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Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 916249875

Centralized Course Allocation

Antonio Romero-Medina* Matteo Triossi†

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Abstract

We present the renegotiable acceptance mechanism in the context of the multi-unit assignment problem. This mechanism combines features of the immediate and deferred acceptance mechanisms and implements the set of stable matchings in both Nash and undominated Nash equilibria under substitutable priorities. In addition, we prove that under slot-specific priorities, the immediate acceptance mechanism also implements the set of stable matchings in Nash and undominated Nash equilibria. Finally, we present modifications of both mechanisms and show that we can dramatically reduce the complexity of the message space when preferences are responsive.

Economic Literature Classification Numbers: C71; C78; D71.

Keywords: renegotiable acceptance, immediate acceptance, multi-unit assignment, stability.

* Departamento de Economía, Universidad Carlos III de Madrid, Calle Madrid 126, 28903 Getafe, Madrid, Spain. E-mail: aromero@eco.uc3m.es.

† Corresponding Author. Centro de Economía Aplicada, Departamento de Ingeniería Industrial, Universidad de Chile, Calle Beaucheff 851, 8370456, Santiago, Chile. E-mail: mtriossi@dii.uchile.

1 Introduction

We are interested in multi-unit assignment problems for which multiple objects are assigned to agents on the basis of priorities. This is the case, for example, of the course allocation problem (see Sönmez and Ünver, 2010; Budish, 2011; Kojima, 2013). The time scheduling problem or the assignment of landing slots (see Schummer and Vohra, 2013; Schummer and Abizada, 2017) are also examples of multi-unit assignment problems.

We focus our attention on fair (or stable) allocations. Under multi-unit demand, no stable and strategy-proof mechanism exists. Additionally, the deferred acceptance mechanism can produce unstable matchings as Nash equilibrium outcomes (see Roth and Sotomayor, 1990; Haeringer and Klijn, 2009). To overcome this difficulty, we concentrate on mechanisms that are able to achieve stable matchings as a result of a strategic interaction. To achieve this objective, we introduce the renegotiable acceptance mechanism. Similar to the immediate acceptance mechanism, the renegotiable acceptance mechanism assigns seats at courses to students who rank them first and then to those who rank them second, and so on.[1] However, similar to the deferred acceptance mechanism, seats are not definitively assigned to students. The allocation can be renegotiated, and a student can lose a (tentatively) assigned course. Our mechanism allows for students to express the intensity of their

[1] We use the terminology of immediate acceptance introduced in Thomson (2018), but the many-to-one version of this mechanism is also known as the "Boston mechanism" (see Abdulkadiroğlu and Sönmez, 2003) and was first analyzed by Alcalde (1996), who calls it the "now-or-never mechanism".

preferences. By ranking a course higher, a student increases her chances of being admitted. Therefore, students have incentives to act strategically. The effects of these manipulations cancel out at equilibrium and the renegotiable acceptance mechanism is able to implement the set stable matchings in both Nash equilibrium and undominated Nash equilibrium under substitutable priorities.

As best as we know, this is the first paper to consider substitutable priorities in a course allocation problem (see Marutani, 2018, for the use of substitutable priorities in the school choice problem). Substitutable priorities account for situations in which admission criteria are slot-specific (see Kominers and Sönmez, 2016). These are situations in which a subgroup of students is given priority for a portion of the seats that are otherwise assigned according to a given criterion. Slot-specific priorities allow for the designer, for example, to introduce diversity in the classroom (see Dur et al. 2016, 2018 for applications to school choice).[2] Restricting our attention to slot-specific priorities, we show that the immediate acceptance mechanism is able to implement the set stable of matchings in Nash equilibrium and undominated Nash equilibrium. In situations in which more general substitutable priorities are required, for example, when students are expected to work in teams of a given size, the immediate acceptance mechanism can result in unstable matchings. Thus, our results extend to the multi-unit assignment problem

[2]Slot-specific priorities also encompass approaches such as majority quotas as defined in Kojima (2012) and minority reserves introduced by Hafalir et al. (2013).

the previous implementation results by Alcalde (1996) and Ergin and Sönmez (2006) for the marriage and school admission problems, respectively.

Finally, note that in the multi-unit assignment problem, describing the entire preference profile requires listing up to 2^n subsets of courses, where n is the number of courses (see Budish et al., 2017). We show that if students' preferences are responsive, then it is possible to simplify the strategy space.[3] We introduce simplified versions of the renegotiable and immediate acceptance mechanisms for which students' strategies are the rankings over individual courses and the maximum number of courses they are willing to take. Then, we run either the renegotiable acceptance or the immediate acceptance mechanism with any responsive extension of the submitted profiles. We prove that the simplified mechanisms preserve the incentive properties of the full mechanisms.

1.1 Related Literature

Our strategy is to relax the equilibrium requirements from dominant strategy to Nash equilibrium to implement the set of stable allocations. Two other approaches have been used to tackle the course allocation problem.

The first approach is to insist on implementing stable allocations in dominant strategies and restrict the set of admissible priorities. Kojima (2013) shows that dominant strategy implementation of stable allocations is possible

[3]The assumption of responsive preferences is common in the literature on course allocation and is used, among others, in Kojima (2013), Kojima and Ünver (2014), and Dogal and Klaus (2018).

if and only if priorities satisfy essential homogeneity. Under essential homogeneity, the mechanism required to implement stable allocations is equivalent to a serial dictatorship in which the order of choice depends on the distribution of seats among courses. If we require the mechanism to be independent of the seat distribution, a stable mechanism that makes true preferences a dominant strategy exists if and only if priorities are acyclical (see Romero-Medina and Triossi, 2018). Essential homogeneity and acyclicity impose severe restrictions on the design of priorities. It is worth noting that both Kojima (2013) and Romero-Medina and Triossi (2018) consider only responsive priorities.

In a model without priorities, Budish (2011) focuses on the efficiency of the final allocation and introduces the approximate competitive equilibrium from equal incomes. This mechanism is efficient and approximately strategy-proof in large markets. However, unstable allocations can survive even in large markets, maintaining the tension between efficiency and fairness (see Budish and Cantillon, 2012).[4] Also, the implementation of the approximate competitive equilibrium from equal incomes is complex and computationally intensive (see Budish et al., 2017).

The paper is organized as follows. Section 2 introduces the model and

[4] Budish (2011) bounds absolute envy in a weak way. More precisely, student a can envy student b if this envy can be removed by kicking student b out of at most one of his assigned courses without altering his bundle. This weak fairness concept is compatible with the existence of multiple blocking pairs. Consider a model in which the preferences of the students are monotonic in the number of assigned courses, and consider any assignment in which all students are assigned the same number of courses. This assignment satisfies Budish's weak no-envy condition but can fail to eliminate justified envy.

notation. Section 3 presents our results. Section 4 concludes.

2 The Model

There are a finite set of courses C and a finite set of students S, with $C \cap S = \emptyset$. Each course c has priorities over the subsets of students, Ch_c. Priorities are described by a choice function $Ch_c : 2^S \to 2^S$, where $Ch_c(S') \subseteq S'$ for all $S' \subseteq S$.[5] We assume that the choice function is **substitutable**. Formally, if $S' \subseteq S$, $s, s' \in S \setminus S'$ and $s \notin Ch_c(S' \cup \{s\})$, then $s \notin Ch_c(S' \cup \{s, s'\})$. In other words, Ch_c is substitutable if, whenever course c rejects a student from a given subset of students, it rejects her when more students become available. We also assume that Ch_c satisfies the **irrelevance of rejected students condition**.[6] Formally, we assume that if $S' \subseteq S$ and $s \notin Ch_c(S' \cup \{s\})$, then $Ch_c(S' \cup \{s\}) = Ch_c(S')$. In other words, Ch_c satisfies the irrelevance of rejected students condition if rejected students do not affect courses' choices. If Ch_c is substitutable and satisfies the irrelevance of rejected students condition, then they are rationalizable by a linear order on 2^S, P_c, which is $Ch_c(S') = \max_{P_c} \{S'' \mid S'' \subseteq S'\}$ for all $S' \subseteq S$ (see Alva, 2018). A priority structure is given by $Ch_C = (Ch_c)_{c \in C}$ or, equivalently, by $P_C = (P_c)_{c \in C}$, where P_c rationalizes Ch_c for all $c \in C$. A particular class of substitutable priorities is the class of **slot-specific pri-**

[5]Given a set X, by 2^X, we denote the set of the subsets of X.

[6]The condition has been previously studied as "irrelevance of rejected contracts" in Aygün and Sönmez (2013) for models of matching with contracts and as "irrelevance of rejected items" in Alva (2018) for general choice models.

orities introduced by Kominers and Sönmez (2016) in a matching model with contracts. Under slot-specific priorities, each course $c \in C$ has a finite set of slots, $\sigma \in \Sigma_c$. Each slot σ has a priority order $\succ_\sigma$, which is a strict, complete, and transitive binary relation over $S \cup \{\emptyset\}$. The higher a student is ranked under $\succ_\sigma$, the stronger the claim that she has for slot σ in course c. If $\emptyset \succ_\sigma s$, student s is not acceptable for slot σ. The total supply of course c is $q_c = |\Sigma_c|$. Let us define q as the vector of supply for each course $q = (q)_{c \in C}$. We assume that the slots in C are ordered according to a linear order of precedence $\rhd_c$. Given two slots $\sigma, \sigma' \in \Sigma_c$, $\sigma \rhd_c \sigma'$ means that slot σ is to be filled before slot σ' whenever possible. For each course c, we assume that slots in Σ_c are ordered in such a way that $\sigma^1 \rhd_c \sigma^2 \rhd_c ... \rhd_c \sigma^{q_c}$. Let $S' \subseteq S$. The choice of school c from S', denoted by $Ch_c(S')$, is obtained as follows: slots at school c are filled one at a time following the order of precedence. The highest-priority acceptable student in S' under $\succ_{\sigma^1}$, for example, student s^1, is chosen for slot σ^1 of school c; the highest-priority acceptable student in $S' \setminus \{s^1\}$ under $\succ_{\sigma^1}$, for example, student s^2, is chosen for slot σ^2 of school c, and so on. The choice function Ch_c satisfies substitutability (see Kominers and Sönmez, 2016) and the irrelevance of rejected students condition. A slot-specific priority structure is a tuple $\left(q, \left(\Sigma_c, (\succ_\sigma)_{\sigma \in \Sigma_c}, \rhd_c\right)_{c \in C}\right)$. Throughout the paper, we assume that priorities are fixed, substitutable, and satisfy the irrelevance of rejected students conditions.

Each student $s \in S$ has a strict preference relation P_s over the set of subsets of C, 2^C. For each $C' \subseteq C$ and each $s \in S$, we denote by $Ch_s(C')$

the **choice set** of student s, which is the favorite combination of courses among the ones belonging to C'. Formally, $Ch_s(C') = \max_{P_s} \{D \mid D \subseteq C'\}$. A subset of courses $C' \subseteq C$ is not acceptable to student s when $\emptyset P_s C'$. We assume that the choice set induced by each P_s is **substitutable** as previously defined for the case of courses' priorities. Let $\mathcal{P}$ be the set of substitutable preferences on 2^C. A more restrictive condition is responsiveness. We say that P_s is **responsive** (see Roth 1985), with **demand** q_s if, for each $C' \subseteq C$ and for all $c, c' \in C \setminus C'$, the following holds: (1) if $|C'| < q_s$, then $C' \cup \{c\} P_s C' \cup \{c'\}$ if and only if $\{c\} P_s \{c'\}$, (2) if $|C'| < q_s$, then $C' \cup \{c\} P_s C'$ if and only if $\{c\} P_s \emptyset$, and (3) if $|C'| > q_s$, then $\emptyset P_s C'$.

For each $S' \subseteq S$, set $P_{S'} = (P_s)_{s \in S'}$. For each $s \in S$, set $P_{-s} = P_{S \setminus \{s\}}$. Given a preference relation P on 2^C, the restriction of P to $C' \subseteq C$, denoted by $P_{|C'}$ is a preference that ranks all subsets in $2^{C'}$ exactly as P does and ranks all other subsets of courses as not acceptable. Formally $P_{|C'}$ is such that, for all $Q, T \subseteq C'$, $QP_{|C'}T$ if and only if QPT and, for all $Q \nsubseteq C'$, $\emptyset P_{|C'} Q$.

A **matching** is a function $\mu : C \cup S \to 2^C \cup 2^S$ such that, for each $s \in S$ and each $c \in C$, $\mu(s) \in 2^C$, $\mu(c) \in 2^S$ and $c \in \mu(s)$ if and only if $s \in \mu(c)$. The set of all matchings is denoted by $\mathcal{M}$. Matching μ is **individually rational** for $x \in C \cup S$ if $Ch(\mu(x)) = \mu(x)$. Matching μ is **blocked** by a pair $(c, s) \in C \times S$ if $s \notin \mu(c)$, $c \in Ch_s(\mu(s) \cup \{c\})$, and $s \in Ch_c(\mu(c) \cup \{s\})$. Finally, a matching μ is **stable** for (S, C, P_S, Ch_C) if it is individually rational for all $x \in C \cup S$ and there exists no pair blocking it.

If P_S and Ch_C are substitutable and Ch_C satisfies the irrelevance of rejected students conditions, then a stable matching exists (see Echenique y Oviedo, 2006).

A **mechanism** is a function that associates a matching to every preference profile for students, $P = (P_s)_{s \in S}$, $\varphi : \mathcal{P}^{|S|} \to \mathcal{M}$. A mechanism is **stable** if $\varphi(P)$ is a stable matching for each P. A mechanism is **strategy-proof** if $\varphi(P) R_s \varphi(P'_s, P_{-s})$ for each P, $s \in S$, and P'_s, where R_s denotes the weak preferences associated to P_s. Given a priority structure Ch_C and preference profile $P \in \mathcal{P}^{|S|}$, a mechanism φ induces a normal form game $\mathcal{G}(P) = \left(S, \mathcal{P}^{|S|}, \varphi, P\right)$, where S is the set of players, $\mathcal{P}^{|S|}$ is the strategy space, φ is the outcome function, and P is the profile of students' preferences. Let $\Phi : \mathcal{P}^{|S|} \rightrightarrows \mathcal{M}$ be a correspondence. We say that mechanism φ **implements** Φ in **Nash equilibrium** (NE from now on) if, for each $P \in \mathcal{P}^{|S|}$, the set of Nash equilibria of $\mathcal{G}(\mathcal{P}) = \left(S, \mathcal{P}^{|S|}, \varphi, P\right)$, $NE(P)$ coincides with $\Phi(P)$. We say that mechanism φ **implements** Φ in **undominated Nash equilibrium** (UNE from now on) if, for each $P \in \mathcal{P}^{|S|}$, the set of undominated Nash equilibria of $\mathcal{G}(\mathcal{P}) = \left(S, \mathcal{P}^{|S|}, \varphi, P\right)$, $UNE(P)$ coincides with $\Phi(P)$.

3 The renegotiable acceptance mechanism

In this section, we introduce the renegotiable acceptance mechanism. The new mechanism has characteristics of both the immediate and the deferred acceptance mechanisms. As in the immediate acceptance mechanism, students are accepted by courses at most once along the mechanism. As in the deferred acceptance mechanism, a course can replace previously accepted students with new ones.

The message space for students in the renegotiable acceptance mechanism is the set of preference profiles on the subsets of courses. In the first stage, only the favorite set of courses of each student is considered. Among the students demanding a given course, the group with the highest priority is chosen. At the end of this stage, all students assigned to at least one course are removed, jointly with the students not demanding any course. At the r^{th} step of the procedure, only the r^{th} choices of the remaining students are considered. Each course considers the students already assigned to it and the new students claiming a seat, and chooses the subset of highest priority. All students who have been assigned at least one course at this stage are removed, jointly with the students not demanding any course. The procedure stops when all students have been removed.

Let $P = (P_s)_{s \in S}$ be a preference profile. Let $s \in S$ and let r be an integer such that $1 \leq r \leq 2^{|S|}$, and let $C^r_{P_s}$ be the r^{th} ranked acceptable set of courses according to P_s, when one exists. Let $C^r_{P_s}$ be empty otherwise.

Given a priority system $(P_c)_{c \in C}$ and a preference profile for students $(P_s)_{s \in S}$, the following procedure describes the **renegotiable acceptance mechanism**.

Step 1: Only the top acceptable choices of students are considered. For each course c, let S_c^1 be the set of students who selected c among their first choices. Formally, $S_c^1 = \left\{s \in S \mid c \in C_{P_s}^1\right\}$. Define $\mu^1(c) = Ch_c(S_c^1)$. Every student in $\mu^1(c)$ is enrolled in course c. Every student in $\mu^1(c)$ and every student s such that $C_{P_s}^1 = \emptyset$ is removed from the market. Set $T^1 = S$. Let T^2 be the set of remaining students.

Step r, $r \geq 2$: Only the r^{th} choices of students in T^r are considered. For each course c, let $S_c^r = \mu^{r-1}(c) \cup \left\{s \in T^r \mid c \in C_{P_s}^r\right\}$ be the set of students enrolled at c at the end of stage r and of the remaining students ranking a set containing c in the r^{th} place. Let $\mu^r(c) = Ch_c(S_c^r)$. Every student in $\mu^r(c)$ and every student s such that $C_{P_s}^r = \emptyset$ is removed from the market. Let T^{r+1} be the set of remaining students.

The procedure stops when all students have been removed. Formally, it stops at $r^* = \min\{r \mid T^{r+1} = \emptyset\}$. Let $RA(P) = \mu^{r^*}$ be the final outcome. Note that the procedure produces an outcome even when preferences are not substitutable.

We first show that in the renegotiable acceptance mechanism, students can obtain any attainable set of courses by ranking them in the first place.

Lemma 1 *Let $P = (P_s)_{s \in S}$ be a preference profile for students, and let $\mu = RA(P)$. If priorities are substitutable, for each $s \in S$ and $C' \subseteq \mu(s)$, $C' = RA\left(P_{s|C'}, P_{-s}\right)(s)$.*

Proof. Let $s \in S$ and let $C' \subseteq \mu(s)$. Let $c \in C'$, let $r(c)$ be the step of the renegotiable acceptance mechanism when c has been assigned to s for the first time along the mechanism, and formally let $r(c) = \min_{r \leq r^*} \{r \mid s \in \mu^r(c)\}$. Note that $r(c) = r(c')$ for all $c, c' \in \mu(s)$ and that $\mu^r(s) = \emptyset$ for all $r < r(c)$. The substitutability of Ch_c implies that $C^{r(c)}_{P_s} P_s C'$; otherwise, $s \in \mu^r(c)$ for some $r < r(c)$. For all $i \leq r(c)$, let $P^{r(c)}_s$ be a preference profile over 2^C such that $C^{r(c)}_{P^{r(c)}_s} = C'$, and for $j \neq r(c)$: $C^{r(c)}_{P^{r(c)}_s} = C^j_{P_s}$ if $C^j_{P_s} \neq C'$ and $C^j_{P^{r(c)}_s} = C^{r(c)}_{P_s}$ if $C^j_{P_s} = C'$. Note that $RA\left(P^{r(c)}_s, P_{-s}\right)(s) = C'$. For all i, $i < r(c)$, let P^i_s be a preference profile over 2^C, such that $C^i_{P^i_s} = C'$, and for $j \neq i$: $C^j_{P^j_s} = C^{j+1}_{P_s}$ if $C^{j+1}_{P^{j+1}_s} \neq C'$ and $C^j_{P^j_s} = C^{j+1}_{P^{j+1}_s}$ if $C^{j+1}_{P^{j+1}_s} = C'$. Intuitively, each P^j_s lifts C' at place j in the preference profile of s without altering the ranking above the j^{th} place.

We prove by contradiction that $RA(P^{i-1}_s, P_{-s})(s) = RA(P^i_s, P_{-s})(s) = C'$ for all i, $1 \leq i < r(c)$. For every preference profile on 2^C, Q_s, let $\mu^j_{Q_s}$ be the outcome at stage j of the mechanism when preferences are (Q_s, P_{-s}). Note that $\mu^i_{P_s} = \mu^i_{P^j_s}$ for all i, j, $2 \leq i < j \leq r(c)$. Thus, to prove that $RA(P^{i-1}_s, P_{-s})(s) = RA(P^i_s, P_{-s})(s)$ for all $i < r(c)$, it suffices to show that $s \in Ch_c\left(\mu^{i-1}_{P_s}(c) \cup \left\{s \in S \mid c \in \bigcup_{s' \neq s} C^{i-1}_{P_{s'}}\right\} \cup \{s\}\right)$ for all i, $2 \leq i \leq r(c)$. By contradiction, assume that it is not the case, and let j be the maximum integer such that

$s \notin Ch_c\left(\mu^{j-1}_{P_s}(c) \cup \left\{s \in S \mid c \in \bigcup_{s' \neq s} C^{i-1}_{P_{s'}}\right\} \cup \{s\}\right)$ and

$s \in Ch_c\left(\mu^{j}_{P_s}(c) \cup \left\{s \in S \mid c \in \bigcup_{s' \neq s} C^{i-1}_{P_{s'}}\right\} \cup \{s\}\right)$. Because P_c is substitutable, $s \in Ch_c\left(\mu^j_{P_s}(c) \cup \{s\}\right)$. The j^{th} step of the mechanism when preferences are (P^j_s, P_{-s}) yields $\mu^j_{P^j_s}(c)$ to course c. We have

$s \notin Ch_c\left(\mu^{j-1}_{P_s}(c) \cup \left\{s \in S \mid c \in \bigcup_{s' \neq s} C^{i-1}_{P_{s'}}\right\} \cup \{s\}\right) = \mu^j_{P^j_s}(c)$. The irrelevance of rejected students condition implies that

$$Ch_c\left(\mu_{P_s}^{j-1}(c) \cup \left\{s \in S \mid c \in \bigcup_{s' \neq s} C_{P_{s'}}^{i-1}\right\} \cup \{s\}\right) = Ch_c\left(\mu_{P_s}^{j}(c) \cup \{s\}\right) = \mu_{P_s}^{j}(c).$$

In particular, $s \notin Ch_c\left(\mu_{P_s}^{j}(c) \cup \{s\}\right)$, which yields a contradiction. Thus, we have $RA\left(P_s^1, P_{-s}\right)(s) = C'$. It follows that $RA\left(P_{s|C'}, P_{-s}\right)(s) = C'$, which concludes the proof of the claim. ■

Lemma 1 implies that each student can obtain her favorite attainable set of courses by listing a reduced amount of options. Thus, we can prove that every Nash equilibrium outcome of the renegotiable acceptance mechanism is stable, and every stable allocation is a Nash equilibrium outcome of the renegotiable acceptance mechanism.

Theorem 1 *The renegotiable acceptance mechanism implements the set of stable matching in NE in the domain of substitutable preferences if priorities are substitutable.*

Proof. (i) We first prove that any NE outcome is a stable matching. Let P^* be a NE of the games induced by the renegotiable acceptance mechanism, and let $\mu = RA\left(P^*\right)$. Matching μ is individually rational for each course by definition. We prove by contradiction that μ is individually rational for students. Assume $Ch_s\left(\mu(s)\right) \neq \mu(s)$ for some $s \in S$. Let $P'_s = P_{s|Ch_s(\mu(s))}$. Because P_s is substitutable, P'_s is substitutable as well. By Lemma 1: $RA\left(P'_s, P^*_{-s}\right)(s) = Ch_s\left(\mu(s)\right)$. Thus, the deviation is profitable to s, which yields a contradiction. Assume that there exists a pair blocking μ, $(c, s) \in C \times S$. Let $P' = P_{s|Ch_s(\mu(s) \cup \{c\})}$. Because $s \in Ch_c\left(\mu(c) \cup \{s\}\right)$, the deviation is profitable to s, which yields a contradiction. It follows that matching μ is individually rational and cannot be blocked by any course-student pair; thus, it is stable.

(ii) Let μ be a stable matching. For each s, let $P_s^* = P_{s|\mu(s)}$. Set $P^* = (P_s^*)_{s\in S}$. We have $RA(P^*) = \mu$. We prove by contradiction that P^* is a Nash equilibrium. Assume that $s \in S$ has a profitable deviation, P'_s, and let $\mu' = RA\left(P'_s, P^*_{-s}\right)$. Let $c \in Ch_s(\mu(s) \cup \mu'(s)) \setminus \mu(s)$. Because P_s is substitutable, $c \in Ch_s(\mu(s) \cup \{c\})$. Let $P''_s = P_{s|Ch_s(\mu(s)\cup\{c\})}$, then $RA\left(P''_s, P^*_{-s}\right)(s) = Ch_s(\mu(s) \cup \{c\})$. It follows that (c, s) blocks μ, which yields a contradiction. ■

The renegotiable acceptance mechanism yields unstable matchings with respect to stated preferences. However, unstable matchings are ruled out by strategic behavior. From Lemma 1, it follows that if pair (c, s) blocks an outcome matching μ, $P_{s|Ch_s(\mu(s)\cup\{c\})}$ is a profitable deviation for s.

Note that the equilibrium strategies defined in part (ii) of the proof of Theorem 1 are undominated. Thus, we have the following result.

Corollary 1 *The renegotiable acceptance mechanism implements the set of stable matching in UNE in the domain of substitutable preferences if priorities are substitutable.*

3.1 Slot-specific priorities

The class of slot-specific priorities is a strict subset of the set of substitutable priorities that allows for a flexible matching of students to courses. Let $\left(q, \left(\Sigma_c, (\succ_\sigma)_{\sigma\in\Sigma_c}, \rhd_c\right)_{c\in C}\right)$ be a slot-specific priority structure. Let $P_C = (P_c)_{c\in C}$ be a profile of linear orders that rationalize the respective choice functions. The hypothesis of Theorem 1 are satisfied by slot-specific priorities. It follows that under these priorities, the renegotiable acceptance mechanism implements the set of stable matching in NE when students' preferences are substitutable.

Under slot-specific priorities, we can adapt the immediate acceptance mechanism to allocate courses, which works as follows. First, all students submit a preference profile. In the first stage, the favorite acceptable set of courses of each student is considered. Among the students claiming a course, those with the highest priorities for any given course are assigned to it. At the end of this stage, all students assigned to at least one course and all assigned seats are removed from the procedure. At the n^{th} stage of the mechanism, only the n^{th} choices of the remaining students are considered, and we repeat the procedure until no more slots or students are remaining.

Given a priority structure $(P_c)_{c\in C}$ and a preference profile for students $(P_s)_{s\in S}$, the following procedure describes the **immediate acceptance mechanism**.

- Step 1: Only the top acceptable choices of students are considered.

For each course c, let S_c^1 be the set of students who selected c among their first choices. Formally, $S_c^1 = \left\{s \in S \mid c \in C_{P_s}^1\right\}$.[7] Define $\mu^1(c) = Ch_c(S_c^1)$. Every student in $\mu^1(c)$ is definitively enrolled in course c. Every student in $\mu^1(c)$ and every student s such that $C_{P_s}^1 = \emptyset$ is removed from the market. Set $T^1 = S$. Let T^2 be the set of remaining students.

- Step r, $r \geq 2$: Only the r^{th} choices of students in T^r are considered. For each course c let $S_c^r = \left\{s \in T^r \mid c \in C_{P_s}^r\right\}$ be the set of students in T^r who selected c among their r^{th} choices. Let $\mu^r(c) = \max_{P_c}\{\mu^{r-1}(c) \cup S' \mid S' \subseteq S_c^r\}$. Every student in $\mu^r(c)$ and every student s such that $C_{P_s}^r = \emptyset$ is removed from the market. Let T^{r+1} be the set of remaining students.

The procedure stops when all students have been removed. Formally, it stops at $r^* = \min\{r \mid T^{r+1} = \emptyset\}$. Let $IA(P) = \mu^{r^*}$ be the final outcome. Note that a student never loses the seat at a course she has been assigned to at some step of the mechanism, but she can be moved to slots of different precedence along the mechanism. Furthermore, all matchings are individually rational for courses.

Under substitutable preferences, all stable matchings are Nash equilibrium outcomes of the immediate acceptance mechanism. However, not all Nash equilibrium outcomes are stable matchings. This is because not all

[7] For each i and each P_s, $C_{P_s}^i$ is defined as for the renegotiable acceptance mechanism.

outcomes of the mechanism are individually rational for courses as can be seen in Example 1.

Example 1 *There are two courses,* $C = \{c_1, c_2\}$ *and four students,* $S = \{s_1, s_2, s_3, s_4\}$*. Each student wants to enroll in exactly one course. The maximal number of students* c_1 *can enroll is three but the ideal number is two. Preferences and priorities are as follows:*

$P_{s_1} : \{c_2\}, \{c_1\}$;

$P_{s_2} : \{c_1\}$;

$P_{s_3} : \{c_1\}$;

$P_{s_4} : \{c_2\}$;

$P_{c_1} : \{s_1, s_3\}, \{s_1, s_2, s_3\}, \{s_2, s_3\}, \{s_1, s_2\}, \{s_1\}, \{s_3\}, \{s_2\}$;

$P_{c_2} : \{s_4\}, \{s_1\}, \{s_2\}, \{s_3\}$.

All priorities are substitutable. Truth telling results in matching μ*, where* $\mu(c_1) = \{s_1, s_2, s_3\}$ *and* $\mu(c_2) = \{s_4\}$*, which is not individually rational because* $Ch_{c_1}(\mu(c_1)) \neq \mu(c_1)$*. However, truth telling is a Nash equilibrium of the immediate acceptance mechanism because any agent but* s_1 *is assigned to her preferred course, and* s_1 *has no profitable deviations.*

The instability of NE allocations comes from the fact that acceptances are definitive. In Example 1, when s_1's application comes, course c_1's priorities prescribe the rejection of the student's application, but it cannot. Unlike the renegotiable acceptance mechanism, the immediate acceptance mechanism does not allow for courses to reject previously accepted students.

When priorities are slot-specific, this is not a concern because all outcomes of the immediate acceptance mechanism are individually rational for courses. Even if the executions of the two mechanisms do not coincide under slot-specific priorities, we can replicate the strategy of the proof of Theorem 1 and prove an analogous of Lemma 1: students can obtain any attainable set of courses by ranking them in the first place when the immediate acceptance mechanism is employed.

Lemma 2 *Let $P = (P_s)_{s \in S}$ be a preference profile for students, and let $\mu = IA(P)$. For each $s \in S$ and $C' \subseteq \mu(s)$ $C' = IA\left(P_{s|C'}, P_{-s}\right)$.*

Proof. Let $s \in C$ and let $C' \subseteq \mu(s)$. Let $c \in C'$, let $r(c)$ be the step of the immediate acceptance mechanism when c has been assigned to s, and formally let $r(c) = \min_{r \leq r^*} \{r \mid s \in \mu^r(c)\}$. Let σ be the slot to which s is assigned at stage $r(c)$. Thus, student s is the highest priority student for slot σ among the ones in $\mu^r(c)$ and who are not assigned to a slot preceding σ. Formally, for each $r \leq r(c)$, if $s' \in \mu^r$and $s' \succ_\sigma s \succ_\sigma$, there exists a slot $\sigma' \in \Sigma_c$, $\sigma' \rhd_c \sigma$ such that $s' \succ_{\sigma'} \sigma$. Thus, $C' = IA\left(P_{s|C'}, P_{-s}\right)$. ■

This result allows us to prove that every Nash equilibrium outcome of the immediate acceptance mechanism is stable and every stable allocation is a Nash equilibrium outcome of the immediate acceptance mechanism under slot-specific priorities.

Theorem 2 *The immediate acceptance mechanism implements the set of stable matching in NE in the domain of substitutable preferences if priorities* *are slot-specific.*

Proof. (i) We first prove that any NE outcome is a stable matching. Let P^* be a NE of $\left(S, \mathcal{P}^{|S|}, IA, P\right)$ and let $\mu = IA\left(P^*\right)$. As observed, μ is individually rational for each course. We prove by contradiction that μ is individually rational for students. Assume $Ch_s\left(\mu\left(s\right)\right) \neq \mu\left(s\right)$ for some $s \in S$. Let $P_s' = P_{s|Ch_s(\mu(s))}$, by Lemma 2: $IA\left(P_s', P_{-s}^*\right)\left(s\right) = Ch_s\left(\mu\left(s\right)\right)$. Thus, the deviation is profitable to s, which yields a contradiction. Assume that there exists a pair blocking μ, $(c, s) \in C \times S$. Let $P' = P_{s|Ch_s(\mu(s)\cup\{c\})}$. Because $s \in Ch_c\left(\mu\left(c\right) \cup \{s\}\right)$, the deviation is profitable to s, which yields a contradiction. Because μ is individually rational and cannot be blocked by a pair, μ is stable.

(ii) Let μ be a stable matching. For each s, let $P_s^* = P_{s|\mu(s)}$. Set $P^* = \left(P_s^*\right)_{s\in S}$. We have $IA\left(P^*\right) = \mu$. We prove by contradiction that P^* is a Nash equilibrium. Assume that $s \in S$ has a profitable deviation, P_s', and let $\mu' = IA\left(P_s', P_{-s}^*\right)$. Let $c \in Ch_s\left(\mu\left(s\right) \cup \mu'\left(s\right)\right) \setminus \mu\left(s\right)$. Because P_s is substitutable, $c \in Ch_s\left(\mu\left(s\right) \cup \{c\}\right)$. Let $P_s'' = P_{s|Ch_s(\mu(s)\cup\{c\})}$, then $IA\left(P_s'', P_{-s}^*\right)\left(s\right) = Ch_s\left(\mu\left(s\right) \cup \{c\}\right)$. It follows that (c, s) blocks μ, which yields a contradiction. ■

The cost of introducing substitutable priorities is to allow for courses to renegotiate their assigned group of students to preserve the individual rationality of the outcome. Theorem 2 proves that this is no longer the case under slot-specific priorities: Nash implementation of stable matchings does not require students to lose their positions along the mechanism.

Note that the equilibrium strategies defined in the part (ii) of the proof of Theorem 2 are undominated. Thus, we obtain the following result.

Corollary 2 *The immediate acceptance mechanism implements the set of stable matching in UNE in the domain of substitutable preferences if priorities are slot-specific.*

3.2 Simplifying the strategy space

The renegotiable and immediate acceptance mechanisms perform well under substitutable and slot-specific priorities, respectively. However, the complexity of the strategy space might hinder its practical implementation (see Budish et al., 2017). We prove that if the preferences of the students are responsive, the message space can be simplified.[8] Our findings can be applied to situations in which course schedules do not overlap, and students have only one possible group to attend to for each course. This is often the case for the courses organized by neighborhood associations and local libraries, and for elective courses at small community colleges and universities.

We next introduce two mechanisms derived from the renegotiable and immediate acceptance mechanisms for which students have to reveal their preferences for individual courses and demands, instead of their full profile of preferences for all possible subsets of courses.

For each $s \in S$, let $M_s = \mathcal{L}(C) \times (\mathbb{N} \cap [0, |C|])$, where $\mathcal{L}(C)$ is the set of

[8] As mentioned in the introduction, the assumption of responsive preferences is standard in the course assignment literature.

linear order on $C \cup \{\emptyset\}$ and $\mathbb{N}$ is the set of non-negative integers. For each $s \in S$, let $(\geq_s, q_s) \in M_s$ and let $P_s = P_s(\geq_s, q_s)$ be a profile of responsive preferences with demand q_s, which coincides with $\geq_s$ on the set of individual courses.[9]

Given a priority system $(Ch_c)_{c \in C}$ and $(\geq_s, q_s)_{s \in S}$, the simplified renegotiable acceptance mechanism is defined by the following outcome function $SRA\left((\geq_s, q_s)_{s \in S}\right) = RA\left(P_s(\geq_s, q_s)_{s \in S}\right)$. In other words, in a simplified mechanism, students play the game induced by the corresponding mechanism with preferences that are responsive to the revealed ones.

Proposition 1 *Assume that students preferences are responsive and priorities are substitutable. The simplified renegotiable acceptance mechanism implements the set of stable matchings in Nash equilibrium.*

Proof. (i) We first prove that any NE outcome is a stable matching. Let $(>^*_s, q^*_s)_{s \in S}$ be a NE of the game induced by the simplified renegotiable acceptance mechanism when students' preferences are given by $(P_s)_{s \in S}$ and let $\mu = SRA\left((>^*_s, q^*_s)_{s \in S}\right)$. Matching μ is individually rational for each course. We prove by contradiction that μ is individually rational for students. Assume that μ is not individually rational for student $s \in S$, which assumes that there exists a course $c \in \mu(s)$ such that $\emptyset P_s c$ or $|\mu(s)| > q_s$, where q_s is the offer of course s according to P_s. Let $>_s$ be the restriction of P_s to individual courses. By Lemma 1, $(>_s, q_s)$ is a profitable deviation for student s, which yields a contradiction. We next prove by contradiction that

[9] This means that $c \geq_s c'$ if and only if $\{c\} P_s \{c'\}$ for all $c, c' \in C$.

μ is not blocked by any pair. Assume that there exists a pair blocking μ, $(c, s) \in C \times S$. Let $>_s$ be the restriction of P_s to the individual courses in $\mu(s) \cup \{c\}$. Because $s \in Ch_c(\mu(s) \cup \{c\})$, the deviation $(>_s, q_s)$ is profitable to s, which yields a contradiction.

(ii) Let μ be a stable matching. For each s, let $>_s$ be the restriction of P_s to the individual courses in $\mu(s)$. Note that $(>_s, q_s)_{s\in S}$ yields μ as outcome. We prove by contradiction that $(>_s, q_s)_{s\in S}$ is a Nash equilibrium. Assume that student s has a profitable deviation, $(>'_s, q'_s)$, and let μ' be the outcome of such a deviation. Let $c \in Ch_s(\mu(s) \cup \mu'(s), P_s(>_s, q_s)) \setminus \mu(s)$. Because P_s is responsive, $c \in Ch_s(\mu(s) \cup \{c\})$. Let $>''_s$ be the restriction of P_s to the individual courses of $\mu(s) \cup \{c\}$. Then, $(>'_s, q_s)$ is a profitable deviation as well, yielding $Ch_s(\mu(s) \cup \{c\}, P_s)$. Thus, the pair (c, s) blocks matching μ, which yields a contradiction. ■

We can also define a simplified version of the immediate acceptance mechanism as follows. Given a priority system $(Ch_c)_{c\in C}$ and $(\geq_s, q_s)_{s\in S}$, the simplified immediate acceptance mechanism is defined by the following outcome function $SIA\left((\geq_s, q_s)_{s\in S}\right) = IA\left(P_s(\geq_s, q_s)_{s\in S}\right)$.

Proposition 2 *Assume that student preferences are responsive and priorities are slot-specific. The simplified immediate acceptance mechanism implements the set of stable matchings in Nash equilibrium.*

The proof of Proposition 2 is similar to the proof of Proposition 1 and is omitted.

4 Conclusions

In this paper, we present the renegotiable acceptance mechanism to allocate courses to students on the basis of priorities. Under substitutable preferences and priorities, the renegotiable acceptance mechanism implements the set of stable matching in Nash equilibrium and in undominated Nash equilibrium. The mechanism produces matchings that are fair, and its practical implementation is not computationally demanding. The renegotiable acceptance mechanism is based on the immediate acceptance mechanism but allows for courses to reject previously accepted students. During the procedure, courses are only tentatively assigned, and the readjustments preserve individual rationality. This makes our new procedure a hybrid between the immediate and the deferred acceptance mechanisms. We also analyze the immediate acceptance mechanism under the assumption of slot-specific priorities and find that it implements the set of stable matching in Nash equilibrium and in undominated Nash equilibrium. The results depend on the fact that both mechanisms provide each student with incentive to top-rank the best achievable subset of courses given the preferences submitted by the other students. This property helps to rule out unstable matchings as equilibrium outcomes.

Finally, we study the possibility of reducing the complexity of the strategy space. We show that this is possible when courses are not complements. In this case, a mechanism that asks each student a ranking on individual courses and the number of courses that she is willing to take implements the set of stable matchings in Nash equilibria.

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