \rho'(s'' + s'/Z_1) - \alpha_2 \rho'(1 + s)/Z_1^2.$$

³In fact,

$$\begin{aligned} (1 + s)s'' - s'(\mu + s') &\geq (1 + s)s'' - (s')^2 \\ &= \eta(\eta + 1)s_0 Z_2^{-\eta-2} + \eta s_0^2 Z_2^{-2\eta-2} > 0 \end{aligned}$$

for all $Z_2 > 0$ and $\mu \geq 0$.

⁴The detailed derivation is available by a request to the author.

We have assumed $\rho' \geq 0$. Thus, the term Ω is negative when $s''(Z_2) + s'(Z_2)/Z_1 > 0$. The condition $s''(Z_2) + s'(Z_2)/Z_1 > 0$ is rewritten as

$$\frac{m}{k} = Z_2/Z_1 < -Z_2 s''/s'.$$

This condition is more likely to hold as the degree of concavity of s is sufficiently stronger or as the amount of money is relatively less than that of capital on the BGP. When $s(m/c) = s_0(m/c)^{-\eta}$ for example, then $-Z_2 s''(Z_2)/s'(Z_2) = \eta + 1$. We assume $\Omega < 0$ within this section.

When $\Omega < 0$ and $\rho' \geq 0$, the monetary expansion raises the ratio of capital to consumption ($dZ_1/d\mu > 0$) and lowers the ratio of money to consumption ($dZ_2/d\mu < 0$). The effect of monetary expansion on economic growth is

$$\frac{d\gamma}{d\mu} = -\rho' \frac{\alpha_1 s'(Z_2) Z_1 + \alpha_2 (1 + s(Z_2))}{Z_1^2 \Omega}. \quad (15)$$

Note that economic growth is independent of the rate of money supply when either $\rho' = 0$ or $\alpha_1 = \alpha_2 = 0$. Except for these cases, the sign depends on the size of α_1 and α_2 . For further analysis, we consider two extreme cases: $\xi = Z_1$ ($\alpha_1 = 1$ and $\alpha_2 = 0$) and $\xi = Z_2$ ($\alpha_1 = 0$ and $\alpha_2 = 1$), and we then return to general cases.

With $\alpha_1 = 1$ and $\alpha_2 = 0$, the effect of monetary expansion on economic growth is $d\gamma/d\mu = -\rho' s'/(Z_1 \Omega)$. When the degree of impatience is an increasing function of the ratio of capital to consumption ($\rho'(K/C) > 0$), the effect of growth rates of money has a negative effect on economic growth. The intuition is as follows. Higher rates of money supply lower real balances, raise the transaction costs and therefore discourage consumption. When the degree of impatience increases in the ratio of capital to consumption, decreasing consumption makes the agent less patient, and thus lowers economic growth $(A - \rho)/\sigma$.

With $\alpha_1 = 0$ and $\alpha_2 = 1$, the effect of monetary expansion on economic growth is the opposite ($d\gamma/d\mu = -\rho'(1+s)/(Z_1^2\Omega)$). When the degree of impatience increases in the ratio of real balances to consumption ($\rho'(M/C) > 0$), higher money rates of money have a positive effect on economic growth. Intuitively, given a common growth rate γ , the higher cost of holding money decreases the ratio of money to consumption Z_2 from (13). When an economic agent is more impatient as the ratio of money to consumption decreases, decreasing Z_2 makes the agent more patient, and thus raises economic growth $(A - \rho)/\sigma$.

In general ($\alpha_1 > 0$ and $\alpha_2 > 0$), both the effects in the cases of $\rho'(K/C) > 0$ and $\rho'(M/C) > 0$ are mixed. The higher rates of money supply lower real balances, raise the transaction costs and therefore discourage consumption, increasing Z_1 . At the same time higher cost of money holdings decreases the ratio of money to consumption Z_2 from (13). When an economic agent is less patient as X/C increases ($\rho' > 0$), increasing Z_1 and decreasing Z_2 makes the agent *either* more patient if the capital effect is stronger *or* less patient if the money effect is stronger. Thus the effect on economic growth depends on which effect is dominant.

Recall that $\lim_{m/c \rightarrow 0} s'(m/c) = -\infty$ by assumption. Thus, a sufficiently high growth rate of money makes $\alpha_1 s'(Z_2)Z_1 + \alpha_2(1 + s(Z_2))$ in (15) negative, and dampens economic growth $\gamma = (A - \rho)/\sigma$. This may cause a hump-shaped relationship between the money supply and economic growth. When the inflation rate is low, monetary expansion has a positive effect on economic growth (a Tobin effect), but sufficiently higher inflation, on the contrary, decreases the economic growth (a reverse Tobin effect).

We formally present the following proposition.

Proposition 3: When the government can set $\mu = (\alpha_1/\alpha_2 - 1)s'(Z_2)$,

such monetary policy locally maximizes the economic growth rates.

Proof: See the Appendix.

When $\alpha_1 \leq \alpha_2$, the optimal rate of money is greater than zero. As discussed in the last part of Section 3, there exists the unique BGP when $\mu \geq 0$ and $s(Z_2) = s_0(Z_2)^{-\eta}$. This specification is used in Section 6.

In particular, when $\alpha_1 = \alpha_2$ ($\xi = \alpha_1(Z_1 + Z_2)$), the economic growth rate γ is maximized by $\mu = 0$. That is, if the degree of impatience increases in the economy-wide average ratio of total assets (the sum of capital and money holdings) to consumption, then a zero rate of growth of the money supply achieves maximized endogenous growth. Notice that the inflation rate is $-\gamma < 0$ and the nominal rates of interests is $A - \gamma$ under the optimal policy.

We should notice that this proposition does not always guarantee that the government is able to print money at the rate of $(\alpha_1/\alpha_2 - 1)s'(Z_2)$. When $\alpha_1 > \alpha_2$, the optimal growth rates of money is negative, and the rate could be smaller than $A(1 - \sigma)/\sigma$. Remember $\mu \geq A(1 - \sigma)/\sigma$ is assumed for a positive nominal rate of interests. Furthermore, the proposition says only the local properties of optimal monetary policy. In Section 6, we conduct numerical examples to demonstrate the global hump-shaped relation with a set of parameters.

Finally, we consider the relationship between money growth and inflation. Because $\pi = \mu - \gamma$,

$$\frac{d\pi}{d\mu} = 1 - \frac{d\gamma}{d\mu} = s'' \frac{\sigma(1+s) + \alpha_1 \rho' Z_1^2}{\sigma s''(1+s) + \alpha_1 \rho'(s'' Z_2 + s' Z_1) + \alpha_2 \rho'(1+s)} = -s'' \frac{dZ_2}{d\mu} > 0.$$

That is, inflation and the money supply always have a positive correlation. When $d\gamma/d\mu > 0$ (resp. < 0), monetary expansion accelerates the rate of inflation more (resp. less) than proportionately. When $d\gamma/d\mu = 0$, then $d\pi/d\mu = 1$. Although Proposition 3 shows the relationship between economic growth and

the money supply rate, Section 6 examines the relationship between economic growth and inflation rates numerically.

5 Local Dynamics

In this section, we briefly mention the local stability of the balanced growth path to examine whether the path is stable in an economic sense.

Remember that $Z_1 = k/c$, $Z_2 = m/c$ and s is a function of Z_2 alone. Manipulating equations (8), (9), and (10) yields the following differential equation system of Z_1 and Z_2 :

$$\begin{bmatrix} \dot{Z}_1/Z_1 \\ \dot{Z}_2/Z_2 \end{bmatrix} = \begin{bmatrix} \sigma & \tau \\ 0 & \sigma + \tau \end{bmatrix}^{-1} \begin{bmatrix} \sigma\{A - \{1 + s(Z_2)\}/Z_1\} - A + \rho(\xi) \\ \sigma\{\mu + A + s'(Z_2)\} - A + \rho(\xi) \end{bmatrix}$$

where

$$\tau(Z_2) = \frac{s''(Z_2)Z_2^2}{1 + s(Z_2) - s'(Z_2)Z_2},$$

and $\xi = \alpha_1 Z_1 + \alpha_2 Z_2$, respectively.

Linearization with respect to Z_1 , and Z_2 leads to

$$\begin{bmatrix} \dot{Z}_1/Z_1 \\ \dot{Z}_2/Z_2 \end{bmatrix} = J \begin{bmatrix} Z_1 - Z_1^* \\ Z_2 - Z_2^* \end{bmatrix},$$

where J is the 2×2 Jacobian matrix of the dynamic system around Z_1^* and Z_2^* .

With some algebra, we can show that

$$\det J = -\Omega/\{\sigma + \tau(Z_2)\}$$

for each case $X = \alpha_1 K + \alpha_2 M$, where Ω is defined in the previous section for

each case.⁵ Since $\sigma > 0$ and $\tau(Z_2) > 0$, $\det J$ and Ω have the opposite signs. Similarly, the trace takes:⁶

$$\text{tr}J = (1 + s)/Z_1^2 + \frac{(\alpha_1 + \alpha_2)\rho' + \sigma s''}{\sigma + \tau}.$$

Our before-reduced dynamic system has two jump variables c and m . Therefore, Z_1 and Z_2 should have unstable roots of the characteristic function for an economic stability. That is, the economic stability condition is $\det J > 0$ and $\text{tr}J > 0$. Both hold when $\Omega < 0$ and $\rho' > 0$. As discussed in the previous section, the term Ω is negative as long as $s''(Z_2) + s'(Z_2)/Z_1 > 0$. In Section 6, we confirm $\Omega < 0$ with a plausible set of parameters.

6 Numerical Exercises

This section conducts numerical exercises using several sets of parameters.⁷ Remember that the felicity function is assumed to be $u(c) = c^{1-\sigma}/(1-\sigma)$, $\sigma > 1$ for a BGP. In addition, the transaction-costs function is specified as $s(m/c) = s_0(m/c)^{-\eta}$ for $m/c > 0$, where $s_0 > 0$ and $\eta > 0$.

Furthermore, ρ is assumed to be linear ($\rho'' = 0$) for a simple analysis. The discount rate function is represented as $\rho(\xi) = \rho_0 + \rho_1(\xi - \xi_0)$, where $\xi = \alpha_1 Z_1 + \alpha_2 Z_2$ and ξ_0 is obtained from the corresponding BGP with a benchmark rate of money growth μ_0 and a constant time preference ρ_0 . Clearly $\rho'(\xi) = \rho_1 > 0$. To satisfy the assumption of Proposition 2, we assume that $\rho(\xi) = \epsilon > 0$ if $\rho_0 + \rho_1(\xi - \xi_0) < \epsilon$ and $\rho(\xi) = A - \epsilon$ if $\rho_0 + \rho_1(\xi - \xi_0) > A - \epsilon$.

We should specify all the parameters σ , ρ_0 , ρ_1 , μ_0 , s_0 , η , A , and ϵ . We calibrate these parameters, so that $Z_1 = 2$, $Z_2 = 1$, $\gamma = 0.1$ and $\pi = 0.05$ on the

⁵The detailed derivation is available by a request to the author.

⁶The detailed derivation is available by a request to the author.

⁷The Matlab codes are available by a request to the author.

σ	ρ_0	ρ_1	μ_0	s_0	η	A	ϵ
5	0.1	0.1	0.15	0.5	1.3	0.6	0.01

Table 1: A Set of Parameters

BGP. The ratio of capital to consumption is two, and the amounts of money and consumption are equivalent on the BGP. The economy is growing at 10%. We set the coefficient of relative risk aversion σ to be five. We assume $\rho_0 = 0.1$. With the endogenous growth rate of 10%, the growth rate of money supply μ_0 should be 15%. The remaining parameters s_0 , η , and A are determined from (12), (13), and (14): $s_0 = 0.5$, $\eta = 1.3$, and $A = 0.6$. Remember that $\Omega < 0$, discussed in Section 4, if $\rho' > 0$ and $s''(Z_2) + s'(Z_2)/Z_1 > 0$. The last condition is equivalent to $Z_2/Z_1 \leq Z_2 s''/s' = \eta + 1$, which is satisfied in our transaction function ($Z_2/Z_1 = 0.5 < \eta + 1 = 2.3$).

When $\rho_1 = 0$ or $\rho = \rho_0$, the endogenous economic growth is five percent, independent of growth rates of the monetary supply or inflation. Apart from $\rho_1 = 0$, we investigate the relationship between inflation and economic growth rates. We assume $\rho_1 = 0.1$, implying that the discounting function increases with $\xi = X/C$. We also set ϵ to be 0.01 so that $0 < \rho(\xi) < A$ for all $\xi > 0$. The benchmark values of the parameters are summarized in Table 1. With this set, we can find a non-degenerating BGP ($\gamma > 0$, $Z_1 > 0$, $Z_2 > 0$) for a sufficiently wide range of μ .

Figure 1 considers the case where $X = K$ ($\alpha_1 = 1$ and $\alpha_2 = 0$) on the BGP. With this benchmark set of parameters, $\Omega < 0$ as discussed in Section 4, or $\det J > 0$ as discussed in Section 5. As examined in Section 4, $\rho'(K/C) = 0.1 > 0$ indicates $d\gamma/d\mu < 0$ and $d\pi/d\mu > 0$. Thus, there exists a negative relationship between inflation and economic growth. Figure 2 depicts the relationship between inflation and economic growth in the case where $X = M$ ($\alpha_1 = 1$ and $\alpha_2 = 1$). As discussed in Section 3, monetary expansion

policy increases the rate of economic growth as well as the rate of inflation.

Figure 3 illustrates the relationship between inflation and economic growth in the case where $X = K + M$ ($\alpha_1 = \alpha_2 = 1$). As shown in Proposition 3, the endogenous economic growth is 10.03% maximized at zero growth rate of money supply. Reducing the rate of money supply from 15% to 0% increases the endogenous growth by 0.03%. Since zero monetary expansion implies that the rate of inflation is equal to the negative rate of growth, Figure 3 shows that the relationship is hump-shaped at the inflation rates of -10.03% with the given set of parameters.

Figure 4 illustrates the relationship between inflation and economic growth in the case where $X = AK + M$ ($\alpha_1 = 0.6$ and $\alpha_2 = 1$). This case implies that the agent is affected by the sum of real balances and income. As presented in Proposition 3, the endogenous growth is maximized when the rate of money supply is $\mu = 0.4\eta Z_2^{-\eta-1} = 0.26$. The maximized economic growth is 10.02%. The corresponding rate of inflation is 15.98%. This numerical example indicates that a certain positive inflation can achieve the optimal economic growth.

We examine the robustness of the parameter regarding the marginal effect of the relative assets on the degree of impatience, ρ_1 . Figure 5 compares the case of $\rho_1 = 0.1$ with that of $\rho_1 = 0.8$. Both cases consider $X = K + M$ ($\alpha_1 = 1$ and $\alpha_2 = 1$), and the optimal policies are both zero growth of money supply. The rate of endogenous growth is 10.03% in the case of $\rho_1 = 0.1$, whereas the rate is 10.14% in the case of $\rho_1 = 0.8$. The larger the marginal effect of the relative asset on the degree of impatience, the higher the correlation between inflation and economic growth. However, even with much higher ρ_1 , optimal monetary policy improves the rate of economic growth only by less than 0.2%.

In sum, we have presented the relationship between inflation and economic growth with several cases of varying discount rates. It should be noted that the

effect of monetary policy is not very large with the benchmark set of parameters. Figures 1 and 2 show that approximately a 50% increase in inflation (in terms of 10 years) raises or lowers economic growth by around 1%, and Figures 3 and 4 demonstrate that approximately a 20% increase in inflation has less than 0.1% impact on economic growth. Money is not superneutral, but the magnitude is not very large in our specification of parameters and functional forms.

7 Comparison to the Model with Internally Determined Discount Rates

This section compares the previous model with a model in which consumption in the discount rates is determined internally by the economic agent. The model environment follows the model we used except for the discount rate being determined by

$$\dot{\Delta} = \rho(X/c), \quad (16)$$

instead of (2), where $X = \alpha_1 K + \alpha_2 M$. The discount rate is determined by the ratio of aggregate assets to individual consumption. We should note that the above discount rate is constant on the BGP.

Given X , the economic agent chooses c , k , and m to maximize (1) subject to (3) and (16). To solve the problem, we consider the Hamiltonian:

$$\begin{aligned} H(c, m, k, a, \Delta, \lambda, \phi, \psi) = & e^{-\Delta} \{c^{1-\sigma}/(1-\sigma) \\ & + \lambda(Ak - \pi m + \nu - [1 + s(m/c)]c) - \phi\rho(X/c) + \psi(a - m - k)\} \end{aligned} \quad (17)$$

where λ and ϕ are the costate variables of (3) and (16), respectively. Pontrya-

gin's maximum principle yields (5), (6), and

$$c^{-\sigma} - \{1 + s(m/c) - s'(m/c)m/c\}\lambda + \phi\rho'X/c^2 = 0 \quad (18)$$

$$\dot{\phi} = -u + \phi\rho.$$

The initial conditions are $k_0 > 0$, $M_0 > 0$ and $P_0 = 1$, and the transversality condition by Michel (1982) is $\lim_{t \rightarrow \infty} H(t) = 0$.

As discussed in Palivos et al. (1997), the Hamiltonian is independent of time on the optimal path, and the above transversality condition implies that the value of the Hamiltonian is zero on the optimal path. Hence,

$$\phi = \frac{u(c) + \lambda(Ak - \pi m + \nu - [1 + s(m/c)]c) + \psi(a - m - k)}{\rho(c/X)}.$$

This is interpreted as the lifetime utility on the optimal path. Using (18), it is easy to show $\phi_c = 0$ or equivalently

$$\lambda = c^{-\sigma} \frac{1 + (\rho'X/c)/\{\rho(1 - \sigma)\}}{1 + s(m/c) - s'(m/c)m/c - (\rho'/\rho)(\dot{a}X/c^2)}.$$

The dynamic system of c , m , and k under a monetary equilibrium is characterized by (6), (8), (10), and

$$\lambda = c^{-\sigma} \frac{1 + (\rho'\xi)/\{\rho(1 - \sigma)\}}{1 + s(Z_2) - s'(Z_2)Z_2 - (\rho'/\rho)(Z_1 + Z_2)\xi(\dot{a}/a)}, \quad (19)$$

with the boundary conditions, where $Z_1 = k/c$, $Z_2 = m/c$, and $\xi = \alpha_1 Z_1 + \alpha_2 Z_2$.

The dynamic system of the model with the externally determined discount rates is characterized by (8), (9), and (10). The only difference from the model with internally determined discount rates is the shadow price, λ . In the case of externally determined discount rates, combining (6) and the time difference with respect to λ yields (9). In the case of internally determined discount rates, it is

hard to describe the time difference with respect to (19) with $\xi = \alpha_1 Z_1 + \alpha_2 Z_2$, and therefore it is difficult to characterize the dynamic system without the shadow price λ .

On the BGP, however, equation (19) indicates that $\dot{\lambda}/\lambda = -\sigma\dot{c}/c = -\sigma\gamma$. Thus, the dynamic system of equations (6), (8), (10), and (19) are reduced to (12), (13), and (14) on the BGP. Thus, we can obtain the following proposition:

Proposition 4: Consider the model with the discount rate function determined by (16). Then, γ , Z_1 , and Z_2 on the balanced-growth path are exactly the same as those of the model the discount rates socially determined by $\rho(X/C)$ where $X = \alpha_1 K + \alpha_2 M$.

The comparative analysis conducted in Section 2 is shared in the model with internally determined discount rates.

Many studies dealing with internally determined discount rates assume that the degree of impatience increases with consumption.⁸ This assumption is less plausible from an empirical viewpoint, but it is imposed for dynamic stability. On the other hand, the previous model assumes the increasing impatience in a ratio of aggregate generalized assets to consumption to produce the hump-shaped relationship between the money supply and economic growth. Replacing economy-wide consumption by individual consumption implies that the degree of impatience is decreasing in individual consumption given average generalized assets, which supports empirical evidence including Becker and Mulligan (1997).

However, we have two caveats to this proposition. First, the equivalent result holds only on the BGP. When we attempt to conduct dynamic analysis, we have to differentiate λ in (19) with respect to time. Even in local stability analysis, the dynamic system is too complicated, and so we have to resort to numerical investigation.

⁸Das (2003) investigated the case the decreasing impatience with consumption. But her interest lay only on the exogenous growth model.

Next, the equivalent result on the proposition does not hold even on the BGP when another individual variable apart from consumption determines the discount rate. For example, consider

$$\dot{\Delta} = \rho(m/c).$$

The degree of impatience is determined by relative individual real balances. Then Pontryagin's maximum principle yields (4), (6), and

$$\lambda(A + \pi + s') + \phi\rho'/c = 0$$

instead of (5). With a bit of algebra,⁹ we can show that the BGP characterizes (12), (14), and

$$\left\{ 1 - \frac{\rho' (1 - \sigma)(Z_1 + Z_2)}{\rho (1 - \sigma) + Z_2\rho'/\rho} \right\} \gamma = A + \mu + s' + \frac{\rho'}{\rho} \left\{ \frac{1 + s(Z_2) - s'(Z_2)Z_2}{(1 - \sigma) + Z_2\rho'/\rho} \right\}$$

instead of (13). Thus, it is much more difficult to find conditions for the existence of BGP.

8 Concluding Remarks

To explore the relationship between inflation and economic growth, we have used the transaction-costs model with a socially determined discount rate and a linear production technology. We have defined a BGP and proved that there uniquely exists a non-degenerate BGP when we assume increasing impatience in the ratio of general assets to consumption and several other conditions, and that such a BGP is locally stable. We have found that inflation affects the endogenous growth in the case of non-constant time preferences. Specifically, if

⁹The detailed derivation is available by a request to the author.

the degree of impatience increases in the economy-wide average total assets to consumption ratio, then a zero rate of growth of the money supply achieves the maximized endogenous growth.

We have conducted several numerical exercises to confirm the hump-shaped relationship between the rate of inflation and the rate of economic growth with a set of plausible parameters. In particular, we have displayed such a hump-shaped relationship, but we have discovered that the impact of inflation on economic growth is quantitatively small. Finally, we have compared our models to the models with the discount rates determined internally by the individual, and we have proved that if the discount rate depends on the individual consumption, the results of the comparative statics on the BGP are observationally equivalent in both models.

We point out two extensions. First, whereas we have used the transaction-cost (TC) model in this study, we would apply our theory to the money-in-the-utility (MIUF) model. Feenstra (1986) demonstrated a functional equivalence between TC and MIUF models with an inelastic labor supply and a constant time preference in an exogenous growth framework. We need to examine whether such functional equivalence establishes in our framework. When a equivalent functional form in the MIUF is not proper, a quantitative equivalence (Wang and Yip, 1992) is worth investigating. When we find a proper functional form, in which a non-degenerating monetary BGP exists, then it would be interesting to examine whether inflation and economic growth still have a hump-shaped relationships.

Second, although we have concluded that the impact of inflation on economic growth is small, our choosing of parameters and functional forms does not capture the real economy very well. In particular, we should estimate the marginal effect of the relative assets on the degree of impatience, ρ_1 , in a more accurate

way. To estimate the marginal effect of assets on time preferences, many empirical studies including Becker and Mulligan (1997) use microeconomic data, and considers an individually determined discount rate. For the calibration of our model, international macroeconomic panel data would be more suitable. Empirical analysis using such data is another future task.

Appendix

Proof of Proposition 2:

From (12), $\gamma = (A - \rho(\xi))/\sigma > (A - \bar{\rho})/\sigma > 0$. We combine (12), (13), and (14) to reduce two equations:

$$\rho(\alpha_1 Z_1 + \alpha_2 Z_2) = (1 - \sigma)A - \sigma\mu - \sigma s'(Z_2), \quad (20)$$

$$-\frac{1 + s(Z_2)}{Z_1} = \mu + s'(Z_2). \quad (21)$$

It suffices for the proof to examine whether there exists a pair of $Z_1 > 0$ and $Z_2 > 0$ satisfying (20) and (21).

Set an arbitrary $\mu \geq (1 - \sigma)A/\sigma$ to be fixed. Then, by the property of s' , there exists a $\bar{Z}_2 > 0$ such that $s'(\bar{Z}_2) + \mu - A(1 - \sigma)/\sigma = 0$. Note that \bar{Z}_2 can take infinity when $\mu = (1 - \sigma)A/\sigma$.

Because $s'(Z_2) + \mu < s'(\bar{Z}_2) + \mu = A(1 - \sigma)/\sigma < 0$ for all $Z_2 \in (0, \bar{Z}_2)$, equation (21) is rewritten as:

$$Z_1 \equiv \zeta(Z_2) = -\frac{1 + s(Z_2)}{\mu + s'(Z_2)}, \quad (22)$$

with $\lim_{Z_2 \rightarrow 0} \zeta(Z_2) = 0$ and

$$\lim_{Z_2 \rightarrow \bar{Z}_2} \zeta(Z_2) = -\frac{1 + s(\bar{Z}_2)}{(\mu + s'(\bar{Z}_2))} = \frac{\sigma\{1 + s(\bar{Z}_2)\}}{(\sigma - 1)A} > 0.$$

The derivative of $\zeta'(Z_2)$ is:

$$\zeta'(Z_2) = \frac{s''(1+s) - s'(\mu+s')}{(\mu+s')^2}.$$

That is, $\zeta'(Z_2) > 0$ or $\zeta(Z_2)$ increases monotonically if $s''(1+s) - s'(\mu+s') > 0$ for all Z_2 .

To examine equation (20), we consider the case with non-constant ρ . Substituting (22) into (20) leads to

$$\rho(\alpha_1\zeta(Z_2) + \alpha_2 Z_2) = (1-\sigma)A - \sigma\mu - \sigma s'(Z_2), \quad (23)$$

Then, the right-hand side of (23) decreases in $Z_2 \in (0, \bar{Z}_2)$ with:

$$\begin{aligned} \lim_{Z_2 \rightarrow 0} (1-\sigma)A - \sigma\mu - \sigma s'(Z_2) &\rightarrow \infty, \\ \lim_{Z_2 \rightarrow \bar{Z}_2} (1-\sigma)A - \sigma\mu - \sigma s'(Z_2) &\rightarrow 0. \end{aligned}$$

Thus, as long as $0 < \rho < A$, there exists at least one Z_2 satisfying (23).

When ρ is constant, or an increasing function only of Z_2 ($\alpha_1 = 0$), then we can obtain a unique $Z_2 \in (0, \bar{Z}_2)$. From (22), we can obtain a unique $Z_1 > 0$.

Next, consider the case where $\alpha_1 > 0$ and $\rho'(\xi) \geq 0$. When $s''(1+s) - s'(\mu+s') > 0$, then $\zeta(Z_2)$ increases monotonically. Therefore, the left-hand side of (23) increases in $Z_2 \in (0, \bar{Z}_2)$ with:

$$\begin{aligned} \lim_{Z_2 \rightarrow 0} \rho(\alpha_1\zeta(Z_2) + \alpha_2 Z_2) &\rightarrow \rho(0) > 0, \\ \lim_{Z_2 \rightarrow \bar{Z}_2} \rho(\alpha_1\zeta(Z_2) + \alpha_2 Z_2) &\rightarrow \rho(\alpha_1\sigma\{1+s(\bar{Z}_2)\}/\{(\sigma-1)A\} + \alpha_2 \bar{Z}_2) > \rho(0). \end{aligned}$$

Thus, there exists a unique Z_2 satisfying (23), giving a unique $Z_1 > 0$ from (22).

Proof of Proposition 3:

As discussed before, the first derivative of γ with respect to μ is zero when $\alpha_2(1 + s(Z_2)) + \alpha_1 s'(Z_2)Z_1 = 0$. Since $\mu + s'(Z_2) + (1 + s(Z_2))/Z_1 = 0$ from (13) and (14), $d\gamma/d\mu = 0$ at $\mu = (\alpha_1/\alpha_2 - 1)s'(Z_2)$. Since $dZ_1/d\mu = (\sigma s'/Z_1 - \alpha_2 \rho')/\Omega$ and $dZ_2/d\mu = (\alpha_1 \rho' + \sigma(1 + s)/Z_1^2)/\Omega$, the second derivative of γ with respect to μ at $\mu = (\alpha_1/\alpha_2 - 1)s'(Z_2)$ is:

$$\begin{aligned} \frac{d^2\gamma}{d\mu^2} &= -\frac{\rho'}{Z_1^2\Omega} \frac{d[\alpha_2(1 + s(Z_2)) + \alpha_1 s'(Z_2)Z_1]}{d\mu} - [\alpha_2(1 + s(Z_2)) + \alpha_1 s'(Z_2)Z_1] \frac{d}{d\mu} \frac{\rho'}{Z_1^2\Omega} \\ &= -\frac{\rho'}{Z_1^2\Omega} \left[\alpha_1 s' \frac{dZ_1}{d\mu} + \{\alpha_2 s' + \alpha_1 s'' Z_1\} \frac{dZ_2}{d\mu} \right] \\ &= -\frac{\rho'}{Z_1^2\Omega^2} [\alpha_1 s'' Z_1 (\alpha_1 \rho' + \sigma(1 + s)/Z_1^2)] = -\frac{\rho'}{Z_1^2\Omega} \alpha_1 s'' Z_1 \frac{dZ_2}{d\mu}. \end{aligned}$$

Since $\Omega < 0$, $\rho' > 0$, and $dZ_2/d\mu < 0$, then $d^2\gamma/d\mu^2$ is negative; γ is locally maximized in the neighborhood of $\mu = 0$.

References

- [1] Becker, G. S., and C. B. Mulligan (1997) "The endogenous determination of time preference." *Quarterly Journal of Economics*, Vol. 122, pp. 729-758.
- [2] Benhabib, J., and R. Farmer (1994) "Indeterminacy and increasing returns." *Journal of Economic Theory*, Vol. 63, pp. 19-41.
- [3] Bullard, J., and J. Keating (1995) "The long-run relationship between inflation and output in postwar economies." *Journal of Monetary Economics*, Vol. 51, pp. 477-496.
- [4] Chen, B.-L., M. Hsu, and C.-H. Lin (2008) "Inflation and growth: impatience and a qualitative equivalent." *Journal of Money, Credit, and Banking*, Vol. 40, pp. 1310-1323.

- [5] Das, M. (2003) "Optimal growth with decreasing marginal impatience." *Journal of Economic Dynamics and Control*, Vol. 27, pp. 1881-1898.
- [6] De Gregorio, J. (1993) "Inflation, taxation, and long-run growth." *Journal of Monetary Economics*, Vol. 31, pp. 271-298.
- [7] Dutta, J., and Kapur, S. (1998) "Liquidity preference and financial intermediation." *Review of Economic Studies*, Vol. 65, pp. 551-572.
- [8] Feenstra, R. C. (1986) "Functional Equivalence between Liquidity Costs and the Utility of Money, Vol. 65, pp. 271-291.
- [9] Fukuda, S. (1996) "Money and economic growth with increasing-returns-to-scale." The Institute of Economic Research, Hitotsubashi University. Discussion Paper no. 320.
- [10] Gomme, P. (1993) "Money and economic growth revised: measuring the costs of inflation in an endogenous growth model." *Journal of Monetary Economics*, Vol. 32, pp. 51-77.
- [11] Itaya, J. and K. Mino (2003) "Inflation, transaction costs and indeterminacy in monetary economies with endogenous growth." *Economica*, Vol. 70, pp. 451-470.
- [12] Jha, S. K., Ping Wang, and Chong K. Yip (2002) "Dynamics in a transactions-based monetary growth model." *Journal of Economic Dynamics and Control*, Vol. 26, pp. 616-635.
- [13] Jones, L. E. and Manuelli, R. E. (1995) "Growth and the effects of inflation." *Journal of Economic Dynamics and Control*, Vol. 19, pp. 1405-1428.
- [14] Kam, E. (2005) "A note on time preference and the Tobin effect." *Economics Letters*, Vol. 89, pp. 127-132.

- [15] Meng, Q. (2006) "Impatience and equilibrium indeterminacy." *Journal of Economic Dynamics and Control*, Vol. 30, pp. 2671-2692.
- [16] Michel, P. (1982) "On the transversality condition in infinite horizon optimal problem." *Econometrica*, Vol. 50, pp. 975-985.
- [17] Palivos, T., P. Wang, and J. Zhang (1997) "On the existence of the balanced growth equilibrium." *International Economic Review*, Vol. 38, pp. 205-224.
- [18] Sidrauski, M. (1967) "Inflation and economic growth, *The Journal of Political Economy*, Vol. 75, pp. 796-810.
- [19] Tobin, J. (1965) "Money and economic growth." *Econometrica*, Vol. 33, pp. 671-684.
- [20] Wang, P., and C. K. Yip (1992) "Alternative approaches to money and growth." *Journal of Money, Credit and Banking*, Vol. 24, pp. 553-562.

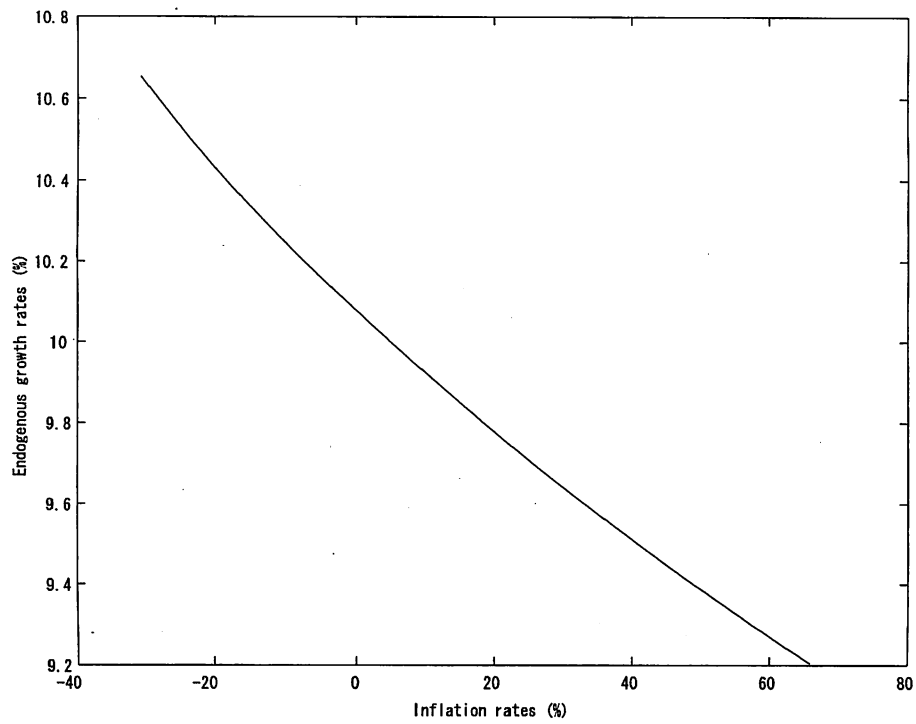


Figure 1: The relation on the BGP between inflation and economic growth in the case of $\alpha_1 = 1$ and $\alpha_2 = 0$

Notes: The rate of economic growth is 10% when the money supply rate is 15%.

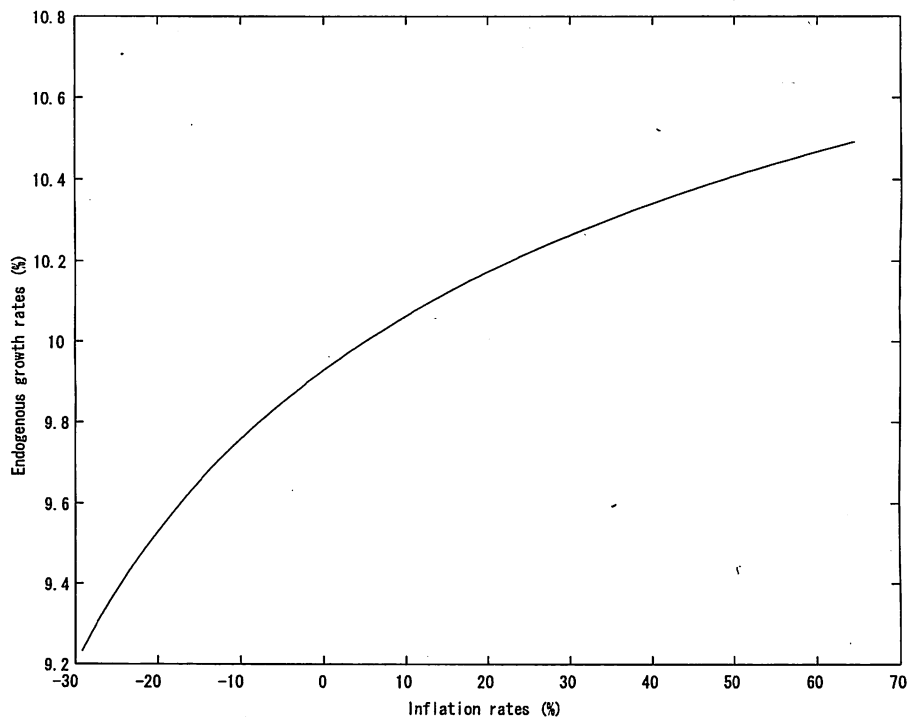


Figure 2: The relation on the BGP between inflation and economic growth in the case of $\alpha_1 = 0$ and $\alpha_2 = 1$

Notes: The rate of economic growth is 10% when the money supply rate is 15%.

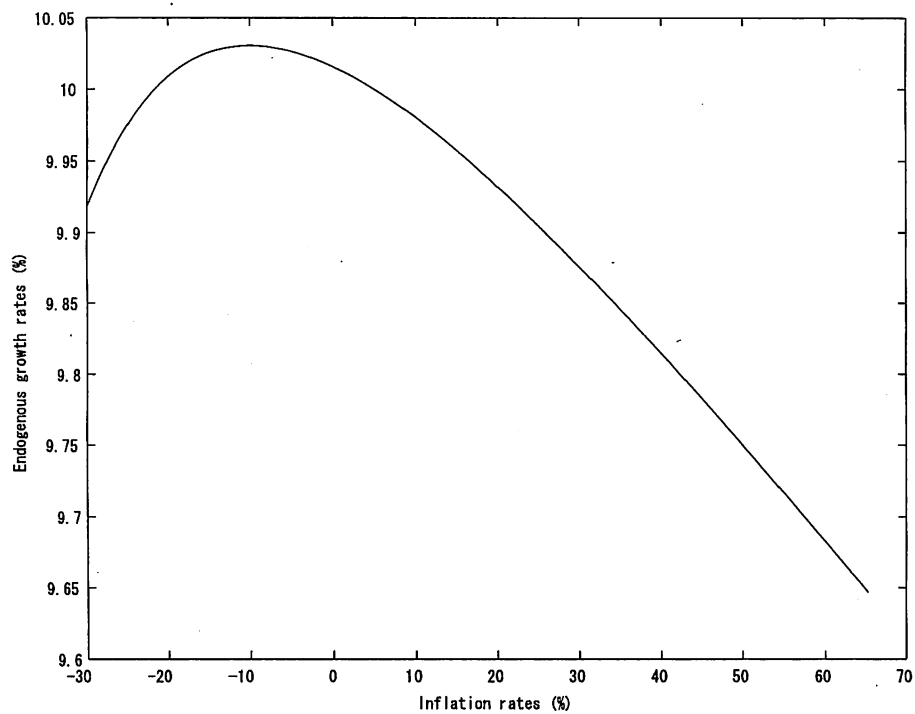


Figure 3: The relation on the BGP between inflation and economic growth in the case of $\alpha_1 = 1$ and $\alpha_2 = 1$

Notes: The rate of economic growth is 10% when the money supply rate is 15%.

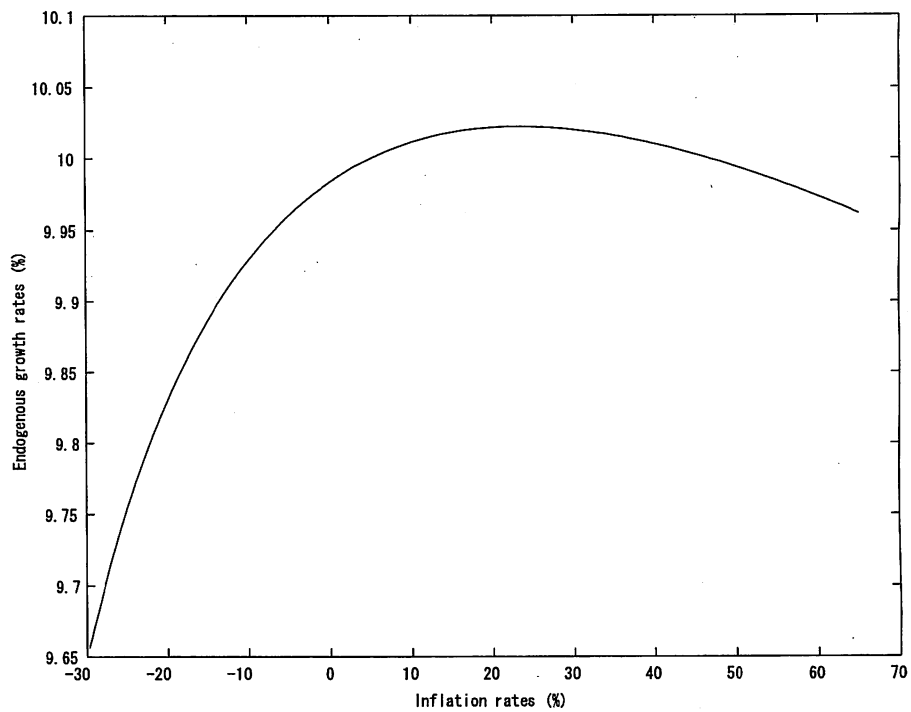


Figure 4: The relation on the BGP between inflation and economic growth in the case of $\alpha_1 = 0.3$ and $\alpha_2 = 1$

Notes: The rate of economic growth is 10% when the money supply rate is 15%.

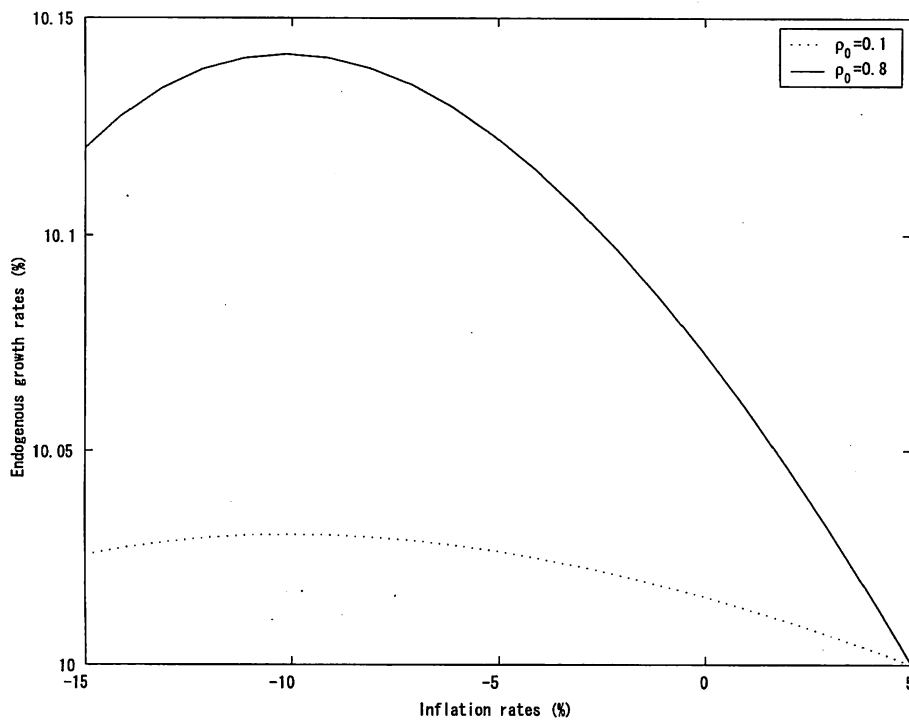


Figure 5: The comparison on the BGP between the cases with $\rho_1 = 0.1$ and $\rho_0 = 0.8$

Notes: The rate of economic growth is 10% when the money supply rate is 15%.