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出版者	Institute of Comparative Economic Studies, Hosei University
journal or publication title	比較経済研究所ワーキングペーパー
volume	127
page range	1-30
year	2007-02-01
URL	http://hdl.handle.net/10114/3977

Representation of Convex Preferences in a Nonatomic Measure Space: ε-Pareto Optimality and ε-Core in Cake Division*

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October 2006

*The authors would like to thank Chiaki Hara, Hidetoshi Komiya and Toru Maruyama for their invaluable technical discussion in an earlier version of this paper. This research is supported by a Grant-in-Aid for Scientific Research (No. 18610003) from the Japan Society for the Promotion of Science, by the Institute Comparative Economic Studies, Hosei University and by the Hosei Institute on Aging, Hosei University.

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Abstract

The purpose of this paper is threefold. First, we represent preference relations on σ-fields in terms of nonadditive set functions that satisfy convexity and continuity in an appropriate sense. To this end, we introduce the convexity and continuity axioms for preferences on a σ-field with a metric topology and show the existence of a utility function representing a convex continuous preference relation. Second, we prove the existence of ε-Pareto-optimal partitions, show how they approximate Pareto-optimal partitions and provide their characterization. Third, we prove the existence of ε-core partitions with nontransferable utility arising in a pure exchange economy and show how they approximate core partitions.

Mathematics Subject Classification 2000: Primary 28A10, 91B16; Secondary 90C29, 91B32.
Journal of Economic Literature Classification: C61, C71.
Key words: Convex continuous preferences; Nonadditive representation; μ-concave function; Cake division; ε-Pareto optimality; ε-Core.

1 Introduction

Dividing fixed resources between members of a society so as to ensure equity and efficiency is a central theme of social decision making. The problem of fair division in a measurable space among finitely many individuals has a long history, although it has attracted more attention in recent years. From the publication of the seminal work by Dubins and Spanier (1961), it has been commonly assumed in the theory of fair division that the preferences of each individual are represented by a nonatomic probability measure. Under this assumption, it is relatively simple to show the existence of Pareto-optimal partitions by a direct application of the Lyapunov convexity theorem, which ensures that the utility possibility set is convex and compact (see Barbanel and Zwicker 1997, Dubins and Spanier 1961, and Sagara 2006).

However, representing a preference relation on a σ-field by a probability measure means that the corresponding utility function is countably additive on the σ-field and consequently assumes a "constant marginal utility". This is obviously a severe restriction on the preference relation that is difficult to justify from an economic viewpoint.

The purpose of this paper is threefold. First, we represent preference relations on σ-fields in terms of nonadditive set functions that satisfy convexity and continuity in an appropriate sense. To this end, we propose a convex-like structure in a nonatomic finite measure space. We introduce convex combinations of measurable sets, and quasiconcave and concave functions on a σ-field and prove Jensen's inequalities, which conform with the standard results in convex analysis. We then introduce the convexity of preference relations on the σ-field and show that a utility function representing the convex preference relation is quasiconcave on the σ-field. The nonadditive utility functions under investigation not only are generalizations of additive preferences but also can capture a "decreasing marginal utility".

We next introduce the continuity axiom for preferences on the σ-field with a metric topology and show the existence of a continuous utility function representing a continuous preference relation by the standard argument of Debreu (1964). Such an approach for the continuous representation of a preference relation on a σ-field is also pursued by Berliant (1986), Berliant and Dunz (2004), and Berliant and ten Raa (1988) using different topologies from the current paper. Unlike the previous works, the metric topology with which we endow the σ-field does not ensure the compactness of the set of partitions although it is mathematically natural. Therefore, the existence of Pareto-optimal partitions is not guaranteed, in general, under the continuity hypothesis on preference relations.

Second, we apply concepts and basic results analogous to those of standard utility theory to the problems of cake division among a finite number of individuals. In particular, we are concerned with the existence of ε-Pareto-optimal partitions. We show that if the preferences of each individual satisfy the continuity hypothesis, an approximation limit of weakly ε-Pareto-optimal partitions is a weakly Pareto-optimal partition. We show that if the preferences of each individual are strictly monotone and continuous, then weak ε-Pareto optimality is equivalent to ε-Pareto optimality. We also provide conditions guaranteeing that every weakly Pareto-optimal partition is a solution to the problem of maximizing a weighted sum of individual utilities. To this end, the convexity of preference relations of each individual plays a significant role in guaranteeing the convexity of the utility possibility set.

Third, under the convexity hypothesis on the preferences of each individual, we show the existence of ε-core partitions with nontransferable utility (NTU) arising in a pure exchange economy in which each individual is endowed with an initial "piece" of the cake. We also show that if the preferences of each individual satisfy the continuity hypothesis, then an approximation limit of ε-core partitions is a core partition.

Berliant (1985) and Berliant and Dunz (2004) introduced a price system into the problem of optimal partitioning and proved the existence of equilibria, which implies the existence of Pareto-optimal partitions. However, the lack of compactness of the set of partitions, and hence the lack of closedness of the utility possibility set, prevent us from using the fixed-point argument to show the existence of equilibria. This is the reason why we present an approximation procedure to obtain the existence of Pareto-optimal partitions and core partitions with NTU without introducing a price system.

The organization of this paper is as follows. In Section 2, we introduce convex combinations of measurable sets, and quasiconcave and concave functions on a σ-field and prove Jensen's inequalities. In Section 3, we define the convexity and continuity of preference relations on the σ-field for the existence of a utility function representing convex continuous preferences. Section 4 is concerned with the existence and characterization of weakly ε-Pareto-optimal partitions and their approximation to weakly Pareto-optimal partitions. Section 5 demonstrates the existence of ε-core partitions with NTU and their approximation to core partitions.

2 Convexity in a Measure Space

In this section, we propose a new concept of convexity in a nonatomic finite measure space. We introduce convex combinations of measurable sets, and concave and quasiconcave functions on a σ-field in conformity with standard convex analysis.

2.1 Convex Combinations of Measurable Sets

Let $(\Omega, \mathscr{F}, \mu)$ be a nonatomic finite measure space, where $\mathscr{F}$ is a σ-field of subsets of Ω and μ is a nonatomic finite measure on $\mathscr{F}$. By the Lyapunov convexity theorem, the range of μ is convex. Therefore, for any $t \in [0, \mu(\Omega)]$ there exists some $A \in \mathscr{F}$ satisfying $\mu(A) = t$. Especially, for any $A \in \mathscr{F}$ and $t \in [0, \mu(A)]$, there exists a measurable subset E of A satisfying $\mu(E) = t$.

Let $A \in \mathscr{F}$ and $t \in [0, 1]$ be given arbitrarily. We define the family $\langle tA \rangle$ of subsets of A by:

$$\langle tA \rangle = \{E \in \mathscr{F} \mid \mu(E) = t\mu(A),\ E \subset A\}.$$

In view of the nonatomicity of μ, it follows that $\langle tA \rangle$ is nonempty for any $A \in \mathscr{F}$ and $t \in [0, 1]$. Note that $E \in \langle tA \rangle$ if and only if $A \setminus E \in \langle (1-t)A \rangle$, and $\mu(A) = 0$ if and only if $\langle tA \rangle$ contains the empty set for any $t \in [0, 1]$.

Theorem 2.1. *For every element A and B in $\mathscr{F}$ and every $t \in [0, 1]$ there exist disjoint elements $E \in \langle tA \rangle$ and $F \in \langle (1-t)B \rangle$.*

Proof. Select $A, B \in \mathscr{F}$ and $t \in [0, 1]$ arbitrarily. Without loss of generality, we may assume that $\mu(B) \leq \mu(A)$. If $\mu(A) = 0$, then it suffices to take $E = \emptyset \in \langle tA \rangle$ and choose any $F \in \langle (1-t)B \rangle$. We thus assume that $\mu(A) > 0$. Take any $F \in \langle (1-t)B \rangle$. We then have $\mu(F) = (1-t)\mu(B) \leq (1-t)\mu(A)$, and hence $\mu(A \cap F) \leq (1-t)\mu(A)$, which is equivalent to $t\mu(A) \leq \mu(A \setminus F)$. Therefore, by the nonatomicity of μ, we can choose a subset E of $A \setminus F$ satisfying $E \in \langle tA \rangle$. By construction, we obtain $E \cap F = \emptyset$. $\square$

Theorem 2.1 guarantees that for every element A and B in $\mathscr{F}$ and every $t \in [0, 1]$, there exists some $C \in \mathscr{F}$ such that C is a union of disjoint sets E and F satisfying $E \in \langle tA \rangle$ and $F \in \langle (1-t)B \rangle$. The family of all such elements C is denoted by $\mathscr{D}_t(A, B)$.

Let Δ^{n-1} denote the $(n-1)$-dimensional unit simplex in $\mathbb{R}^n$; that is:

$$\Delta^{n-1} = \left\{ (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} \alpha_i = 1 \text{ and } \alpha_i \geq 0,\ i = 1, \dots, n \right\}.$$

Lemma 2.1. *Let $A_1, \dots, A_n$ be a finite collection of elements in $\mathscr{F}$ and $(t_1, \dots, t_n) \in \Delta^{n-1}$. If j is such that $\mu(A_i) \leq \mu(A_j)$ for each $i = 1, \dots, n$, then for every collection $\{E_i\}_{i \neq j}$ with $E_i \in \langle t_i A_i \rangle$ for each $i \neq j$, there exists some $E_j \in \langle t_j A_j \rangle$ such that $E_j \cap \bigcup_{i \neq j} E_i = \emptyset$.*

Proof. Let $A_1, \ldots, A_n$ be elements in $\mathscr{F}$, $(t_1, \ldots, t_n) \in \Delta^{n-1}$ and $\mu(A_i) \leq \mu(A_j)$ for each i. Select any $E_i \in \langle t_i A_i \rangle$ for $i \neq j$. If $\mu(A_j) = 0$, then it suffices to define $E_j = \emptyset \in \langle t_j A_j \rangle$. We thus assume that $\mu(A_j) > 0$. Because $\mu(E_i) = t_i \mu(A_i) \leq t_i \mu(A_j)$ for $i \neq j$, we have $\mu(\bigcup_{i \neq j}^n E_i) \leq \sum_{i \neq j}^n \mu(E_i) \leq \mu(A_j) \sum_{i \neq j}^n t_i$. This implies the inequality $\mu(A_j \cap \bigcup_{i \neq j}^n E_i) \leq \mu(A_j) \sum_{i \neq j}^n t_i$, and hence:

$$\frac{\mu(A_j \setminus \bigcup_{i \neq j}^n E_i)}{\mu(A_j)} \geq 1 - \sum_{i \neq j}^n t_i = t_j.$$

Therefore, we can take $E_j \in \langle t_j A_j \rangle$ with $E_j \subset A_j \setminus \bigcup_{i \neq j}^n E_i$ by the nonatomicity of μ. By construction, $E_j \cap \bigcup_{i \neq j} E_i = \emptyset$. $\square$

The following result is an obvious extension of Theorem 2.1.

Theorem 2.2. *For every finite collection of elements $A_1, \ldots, A_n$ in $\mathscr{F}$ and every $(t_1, \ldots, t_n) \in \Delta^{n-1}$, there exist disjoint elements $E_1 \in \langle t_1 A_1 \rangle, \ldots, E_n \in \langle t_n A_n \rangle$.*

Proof. The argument is based on induction. For $n = 2$, the result is reduced to Theorem 2.1. Suppose that the result is true for $n \geq 2$. Let $A_1, \ldots, A_{n+1}$ be elements in $\mathscr{F}$ and $(t_1, \ldots, t_{n+1}) \in \Delta^n$. Without loss of generality, we may assume that $\mu(A_i) \leq \mu(A_{n+1})$ for each $i = 1, \ldots, n$. If $t_{n+1} = 1$, then it suffices to take $E_i = \emptyset \in \langle t_i A_i \rangle$ for $i = 1, \ldots, n$ and choose any $E_{n+1} \in \langle t_{n+1} A_{n+1} \rangle$. We thus further assume that $1 - t_{n+1} > 0$. Define the real numbers s_i by $s_i = (1 - t_{n+1})^{-1} t_i$ for $i = 1, \ldots, n$. In view of $(s_1, \ldots, s_n) \in \Delta^{n-1}$, the induction hypothesis implies the existence of $F_i \in \langle s_i A_i \rangle$ for $i = 1, \ldots, n$ such that $F_i \cap F_j = \emptyset$ for $i \neq j$. Take any $E_i \in \langle (1 - t_{n+1}) F_i \rangle$ for $i = 1, \ldots, n$. We then have $\mu(E_i) = (1 - t_{n+1}) \mu(F_i) = (1 - t_{n+1}) s_i \mu(A_i) = t_i \mu(A_i)$. Therefore, we have $E_i \in \langle t_i A_i \rangle$ for each $i = 1, \ldots, n$. By Lemma 2.1, we can take $E_{n+1} \in \langle t_{n+1} A_{n+1} \rangle$ such that $E_i \cap E_{n+1} = \emptyset$ for each $i \neq n + 1$, and hence $E_i \cap E_j = \emptyset$ for each $i, j = 1, \ldots, n + 1$ with $i \neq j$. $\square$

Theorem 2.2 guarantees that for every finite collection of elements $A_1, \ldots, A_n$ in $\mathscr{F}$ and any $(t_1, \ldots, t_n) \in \Delta^{n-1}$, there exists some E in $\mathscr{F}$ such that E is a union of disjoint sets $E_1, \ldots, E_n$ satisfying $E_i \in \langle t_i A_i \rangle$ for each $i = 1, \ldots, n$. The family of all such elements E is denoted by $\mathscr{D}_{t_1, \ldots, t_n}(A_1, \ldots, A_n)$. When $n = 2$, we adhere to using $\mathscr{D}_t(A, B)$ instead of $\mathscr{D}_{t, 1-t}(A, B)$.

By a *partition* we always mean an ordered finite collection of disjoint elements in $\mathscr{F}$ whose union is Ω. A partition is called an *n-partition* if the number of its members is n.

Theorem 2.3. *Let $(X_1, \ldots, X_m)$ be an m-partition. For every finite collection of n-partitions $(A_1^1, \ldots, A_n^1), \ldots, (A_1^l, \ldots, A_n^l)$ and every $(t_1, \ldots, t_l) \in$*

Δ^{l-1}, *there exists some* $A_{ij} \in \mathscr{D}_{t_1,\dots,t_l}(A_i^1 \cap X_j, \dots, A_i^l \cap X_j)$ *for* $i = 1, \dots, n$ *and* $j = 1, \dots, m$ *such that* $(\bigcup_{j=1}^m A_{1j}, \dots, \bigcup_{j=1}^m A_{nj})$ *is an* n*-partition satisfying* $\bigcup_{j=1}^m A_{ij} \in \mathscr{D}_{t_1,\dots,t_l}(A_i^1, \dots, A_i^l)$ *for each* $i = 1, \dots, n$.

Proof. Select any $A_{ij} \in \mathscr{D}_t(A_i^1 \cap X_j, \dots, A_i^l \cap X_j)$ for each i and j. Then $A_{ij} = E_{ij}^1 \cup \dots \cup E_{ij}^l$ with $E_{ij}^1 \in \langle t_1(A_i^1 \cap X_j) \rangle, \dots, E_{ij}^l \in \langle t_l(A_i^l \cap X_j) \rangle$ and $E_{ij}^k \cap E_{ij}^{k'} \neq \emptyset$ for $k \neq k'$. Because $\{E_{ij}^k\}$ are mutually disjoint, we have $\mu(\bigcup_{j=1}^m E_{ij}^k) = \sum_{j=1}^m t_k \mu(A_i^k \cap X_j) = t_k \mu(A_i^k)$ for each i and k, and hence $\bigcup_{j=1}^m A_{ij} \in \mathscr{D}_{t_1,\dots,t_l}(A_i^1, \dots, A_i^l)$ for each i. Note also that:

$$\mu\left(\bigcup_{i=1}^n \bigcup_{j=1}^m A_{ij}\right) = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^l \mu(E_{ij}^k) = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^l t_k \mu(A_i^k \cap X_j)$$
$$= \sum_{i=1}^n \sum_{k=1}^l t_k \mu(A_i^k) = \sum_{k=1}^l t_k \mu(\Omega) = \mu(\Omega).$$

By joining the null set $\Omega \setminus \bigcup_{i=1}^n \bigcup_{j=1}^m A_{ij}$ to any A_{ij}, the desired partition is easily constructed. □

Corollary 2.1. *Let* $(X_1, \dots, X_m)$ *be an* m*-partition. For every pair of* n*-partitions* $(A_1, \dots, A_n)$ *and* $(B_1, \dots, B_n)$ *and every* $t \in [0, 1]$*, there exists some* $C_{ij} \in \mathscr{D}_t(A_i \cap X_j, B_i \cap X_j)$ *for* $i = 1, \dots, n$ *and* $j = 1, \dots, m$ *such that* $(\bigcup_{j=1}^m C_{1j}, \dots, \bigcup_{j=1}^m C_{nj})$ *is an* n*-partition satisfying* $\bigcup_{j=1}^m C_{ij} \in \mathscr{D}_t(A_i, B_i)$ *for each* $i = 1, \dots, n$.

2.2 Concave Functions on a σ-Field

The following definitions of the (strict) μ-quasiconcavity and (strict) μ-concavity of functions on $\mathscr{F}$ are analogues of the standard definitions in convex analysis.

Definition 2.1. Let $A \triangle B = (A \cup B) \setminus (A \cap B)$ be the symmetric difference of A and B. A function f on $\mathscr{F}$ is:

(i) *μ-quasiconcave* if $A, B \in \mathscr{F}$ and $t \in (0, 1)$ imply

$$\min\{f(A), f(B)\} \le f(C) \quad \text{for any } C \in \mathscr{D}_t(A, B);$$

(ii) *strictly μ-quasiconcave* if $\mu(A \triangle B) > 0$ and $t \in (0, 1)$ imply

$$\min\{f(A), f(B)\} < f(C) \quad \text{for any } C \in \mathscr{D}_t(A, B);$$

(iii) *μ-concave* if $A, B \in \mathscr{F}$ and $t \in (0,1)$ imply

$$tf(A) + (1-t)f(B) \le f(C) \quad \text{for any } C \in \mathscr{D}_t(A,B);$$

(iv) *strictly μ-concave* if $\mu(A \triangle B) > 0$ and $t \in (0,1)$ imply

$$tf(A) + (1-t)f(B) < f(C) \quad \text{for any } C \in \mathscr{D}_t(A,B).$$

A function f on $\mathscr{F}$ is said to be (*strictly*) *μ-quasiconvex* if $-f$ is (strictly) μ-quasiconcave, and f is said to be (*strictly*) *μ-convex* if $-f$ is (strictly) μ-concave.

Example 2.1. A trivial example of a μ-concave and also μ-convex function on $\mathscr{F}$ is μ itself. It is immediate that μ is neither strictly μ-quasiconcave, strictly μ-quasiconvex, strictly μ-concave, nor strictly μ-convex by its additivity.

Example 2.2. Let φ be a function on the closed interval $[0, \mu(\Omega)]$. Define the function f_φ on $\mathscr{F}$ by $f_\varphi(A) = \varphi(\mu(A))$. Because $C \in \mathscr{D}_t(A,B)$ implies $\mu(C) = t\mu(A) + (1-t)\mu(B)$, if φ is quasiconcave, then we have:

$$\begin{aligned} f_\varphi(C) = \varphi(t\mu(A) + (1-t)\mu(B)) &\ge \min\{\varphi(\mu(A)), \varphi(\mu(B))\} \\ &= \min\{f_\varphi(A), f_\varphi(B)\}, \end{aligned}$$

for any $C \in \mathscr{D}_t(A,B)$ and $t \in (0,1)$, and hence f_φ is μ-quasiconcave on $\mathscr{F}$. Conversely, suppose that f_φ is μ-quasiconcave on $\mathscr{F}$. Choose $a, b \in [0, \mu(\Omega)]$ and $t \in (0,1)$ arbitrarily. By the nonatomicity of μ, there exist A and B in $\mathscr{F}$ such that $\mu(A) = a$ and $\mu(B) = b$. Then by Theorem 2.1, there exist $E \in \langle tA \rangle$ and $F \in \langle (1-t)B \rangle$ such that $E \cap F = \emptyset$. We then have:

$$\begin{aligned} \varphi(ta + (1-t)b) = \varphi(t\mu(A) + (1-t)\mu(B)) = \varphi(\mu(E) + \mu(F)) = f_\varphi(E \cup F) \\ \ge \min\{f_\varphi(A), f_\varphi(B)\} = \min\{\varphi(a), \varphi(b)\}, \end{aligned}$$

and hence φ is quasiconcave on $[0, \mu(\Omega)]$. Consequently, f_φ is μ-quasiconcave on $\mathscr{F}$ if and only if φ is quasiconcave on $[0, \mu(\Omega)]$. Similarly, f_φ is strictly μ-quasiconcave [resp. (strictly) μ-concave] if and only if φ is strictly quasiconcave [resp. (strictly) concave].

Recall that a continuous function φ is concave if and only if φ has *decreasing differences*: $x, y \in [0, \mu(\Omega)]$, $x < y$, $x+v, y+v \in [0, \mu(\Omega)]$ and $v > 0$ imply $\varphi(y+v) - \varphi(y) \le \varphi(x+v) - \varphi(x)$. Therefore, for any continuous function φ, it follows that f_φ is *submodular* on $\mathscr{F}$: $f_\varphi(A \cup B) + f_\varphi(A \cap B) \le f_\varphi(A) + f_\varphi(B)$ for any $A, B \in \mathscr{F}$ if and only if φ is concave. As a consequence, the submodularity of f_φ is equivalent to the μ-concavity of f_φ. Note however, that this is not true when φ is defined on a convex subset of a multidimensional Euclidean space (see Example 2.4).

A partition $(X_1, \dots, X_n)$ is *μ-positive* if $\mu(X_i) > 0$ for each $i = 1, \dots, n$.

Definition 2.2. Let $(X_1, \dots, X_n)$ be a μ-positive partition. A function f on $\mathscr{F}$ is:

(i) *μ-quasiconcave at* $(X_1, \dots, X_n)$ if $A, B \in \mathscr{F}$, $t \in (0,1)$ and $C_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each $i = 1, \dots, n$ imply

$$\min\{f(A), f(B)\} \leq f\left(\bigcup_{i=1}^{n} C_i\right);$$

(ii) *strictly μ-quasiconcave at* $(X_1, \dots, X_n)$ if $\mu(A \triangle B) > 0$, $t \in (0,1)$ and $C_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each $i = 1, \dots, n$ imply

$$\min\{f(A), f(B)\} < f\left(\bigcup_{i=1}^{n} C_i\right);$$

(iii) *μ-concave at* $(X_1, \dots, X_n)$ if $A, B \in \mathscr{F}$, $t \in (0,1)$ and $C_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each $i = 1, \dots, n$ imply

$$tf(A) + (1-t)f(B) \leq f\left(\bigcup_{i=1}^{n} C_i\right);$$

(iv) *strictly μ-concave at* $(X_1, \dots, X_n)$ if $\mu(A \triangle B) > 0$, $t \in (0,1)$ and $C_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each $i = 1, \dots, n$ imply

$$tf(A) + (1-t)f(B) < f\left(\bigcup_{i=1}^{n} C_i\right).$$

(Strict) μ-quasiconcavity [resp. (strict) μ-concavity] implies (strict) μ-quasiconcavity [resp. (strict) μ-concavity] at $(X_1, \dots, X_n)$. To show this, it suffices to demonstrate that for every μ-positive n-partition $(X_1, \dots, X_n)$, it follows that $\bigcup_{i=1}^n \mathscr{D}_t(A \cap X_i, B \cap X_i) \subset \mathscr{D}_t(A, B)$ for any $t \in (0,1)$ and $A, B \in \mathscr{F}$. Let $t \in (0,1)$, and A and B be elements in $\mathscr{F}$. Choose any $C_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each i. Define $\alpha_i = \mu(A)^{-1}\mu(A \cap X_i)$ and $\beta_i = \mu(B)^{-1}\mu(B \cap X_i)$ if $\mu(A), \mu(B) > 0$. We then have $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n) \in \Delta^{n-1}$. If $\mu(A) = 0$, by taking $(\alpha_1, \dots, \alpha_n) \in \Delta^{n-1}$ arbitrarily, we have $A \cap X_i \in \langle \alpha_i A \rangle$, and similarly, if $\mu(B) = 0$, then for any choice of $(\beta_1, \dots, \beta_n) \in$

Δ^{n-1}, we have $B \cap X_i \in \langle \beta_i B \rangle$. Because $C_i = E_i \cup F_i$ with $E_i \in \langle t(A \cap X_i) \rangle$, $F_i \in \langle (1-t)(B \cap X_i) \rangle$ and $E_i \cup F_i \neq \emptyset$ for each i, we have:

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i) = t \sum_{i=1}^{n} \mu(A \cap X_i) = t \sum_{i=1}^{n} \alpha_i \mu(A) = t\mu(A),$$

and similarly:

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{n} F_i\right) = \sum_{i=1}^{n} \mu(F_i) &= (1-t) \sum_{i=1}^{n} \mu(B \cap X_i) \\ &= (1-t) \sum_{i=1}^{n} \beta_i \mu(B) = (1-t)\mu(B). \end{aligned}$$

Therefore, $\bigcup_{i=1}^{n} C_i \in \mathscr{D}_t(A, B)$.

However, for arbitrary $n \geq 2$ and for any $A, B \in \mathscr{F}$ and $t \in (0, 1)$, we can easily find an n-partition $(X_1, \ldots, X_n)$ such that $\mathscr{D}_t(A, B) \not\subset \bigcup_{i=1}^{n} \mathscr{D}_t(A \cap X_i, B \cap X_i)$. Thus, (strict) μ-quasiconcavity [resp. (strict) μ-concavity] at some μ-positive partition does not imply (strict) μ-quasiconcavity [resp. (strict) μ-concavity]. The former is a "local" property whereas the latter is "global". When $n = 1$, Definition 2.2 is equivalent to Definition 2.1. See also Example 2.4.

Theorem 2.4. *A function on $\mathscr{F}$ is μ-quasiconcave if and only if it is μ-quasiconcave at each μ-positive n-partition.*

Proof. Let n be fixed. Suppose that a function on $\mathscr{F}$ is μ-quasiconcave at each μ-positive n-partition. To prove the μ-quasiconcavity, it suffices to show that for any $A, B \in \mathscr{F}$ and $t \in (0, 1)$, there exists a μ-positive partition $(X_1, \ldots, X_n)$ such that $\mathscr{D}_t(A, B) \subset \bigcup_{i=1}^{n} \mathscr{D}_t(A \cap X_i, B \cap X_i)$. Take any $C \in \mathscr{D}_t(A, B)$. Then $C = E \cup F$ with $E \in \langle tA \rangle$, $F \in \langle (1-t)B \rangle$ and $E \cap F = \emptyset$. Decompose the $A \cap B$ into disjoint sets $G_1 = A \cap B \cap E$, $G_2 = A \cap B \cap F$ and $G_3 = (A \cap B) \setminus (E \cup F)$, the set $A \setminus (E \cap F)$ into disjoint sets $H_1 = E \setminus (A \cap B)$ and $H_2 = A \setminus (A \cap B) \cup E)$ and the set $B \setminus (E \cap F)$ into disjoint sets $K_1 = F \setminus (A \cap B)$ and $K_2 = B \setminus (A \cap B) \cup F)$. These decompositions compose a decomposition of $A \cup B$ (see Figure 2.1). By the nonatomicity of μ, we can decompose the set G_j into $G_{j1}, \ldots, G_{jn}$ with $\mu(G_{ji}) = \frac{1}{n}\mu(G_j)$ for $j = 1, 2, 3$ and $i = 1, \ldots, n$, the set H_j into disjoint sets $H_{j1}, \ldots, H_{jn}$ with $\mu(H_{ji}) = \frac{1}{n}\mu(H_j)$ for $j = 1, 2$ and $i = 1, \ldots, n$ and the set K_j into disjoint sets $K_{j1}, \ldots, K_{jn}$ with $\mu(K_{ji}) = \frac{1}{n}\mu(K_j)$ for $j = 1, 2$ and $i = 1, \ldots, n$. We then have $E = \bigcup_{i=1}^{n}(G_{1i} \cup H_{1i})$ and $G_{1i} \cup H_{1i} \in \langle \frac{1}{n}E \rangle$ for each i, and $F = \bigcup_{i=1}^{n}(G_{2i} \cup K_{2i})$ and $G_{2i} \cup H_{2i} \in \langle \frac{1}{n}E \rangle$ for each i.

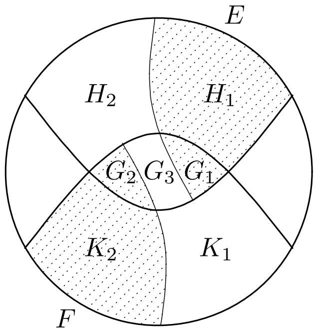

Figure 2.1: Decomposition of $A \cup B$

Define $\Omega_i = G_{1i} \cup G_{2i} \cup G_{3i} \cup H_{1i} \cup H_{2i} \cup K_{1i} \cup K_{2i}$. By construction, $\Omega_1, \ldots, \Omega_n$ are mutually disjoint and $A \cap \Omega_i = G_{1i} \cup G_{2i} \cup G_{3i} \cup H_{1i} \cup H_{2i}$, $B \cap \Omega_i = G_{1i} \cup G_{2i} \cup G_{3i} \cup K_{1i} \cup K_{2i}$ and $(\Omega \setminus (A \cup B)) \cap \Omega_i = \emptyset$ for each i. Decompose $\Omega \setminus (A \cup B)$ into disjoint sets $\Omega'_1, \ldots, \Omega'_n$ with $\mu(\Omega'_i) = \frac{1}{n}\mu(\Omega \setminus (A \cup B))$ and define $X_i = \Omega_i \cup \Omega'_i$ for each i. Then $(X_1, \ldots, X_n)$ is a μ-positive n-partition such that $A \cap X_i = G_{1i} \cup G_{2i} \cup G_{3i} \cup H_{1i} \cup H_{2i}$ and $B \cap X_i = G_{1i} \cup G_{2i} \cup G_{3i} \cup K_{1i} \cup K_{2i}$ for each i. Therefore:

$$\begin{aligned}
\mu(A \cap X_i) &= \mu(G_{1i}) + \mu(G_{2i}) + \mu(G_{3i}) + \mu(H_{1i}) + \mu(H_{2i}) \\
&= \frac{1}{n}[\mu(G_1) + \mu(G_2) + \mu(G_3) + \mu(H_1) + \mu(H_2)] \\
&= \frac{1}{n}[\mu(A \cap B) + \mu(A \setminus (A \cap B))] = \frac{1}{n}\mu(A)
\end{aligned}$$

and:

$$\begin{aligned}
\mu(B \cap X_i) &= \mu(G_{1i}) + \mu(G_{2i}) + \mu(G_{3i}) + \mu(K_{1i}) + \mu(K_{2i}) \\
&= \frac{1}{n}[\mu(G_1) + \mu(G_2) + \mu(G_3) + \mu(K_1) + \mu(K_2)] \\
&= \frac{1}{n}[\mu(A \cap B) + \mu(B \setminus (A \cap B))] = \frac{1}{n}\mu(B).
\end{aligned}$$

Because $\mu(G_{1i} \cup H_{1i}) = \langle \frac{1}{n}E \rangle = \frac{1}{n}t\mu(A)$ and $\mu(G_{2i} \cup H_{2i}) = \langle \frac{1}{n}F \rangle = \frac{1}{n}(1 - t)\mu(B)$, we obtain $G_{1i} \cup H_{1i} \in \langle t(A \cap X_i) \rangle$ and $G_{2i} \cup H_{2i} \in \langle (1-t)(B \cap X_i) \rangle$, and hence $G_{1i} \cup H_{1i} \cup G_{2i} \cup H_{2i} \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each i. Therefore, $C = \bigcup_{i=1}^n (G_{1i} \cup H_{1i} \cup G_{2i} \cup H_{2i}) \in \bigcup_{i=1}^n \mathscr{D}_t(A \cap X_i, B \cap X_i)$. □

Example 2.3. Let $(X_1, \ldots, X_n)$ be a μ-positive partition, and let φ be a function on the product $[0, \mu(X_1)] \times \cdots \times [0, \mu(X_n)]$ of the closed intervals. Define the function f_φ on $\mathscr{F}$ by:

$$f_\varphi(A) = \varphi(\mu(A \cap X_1), \ldots, \mu(A \cap X_n)).$$

When $n=1$, this case reduces to Example 2.2. Define the set S by:

$$S=\{(\mu(A\cap X_1),\dots,\mu(A\cap X_n))\in\mathbb{R}^n \mid A\in\mathscr{F}\}.$$

Because the measure μ_i defined by $\mu_i(A)=\mu(A\cap X_i)$ is nonatomic and S is the range of the vector measure $(\mu_1,\dots,\mu_n)$, by the Lyapunov convexity theorem, it follows that S is convex and compact in $\mathbb{R}^n$.

Suppose that φ is quasiconcave on S. Because $C_i\in\mathscr{D}_t(A\cap X_i,B\cap X_i)$ implies $\mu(C_i)=t\mu(A\cap X_i)+(1-t)\mu(B\cap X_i)$, it follows that $C_i\in\mathscr{D}_t(A\cap X_i,B\cap X_i)$ for each $i=1,\dots,n$ and $t\in(0,1)$ imply:

$$\begin{aligned}
&f_\varphi\left(\bigcup_{i=1}^n C_i\right)=\varphi(\mu(C_1),\dots,\mu(C_n))\\
&\geq \min\{\varphi(\mu(A\cap X_1),\dots,\mu(A\cap X_n)),\varphi(\mu(B\cap X_1),\dots,\mu(B\cap X_n))\}\\
&=\min\{f_\varphi(A),f_\varphi(B)\}.
\end{aligned}$$

Hence, f_φ is μ-quasiconcave at $(X_1,\dots,X_n)$. Conversely, suppose that f_φ is μ-quasiconcave at $(X_1,\dots,X_n)$. Choose $(a_1,\dots,a_n),(b_1,\dots,b_n)\in S$ and $t\in(0,1)$ arbitrarily. Then there exist A and B in $\mathscr{F}$ such that $\mu(A\cap X_i)=a_i$ and $\mu(B\cap X_i)=b_i$ for each i. Then by Theorem 2.1, there exist $E_i\in\langle t(A\cap X_i)\rangle$ and $F_i\in\langle(1-t)(B\cap X_i)\rangle$ such that $E_i\cap F_i=\emptyset$. We then have:

$$\begin{aligned}
&\varphi(ta_1+(1-t)b_1,\dots,ta_n+(1-t)b_n)\\
&=\varphi(t\mu(A\cap X_1)+(1-t)\mu(B\cap X_1),\dots,t\mu(A\cap X_n)+(1-t)\mu(B\cap X_n))\\
&=\varphi(\mu(E_1\cup F_1),\dots,\mu(E_n\cup F_n))=f_\varphi\left(\bigcup_{i=1}^n(E_i\cup F_i)\right)\\
&\geq\min\{f_\varphi(A),f_\varphi(B)\}=\min\{\varphi(a_1,\dots,a_n),\varphi(b_1,\dots,b_n)\},
\end{aligned}$$

in view of $E_i\cup F_i\in\mathscr{D}_t(A\cap X_i,B\cap X_i)$ for each i and the μ-quasiconcavity of f_φ at $(X_1,\dots,X_n)$. Hence, φ is quasiconcave on $[0,\mu(\Omega)]$.

Consequently, f_φ is μ-quasiconcave on $\mathscr{F}$ at $(X_1,\dots,X_n)$ if and only if φ is quasiconcave on S. Similarly, f_φ is strictly μ-quasiconcave [resp. (strictly) μ-concave] at $(X_1,\dots,X_n)$ if and only if φ is strictly quasiconcave [resp. (strictly) concave] on S.

Example 2.4. Consider the case for $n=2$ in Example 2.3. Let φ be a concave function on $[0,\mu(X_1)]\times[0,\mu(X_2)]$ given by $\varphi(x_1,x_2)=\sqrt{x_1x_2}$. Then $f_\varphi(A)=\sqrt{\mu(A\cap X_1)\mu(A\cap X_2)}$ is μ-concave at (X_1,X_2) as shown in Example 2.3. We shall show that f_φ is not μ-concave. To this end, let A and B be measurable sets with positive measure such that $\mu(A\cap X_1)=\frac{1}{2}\mu(A)$

and $\mu(B \cap X_1) = \frac{1}{2}\mu(B)$. Let $C = (A \cap X_1) \cup (B \cap X_1)$. By construction, $C \in \mathscr{D}_{\frac{1}{2}}(A, B)$ and $C \cap X_2 = \emptyset$. We then have $f_\varphi(C) = 0$, $f_\varphi(A) = \frac{1}{2}\mu(A)$ and $f_\varphi(B) = \frac{1}{2}\mu(B)$, and hence $f_\varphi(C) < \frac{1}{2}f_\varphi(A) + \frac{1}{2}f_\varphi(B)$. Therefore, f_φ is not μ-concave. Because φ has decreasing differences, this example also shows that a concave continuous function φ with decreasing differences does not imply the μ-concavity of f_φ in multidimensional cases.

Recall that if a function on a vector space is both concave and convex, then it is an additive function. A similar property holds for a function on $\mathscr{F}$ that is both μ-concave and μ-convex at some μ-positive n-partition.

Theorem 2.5. *If f is both μ-concave and μ-convex at some μ-positive partition and $f(\emptyset) = 0$, then f is finitely additive on $\mathscr{F}$.*

Proof. Let f be both μ-concave and μ-convex at some μ-positive partition $(X_1, \ldots, X_n)$. Suppose that A and B are disjoint elements in $\mathscr{F}$. It suffices to show that $f(A \cup B) = f(A) + f(B)$. By the nonatomicity of μ, we can decompose the set $A \cap X_i$ into disjoint subsets E_{i1} and E_{i2} of $A \cap X_i$, and the set $B \cap X_i$ into F_{i1} and F_{i2} of $B \cap X_i$ such that $\mu(E_{i1}) = \mu(E_{i2}) = \frac{1}{2}\mu(A \cap X_i)$ and $\mu(F_{i1}) = \mu(F_{i2}) = \frac{1}{2}\mu(B \cap X_i)$. Because $\mu(E_{i1} \cup F_{i1}) = \mu(E_{i2} \cup F_{i2}) = \frac{1}{2}\mu((A \cap X_i) \cup (B \cap X_i))$, and $E_{i1} \cup F_{i1}$ and $E_{i2} \cup F_{i2}$ belong to $\mathscr{D}_{\frac{1}{2}}((A \cap X_i) \cup (B \cap X_i), \emptyset)$, we have $f(\bigcup_{i=1}^n (E_{i1} \cup F_{i1})) = f(\bigcup_{i=1}^n (E_{i2} \cup F_{i2})) = \frac{1}{2}(f(A \cup B) + f(\emptyset)) = \frac{1}{2}f(A \cup B)$ by the μ-concavity and μ-convexity of f at $(X_1, \ldots, X_n)$ and the fact that $f(\emptyset) = 0$. Because $E_{i1} \cup F_{i1}$ and $E_{i2} \cup F_{i2}$ belong to $\mathscr{D}_{\frac{1}{2}}(A \cap X_i, B \cap X_i)$, it follows that $\bigcup_{i=1}^n (E_{i1} \cup F_{i1})$ and $\bigcup_{i=1}^n (E_{i2} \cup F_{i2})$ also belong to $\mathscr{D}_{\frac{1}{2}}(A, B)$. We thus have $f(\bigcup_{i=1}^n (E_{i1} \cup F_{i1})) = f(\bigcup_{i=1}^n (E_{i2} \cup F_{i2})) = \frac{1}{2}(f(A) + f(B))$ again by the μ-concavity and μ-convexity of f at $(X_1, \ldots, X_n)$. Therefore, we have:

$$\begin{aligned} f(A) + f(B) &= f\left(\bigcup_{i=1}^n (E_{i1} \cup F_{i1})\right) + f\left(\bigcup_{i=1}^n (E_{i2} \cup F_{i2})\right) \\ &= \frac{1}{2}f(A \cup B) + \frac{1}{2}f(A \cup B) = f(A \cup B). \end{aligned}$$

□

Lemma 2.2. *Let $A_1, \ldots, A_n$ be a finite collection of elements in $\mathscr{F}$, and let $t_1, \ldots, t_n$ be nonnegative real numbers satisfying $\sum_{i=1}^n t_i \leq 1$. If $E_1 \in \langle t_1 A_1 \rangle$, $\ldots$, $E_n \in \langle t_n A_n \rangle$ are disjoint, then for every real number $s_1, \ldots, s_n$ satisfying $\sum_{i=1}^n s_i \leq 1$ and $t_i \leq s_i$ for each $i = 1, \ldots, n$, there exist disjoint elements $F_1 \in \langle s_1 A_1 \rangle, \ldots, F_n \in \langle s_n A_n \rangle$ such that $\bigcup_{i=1}^n E_i \subset \bigcup_{i=1}^n F_i$.*

Proof. The argument is based on induction. Let A_1 and A_2 be elements in $\mathscr{F}$, t_1 and t_2 be nonnegative real numbers satisfying $t_1 + t_2 \leq 1$, $E_1 \in \langle t_1 A_1 \rangle$ and $E_2 \in \langle t_2 A_2 \rangle$ be disjoint elements, and s_1, s_2 be real numbers satisfying $s_1 + s_2 \leq 1$, $t_1 \leq s_1$ and $t_2 \leq s_2$. Without loss of generality we may assume that $\mu(A_1) \leq \mu(A_2)$. By the nonatomicity of μ, there exists some $F_1 \in \langle s_1 A_1 \rangle$ such that $E_1 \subset F_1$. We then have:

$$\begin{aligned}\mu(A_2 \setminus F_1) \geq \mu(A_2) - \mu(F_1) &= \mu(A_2) - s_1\mu(A_1) \\ &\geq \mu(A_2) - s_1\mu(A_2) \geq s_2\mu(A_2).\end{aligned}$$

By the nonatomicity of μ, there exists some $F_2 \in \langle s_2 A_2 \rangle$ such that $E_2 \setminus F_1 \subset F_2 \subset A_2 \setminus F_1$. By construction, we have $F_1 \cap F_2 = \emptyset$ and $E_1 \cup E_2 \subset F_1 \cup F_2$. Thus, the result is true for $n = 2$.

Suppose that the result is true for $n \geq 2$. Let $A_1, \dots, A_{n+1}$ be elements in $\mathscr{F}$, $t_1, \dots, t_{n+1}$ be nonnegative real numbers satisfying $\sum_{i=1}^{n+1} t_i \leq 1$, $E_1 \in \langle t_1 A_1 \rangle, \dots, E_{n+1} \in \langle t_{n+1} A_{n+1} \rangle$ be disjoint elements, and $s_1, \dots, s_{n+1}$ be real numbers with $\sum_{i=1}^{n+1} s_i \leq 1$ and $t_i \leq s_i$ for each $i = 1, \dots, n+1$. Without loss of generality, we may assume that $\mu(A_i) \leq \mu(A_{n+1})$ for each $i = 1, \dots, n$. By the induction hypothesis, for each $i = 1, \dots, n$, there exist disjoint elements $F_1, \dots, F_n$ in $\mathscr{F}$ such that $F_i \in \langle s_i A_i \rangle$ for each $i = 1, \dots, n$ and $\bigcup_{i=1}^n E_i \subset \bigcup_{i=1}^n F_i$. Because it follows that $\mu(E_{n+1}) \leq s_{n+1}\mu(A_{n+1})$ and:

$$\begin{aligned}\mu\left(A_{n+1} \setminus \bigcup_{i=1}^n F_i\right) &\geq \mu(A_{n+1}) - \mu\left(\bigcup_{i=1}^n F_i\right) = \mu(A_{n+1}) - \sum_{i=1}^n \mu(F_i) \\ &= \mu(A_{n+1}) - \sum_{i=1}^n s_i\mu(A_i) \geq \mu(A_{n+1}) - \sum_{i=1}^n s_i\mu(A_{n+1}) \\ &\geq s_{n+1}\mu(A_{n+1}),\end{aligned}$$

there exists some $F_{n+1} \in \langle s_{n+1} A_{n+1} \rangle$ such that $E_{n+1} \setminus \bigcup_{i=1}^n F_i \subset F_{n+1} \subset A_{n+1} \setminus \bigcup_{i=1}^n F_i$ by the nonatomicity of μ. Then the elements $F_1 \in \langle s_1 A_1 \rangle, \dots, F_{n+1} \in \langle s_{n+1} A_{n+1} \rangle$ are disjoint and satisfy $\bigcup_{i=1}^{n+1} E_i \subset \bigcup_{i=1}^{n+1} F_i$ by construction. Therefore, the result is true for $n + 1$ and the proof is complete. $\square$

Denote the interior of Δ^{n-1} by:

$$\operatorname{int} \Delta^{n-1} = \{(\alpha_1, \dots, \alpha_n) \in \Delta^{n-1} \mid \alpha_i > 0,\ i = 1, \dots, n\}.$$

The following result, a variant of Jensen's inequality, also justifies the introduction of the μ-quasiconcavity and μ-concavity of functions on $\mathscr{F}$.

Theorem 2.6 (Jensen's inequality). *Let $(X_1, \dots, X_m)$ be a μ-positive m-partition. A function f on $\mathscr{F}$ is:*

(i) *μ-quasiconcave if and only if for every finite collection of elements $A_1, \dots, A_n$ in $\mathscr{F}$ and every $(t_1, \dots, t_n) \in \operatorname{int}\Delta^{n-1}$,*

$$\min_{1\le i\le n}\{f(A_i)\} \le f(Y) \quad \textit{for any } Y \in \mathscr{D}_{t_1,\dots,t_n}(A_1, \dots, A_n);$$

(ii) *μ-concave if and only if for every finite collection of elements $A_1, \dots, A_n$ in $\mathscr{F}$ and every $(t_1, \dots, t_n) \in \operatorname{int}\Delta^{n-1}$,*

$$\sum_{i=1}^{n} t_i f(A_i) \le f(Y) \quad \textit{for any } Y \in \mathscr{D}_{t_1,\dots,t_n}(A_1, \dots, A_n);$$

(iii) *μ-quasiconcave at $(X_1, \dots, X_m)$ if and only if for every finite collection of elements $A_1, \dots, A_n$ in $\mathscr{F}$ and every $(t_1, \dots, t_n) \in \operatorname{int}\Delta^{n-1}$, $Y_j \in \mathscr{D}_{t_1,\dots,t_n}(A_1 \cap X_j, \dots, A_n \cap X_j)$ for each $j = 1, \dots, m$ implies*

$$\min_{1\le i\le n}\{f(A_i)\} \le f\left(\bigcup_{j=1}^{m} Y_j\right);$$

(iv) *μ-concave at $(X_1, \dots, X_m)$ if and only if for every finite collection of elements $A_1, \dots, A_n$ in $\mathscr{F}$ and every $(t_1, \dots, t_n) \in \operatorname{int}\Delta^{n-1}$, $Y_j \in \mathscr{D}_{t_1,\dots,t_n}(A_1 \cap X_j, \dots, A_n \cap X_j)$ for each $j = 1, \dots, m$ implies*

$$\sum_{i=1}^{n} t_i f(A_i) \le f\left(\bigcup_{j=1}^{m} Y_j\right).$$

Proof. Obviously, (i) and (ii) are respectively equivalent to the special cases of (iii) and (iv) obtained for $m = 1$. Thus it is sufficient to prove only (iii) and (iv).

(iii) Because the sufficiency is obvious, we only need to prove the necessity, for which we use the induction on n. Let f be μ-quasiconcave at $(X_1, \dots, X_m)$. For $n = 2$, the result immediately follows from Definition 2.2(i). Suppose that the result is true for $n \ge 2$. Let $A_1, \dots, A_{n+1}$ be elements in $\mathscr{F}$ and $(t_1, \dots, t_{n+1}) \in \operatorname{int}\Delta^n$. Choose any $Y_j \in \mathscr{D}_{t_1,\dots,t_{n+1}}(A_1 \cap X_j, \dots, A_{n+1} \cap X_j)$ for each j. Then $Y_j = \bigcup_{i=1}^{n+1} E_{ij}$ is the union of disjoint sets $E_{1j} \in \langle t_1(A_1 \cap X_j)\rangle, \dots, E_{n+1j} \in \langle t_{n+1}(A_{n+1} \cap X_j)\rangle$. Define $s_i = (1 - t_{n+1})^{-1} t_i$ for each $i = 1, \dots, n$. We then have $\sum_{i=1}^{n} s_i = 1$ and $t_i \le s_i$ for each $i = 1, \dots, n$. By Lemma 2.2, there exist disjoint elements $F_{1j} \in \langle s_1(A_1 \cap X_j)\rangle, \dots, F_{nj} \in \langle s_n(A_n \cap X_j)\rangle$ such that $\bigcup_{i=1}^{n} E_{ij} \subset \bigcup_{i=1}^{n} F_{ij}$. In view of:

$$\mu\left(\bigcup_{i=1}^{n} E_{ij}\right) = \sum_{i=1}^{n} \mu(E_{ij}) = \sum_{i=1}^{n} t_i \mu(A_i \cap X_j) = (1 - t_{n+1}) \sum_{i=1}^{n} s_i \mu(A_i \cap X_j)$$

$$= (1 - t_{n+1}) \sum_{i=1}^{n} \mu(F_{ij}) = (1 - t_{n+1}) \mu \left(\bigcup_{i=1}^{n} F_{ij} \right),$$

we have $\bigcup_{i=1}^n E_{ij} \in \langle (1 - t_{n+1}) \bigcup_{i=1}^n F_{ij} \rangle \rangle$. Note that $(\bigcup_{j=1}^m F_{ij}) \cap X_j = F_{ij}$ for each i and j. Because $\bigcup_{i=1}^n E_{ij} \cup E_{n+1j} \in \mathscr{D}_{1-t_{n+1}}((\bigcup_{i=1}^n \bigcup_{j=1}^m F_{ij}) \cap X_j, A_{n+1} \cap X_j)$ and $\bigcup_{i=1}^n F_{ij} \in \mathscr{D}_{s_1,\ldots,s_n}(A_1 \cap X_j, \ldots, A_n \cap X_j)$ for each j, by the μ-quasiconcavity of f at $(X_1, \ldots, X_m)$ and the induction hypothesis, we have:

$$\begin{aligned}
f \left(\bigcup_{j=1}^{m} Y_j \right) &= f \left(\bigcup_{j=1}^{m} \bigcup_{i=1}^{n} E_{ij} \cup \bigcup_{j=1}^{m} E_{n+1j} \right) \\
&\geq \min \left\{ f \left(\bigcup_{j=1}^{m} \bigcup_{i=1}^{n} F_{ij} \right), f(A_{n+1}) \right\} \\
&\geq \min\{\min\{f(A_1), \ldots, f(A_n)\}, f(A_{n+1})\} \\
&= \min\{f(A_1), \ldots, f(A_{n+1})\}.
\end{aligned}$$

Hence, the result is true for $n + 1$.

(iv) Let f be μ-concave at $(X_1, \ldots, X_m)$. Sufficiency is again obvious. The argument is based on induction. The same argument applies here as in the proof of part (iii) except for the last inequalities, which are replaced by:

$$\begin{aligned}
f \left(\bigcup_{j=1}^{m} Y_j \right) &= f \left(\bigcup_{j=1}^{m} \bigcup_{i=1}^{n} E_{ij} \cup \bigcup_{j=1}^{m} E_{n+1j} \right) \\
&\geq (1 - t_{n+1}) f \left(\bigcup_{j=1}^{m} \bigcup_{i=1}^{n} F_{ij} \right) + t_{n+1} f(A_{n+1}) \\
&\geq (1 - t_{n+1}) \sum_{i=1}^{n} s_i f(A_i) + t_{n+1} f(A_{n+1}) = \sum_{i=1}^{n+1} t_i f(A_i).
\end{aligned}$$

□

It is obvious from the above proof that Jensen's inequality is also valid for strictly μ-quasiconcave and strictly μ-concave functions by replacing the inequalities in Theorem 2.6 with strict inequalities and adding the condition that $\mu(A_i \triangle A_j) > 0$ for some $i \neq j$.

3 Preference Relations on a σ-Field

In this section, we first define the convexity of preference relations on $\mathscr{F}$. Convex preferences are in conformity with the representation by a μ-quasiconcave function discussed in Subsection 2.2. We then show that maximal elements in $\mathscr{F}$ are essentially unique with respect to the strict μ-convex preferences. We next introduce a metric on $\mathscr{F}$ and define the continuity of preference relations on $\mathscr{F}$ under which the existence of a continuous utility function representing the continuous preferences is guaranteed when $\mathscr{F}$ is countably generated.

3.1 Convexity of Preference Relations

A *preference relation* $\succsim$ on $\mathscr{F}$ is a complete transitive binary relation on $\mathscr{F}$. The strict preference $A \succ B$ means that $A \succsim B$ and $B \not\succsim A$. The indifference $A \sim B$ means that $A \succsim B$ and $B \succsim A$. A real-valued set function u on $\mathscr{F}$ *represents* $\succsim$ if $u(A) \geq u(B)$ holds if and only if $A \succsim B$ does, and such u is called a *utility function* representing $\succsim$.

The following definition of the (strictly) μ-convexity of preference relations are analogues of the (strict) convexity of preference relations on a standard commodity space.

Definition 3.1. A preference relation $\succsim$ on $\mathscr{F}$ is:

(i) *μ-convex* if $A \succsim C$, $B \succsim C$, and $t \in (0,1)$ imply $D \succsim C$ for any $D \in \mathscr{D}_t(A,B)$;

(ii) *strictly μ-convex* if $A \succsim C$, $B \succsim C$, $\mu(A \triangle B) > 0$, and $t \in (0,1)$ imply $D \succ C$ for any $D \in \mathscr{D}_t(A,B)$.

Definition 3.2. Let $(X_1, \dots, X_n)$ be a μ-positive partition. A preference relation $\succsim$ on $\mathscr{F}$ is:

(i) *μ-convex at* $(X_1, \dots, X_n)$ if $A \succsim C$, $B \succsim C$, $t \in (0,1)$, and $D_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each $i = 1, \dots, n$ imply $\bigcup_{i=1}^n D_i \succsim C$;

(ii) *strictly μ-convex at* $(X_1, \dots, X_n)$ if $A \succsim C$, $B \succsim C$, $\mu(A \triangle B) > 0$, $t \in (0,1)$, and $D_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each $i = 1, \dots, n$ imply $\bigcup_{i=1}^n D_i \succ C$.

Theorem 3.1. *A preference relation is (strictly) μ-convex if and only if it is (strictly) μ-convex at each μ-positive n-partition.*

Proof. This follows from the observation in the proof of Theorem 2.4 that for each fixed n and for any $A, B \in \mathscr{F}$ and $t \in (0,1)$, there exists a μ-positive partition $(X_1, \dots, X_n)$ such that $\mathscr{D}_t(A,B) = \bigcup_{i=1}^n \mathscr{D}_t(A \cap X_i, B \cap X_i)$. $\square$

The following result characterizes (strictly) μ-quasiconcave and (strictly) μ-concave utility functions.

Theorem 3.2. *Let $(X_1, \ldots, X_n)$ be a μ-positive partition. A utility function representing a preference relation $\succsim$ is:*

(i) *(strictly) μ-quasiconcave if and only if $\succsim$ is (strictly) μ-convex;*

(ii) *(strictly) μ-concave at $(X_1, \ldots, X_n)$ if and only if $\succsim$ is (strictly) μ-convex at $(X_1, \ldots, X_n)$.*

Proof. (i) Let u be a utility function representing a preference relation $\succsim$ on $\mathscr{F}$. Suppose that u is μ-quasiconcave. Let $A \succsim C$, $B \succsim C$, and $t \in (0,1)$. Because $u(D) \geq \min\{u(A), u(B)\} \geq u(C)$ for any $D \in \mathscr{D}_t(A, B)$, we then have $D \succsim C$. Thus, $\succsim$ is μ-convex. Conversely, suppose that $\succsim$ is μ-convex. Without loss of generality, we may assume $A \succsim B$. Because the μ-convexity of $\succsim$ implies $C \succsim B$ for any $C \in \mathscr{D}_t(A, B)$ and $t \in (0,1)$, we then have $u(C) \geq u(B) = \min\{u(A), u(B)\}$. Therefore, u is μ-quasiconcave. The same argument applies to the case for the strict μ-quasiconcavity of u and the strict μ-convexity of $\succsim$.

(ii) Let u be a utility function representing a preference relation $\succsim$ on $\mathscr{F}$. Suppose that u is μ-quasiconcave at $(X_1, \ldots, X_n)$. Let $A \succsim C$, $B \succsim C$ and $t \in (0,1)$. Because $D_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$, $i = 1, \ldots, n$ implies $u(\bigcup_{i=1}^n D_i) \geq \min\{u(A), u(B)\} \geq u(C)$, we have $\bigcup_{i=1}^n D_i \succsim C$. Thus, $\succsim$ is μ-convex at $(X_1, \ldots, X_n)$. Conversely, suppose that $\succsim$ is μ-convex at $(X_1, \ldots, X_n)$. Without loss of generality, we may assume $A \succsim B$. Because $t \in (0,1)$ and $C_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$, $i = 1, \ldots, n$ imply $\bigcup_{i=1}^n C_i \succsim B$, we have $u(\bigcup_{i=1}^n C_i) \geq u(B) = \min\{u(A), u(B)\}$. Therefore, u is μ-quasiconcave at $(X_1, \ldots, X_n)$. The same argument applies to the case for the strict μ-quasiconcavity of u and the strict μ-convexity of $\succsim$ at $(X_1, \ldots, X_n)$. $\square$

An element $A \in \mathscr{F}$ is *maximal* with respect to $\succsim$ if there exists no element $B \in \mathscr{F}$ such that $B \succ A$. Because $\succsim$ is complete, this is equivalent to saying that $A \succsim B$ for every $B \in \mathscr{F}$.

Two measurable sets A and B in $\mathscr{F}$ are *μ-equivalent* if $\mu(A \triangle B) = 0$. It can easily be seen that the μ-equivalence is an equivalence relation on $\mathscr{F}$.

Theorem 3.3. *If a preference relation on $\mathscr{F}$ is strictly μ-convex at some μ-positive partition, then its maximal element is unique up to μ-equivalence.*

Proof. Let $\succsim$ be strictly μ-convex preference at some μ-positive partition $(X_1, \ldots, X_n)$, and let A and B be maximal elements with respect to $\succsim$. Suppose that $\mu(A \triangle B) > 0$. Because $A, B \succsim C$ for any $C \in \mathscr{F}$, it follows that $t \in (0,1)$ and $D_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for $i = 1, \ldots, n$ imply $\bigcup_{i=1}^n D_i \succ C$ by the strict μ-convexity of $\succsim$. This implies the existence of an element $\bigcup_{i=1}^n D_i \in \mathscr{F}$ satisfying $\bigcup_{i=1}^n D_i \succ A$, which contradicts the maximality of A. Therefore, we have $\mu(A \triangle B) = 0$. $\square$

Remark 3.1. In this paper, we have not pursued the representability of μ-convex preferences by a μ-concave utility function. The situation here is similar to the possibility in which convex preferences may not be representable by a concave utility function on a standard commodity space. For a finite dimensional commodity space, Kannai (1977) characterized the representability of convex preferences by a concave utility function. At present, we do not know whether the approach of Kannai is applicable to the convex preferences on measure spaces in our framework.

3.2 Continuity of Preference Relations

Let $\mathscr{F}$ be countably generated. We denote the μ-equivalence class of $A \in \mathscr{F}$ by $[A]$ and the set of μ-equivalence classes in $\mathscr{F}$ by $\mathscr{F}[\mu]$. If, for any two μ-equivalence classes $\mathbf{A}$ and $\mathbf{B}$, we define the metric d by $d(\mathbf{A}, \mathbf{B}) = \mu(A \triangle B)$, where A and B are arbitrarily selected elements of $\mathbf{A}$ and $\mathbf{B}$, then $\mathscr{F}[\mu]$ becomes a complete separable metric space (see Dunford and Schwartz, 1958, Lemma III.7.1 and Halmos, 1950, Theorem 40.B).

Definition 3.3. A preference relation $\succsim$ on $\mathscr{F}$ is *μ-indifferent* if $\mu(A \triangle B) = 0$ implies $A \sim B$.

A μ-indifferent preference relation $\succsim$ *induces* a preference relation $\succsim_\mu$ on $\mathscr{F}[\mu]$ defined by $\mathbf{A} \succsim_\mu \mathbf{B}$ if and only if there exist $A \in \mathbf{A}$ and $B \in \mathbf{B}$ such that $A \succsim B$. This is equivalent to saying that $\mathbf{A} \succsim_\mu \mathbf{B}$ if and only if $A \succsim B$ for any $A \in \mathbf{A}$ and $B \in \mathbf{B}$. Thus, any utility function u representing $\succsim$ on $\mathscr{F}$ induces a utility function u_μ representing $\succsim_\mu$ on $\mathscr{F}[\mu]$ by $u_\mu(\mathbf{A}) = u(A)$, where A is an arbitrary element in $\mathbf{A}$.

Definition 3.4. A preference relation $\succsim$ on $\mathscr{F}$ is *μ-continuous* if it is μ-indifferent and for every $\mathbf{A} \in \mathscr{F}[\mu]$ both the upper contour set $\{\mathbf{B} \in \mathscr{F}[\mu] \mid \mathbf{B} \succsim_\mu \mathbf{A}\}$ and the lower contour set $\{\mathbf{B} \in \mathscr{F}[\mu] \mid \mathbf{A} \succsim_\mu \mathbf{B}\}$ are closed in $\mathscr{F}[\mu]$.

The μ-continuity of $\succsim$ states that the preference relation $\succsim_\mu$ induced by $\succsim$ satisfies the standard continuity axiom for preferences.

Definition 3.5. A function f on $\mathscr{F}$ is:

(i) *μ-indifferent* if $\mu(A \triangle B) = 0$ implies $f(A) = f(B)$;

(ii) *μ-continuous* if it is μ-indifferent and induces a continuous function f_μ on $\mathscr{F}[\mu]$.

It is obvious that a utility function representing a preference relation is μ-continuous if and only if the preference relation is μ-continuous. The next theorem guarantees the existence of a μ-continuous utility function representing μ-continuous preferences.

Theorem 3.4. *Let $\mathscr{F}$ be countably generated. Then for any μ-continuous preference relation $\succsim$ on $\mathscr{F}$, there exists a μ-continuous utility function representing $\succsim$.*

Proof. Note that $\mathscr{F}[\mu]$ has a countable base of open sets because it is a separable metric space. Because $\succsim_\mu$ is a continuous preference relation on $\mathscr{F}[\mu]$, by virtue of the celebrated theorem of Debreu (1964), there exists a continuous utility function $\tilde{u}$ on $\mathscr{F}[\mu]$ representing $\succsim_\mu$. Define $u(A) = \tilde{u}(\mathbf{A})$ for $A \in \mathbf{A}$. Then u is a μ-continuous utility function on $\mathscr{F}$ representing $\succsim$. □

Example 3.1. Let $\mu_1, \dots, \mu_n$ be finite measures of a measurable space $(\Omega, \mathscr{F})$. Define $\mu = \frac{1}{n}\sum_{i=1}^n \mu_i$. Then each μ_i is absolutely continuous with respect to μ. Let φ be a function on $[0, \mu_1(\Omega)] \times \cdots \times [0, \mu_n(\Omega)]$. A preference relation on $\mathscr{F}$ defined by:

$$A \succsim B \overset{\text{def}}{\Longleftrightarrow} \varphi(\mu_1(A), \dots, \mu_n(A)) \geq \varphi(\mu_1(B), \dots, \mu_n(B))$$

is μ-continuous if φ is continuous. To see this, we show that the function f_φ defined by $f_\varphi(A) = \varphi(\mu_1(A), \dots, \mu_n(A))$ is μ-continuous. If $\mu(A \triangle B) = 0$, then $\mu_i(A \triangle B) = 0$ by the absolute continuity, and hence $\mu_i(A \cup B) = \mu_i(A \cap B) = \mu_i(A) = \mu_i(B)$ for each i. Thus, f_φ is μ-indifferent and induces a function $\tilde{f}_\varphi$ on $\mathscr{F}[\mu]$ via the formula $\tilde{f}_\varphi(\mathbf{A}) = f_\varphi(A)$ with $A \in \mathbf{A}$. Let $\{\mathbf{A}^\nu\}$ be a sequence in $\mathscr{F}[\mu]$ converging to $\mathbf{A}$, and take $A^\nu \in \mathbf{A}^\nu$ and $A \in \mathbf{A}$ arbitrarily. Because $\mu(A^\nu \triangle A) \to 0$ implies $\mu_i(A^\nu \triangle A) \to 0$ for each i by the absolute continuity, we have $\lim_\nu \mu_i(A^\nu \cup A) = \lim_\nu \mu_i(A^\nu \cap A)$. For any $\varepsilon > 0$, there exists an integer ν_0 such that $\mu_i(A^\nu \cup A) < \mu_i(A^\nu \cap A) + \varepsilon$ for each $\nu \geq \nu_0$. We thus have $\mu_i(A^\nu) \leq \mu_i(A^\nu \cup A) < \mu_i(A^\nu \cap A) + \varepsilon \leq \mu_i(A) + \varepsilon$ for each $\nu \geq \nu_0$. Thus, $\limsup_\nu \mu_i(A^\nu) \leq \mu_i(A) + \varepsilon$. Because ε is arbitrary, we have $\limsup_\nu \mu_i(A^\nu) \leq \mu_i(A)$. Similarly, because $\mu_i(A) \leq \mu_i(A^\nu \cup A) < \mu_i(A^\nu \cap A) + \varepsilon \leq \mu_i(A^\nu) + \varepsilon$ for each $\nu \geq \nu_0$, we have $\mu_i(A) \leq \liminf_\nu \mu_i(A^\nu)$. Therefore, $\lim_\nu \tilde{f}_\varphi(\mathbf{A}^\nu) = \lim_\nu \varphi(\mu_1(A^\nu), \dots, \mu_n(A^\nu)) = \varphi(\mu_1(A), \dots, \mu_n(A)) = \tilde{f}_\varphi(\mathbf{A})$, and hence f_φ is μ-continuous.

When $n = 1$ and μ is a nonatomic finite measure, the converse implication is also true; f_φ is μ-continuous if and only if φ is continuous on $[0, \mu(\Omega)]$. To this end, suppose that f_φ is μ-continuous. Let $\{a^\nu\}$ be a sequence converging to some point a in the open interval $(0, \mu(\Omega))$. Then there exists a subsequence $\{a^{\nu_k}\}$ satisfying $|a^{\nu_k} - a| < \frac{1}{k}$ and $a^{\nu_k} \pm \frac{1}{k} \in (0, \mu(\Omega))$ for each $k = 1, 2, \ldots$. By the nonatomicity of μ, there exist A and $A^{\nu_k} \subset A$ such that $\mu(A) = a$ and $\mu(A^{\nu_k}) = a^{\nu_k} - \frac{1}{k}$, and hence $\mu(A^{\nu_k} \bigtriangleup A) < \frac{2}{k} \to 0$. Therefore, $|\varphi(a^{\nu_k}) - \varphi(a)| = |f_\varphi(A^{\nu_k}) - f_\varphi(A)| \to 0$ by the μ-continuity of f_φ. This implies that φ is continuous on $(0, \mu(\Omega))$. If $a^\nu \to 0$ and $a^\nu \in (0, \mu(\Omega))$ for each ν, then there exists some A^ν such that $\mu(A^\nu) = a^\nu$ by the nonatomicity of μ. Because $\mu(A^\nu \bigtriangleup \emptyset) = \mu(A^\nu) \to 0$, we have $|\varphi(a^\nu) - \varphi(0)| = |f_\varphi(A^\nu) - f_\varphi(\emptyset)| \to 0$. Thus, φ is continuous at the origin. Similarly, if $a^\nu \to \mu(\Omega)$ and $a^\nu \in (0, \mu(\Omega))$ for each ν, then there exists some A^ν such that $\mu(A^\nu) = a^\nu$. Because $\mu(A^\nu \bigtriangleup \Omega) = \mu(\Omega \setminus A^\nu) \to 0$, we have $|\varphi(a^\nu) - \varphi(\mu(\Omega))| = |f_\varphi(A^\nu) - f_\varphi(\Omega)| \to 0$. Thus, φ is continuous at $\mu(\Omega)$.

Example 3.2. Let $\mu_1, \ldots, \mu_n$ and μ be defined as in Example 3.1, and let $(X_1, \ldots, X_n)$ be a partition. Let φ be a continuous function on $[0, \mu_1(X_1)] \times \cdots \times [0, \mu_n(X_n)]$. Consider a preference relation on $\mathscr{F}$ defined by:

$$A \succsim B \overset{\text{def}}{\iff} \varphi(\mu_1(A \cap X_1), \ldots, \mu_n(A \cap X_n)) \geq \varphi(\mu_1(B \cap X_1), \ldots, \mu_n(B \cap X_n)).$$

This is the numerical representation of preference relations studied by Sprumont (2004). As in Example 3.1, it can be shown that $\succsim$ is μ-continuous.

Example 3.3. Berliant and ten Raa (1988) introduced a topology on $\mathscr{F}$ that makes $\mathscr{F}$ a compact metric space. Let Ω be a compact subset of $\mathbb{R}^n$ with nonempty interior, and let $(\Omega, \mathscr{F}, \mu)$ be a Lebesgue measure space with $\mathscr{F}$ the σ-field of Borel subsets of Ω. A semimetric $\tilde{d}$ on $\mathscr{F}$ is defined by:

$$\tilde{d}(A, B) = H(\Omega \setminus \operatorname{int} A, \Omega \setminus \operatorname{int} B) + \left| \int_A h(x) d\mu - \int_B h(x) d\mu \right|,$$

where H is the Hausdorff distance, $\operatorname{int} A$ denotes the interior of the set A and h is an element in $L^1(\Omega, \mathscr{F}, \mu)$. If an equivalence class of $\mathscr{F}$ is defined to be a family of elements of $\mathscr{F}$, where each pair has semimetric zero, then $\tilde{d}$ defines a metric on such equivalence classes $\tilde{\mathscr{F}}$ of $\mathscr{F}$. Berliant and ten Raa (1988) proved that $\tilde{\mathscr{F}}$ is a compact metric space.

Denote the ε-open ball with center $x \in \mathbb{R}^n$ by $B_\varepsilon(x)$. Define the function u on $\mathscr{B}_\Omega$ by:

$$u(A) = \sup\{\varepsilon > 0 \mid B_\varepsilon(x) \subset A,\ x \in A\}.$$

If the preference relation $\succsim$ on $\mathscr{F}$ is defined by $A \succsim B$ if and only if $u(A) \geq u(B)$, then $\succsim$ induces a continuous preference on $\tilde{\mathscr{F}}$; that is, u induces a continuous function on $\tilde{\mathscr{F}}$ (see Berliant and Dunz 2004, p. 606). However, this preference relation is not μ-continuous. To show this, it suffices to compare the utility of $B_\varepsilon(x)$ and $B_\varepsilon(x) \setminus \{x\}$. We have $u(B_\varepsilon(x)) = \varepsilon$ and $u(B_\varepsilon(x) \setminus \{x\}) = \frac{\varepsilon}{2}$, but $\mu(B_\varepsilon(x) \bigtriangleup (B_\varepsilon(x) \setminus \{x\})) = 0$. Thus, u cannot be μ-indifferent, and hence $\succsim$ cannot be μ-continuous.

The (strict) μ-monotonicity of preference relations on $\mathscr{F}$ in the following definition are analogues of the (strict) monotonicity of preference relations on a standard commodity space.

Definition 3.6. A preference relation $\succsim$ on $\mathscr{F}$ is:

(i) *μ-monotone* if $A \supset B$ and $\mu(A) > \mu(B)$ implies $A \succsim B$;

(ii) *strictly μ-monotone* if $A \supset B$ and $\mu(A) > \mu(B)$ implies $A \succ B$.

Similarly to Definition 3.6, the (strict) μ-monotonicity of functions on $\mathscr{F}$ is defined as follows.

Definition 3.7. A function f on $\mathscr{F}$ is:

(i) *μ-monotone* if $A \supset B$ and $\mu(A) > \mu(B)$ implies $f(A) \geq f(B)$;

(ii) *strictly μ-monotone* if $A \supset B$ and $\mu(A) > \mu(B)$ implies $f(A) > f(B)$.

Example 3.4. Let f_φ be a set function on $\mathscr{F}$ introduced in Example 2.3. It is evident that if φ is increasing on S, then f_φ is μ-monotone on $\mathscr{F}$. Conversely, suppose that f_φ is μ-monotone on $\mathscr{F}$. Choose any $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ in S satisfying $a_i \leq b_i$ for each i. By the nonatomicity of μ, there exist A and B in $\mathscr{F}$ such that $\mu(A \cap X_i) = a_i$, $\mu(B \cap X_i) = b_i$ and $A \subset B$. We then have $\varphi(a_1, \dots, a_n) = f_\varphi(A) \leq f_\varphi(B) = \varphi(b_1, \dots, b_n)$, and hence φ is increasing on S. Consequently, f_φ is μ-monotone on $\mathscr{F}$ if and only if φ is increasing on S. Similarly, f_φ is strictly μ-increasing if and only if φ is strictly increasing.

Note that preference relations on a standard commodity space are strictly monotone if they are continuous, monotone and strictly convex. As the following result shows, a similar property holds for preference relations on $\mathscr{F}$.

Theorem 3.5. *If a preference relation is μ-continuous, μ-monotone, and strictly μ-convex at some μ-positive partition, then it is strictly μ-monotone.*

Proof. Let $\succsim$ be μ-continuous, μ-monotone and strictly μ-convex at some μ-positive partition $(X_1, \dots, X_n)$. Suppose to the contrary that $\succsim$ is not strictly μ-monotone. Then, for some A and B in $\mathscr{F}$ satisfying $B \subset A$ and $\mu(B) < \mu(A)$, we have $B \succsim A$. By the strict μ-convexity of $\succsim$, it follows that $C_i(t) \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each $i = 1, \dots, n$ and $t \in (0, 1)$ imply $\bigcup_{i=1}^n C_i(t) \succ A$. Denote $C_i(t)$ as a union $C_i(t) = E_i(t) \cup F_i(t)$ of disjoint sets $E_i(t) \in \langle t(A \cap X_i) \rangle$ and $F_i(t) \in \langle (1-t)(B \cap X_i) \rangle$. We then have:

$$\mu\left(\bigcup_{i=1}^n C_i(t) \mathbin{\triangle} B\right) = \mu\left(\bigcup_{i=1}^n C_i(t) \cup B\right) - \mu\left(\bigcup_{i=1}^n C_i(t) \cap B\right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}[\mu(C_i(t) \cup (B \cap X_i)) - \mu(C_i(t) \cap (B \cap X_i))] \\
&\leq \sum_{i=1}^{n}[\mu(E_i(t) \cup (B \cap X_i)) - \mu(F_i(t))] \\
&\leq \sum_{i=1}^{n}[\mu(E_i(t)) + \mu(B \cap X_i) - \mu(F_i(t))] \\
&\leq \sum_{i=1}^{n}[t\mu(A \cap X_i) + \mu(B \cap X_i) - (1-t)\mu(B \cap X_i)] \\
&= t(\mu(A) + \mu(B)),
\end{aligned}
$$

where the third line uses $F_i(t) \subset B \cap X_i$ for each i. Hence, the μ-equivalence classes $[\bigcup_{i=1}^{n} C_i(t)]$ and $[B]$ are close enough in the metric d for any sufficiently small $t \in (0,1)$. Thus, the μ-continuity of $\succsim$ and $\bigcup_{i=1}^{n} C_i(t) \succ A$ for any $t \in (0,1)$ imply $B \succ A$, which is a contradiction. Therefore, $\succsim$ is strictly μ-monotone. $\square$

4 ε-Pareto-optimal Partitions

In this section, we are concerned with ε-Pareto-optimal partitions. We prove their existence, show how they approximate Pareto-optimal partitions, and provide their characterization. The existence of weakly ε-Pareto-optimal partitions follows if the utility function of each individual is bounded. We show that if an approximation limit of the weakly ε-partitions exists, then the limit point is weakly Pareto optimal. It is shown that if each individual has a μ-continuous and strictly μ-monotone utility function, then weak ε-Pareto optimality is equivalent to ε-Pareto optimality. We also show that if each individual has a μ-concave utility function, then the utility possibility set is a convex set, and consequently every weakly Pareto-optimal partition is a solution to the maximization problem of a weighted utility sum of each individual by the supporting hyperplane theorem.

Note that a preference relation is represented by a (strict) μ-monotone utility function if and only if the preference relation is (strictly) μ-monotone. By Proposition 3.4, a preference relation is represented by a μ-continuous utility function if and only if the preference relation is μ-continuous, and by Theorem 3.2, a preference relation is represented by a (strict) μ-quasi-concave utility function if and only if the preference relation is (strictly) μ-convex. Therefore, it is legitimate in the sequel to employ utility functions of individuals instead of their preference relations.

4.1 Approximation of Pareto-optimal Partitions

Denote the finite set of individuals by $I = \{1, \dots, n\}$. A utility function of individual $i \in I$ on $\mathscr{F}$ is denoted by u_i and the set of n-partitions of Ω by $\mathscr{P}_n$.

Definition 4.1. Let $\varepsilon \geq 0$. A partition $(A_1, \dots, A_n)$ is:

(i) *weakly ε-Pareto optimal* if there exists no partition $(B_1, \dots, B_n)$ such that $u_i(A_i) + \varepsilon < u_i(B_i)$ for each $i \in I$. A weakly ε-Pareto-optimal partition for $\varepsilon = 0$ is said to be *weakly Pareto optimal*;

(ii) *ε-Pareto optimal* if there exists no partition $(B_1, \dots, B_n)$ such that $u_i(A_i) + \varepsilon \leq u_i(B_i)$ for each $i \in I$ and $u_i(A_i) + \varepsilon < u_i(B_i)$ for some $i \in I$. An ε-Pareto-optimal partition for $\varepsilon = 0$ is said to be *Pareto optimal.*

We denote the n-time Cartesian product of $\mathscr{F}[\mu]$ by $\mathscr{F}^n[\mu]$ and define the set $\mathscr{P}_n[\mu]$ of μ-equivalence classes of partitions by:

$$\mathscr{P}_n[\mu] = \{(\mathbf{A}_1, \dots, \mathbf{A}_n) \in \mathscr{F}^n[\mu] \mid \exists (A_1, \dots, A_n) \in \mathscr{P}_n : A_i \in \mathbf{A}_i \ \forall i \in I\}.$$

As illustrated in the following example, one cannot expect that U is closed in $\mathbb{R}^n$ even if each u_i is μ-continuous because $\mathscr{P}_n[\mu]$ is not necessarily compact in the product topology of $\mathscr{F}^n[\mu]$. This is the reason why we demonstrate the existence of ε-Pareto-optimal and ε-core partitions instead of the existence of Pareto-optimal and core partitions in the sequel.

Example 4.1. The following example, a variant of Berliant and ten Raa (1988), is suggested by Chiaki Hara. Consider the Lebesgue measure space of the unit interval $\Omega = [0, 1]$. Let $\{A^\nu\}$ be a sequence of measurable subsets in Ω defined by:

$$A^\nu = \left[0, \frac{1}{2^\nu}\right] \cup \left[\frac{2}{2^\nu}, \frac{3}{2^\nu}\right] \cup \cdots \cup \left[\frac{2^\nu - 2}{2^\nu}, \frac{2^\nu - 1}{2^\nu}\right], \quad \nu = 1, 2, \dots.$$

We claim that the sequence of μ-equivalence classes $\{[A^\nu]\}$ has no convergent subsequence in $\mathscr{F}[\mu]$, and hence the μ-equivalence classes of partitions $\mathscr{P}_2[\mu] = \{([A], [\Omega \setminus A]) \in \mathscr{F}^2[\mu] \mid A \in \mathscr{F}\}$ is not compact in $\mathscr{F}^2[\mu]$. To this end, it suffices to show that:

$$\lim_{\nu \to \infty} \mu(A^\nu \triangle E) = \lim_{\nu \to \infty} \mu((\Omega \setminus A^\nu) \triangle E) = \frac{1}{2} \quad \text{for any } E \in \mathscr{F}.$$

This equality follows from:

$$\lim_{\nu\to\infty}\mu(A^\nu\cap E)=\lim_{\nu\to\infty}\mu((\Omega\setminus A^\nu)\cap E)=\frac{1}{2}\mu(E)\quad\text{for any } E\in\mathscr{F}\tag{4.1}$$

because $A^\nu \triangle E=((\Omega\setminus A^\nu)\cap E)\cup(A^\nu\cap(\Omega\setminus E))$ and:

$$\begin{aligned}\lim_{\nu\to\infty}\mu(A^\nu\triangle E)&=\lim_{n\to\infty}\mu((\Omega\setminus A^\nu)\cap E)+\lim_{n\to\infty}\mu(A^\nu\cap(\Omega\setminus E))\\&=\frac{1}{2}\mu(E)+\frac{1}{2}\mu(\Omega\setminus E)=\frac{1}{2}.\end{aligned}$$

Therefore, we shall show (4.1).

If the Lebesgue measure of E is zero, then (4.1) is trivially true. Suppose that the Lebesgue measure of E is positive. It suffices to show that (4.1) is true for every closed interval $E=[a,b]$ with $0\le a<b\le 1$. Consider the following two cases. (i) The end points a and b are of the form $a=\frac{i}{2^\nu}$, $b=\frac{j}{2^\nu}$, $i,j=0,1,\dots,2^\nu$: because for each $k\ge\nu$ it follows that $\mu(A^k\cap E)=\mu((\Omega\setminus A^k)\cap E)$, we obtain $\mu(A^k\cap E)=\frac{1}{2}\mu(E)$ for each $k\ge\nu$ in view of $\mu(E)=\mu(A^k\cap E)+\mu((\Omega\setminus A^k)\cap E)$. Thus, (4.1) holds. (ii) The end points a and b are arbitrary: because a and b can be approximated to whatever degree of accuracy is required by rational points $\frac{i}{2^\nu}$, $i=0,1,\dots,2^\nu$ with large enough value of ν, the Lebesgue measure of the intervals $[\frac{i}{2^\nu},\frac{i+1}{2^\nu}]$ and $[\frac{j}{2^\nu},\frac{j+1}{2^\nu}]$ with $a\in[\frac{i}{2^\nu},\frac{i+1}{2^\nu}]$ and $b\in[\frac{j}{2^\nu},\frac{j+1}{2^\nu}]$, which is equal to $\frac{1}{2^\nu}$, can be made arbitrarily small. Because $\lim_\nu\mu([\frac{i}{2^\nu},\frac{i+1}{2^\nu}])=\lim_\nu\mu([\frac{j-1}{2^\nu},\frac{j}{2^\nu}])=0$, it follows that $E=[a,b]\subset[\frac{i}{2^\nu},\frac{i+1}{2^\nu}]\cup[\frac{i+1}{2^\nu},\frac{j-1}{2^\nu}]\cup[\frac{j}{2^\nu},\frac{j+1}{2^\nu}]$, $a\in[\frac{i}{2^\nu},\frac{i+1}{2^\nu}]$ and $b\in[\frac{j}{2^\nu},\frac{j+1}{2^\nu}]$ for each ν imply $\lim_\nu\mu([\frac{i+1}{2^\nu},\frac{j-1}{2^\nu}])=\mu(E)$. By virtue of (i), we have:

$$\begin{aligned}\lim_{\nu\to\infty}\mu(A^\nu\cap E)&=\lim_{\nu\to\infty}\mu\left(A^\nu\cap\left[\frac{i+1}{2^\nu},\frac{j-1}{2^\nu}\right]\right)\\&=\frac{1}{2}\lim_{\nu\to\infty}\mu\left(\left[\frac{i+1}{2^\nu},\frac{j-1}{2^\nu}\right]\right)=\frac{1}{2}\mu(E).\end{aligned}$$

Because it follows that:

$$\Omega\setminus A^\nu=\left(\frac{1}{2^\nu},\frac{2}{2^\nu}\right)\cup\left(\frac{3}{2^\nu},\frac{4}{2^\nu}\right)\cup\cdots\cup\left(\frac{2^\nu-1}{2^\nu},1\right),\quad \nu=1,2,\dots,$$

we obtain as in the above:

$$\lim_{\nu\to\infty}\mu((\Omega\setminus A^\nu)\cap E)=\frac{1}{2}\mu(E)\quad\text{for any } E\in\mathscr{F}.$$

Theorem 4.1. (i) *If u_i is bounded for each $i\in I$, then for every $\varepsilon>0$, there exists a weakly ε-Pareto-optimal partition.*

(ii) *If u_i is μ-continuous and strictly μ-monotone for each $i \in I$, then for every $\varepsilon \geq 0$, a partition is ε-Pareto optimal if and only if it is weakly ε-Pareto optimal.*

Proof. (i) Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \Delta^{n-1}$ be arbitrary. Because each u_i is bounded, for every $\varepsilon > 0$, there exists a partition $(A_1^\varepsilon, \dots, A_n^\varepsilon)$ such that:

$$\sum_{i \in I} \alpha_i u_i(A_i^\varepsilon) > \sup \left\{ \sum_{i \in I} \alpha_i u_i(A_i) \mid (A_1, \dots, A_n) \in \mathscr{P}_n \right\} - \varepsilon. \tag{4.2}$$

We claim that $(A_1^\varepsilon, \dots, A_n^\varepsilon)$ is weakly ε-Pareto optimal. Suppose to the contrary that $(A_1^\varepsilon, \dots, A_n^\varepsilon)$ is not weakly ε-Pareto optimal. Then there exists a partition $(B_1, \dots, B_n)$ such that $u_i(A_i^\varepsilon) + \varepsilon < u_i(B_i)$ for each $i \in I$. We thus obtain $\sum_{i \in I} \alpha_i u_i(A_i^\varepsilon) + \varepsilon < \sum_{i \in I} \alpha_i u_i(B_i)$, which contradicts (4.2).

(ii) It is immediate that ε-Pareto optimality implies weak ε-Pareto optimality. We show the converse implication. Let u_i be μ-continuous and strictly μ-monotone for each $i \in I$, and let $(A_1^\varepsilon, \dots, A_n^\varepsilon)$ be a weakly ε-Pareto-optimal partition. Suppose that $(A_1^\varepsilon, \dots, A_n^\varepsilon)$ is not ε-Pareto optimal. There then exists a partition $(B_1, \dots, B_n)$ such that $u_i(A_i^\varepsilon) + \varepsilon \leq u_i(B_i)$ for each $i \in I$ and $u_j(A_j^\varepsilon) + \varepsilon < u_j(B_j)$ for some $j \in I$. The μ-continuity of u_j and the nonatomicity of μ imply that there exists some $C_j \subset B_j$ satisfying $\mu(B_j \setminus C_j) > 0$ such that $u_j(A_j^\varepsilon) + \varepsilon < u_j(C_j)$. Decompose $B_j \setminus C_j$ into $n-1$ disjoint sets B_i' for $i \in I \setminus \{j\}$ such that $\bigcup_{i \in I \setminus \{j\}} B_i' = B_j \setminus C_j$ and $\mu(B_i') > 0$ for each $i \in I \setminus \{j\}$. Let $C_i = B_i \cup B_i'$ for each $i \in I \setminus \{j\}$. Then the resulting partition $(C_1, \dots, C_n)$ satisfies $u_i(A_i^\varepsilon) + \varepsilon < u_i(C_i)$ for each $i \in I$ by the strict μ-monotonicity of u_i. This contradicts the weak ε-Pareto optimality of $(A_1^\varepsilon, \dots, A_n^\varepsilon)$. $\square$

Corollary 4.1. *Let $(A_1^\nu, \dots, A_n^\nu)$ be a weakly $\frac{1}{\nu}$-Pareto-optimal partition for each $\nu = 1, 2, \dots$. Then each cluster point of the sequence $\{(A_1^\nu, \dots, A_n^\nu)\}$ belonging to $\mathscr{P}_n[\mu]$ is weakly Pareto optimal whenever u_i is μ-continuous for each $i \in I$. Moreover, if, in addition, u_i is strictly μ-monotone for each $i \in I$, then it is Pareto optimal.*

4.2 Characterization of Pareto Optimality

Define the *utility possibility set U* by:

$$U = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \exists (A_1, \dots, A_n) \in \mathscr{P}_n : x_i \leq u_i(A_i) \ \forall i \in I\}.$$

Note that if u_i is a nonatomic finite measure for each $i \in I$, then the convexity of U trivially follows from the Lyapunov convexity theorem without imposing any concavity on u_i. Thus, the next theorem is regarded as a variant of this result for the case that u_i is not necessarily additive for each $i \in I$.

Theorem 4.2. *If u_i is μ-concave at some μ-positive partition for each $i \in I$, then U is a convex subset of $\mathbb{R}^n$.*

Proof. Let u_i be μ-concave at some μ-positive m-partition $(X_1, \ldots, X_m)$ for each $i \in I$. Take $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in U, and $t \in (0, 1)$ arbitrarily. Let $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ be partitions satisfying $x_i \leq u_i(A_i)$ and $y_i \leq u_i(B_i)$ for each $i \in I$. By Corollary 2.1, there exists some $C_{ij} \in \mathscr{D}_t(A_i \cap X_j, B_i \cap X_j)$ for each $i \in I$ and j such that $(\bigcup_{j=1}^m C_{1j}, \ldots, \bigcup_{j=1}^m C_{nj})$ is a partition. The μ-concavity of u_i implies that $tx_i + (1-t)y_i \leq tu_i(A_i) + (1-t)u_i(B_i) \leq u_i(\bigcup_{j=1}^m C_{ij})$ for each $i \in I$. Therefore, $tx + (1-t)y \in U$. □

Theorem 4.3. *If u_i is μ-concave at some μ-positive partition for each $i \in I$, then a partition is weakly Pareto optimal if and only if it solves the problem:*

$$\sup \left\{ \sum_{i \in I} \alpha_i u_i(A_i) \mid (A_1, \ldots, A_n) \in \mathscr{P}_n \right\} \tag{P_α}$$

for some $\alpha \in \Delta^{n-1}$.

Proof. Let $(A_1, \ldots, A_n)$ be a weakly Pareto-optimal partition. Because the closure $\operatorname{cl} U$ of U is convex by Theorem 4.2 and the utility vector $(u_1(A_1), \ldots, u_n(A_n))$ is in the boundary of U, by the supporting hyperplane theorem, there exists a nonzero vector $(\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ such that $\sum_{i \in I} \beta_i x_i \leq \sum_{i \in I} \beta_i u_i(A_i)$ for any $(x_1, \ldots, x_n) \in \operatorname{cl} U$. Because U is unbounded from below, we can assume $\beta_i \geq 0$ for each $i \in I$. Normalizing $\alpha_i = (\sum_{i \in I} \beta_i)^{-1} \beta_i$ for each $i \in I$ yields $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Delta^{n-1}$ and $\sum_{i \in I} \alpha_i x_i \leq \sum_{i \in I} \alpha_i u_i(A_i)$ for any $(x_1, \ldots, x_n) \in \operatorname{cl} U$. Because $(u_1(B_1), \ldots, u_n(B_n)) \in U$ for any $(B_1, \ldots, B_n) \in \mathscr{P}_n$, we obtain

$$\sum_{i \in I} \alpha_i u_i(B_i) \leq \sum_{i \in I} \alpha_i u_i(A_i) \quad \text{for any } (B_1, \ldots, B_n) \in \mathscr{P}_n.$$

Therefore, $(A_1, \ldots, A_n)$ solves (P_α). The converse implication is obvious. □

Example 4.2. Let $(\Omega, \mathscr{F}, \mu)$ be a Lebesgue measure space with Ω a compact subset of $\mathbb{R}^l$ and $\mathscr{F}$ the σ-field of Borel subsets of Ω. Suppose that Ω is decomposed into disjoint sets $X_1, \ldots, X_m$ with $\mu(X_1), \ldots, \mu(X_m) > 0$. Let utility functions of each individual be given by:

$$u_i(A) = \varphi_i(\mu(A \cap X_1), \ldots, \mu(A \cap X_m)),$$

where φ_i is a function defined on $[0, \mu(X_1)] \times \cdots \times [0, \mu(X_m)]$ for each $i \in I$. This representation of preferences is a special case of Example 3.2. Note that this economy is analogous to a pure exchange economy with n individuals, m commodities and total endowment Ω. If φ_i is continuous, then u_i is μ-continuous (Example 3.2). Define the set S by:

$$S = \{(\mu(A \cap X_1), \dots, \mu(A \cap X_m)) \in \mathbb{R}^m \mid A \in \mathscr{F}\}.$$

Then S is convex and compact, and φ_i is concave and strictly increasing on S if and only if u_i is strictly μ-concave at $(X_1, \dots, X_n)$ and strictly μ-monotone (Examples 2.3 and 3.4). Therefore, Theorems 4.1 to 4.3 are true for this economy. Take, for instance, $l = 1$, $m = 2$ and $n = 2$. This is a 2×2 pure exchange economy illustrated by Berliant et al. (1992), which is analogous to an Edgeworth box economy with continuous preferences.

Remark 4.1. The existence of a weakly Pareto-optimal partition was established first by Dubins and Spanier (1961) for the case of additive preferences represented by a nonatomic finite measure. The equivalence between Pareto optimality and weak Pareto optimality is guaranteed for the case of additive preferences if a nonatomic finite measure of each individual is mutually absolutely continuous (see Sagara 2006). A characterization of weak Pareto optimality in terms of the maximization problem of a weighted utility sum using the supporting hyperplane theorem was provided by Barbanel and Zwicker (1997) for the case of additive preferences. Without imposing any topological structure on a σ-field, Sagara (2006) extended these results for the case of nonadditive preferences with a concave transformation of a nonatomic finite measure by employing Lyapunov's convexity theorem.

5 ε-Core Partitions with NTU

This section introduces a cooperative game with NTU in a pure exchange economy in which the initial individual endowments form a partition. We show the existence of ε-core partitions with NTU under the assumption that the utility function of each individual is bounded and μ-quasiconcave at some μ-positive partition. We also show that if an approximation limit of the ε-core partitions exists, then the limit point is a core partition whenever the utility function of each individual is μ-continuous.

5.1 Approximation of Core Partitions

A nonempty subset of I is called a *coalition*. We denote the collection of coalitions by $\mathscr{N}$. Let $(\Omega_1, \dots, \Omega_n) \in \mathscr{P}_n$ be an *initial partition* in which individual $i \in I$ is endowed with a measurable subset Ω_i of Ω. A partition $(A_1, \dots, A_n)$ is an *S-partition* if $\bigcup_{i \in S} A_i = \bigcup_{i \in S} \Omega_i$ for coalition S.

Definition 5.1. Let $\varepsilon \geq 0$. A coalition S *ε-improves upon* a partition $(A_1, \dots, A_n)$ if there exists some S-partition $(B_1, \dots, B_n)$ such that $u_i(A_i) + \varepsilon < u_i(B_i)$ for each $i \in S$. A partition that cannot be ε-improved upon by any coalition is an *ε-core partition.* An ε-core partition for $\varepsilon = 0$ is said to be a *core partition.*

It is obvious from the definitions that an ε-core partition is weakly ε-Pareto optimal. Note that if u_i is μ-continuous and strictly μ-monotone for each $i \in I$, then an ε-core partition is also ε-Pareto optimal by Theorem 4.1(ii).

Theorem 5.1. *If u_i is bounded and μ-quasiconcave at some μ-positive partition for each $i \in I$, then for every $\varepsilon > 0$, there exists an ε-core partition.*

Proof. Define the n-person game $V : \mathscr{N} \to 2^{\mathbb{R}^n}$ with NTU by:

$$V(S) = \mathrm{cl}\left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, S\text{-partition } (A_1, \dots, A_n) : \\ x_i \leq u_i(A_i)\ \forall i \in S \end{array} \right\}.$$

The core of V, denoted by $\mathrm{Core}(V)$, is defined by:

$$\mathrm{Core}(V) = \{ x \in V(I) \mid \nexists S \in \mathscr{N}\ \nexists y \in V(S) : x_i < y_i\ \ \forall i \in S \}.$$

We show that V is a balanced game. To this end, let $\mathscr{B}$ be a balanced family with balanced weights $\{\lambda^S \geq 0 \mid S \in \mathscr{B}\}$ and let $\mathscr{B}_i = \{S \in \mathscr{B} \mid i \in S\}$. We then have $\sum_{S \in \mathscr{B}_i} \lambda^S = 1$ for each $i \in I$. Define:

$$\chi_i^S = \begin{cases} 1 & \text{if } S \in \mathscr{B}_i, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad t^S = \frac{1}{n} \sum_{i \in I} \lambda^S \chi_i^S.$$

We then have:

$$\sum_{S \in \mathscr{B}} t^S = \frac{1}{n} \sum_{S \in \mathscr{B}} \sum_{i \in I} \lambda^S \chi_i^S = \frac{1}{n} \sum_{i \in I} \sum_{S \in \mathscr{B}_i} \lambda^S = 1.$$

Let u_i be μ-quasiconcave at some μ-positive m-partition $(X_1, \dots, X_m)$ for each $i \in I$. Choose any $(x_1, \dots, x_n) \in \bigcap_{S \in \mathscr{B}} V(S)$. Then there exists a sequence $\{(x_1^\nu, \dots, x_n^\nu)\}$ in $\mathbb{R}^n$ converging to $(x_1, \dots, x_n)$ such that for each $S \in \mathscr{B}$, there exists a sequence $\{(A_1^{S,\nu}, \dots, A_n^{S,\nu})\}$ of S-partitions satisfying $x_i^\nu \leq u_i(A_i^{S,\nu})$ for each $i \in S$, $\nu = 1, 2, \dots$. Enumerate each element in $\mathscr{B}$ by $\mathscr{B} = \{S_1, \dots, S_l\}$ and let $t_k = t^{S_k}$ and $A_i^{k,\nu} = A_i^{S_k,\nu}$ for each $k = 1, \dots, l$. By virtue of Theorem 2.3, there exists some $A_{ij}^\nu \in \mathscr{D}_{t_1, \dots, t_l}(A_i^{1,\nu} \cap X_j, \dots, A_i^{l,\nu} \cap X_j)$ for each $i \in I$ and $j = 1, \dots, m$ such that $(\bigcup_{j=1}^m A_{1j}^\nu, \dots, \bigcup_{j=1}^m A_{nj}^\nu)$ is a partition. By the μ-quasiconcavity of u_i at $(X_1, \dots, X_m)$ and Theorem 2.6(i), we have $x_i^\nu \le \min_{1\le k\le l}\{u_i(A_i^{k,\nu})\} \le u_i(\bigcup_{j=1}^m A_{ij}^\nu)$ for each $i \in I$. We thus obtain $(x_1^\nu, \dots, x_n^\nu) \in V(I)$ for each $\nu = 1, 2, \dots$. Because $V(I)$ is closed and $(x_1^\nu, \dots, x_n^\nu) \to (x_1, \dots, x_n)$, we obtain $(x_1, \dots, x_n) \in V(I)$. Therefore, $\bigcap_{S\in\mathscr{B}} V(S) \subset V(I)$, and consequently V is balanced.

Because the balanced game V obviously satisfies other sufficient conditions guaranteeing the nonemptiness of the core of V (see Scarf 1967), we can select an element $(x_1, \dots, x_n)$ in Core(V). Let $\varepsilon > 0$ be arbitrary. Then there exist some $(y_1, \dots, y_n) \in \mathbb{R}^n$ with $|x_i - y_i| < \varepsilon$ for each $i \in I$ and a partition $(A_1, \dots, A_n)$ such that $y_i \le u_i(A_i)$ for each $i \in I$. Suppose that $(A_1, \dots, A_n)$ is not an ε-core partition. Then there exists some S-partition $(B_1, \dots, B_n)$ such that $u_i(A_i) + \varepsilon < u_i(B_i)$ for each $i \in S$. We then have $(u_1(B_1), \dots, u_n(B_n)) \in V(S)$ and $x_i < u_i(B_i)$ for each $i \in S$, which contradicts the fact that $(x_1, \dots, x_n)$ is in Core(V). $\square$

Corollary 5.1. *Let $(A_1^\nu, \dots, A_n^\nu)$ be a $\frac{1}{\nu}$-core partition for each $\nu = 1, 2, \dots$. Then each cluster point of the sequence $\{(A_1^\nu, \dots, A_n^\nu)\}$ belonging to $\mathscr{P}_n[\mu]$ is a core partition whenever u_i is μ-continuous for each $i \in I$.*

Remark 5.1. Berliant (1985) identified a measurable set with a characteristic function in L^∞ and introduced a price system in L^1 as a weak* continuous linear functional on a commodity space in L^∞ to show the existence of an equilibrium for the case of additive preferences by the standard argument of Bewley (1972). The existence of an equilibrium implies the nonemptiness of a core partition with NTU. Berliant and Dunz (2004) embedded characteristic functions in L^1 with a price system in L^∞ as the norm dual of a commodity space in L^1 to show the existence of an equilibrium for the case of nonadditive preferences by the fixed-point argument under the continuity assumption of preferences and the strong convexity assumption that the upper contour set is separated by hyperplanes in L^∞. Dunz (1991) proved balancedness of the NTU game for the case of nonadditive preferences with a specific integral form, and Sagara (2006) also gave a proof of the balancedness for the case of nonadditive preferences with a concave transformation of a nonatomic finite measure.

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