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Department of Management

Working Paper Series

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Working Paper n. 6/2018
July 2018

ISSN: 2239-2734

Hotelling-Bertrand duopoly competition under firm-specific network effects*

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(June 2018)

Abstract. When dealing with consumer choices, social pressure plays a crucial role; also in the context of market competition, the impact of network/social effects has been largely recognized. However, the effects of firm-specific social recognition on market equilibria has never been addressed so far. In this paper, we consider a duopoly where competing firms are differentiated solely by the level of social (or network) externality they induce on consumers' perceived utility. We fully characterize the subgame perfect Nash equilibria in locations, prices and market shares. Under a scenario of weak social externality, the firms opt for maximal differentiation and the one with the highest social recognition has a relative advantage in terms of profits. Surprisingly, this outcome is not persistent; excessive social recognition may lead to adverse coordination of consumers: the strongest firm can eventually be thrown out of the market with positive probability. This scenario is related to a Pareto inefficient trap of no differentiation.

Keywords: Consumer choice game, Duopoly price competition, Hotelling Location model, Network Externalities, Large Games, Social interaction.

JEL Classification Numbers: L13, C72, C63, D71.

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* We thank Fulvio Fontini, Marco LiCalzi, Simone Marsiglio and participants to seminars held in Ca' Foscari and Pisa Sant'Anna. We also acknowledge the financial support from Ca' Foscari University Venice under the grant *"Interactions in complex economic systems: contagion, innovation and crises"*.

1 Introduction

Consider an economy where a large population of consumers, located on a one-dimensional preference interval, is in charge to choose among two alternative goods issued by two competitors, say A and B. The two firms compete on prices in a Bertrand duopoly setting, after having chosen strategically their location on the consumer interval. Suppose that the two firms are characterized by a different level of *social recognition* they are able to imprint in consumers' mind; consider two parameters measuring social recognition, say J_A and J_B, and let A be the *brand leader*, in that $J_A > J_B$.[1] *Ceteris paribus*, one expects the relative market positioning for A to increase in J_A. This is exactly what happens under a standard scenario of *weak network effects*; moreover, under this scenario firms decide to locate at the extremes of the consumer preference interval in order to benefit from *maximal (horizontal) differentiation*. Now, what happens if $J_A \gg J_B$? Do the market share for A and its revenues increase accordingly? And what are the effects in terms of horizontal (or vertical) differentiation?[2]

The main goal of this paper is to analyze the impact of firm-specific levels of social recognition on the Nash equilibria expressed in terms of locations, prices and market shares.[3] In particular, for high levels of network externalities, we show the emergence of a new scenario: monotonicity is lost and both the location and price strategies change abruptly. We show that this new scenario is related to the appearance of a Pareto inefficient trap of *no (horizontal) differentiation*: firms decide to overlap at the center of the preference interval. In this case, the brand leader can eventually result to be out of the market with positive probability. These somewhat paradoxical results can be explained in terms of social interactions and coordination of consumers.

The paramount role of social externalities, often referred to as network effects, has been widely investigated during the last decades. Among the others, Schelling (1971) explores patterns of residential segregation, Becker (1974)

[1] In the marketing context, this could be interpreted as *brand awareness*. For a detailed discussion about the effects of brands on consumer choices, we refer the reader to Reimann et al. (2012) and Wood and Hayes (2012).

[2] As already noticed in Grilo et al. (2001), the introduction of social/network externalities in location models makes the difference between horizontal and vertical differentiation more blurred.

[3] In a dynamic context, in Lamberson and Page (2018) it is discussed the impact of firm-specific *transition weights* on the (long run) equilibrium market shares of competitors in a stylized oligopoly. In that setting, the transition weights are a proxy of the *ex-ante* perceived quality of the different products.

households' income distribution, Katz and Shapiro (1985) competition and compatibility among technologies, Akerlof (1997) studies how social distance impacts on decisions. Glaeser et al. (1996) empirically studies the impact of social interactions on crime rates in different regions. One of the main consequences of network effects is the appearance of what are commonly called *social traps*. Quoting Akerlof (1997), *"These externalities [...] will create long-run low-level equilibrium traps that are far from socially optimal."* To the best of our knowledge, the first attempt to incorporate network effects in the context of a duopoly competition is Grilo et al. (2001), which extends classical models of horizontal differentiation following Hotelling (1929). In this paper, the authors show how the level of conformity (i.e., the strength of the network effects) impacts on the pricing policies of the two competitors. The stronger the network effects, the higher is the competition and the lower are prices; eventually, assuming very high network effects, a *winner-takes-all* effect prevails and one of the companies may become a monopolist. In line with Grilo et al. (2001), we incorporate network effects in the classical setting of d'Aspremont et al. (1979); however, we let the social recognition parameters to be firm-specific. This allows us to identify a brand leader and to study the asymmetric effects of social externalities on the two competitors.

A second strand of literature in the field of social interactions devotes its attention to the study of large populations of heterogeneous agents linked by social ties. A pioneering contribution is Granovetter (1978) where riots' formation is analyzed in relation to the characteristics of the underlying population. In the context of discrete choices, Brock and Durlauf (2001) describe the aggregate outcomes of the economy when the size of the population increases to infinity. In this case, it is easier to obtain closed-form solutions and characterize the equilibria of the economy. A similar asymptotic perspective is fruitful when players act strategically as in Kalai (2004) who identifies a general class of large games for which existence and robustness of pure-strategy Bayesian Nash equilibria is guaranteed.

Our aim is to unite these two branches of literature in modeling a duopoly competition where network effects are firm-specific and where demand arises as in a consumer game with many players. More precisely, each consumer is located on a one-dimensional *preference interval* and has to choose between two products. In taking decisions, agents weight public signals (prices, locations), their personal taste (signaled by their type) and social norms (being the majority is relevant). In this respect, our approach resembles the *computer-choice game* in Kalai (2004), with one main novelty: the levels of network externality are firm-specific. Eventually, as an outcome of the large game played by consumers, we obtain a piecewise linear demand curve which straight connects our approach to classical models on Bertrand competition in a differentiated duopoly (see, Singh and Vives (1984)). Concerning supply, two firms (A and B) compete on prices to maximize their profits. They form an expectation about consumers' actions, choose a location on the preference interval and set prices. Denote by α and β the location of firms A and B, respectively, and by p_A and p_B the prices for two goods; locations and prices are set by means of two subsequent stages of a non-cooperative game played by the two competitors. Finally, denote by q the proportion of agents choosing A. We analyze the subgame perfect Nash equilibria $(\alpha^*, \beta^*, p_A^*, p_B^*, q^*)$ emerging in the economy.

Under a simplifying setting where locations are fixed, we show that the brand leader takes advantage of his dominant position, increases its market share and, eventually, forces the competitor out of the market. However, in case of large network externalities, multiple equilibria emerge and, for some of them, the brand leader suffers adverse coordination of consumers and, eventually, ends up out of the market with positive probability. As far as we know, this non-linearity in market outcomes due to social interactions has never been documented in the context of market competition.

In the general setting where locations are strategically set by firms, we again recognize two opposite situations: under *weak network effects*, the two firms differentiate as much as possible and locate themselves at the extremes of the preference interval. Conversely, under *strong network effects*, firms converge to the center of the interval, satisfying the interest of the median consumer.[4] Moreover, under this latter scenario, only one firm survives; we show, however, that the *winner-takes-all* situation is intended in a *probabilistic sense*: both the competitors have a positive probability to succeed. However, the brand leader maintains a higher chance to monopolize the market, exerts higher prices, and, eventually, expects higher profits.

The paper is organized as follows. Section 2 introduces the preference space and provides the definition of *marginal consumer* as a formal limit of the n-player consumer game. In section 3 we study in details a simplified version of the model where locations are fixed under the assumption of maximal differentiation. Section 4 is devoted to the discussion of the general framework where locations are now endogenous. In section 5 we draw some conclusions, whereas Appendix A contains all the technical proofs.

[4] This latter outcome is in line with the so called *median voter evidence* under which two political parties with opposite ideologies both converge to a centered positioning. See Downs (1957) and Enelow and Hinich (1984) for more details.

2 The consumer choice game

Consider a large population of n consumers, whose preferences are uniformly distributed over the interval $[-\sigma, \sigma]$; for $i = 1, \ldots, n$ we denote agent's i type by $t_i \in [-\sigma, \sigma]$. The supply side of the market is formed by two firms, A and B, which position themselves on the preference interval at points α and β. We assume that firm A plays on the left part of the interval (from $-\sigma$ to 0), whereas B plays on the right (from 0 to σ). As depicted in Figure 1, both α and β take values in $[0, \sigma]$ and signal the distance of the firm from the center of the interval.

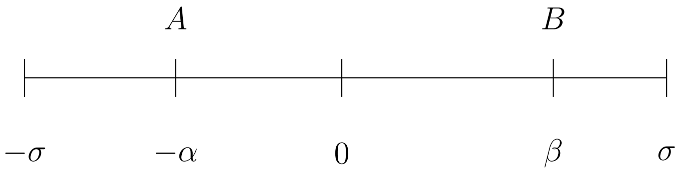

Figure 1: Locations of firms A and B with respect to the consumer preference interval $[-\sigma, \sigma]$.

In Figure 2 we represent the two extreme situations: *maximal differentiation*, where $\alpha = \sigma$ and $\beta = \sigma$ (left panel), and *no differentiation*, where firms converge to the center of the interval, hence $\alpha = 0$ and $\beta = 0$ (right panel).[5]

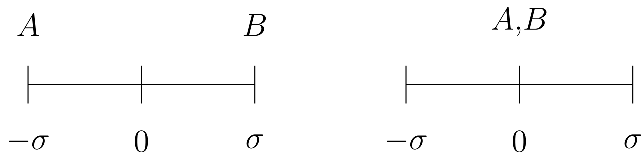

Figure 2: *Maximal differentiation* (left) and *no differentiation* (right).

As suggested by the literature on large n-player games (see, for example, Kalai (2004)) and discrete choices (see Brock and Durlauf (2001)), it is convenient to explore the behavior of the system when the number of agents is getting larger and larger. In this case, the effect of the single agent on the market outcomes (such as the expectations q^i) becomes negligible and aggregate statistics are easier to determine in closed form. Each agent has to decide between two mutually exclusive actions $\omega_i \in \{A; B\}$: buying product

[5] In the rest of the paper, when referring to maximal differentiation as opposed to no differentiation, we mean differentiation expressed in terms of locations.

produced by firm A or by firm B; at this stage of the game, prices and locations are fixed. As previously discussed, we want to consider social effects, in the sense that, the larger is the (expected) share for one product, the higher is its social recognition and the personal/cognitive advantage for the agent in adopting it. Therefore, we make the payoff depend upon the *expected market share* of each product.

We model the (indirect) utility $V_i(\omega_i, t_i)$ of each agent i as the weighted sum of three components: public effects (prices), social effects and, finally, personal taste/preference as follows:

$$\begin{aligned} V_i(A, t_i) &= Y - p_A + J_A q^i - \tau(t_i + \alpha)^2; \\ V_i(B, t_i) &= Y - p_B + J_B(1 - q^i) - \tau(t_i - \beta)^2. \end{aligned} \tag{1}$$

Concerning public effects, they positively depends on the reservation value Y of the good and negatively on its price.[6] The second component encompasses social/network effects; in line with literature on social interactions (see Granovetter (1978) and Brock and Durlauf (2001)) this is expressed as $J_A\, q^i$ for firm A and $J_B\,(1 - q^i)$ for firm B, where (J_A, J_B) measure the level of firm-specific social recognition and where q^i (resp., $1-q^i$) denotes the expectation of agent i about the market share of product issued by firm A (B). In this respect, a higher J reflects a higher social value assigned by consumers to the brand/product, which in turn translates into a positive externality on the associated utility.[7] The latter term of the utility represents a quadratic cost of *cognitive dissonance* and is given by the euclidean distance between the consumer's ideal preference point (signaled by the type t_i) and the location of each good.[8] The parameter $\tau > 0$ measures the consumer sensibility to distance and can be interpreted as the (unitary) cognitive dissonance that a consumer faces when buying a product far from her own individual taste.

The decision faced by each consumer depends on the action of other agents through the participation share shaping the social component of the utility.

[6]We assume that Y is constant across goods and large enough so that each consumer enters the market. These assumptions are standard in the context of horizontal product differentiation models; see, Belleflamme and Peitz (2015). An interpretation can be found in Anderson et al. (1992) where the authors assume Y to be a constant income across all individuals; therefore, the payoff function is interpreted as an indirect utility function.

[7]As said, $J_A - J_B$ can be interpreted as a degree of *perceived differentiation* among the two firms. We will see that, if J_A is large enough compared to J_B, this actually translates into *vertical differentiation* in the sense of Belleflamme and Peitz (2015).

[8]The positive sign in the term $(t_i + \alpha)$ is simply due to the fact that α measures the distance of firm A from the origin; being A located on the left of the origin, its position is, actually, $-\alpha$.

This eventually results in setting a non-cooperative game, in which each agent makes her choice given an expectation of the population outcome. It is assumed that the agents know the characteristics of the economy (prices and parameters) as well as the probability distribution for the types. Moreover, they know the structure of the individual choice problems. Evidently,

$$V_i(A, t_i) > V_i(B, t_i) \iff \\ - p_A + p_B + \tau \left[(t_i - \beta)^2 - (t_i + \alpha)^2 \right] + J_A \, q^i - J_B \, (1 - q^i) > 0.$$

First of all, note that when $\alpha = \beta = 0$, it is immediate to derive a threshold value for q^i such that $V_i(A, t_i) > V_i(B, t_i) \iff q_i < (p_A - p_B + J_B)/(J_A + J_B)$. In this case, when the number of agents tends to infinity, it is easy to see that the unique market share q consistent with the consumer choice game is $q = (p_A - p_B + J_B)/(J_A + J_B)$.[9] On the opposite, in the general case, it is not so straightforward to derive the equilibrium market share since we are left with an implicit problem. By defining agent-specific threshold levels

$$t_i^{th} = \frac{p_B - p_A + \tau \, [\beta^2 - \alpha^2] + J_A \, q^i - J_B \, (1 - q^i)}{2\tau(\alpha + \beta)},$$

it turns out that, for each $i = 1, \ldots, n$,

$$\omega_i = A \iff t_i < t_i^{th}. \tag{2}$$

Note that $\mathbb{P}(t_i = t_i^{th}) = 0$, because the distribution of types is continuous; therefore, decision under equality is immaterial. In the next proposition, we characterize the Nash equilibrium of the consumer choice game. Proofs are postponed to Appendix A.

Proposition 2.1. *Consider the n-player game with payoff structure expressed by thresholds as in (2) and where the types $(t_i)_{i=1,\ldots,n}$ are independent and uniformly distributed on $[-\sigma, \sigma]$. Then, there exists at least one Nash equilibrium in pure strategies.*

Moreover, if $\alpha + \beta \neq 0$, when $n \to \infty$, the marginal agent, indifferent between A or B, has type t_m equal to

$$t_m(q) = \frac{p_B - p_A + \tau \, [\beta^2 - \alpha^2] + J_A \, q - J_B \, (1 - q)}{2\tau(\alpha + \beta)}, \tag{3}$$

[9] Of course, being $q \in [0, 1]$, if $(p_A - p_B + J_B)/(J_A + J_B) \leq 0$, we will have $q = 0$ and, on the opposite, if $(p_A - p_B + J_B)/(J_A + J_B) > 1$, $q = 1$. This means that, under these values of the parameters, the market turns into a monopoly. Details are postponed to the next sections.

where q solves the consistency (fixed point) equation

$$F(t_m(q)) = q, \tag{4}$$

and where

$$F(z) = \begin{cases} 0 & \text{if} \quad z \leq -\sigma \\ \frac{z+\sigma}{2\sigma} & \text{if} \quad -\sigma \geq z \geq \sigma \\ 1 & \text{if} \quad \sigma \geq z \end{cases} . \tag{5}$$

We benefit from rational expectations: each agent shares the same expectation about other players' actions and this expectation, when $n \to \infty$, matches the limiting value q. Eventually, the decision for an action is based upon a comparison between the individual type t_i and the threshold t_m: the population splits so that all agents with $t_i < t_m$ choose product A, and all agents with $t_j \geq t_m$ will choose B.[10]

The rest of the paper is devoted to the analysis of equilibria in locations, prices and market shares as emerging from (4). For sake of clarity, a general treatment where locations are endogenously determined is postponed to section 4. We now turn our attention to a simpler setup where firms locate at the extremes of the consumer interval (maximal differentiation). In this simplified setting, it is easier to capture some key features of the model.

3 The Maximal differentiation case

Maximal differentiation translates into the following assumption: $\alpha = \beta = \sigma$. Moreover, having fixed symmetric locations, without loss of generality we normalize the transportation costs by assuming that $2\tau(\alpha + \beta) = 4\tau\sigma = 1$. Under these assumptions, (3) reads

$$t_m(q) = p_B - p_A + J_A\, q - J_B\, (1 - q). \tag{6}$$

The marginal agent moves to the right, hence, diminishing the market share for B when p_B increases and/or J_B decreases. The opposite effect happens for p_A and J_A.[11]

In the next section, we discuss in details the *consumer game*, showing how it gives rise to piece-wise linear demand curves for the two products.

[10] As said, being the distribution of t_i continuous, it is not relevant where we put the equality sign.

[11] It is worth noting that the same marginal consumer t_m as in (6) would emerge in the

3.1 From the consumer game to a linear demand curve

Depending on the values of the parameters, it is not difficult to derive all the fixed points of (4). To this aim, we introduce firm-specific quantities called *percevied distances*:

$$c_A = \sigma - J_A; \quad c_B = \sigma - J_B. \tag{7}$$

The interpretation is as follows: the *ex-ante median consumer*, positioned at the origin, suffers a distance 'σ' to reach each one of the two firms located at the extremes of the interval. To it, we subtract the social recognition (brand) value of the firm/product expressed by J_K, for $K \in \{A; B\}$. In this respect, J_K can be interpreted as a "discount" on the cognitive distance due to the social recognition level.

It is convenient to separate the analysis in two cases. The first case, referred to as *weak network effects* (WNE),[12] describes the situation where the sum (or the average) of the perceived distances is positive:

$$c_A + c_B > 0. \tag{8}$$

Note that (8) is equivalent to $(J_A + J_B)/2 < \sigma$; interpretation is as follows: the average social recognition parameter is smaller than the distance to be covered by the *ex-ante* median consumer. The opposite situation, $c_A + c_B < 0$, is called *strong network effect* scenario (SNE).

Proposition 3.1. *Consider the infinite-player consumer game resulting in the marginal consumer as in* (6). *Assume that firms are located at the extreme of the consumers interval and marginal costs as in* (7)*; set* $4\tau\sigma = 1$*, and, finally, define*

$$\theta = \frac{p_B - p_A + c_B}{c_A + c_B}. \tag{9}$$

The self-consistent Nash equilibria of the consumer game are as follows:

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case of utilities for product $K \in \{A; B\}$ defined as

$$V_i(K, t_i) = Y - p_K - 0.5\, t_i + J_K\, q^i.$$

This formulation matches payoffs typical of large games; see, for instance, the computer-choice game in Kalai (2004), where consumers have separable and linear preferences for public, private and social effects. This parallelism between industrial organization models and large games seems to be novel.

[12]This terminology and classification is rather standard in the literature, see Grilo et al. (2001) or Belleflamme and Peitz (2015).

- *under WNE, i.e., when* $c_A + c_B > 0$,

$$q = \begin{cases} 0 & \text{if} \quad p_B - p_A \le -c_B \\ \theta & \text{if} \quad -c_B < p_B - p_A \le c_A \\ 1 & \text{if} \quad p_B - p_A > c_A \end{cases} \tag{10}$$

- *under SNE, i.e., when* $c_A + c_B < 0$,

$$q = \begin{cases} 0 & \text{if } p_B - p_A \le -c_B \\ \{0; \theta; 1\} & \text{if } -c_B < p_B - p_A \le c_A \\ 1 & \text{if } p_B - p_A > c_A \end{cases} \tag{11}$$

Under WNE, once prices have been fixed, the equilibrium level q for the market share is unique. Figure 3 (left panel) describes the demand of product A, according to equation (10). Demand q decreases with p_A and increases with p_B, yielding negative own-price elasticity and positive cross-price elasticity. The situation drastically changes under SNE. As depicted by Figure 3 (right panel), for intermediate prices ($\sigma - J_A < p_B - p_A \le J_B - \sigma$), three self-consistent equilibria coexist: two extreme equilibria ($q = 0$ and $q = 1$) and an intermediate equilibrium $\theta \in (0, 1)$. As an example, fix parameters and p_B. Suppose $p_A < p_B - J_B + \sigma$; in this case, the only possible equilibrium demand is $q = 1$. Suppose now that p_A increases so that $p_A > p_B - J_B + \sigma$. We enter in the region where firm A may lose its market power and even go out of the market (in case the equilibrium $q = 0$ prevails). Note that, differently from WNE, this transition is non-smooth.[13]

What is the connection between Proposition 3.1 and standard Bertrand competition models? Let us focus on the case of a proper duopoly; in this case, the market share for A is given by (9). Working on it, we obtain the following demand curves:

$$\begin{aligned} q_A(p_A, p_B) &= \quad \gamma_A \quad - \delta p_A + \delta p_B, \\ q_B(p_A, p_B) &= 1 - \gamma_A + \delta p_A - \delta p_B \end{aligned} \tag{12}$$

where $\gamma_A = c_B/(c_A + c_B)$, $\gamma_B = c_A/(c_A + c_B)$, $\delta = 1/(c_A + c_B)$, and where c_A and c_B are as defined in (7). These expressions provide a clear interpretation of c_A and c_B in terms of classical linear demand curves: they signal the *ex-ante demand* for the two products.[14] The higher is γ_A (or equivalently, the higher is c_B relatively to c_A), the higher is the *ex-ante* market share for firm A. If γ_A is higher than γ_B (equivalently, J_A higher than J_B), firm A benefits

[13] We discuss how the two firms take this *uncertainty* into account in the next section.

[14] By *ex-ante* demand we mean the demand obtained in case of $p_A = p_B = 0$.

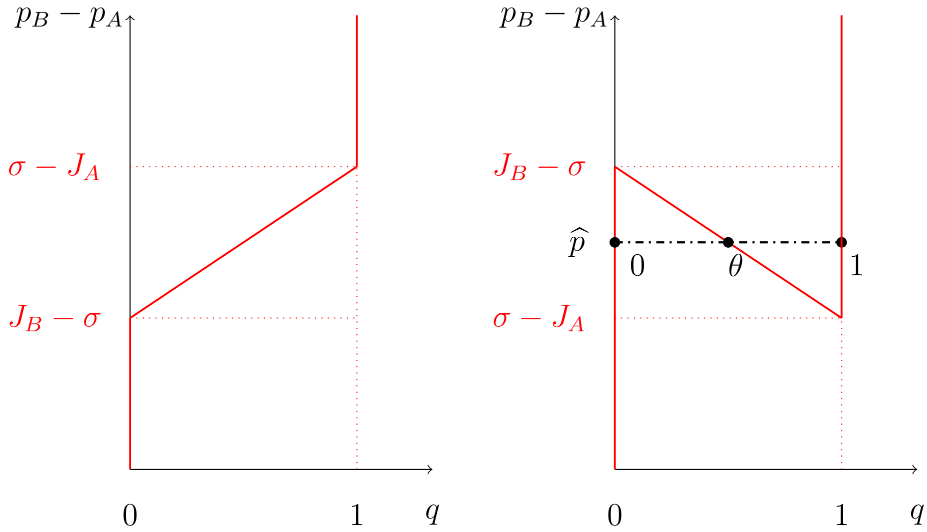

Figure 3: Demand q under weak (left) and strong (right) network effects.

from a relative advantage. Eventually, $\gamma_A \geq 1$, corresponds to the situation of a general consensus about *ex-ante* propensity for product A. A symmetric argument holds for product B. According to Belleflamme and Peitz (2015), the *ex-ante* consensus for a product is the proper definition for the presence of *vertical differentiation*. Therefore, assuming that firm A is the market leader, $J_A > J_B$,[15] we obtain the following result.

Corollary 3.1. *Consider the case where* $\alpha = \beta = \sigma$ *and assume* $4\tau\sigma = 1$*; suppose, moreover, that* A *is the brand leader in that* $J_A > J_B$*. Vertical differentiation applies when* $J_A \geq \sigma$*.*

3.2 The two-player Bertrand competition

Up to now, the prices of the two products have been treated as fixed. We now model the stage of the game where firms are in charge to simultaneously choose their prices by means of a two-player non-cooperative game. As before, it is convenient to distinguish two cases based on the strength of the network effects.

[15] A symmetric argument applies in case of $J_B > J_A$.

Weak network effects

Each firm selects a price, $p_k \geq 0$, where $K \in \{A, B\}$. We denote by π_A and π_B the normalized per-capita profit of firm A and B, respectively:[16]

$$\pi_A = p_A \cdot q\,; \quad \pi_B = p_B \cdot (1-q). \tag{13}$$

We look for a Nash equilibrium, $(p_A^*,\, p_B^*)$, in pure strategies, such that:

$$\begin{aligned}\pi_A(p_A^*, p_B^*) &\geq \pi_A(p_A, p_B^*), \quad \text{for all } p_A \geq 0;\\ \pi_B(p_B^*, p_A^*) &\geq \pi_B(p_B, p_A^*), \quad \text{for all } p_B \geq 0.\end{aligned}$$

Replacing (10) into (13), we get quadratic and concave functions.

Proposition 3.2. *Consider the model where q is described by equation* (10) *and two firms, A and B, simultaneously maximize profits according to* (13). *Under WNE, the unique subgame perfect Nash equilibrium $(p_A^*,\, p_B^*,\, q^*)$ can be described as follows.*

1. *If $c_A + 2c_B > 0$ and $2c_A + c_B > 0$, then*

$$p_A^* = \frac{c_A + 2c_B}{3}; \quad p_B^* = \frac{2c_A + c_B}{3}; \quad q^* = \frac{1}{3} + \frac{c_B}{3(c_A + c_B)} \tag{14}$$

   and the market is a proper duopoly. Moreover,

   – *if $J_A \leq J_B$, then $p_B^* > p_A^*$ and $0 < q^* \leq \frac{1}{2}$;*

   – *if $J_A \geq J_B$, then $p_A^* > p_B^*$ and $\frac{1}{2} \leq q^* < 1$.*

2. *If $2c_A + c_B \leq 0$, then $p_A^* = p_A^M = J_A - \sigma$, $p_B^* = 0$ and $q^* = 1$. Therefore, firm A monopolizes the market.*

3. *If $c_A + 2c_B \leq 0$, then $p_B^* = p_B^M = J_B - \sigma$; $p_A^* = 0$ and $q^* = 0$. Therefore, firm B monopolizes the market.*

Three configurations are possible. In Figure 4 we show an illustrative example by fixing $\sigma = 4$ and $J_B = 2$ and letting J_A vary. We recognize: duopoly (denoted by $\mathcal{D}$ and represented by a shaded area), monopoly of A ($\mathcal{M}_A$), monopoly of B ($\mathcal{M}_B$). The green line represents the *difference in optimal prices*, $p_B^* - p_A^*$, as resulting from Proposition 3.2. Finally, the dashed line, corresponding to equation $p_B^h - p_A^h = (J_B - J_A)/2$, shows to the level of prices under which $q = 1/2$.

[16] To focus on the effects of network externalities, we set the marginal cost for each firm equal to zero, and we assume no fixed costs.

When $J_A < J_B = 2$, $p_B^* > p_A^*$ and $q^* < 1/2$. In this case, firm B takes advantage of the higher network parameter, exerts a higher price, and it is still able to obtain a higher market share compared to its competitor. When $J_A = J_B = 2$, the model is perfectly symmetric: $p_A^* = p_B^*$ and $q^* = \frac{1}{2}$. When $J_A > J_B$, then $p_A^* > p_B^*$ and $q^* > 1/2$. Finally, when $J_A \geq (3\sigma - J_B)/2 = 5$, firm A monopolizes the market ($q^* = 1$) and charges the monopoly price $p_A^M = J_A - \sigma$.

A final remark. As prescribed by the theory on competition in a differentiated duopoly (see Singh and Vives (1984)), equilibrium prices result to be a convex combination of *ex-ante* demands γ_A and γ_B as defined in (12).[17]

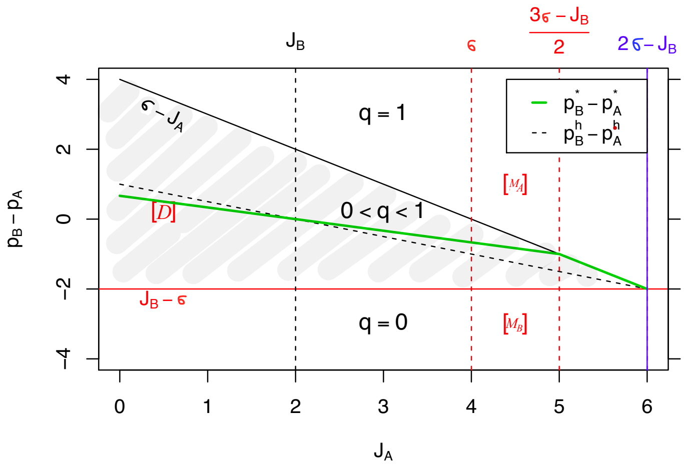

Figure 4: Phase Diagram for q under WNE, $\sigma = 4$, $J_B = 2$.

Strong network effects

Under SNE, as seen in Proposition 3.1, the prevailing market share may be multiply defined, which makes more difficult to properly define profits. The firms, when setting their pricing strategy, must therefore *forecast* the possible equilibrium selected during the consumer choice subgame. Therefore, in order to form an expectation, the two firms consider the emerging equilibrium in a *probabilistic sense*, by assigning a certain probability to the three Nash equilibria. As a modeling assumption, we suppose that the two firms consider as plausible only the equilibria that can be reached during the consumer

[17]Being costs fixed at zero, the equilibrium prices must be convex combinations of *ex-ante* demands. See Table 1 (pg 549) in Singh and Vives (1984) for a similar expression.

choice game as steady states of the best response map iteration procedure. In essence, by starting from an initial configuration $q(0)$, we iterate the map $q(t) = F(t_m(q_{t-1}))$, where F is as defined in (5) and we consider as plausible equilibria only the (stable) steady states of such dynamics. It remains to identify a good candidate for the likelihood of the different equilibria. To this aim, we assume that the probability of a steady state (i.e., a Nash equilibrium) to occur is proportional to the size of its *basin of attraction*.[18] Although our setting is static, this argument is made mathematically precise in the following lemma.

Lemma 3.1. *The solutions to* (4)*, found in Proposition 3.1, can be interpreted as the long-run attractors of the map* $q \mapsto F(t_m(q))$*, where* F *is as defined in* (5)*. In this respect, under SNE and when* $c_A < p_B - p_A \leq -c_B$*,*

- $q = \frac{p_B - p_A + c_A}{c_A + c_B}$ *is a linearly unstable equilibrium;*
- $q = 0$ *and* $q = 1$ *are locally stable and their domains of attraction are, respectively, of size* θ *and* $1 - \theta$*, where* $\theta = \frac{p_B - p_A - c_B}{c_A + c_B}$.

Lemma 3.1 says that the intermediate equilibrium is *not reachable* by agents best-responding to the actions of other players. Put differently, we are postulating that, the two firms evaluate the market share as follows:

$$q = \begin{cases} 0 & \text{if } p_B - p_A \leq -c_B \\ Q & \text{if } -c_B < p_B - p_A \leq c_A \\ 1 & \text{if } p_B - p_A > c_A \end{cases} \tag{15}$$

where

$$Q = \begin{cases} 0 & \text{with probability} \quad \theta \\ 1 & \text{with probability} \quad 1 - \theta \end{cases} \tag{16}$$

is the realization of a Bernoullian random variable.[19] In this sense, under SNE, both the two locally stable equilibria correspond to a *winner-takes-all* situation where the entire population purchases one of the two goods.[20] In

[18]We say that q_0 belongs to the basin of attraction of the equilibrium $\bar{q}$, if, when starting at q_0 and iterating the map $q(t) = F(t_m(q_{t-1}))$, for $t \geq 1$, the system converges to $\bar{q}$. To our opinion, the size of the domains of attraction is a meaningful choice for the probability to assign to the different equilibria.

[19]In this case, we denote the equilibrium demand by a capital letter to emphasize its random nature. Note that the choice we made about the probability θ presumes that the iteration procedure of Lemma 3.1 starts by a uniformly randomly chosen $q_0 \in [0, 1]$.

[20]The fact that $q \in \{0; 1\}$ strongly depends on the choice we made about the uniform distribution of types. By considering any other continuous and unimodal probability distribution, we would still see three equilibria where the intermediate one is unstable, but the two external ones would belong to the open set $(0, 1)$.

particular, for intermediate prices, the social influence is strong enough to let the population coordinate and end up buying one of the goods unanimously, although we cannot predict with certainty which one of the two prevails. This *paradoxical result* is related to social interactions: when coordination is huge, the direction of the coordination is unclear but crucial. Both firms, especially the brand leader, are aware of this intrinsic uncertainty. As a consequence, firms A and B maximize their expected profits, which turn out to be, respectively,[21]

$$\mathbb{E}(\pi_A) = p_A \cdot (\theta \cdot 0 + (1-\theta) \cdot 1) = p_A \cdot \left(\frac{p_A - p_B + c_A}{c_A + c_B} \right); \tag{17}$$

$$\mathbb{E}(\pi_B) = p_B \cdot (1 - (\theta \cdot 0 + (1-\theta) \cdot 1)) = p_B \cdot \left(\frac{p_B - p_A + c_B}{c_A + c_B} \right). \tag{18}$$

The following proposition summarizes the possible outcomes of the model under strong network effects.

Proposition 3.3. *Consider the model where q is described by* (15) *and two firms, A and B, simultaneously optimize their expected profits as in* (17)-(18)*. Assume, finally, that $c_A + c_B < 0$ (SNE).*

1. *If $c_A + 2c_B < 0$ and $2c_A + c_B < 0$, then $E^*[Q] = 1 - \theta^*$, where*

$$\theta^* = \frac{1}{3} + \frac{c_B}{3(c_A + c_B)}, \tag{19}$$

   and optimal prices are

$$p_A^* = \frac{-(2c_A + c_B)}{3}; \; p_B^* = \frac{-(c_A + 2c_B)}{3}.$$

   Moreover, with probability θ^*, firm B monopolizes the market and, with probability $1 - \theta^*$, firm A monopolizes the market. Finally,

   – *if $J_A \leq J_B$, then $p_B^* \geq p_A^*$ and $\theta^* \geq \frac{1}{2}$;*

   – *if $J_A \geq J_B$, then $p_A^* \geq p_B^*$ and $\theta^* \leq \frac{1}{2}$.*

2. *If $c_A + 2c_B \geq 0$, then $p_A^* = p_A^M = J_A - \sigma$, $p_B^* = 0$ and $q^* = 1$. Therefore, firm A monopolize the market.*

3. *If $2c_A + c_B \geq 0$, then $p_B^* = p_B^M = J_B - \sigma$, $p_A^* = 0$ and $q^* = 0$. Therefore, firm B monopolizes the market.*

[21]For simplicity we assume that firms are risk neutral.

In Figure (5), we summarize the phase diagram of q^* as a function of J_A and J_B (here $\sigma = 4$ is fixed). Under WNE (left panel), the admissible values of J_A and J_B are represented by the area below the blue line corresponding to equation $J_A + J_B = 2\sigma$. We recognize the three regions corresponding to situations of Proposition 3.2 (duopoly and monopoly). In the right panel we see the situation when SNE holds. In region $\mathcal{M}_B$, $q^* = 0$; in region $\mathcal{M}_A$, $q^* = 1$, and, finally, in region $\mathcal{M}_\theta$, the two extreme equilibria coexist and $E^*[Q] = 1 - \theta^* \in (0, 1)$.

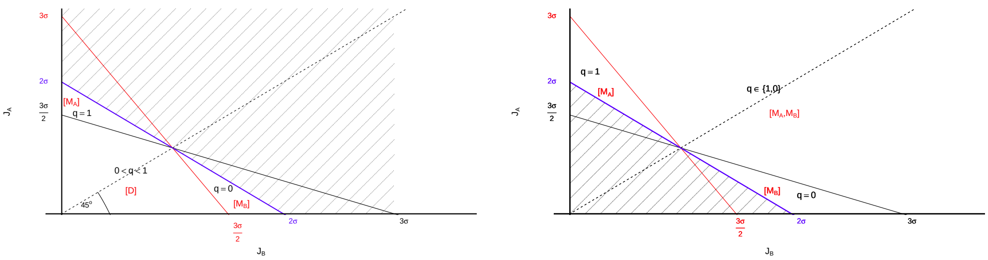

Figure 5: Phase diagram for q^* under WNE (left) and SNE (right).

3.3 Discussion of market equilibria

In this section we discuss some market implications in terms of market shares, prices and profits. Figure 6 depicts the market share q^* for firm A for different values of J_A, assuming that $\sigma = 4$ and $J_B = 2$. The blue dashed vertical line divides the graph into WNE (on the left) and SNE (on the right). Under WNE, firm A market share increases with J_A up to a point where the firm monopolizes the market. Under SNE, at some point we see a *bifurcation* due to the presence of multiple equilibria. Continuous red lines mark the three equilibria $q^* \in \{0; \theta; 1\}$ (the intermediate being unfeasible) and the dashed line the expected market share, $E[Q] = 1 - \theta^*$, as described in Proposition 3.3. Recall the interpretation: with probability θ^*, firm B monopolizes the market. Moreover, it is not difficult to see that, by (19), when $J_A > J_B$, $\lim_{J_A \to \infty} \mathbb{E}^*(1 - Q) = \theta^* = 1/3$.

It might seem counter-intuitive that the market share of firm A decreases with J_A. A closer look to the microstructure of the consumers' decision process shows that this phenomenon can be explained by the contribution due to network externalities. Go back to the discussion about the marginal

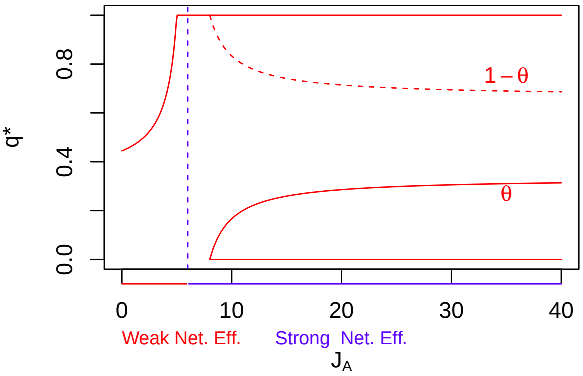

Figure 6: Values of q^* varying J_A, for $\sigma = 4$ and $J_B = 2$.

consumer as seen in (6) and replace $p_B^* - p_A^* = (J_B - J_A)/3$. We obtain

$$V_i(A, t_i) > V_i(B, t_i) \iff \underbrace{\frac{J_B - J_A}{3}}_{\text{Price Impact (-)}} + \underbrace{J_A q^* - J_B(1 - q^*)}_{\text{Social Impact (+)}} > t_i. \tag{20}$$

We can now distinguish two occurrences in J_A, with opposite sign: a "positive contribution" (positive externality) on the social component of the utility and a "negative contribution" (negative externality) in terms of the price (due to the fact that p_A increases with J_A under the strong network effects). In this respect, we obtain the somewhat unexpected result that, under SNE, an *ex-ante* relative advantage in terms of perception of quality has a negative "secondary effect" on the utility of consumers, hence, on the brand leader's market positioning.

Concerning optimal pricing policies, they are depicted in Figure 7. We can identify three regions corresponding to three different scenarios. The first one corresponds to the region where $J_A < (3\sigma - J_B)/2$; here, prices decrease with J_A. This is in line with Grilo et al. (2001) where a similar monotonicity of prices in network externalities is found. One remark may be useful. In that paper, authors motivate this evidence as follows: an increase in network effects signals a fiercer market competition and, finally, a reduction in prices. In our model, where network power is firm-specific, the interpretation of network effects as *market competition* is still plausible but the effect of a fiercer competition on the two firms is disentangled. Firm A, thanks to a higher social recognition, has a relative advantage: its price decreases sensibly

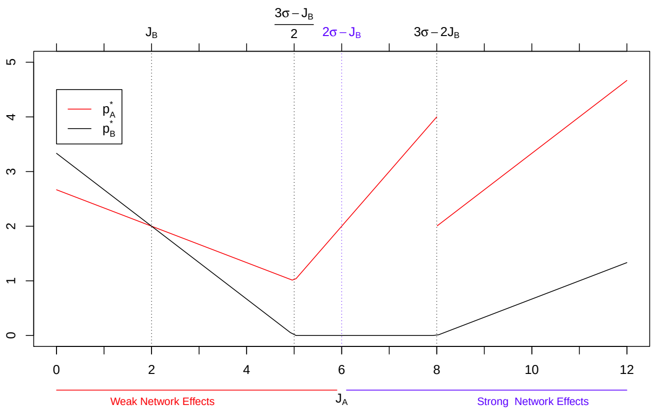

Figure 7: Optimal prices: p_A^* (red) and p_B^* (black), varying J_A.

less compared to the competitor. When $(3\sigma - J_B)/2 < J_A < 3\sigma - 2J_B$, firm A acts as a monopolist and its price increases in J_A. Finally, for $J_A > 3\sigma - 2J_B$, we enter a region of multiple equilibria. As discussed before, there is a positive probability for firm A to lose its market power. As a consequence, it is no longer optimal for the brand leader (firm A) to apply the monopolistic price $p_A^M = J_A - \sigma$. Strictly speaking, the risk of ending up in a *bad scenario* makes the brand leader adjust its price consequently.[22] Conversely, under this latter scenario, firm B takes advantage of this uncertainty and, back to the market, charges a non-zero (although relatively small) price.

Concerning profits, in Figure 8 we see how both firms decrease their profits under the first scenario ($J_A < (3\sigma - J_B)/2$). Under WNE, an increase in J_A signals an increase in competition, hence a decrease in prices and, eventually, a decrease in profits. On the other hand, as noticed before, the fiercer competition has a non-symmetric effect on the two firms: the brand leader (A in this case) is able to maintain almost constant profits whereas firm B loses large part of its. Therefore, even if slightly loosing profits, firm A increases its market power. When entering in the second scenario where A acts as a monopolist, the picture changes completely: here profits π_A^* increase in J_A, since now $p_A^* = p_A^M = J_A - \sigma$ (and $q^* = 1$ is constant). Finally, under the third scenario, there is a discontinuity in (expected) profits for firm A, due to the appearance of multiple equilibria. Nevertheless, profits still grow in J_A due to the increase in the optimal price p_A^*, which offsets the decrease in

[22] If we interpret an option value as the value that is placed on firm A for preserving a chance to monopolize the market in the future; then, the difference between the monopolistic price p^M and the optimal price p_A^* after the (downward) jump, could express an option value, a situation already found in Wirl (2008), yet in a different framework.

the (expected) market share.

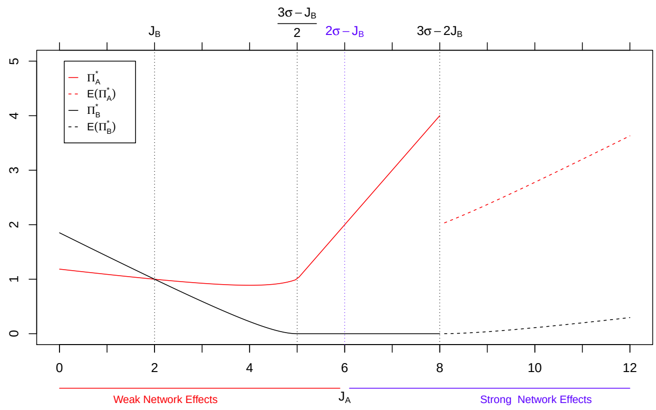

Figure 8: Optimal profits π_A^* (red) and π_B^* (black), varying J_A.

4 The general case with endogenous locations

In this section we generalize the previous results to the case where firms are in charge to decide their location on the consumers preference interval. From a modeling viewpoint, this results in the specification of a preliminary phase of the game where locations are chosen according to a two-player simultaneous non-cooperative game. The general picture is as depicted in Table 1.

Location game	Price Competition	Consumer-choice game
(α, β)	(p_A, p_B)	q

Table 1: The three consecutive subgames.

In line with the analysis developed in the previous simpler framework, we proceed backward: (i) solve for the optimal q as if prices and locations were given; (ii) solve for optimal prices assuming locations as given; (iii) finally, solve for optimal locations.

To solve (i) and (ii), we need to generalize all the previous results on consumer-choice game and price competition for firms locating at a generic point $(\alpha, \beta) \in [0, \sigma] \times [0, \sigma]$. This is done in section 4.1. The identification of the optimal location strategies is postponed to section 4.2.

4.1 Generalizing previous findings to a generic $(\alpha,\ \beta)$

First, we extend the notion of percevied distances as introduced in (7) to account for the differentiation in locations. With a slight abuse of notation, we still call them c_A and c_B:

$$\begin{aligned} c_A &= 2\tau(\alpha+\beta)\cdot\left(\sigma - \tfrac{\beta-\alpha}{2}\right) - J_A; \\ c_B &= 2\tau(\alpha+\beta)\cdot\left(\sigma + \tfrac{\beta-\alpha}{2}\right) - J_B. \end{aligned} \tag{21}$$

Note that this new definition is consistent with the simplified version used in the previous section; by putting $\alpha = \beta = \sigma$ and $4\tau\sigma = 1$, we are back to the simplest form. It also preserves and enriches the economic intuition; as before, perceived distances decrease with J_K. What is new is that the *ex-ante* marginal consumer now moves to the *midpoint* of the segment connecting A and B, whose coordinate is $m = (\beta - \alpha)/2$. Assume $2\tau(\alpha + \beta) = 1$ (in case $\alpha = \beta = 0$, the value of τ is irrelevant), and recall the definition of ex-ante market share γ_K, for $K \in \{A, B\}$, as defined in (12). We immediately see that $\gamma_A = c_B/(c_A + c_B)$ increases in m and decreases in J_B. Therefore, we obtain the following result, which generalizes Corollary 3.1.

Corollary 4.1. *Assume* $2\tau(\alpha + \beta) = 1$*, if* $\alpha + \beta \neq 0$*; suppose, moreover, that* $J_A \geq J_B$*. Vertical differentiation applies when* $J_A \geq \sigma - m$*, where* m *is the midpoint of the segment connecting the locations of the two competitors.*

To clarify this result, suppose $\alpha = 0$ and $\beta = \sigma$, so that $m = \sigma/2$ is positive. Suppose, moreover, $J_A > J_B$. In this case, firm A has a "double advantage": a higher social recognition and a more central positioning. The threshold level for J_A needed to have *ex-ante* consensus about propensity for product A is lower compared to the case where $\alpha = \beta = \sigma$ as in Corollary 3.1.

The discussion about subgame perfect equilibria in prices and market shares is in line with previous analysis; we again identify four situations corresponding to: proper duopoly ($\mathcal{D}$), proper monopoly ($\mathcal{M}_A$ or $\mathcal{M}_B$), and probabilistic monopoly ($\mathcal{M}_\theta$). To have a visual representation of the possible outlooks of the market, we fix all the parameters, apart from locations (α, β) and we partition the space $[0, \sigma] \times [0, \sigma]$, into the four (possibly empty) regions. In Figure 9, we fix $\sigma = 2$, $\tau = 0.5$, $J_B = 2$ and we choose different levels for J_A. In panel A ($J_A = 2$), we have perfect symmetry and all the four regions are present. In panel B ($J_A = 3$), again all the situations are present but $\mathcal{M}_B$ is much smaller. In panel C ($J_A = 4$), $\mathcal{M}_B$ disappears. Finally, in panel D ($J_A = 5$), both $\mathcal{D}$ and $\mathcal{M}_B$ are not feasible. In all diagrams, the red continuous line marks the frontier between the two regimes; indeed, the weak (strong) is above (below) the line.

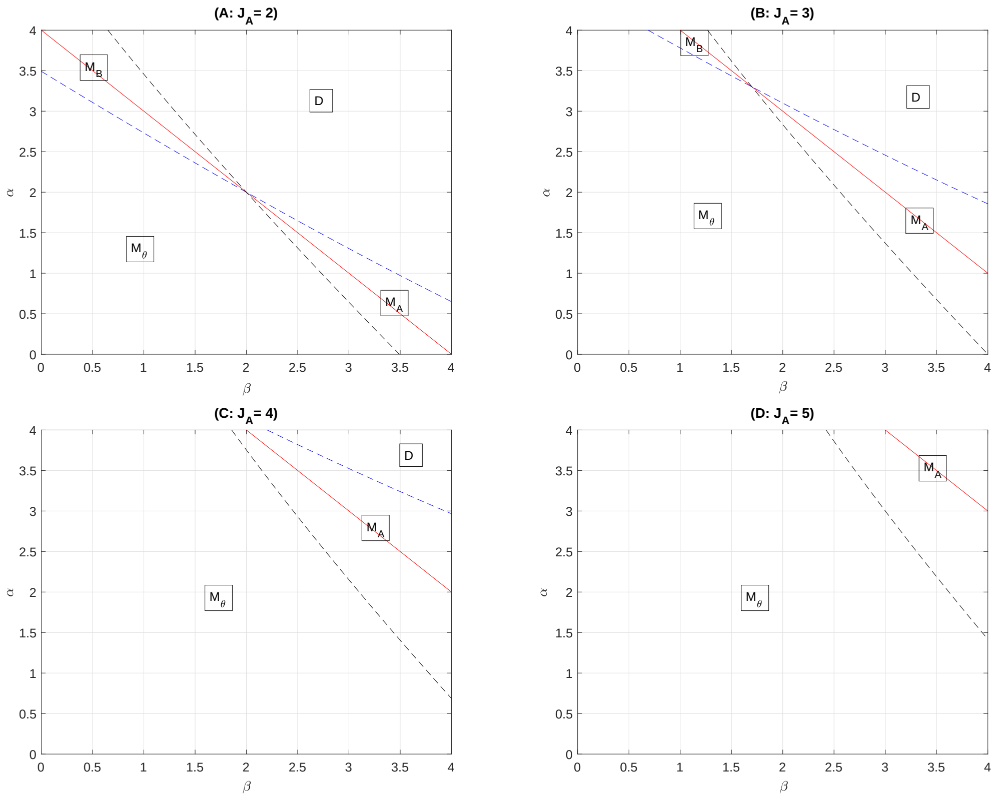

Figure 9: Monopoly/duopoly domains, for different values of J_A.

Algebraically, conditions identifying the four regions are described in the next proposition, together with expressions of prices and market share at the equilibrium. As usual, a brief derivation is postponed to Appendix A.

Proposition 4.1 (Subgame perfect equilibria in prices and market share)**.** *Consider the model where consumers play in line with the marginal agent as described by* (3) *and firms maximize their (expected) profits; assume, moreover, that equation* (21) *holds, and that* α *and* β *are fixed. Then, the following alternative situations may apply:*

1. *If* $2c_A + c_B > 0$ *and* $c_A + 2c_B > 0$*, we are in region* $\mathcal{D}$*; the market is a proper duopoly and the unique equilibrium is* (p_A^*, p_B^*, q^*)*, where*

$$p_A^* = \frac{c_A + 2c_B}{3}, \quad p_B^* = \frac{2c_A + c_B}{3},$$

and

$$q^* = \frac{c_A + 2c_B}{3(c_A + c_B)}.$$

2. *If* $2c_A + c_B \leq 0$ *and* $c_A + 2c_B \geq 0$ *we are in region* $\mathcal{M}_A$*;* $q^* = 1$ *and firm A monopolizes the market. Moreover,* $p_A^* = p_A^M = -c_A$ *and* $p_B^* = 0$*.*

3. *If* $2c_A + c_B \geq 0$ *and* $c_A + 2c_B \leq 0$ *we are in region* $\mathcal{M}_B$*;* $q^* = 0$ *and firm B monopolizes the market. Moreover,* $p_B^* = p_B^M = -c_B$ *and* $p_A^* = 0$*.*

4. *If* $2c_A + c_B < 0$ *and* $c_A + 2c_B < 0$*, we are in region* $\mathcal{M}_\theta$*; then*

$$p_A^* = \frac{-2c_A - c_B}{3}; \quad p_B^* = \frac{-c_A - 2c_B}{3}.$$

Moreover, with probability

$$\theta^* = \frac{c_A + 2c_B}{3(c_A + c_B)}, \tag{22}$$

firm B monopolizes the market and, with probability $1 - \theta^*$*, firm A monopolizes the market.*

Proposition 4.1 generalizes results described in Propositions 3.2 and 3.3. Expressions for prices and market shares written in terms of c_A and c_B are exactly the same as in the previous section, although the perceived distances are now depending on locations as well. These findings are now used to discuss the final part of the game, namely, the location game.

4.2 The location game

Up to now we have considered (α, β) as given; we now turn to the more general situation where firms simultaneously choose their location.

By virtue of Proposition 4.1, four alternative situations are possible. Two of them, $\mathcal{M}_A$ and $\mathcal{M}_B$ are not plausible as outcomes of the location game. Intuitively, the firm resulting out of the market, would react to exit this region in order to have at least a chance of benefit from a positive market share. Therefore, we are left with only two opposite situations: the proper duopoly, $\mathcal{D}$ and the probabilistic monopoly $\mathcal{M}_\theta$. In the next proposition, we state that only the two extreme location strategies (maximal differentiation and no differentiation) are locations consistent with subgame perfection.

Proposition 4.2. *The locations consistent with subgame perfection are* (σ, σ) *(maximal differentiation) and* $(0, 0)$ *(no differentiation).*

As a matter of fact, not necessarily both the previous situations are Nash equilibria; depending on the values of the parameters, we can see one of them or, possibly, both. It turns out to be difficult to provide closed-form conditions identifying all the possible situations. Here below, a general heuristic result is proposed in form of remark. The same statement is made mathematically precise in the subsequent proposition, where the two firms are equal in terms of network externality (i.e., $J_A = J_B$).

Remark 4.1 (Subgame perfect equilibria for locations)**.** *If network effects are small enough, maximal differentiation is the unique subgame perfect Nash equilibrium; if network effects are large enough, the no differentiation scenario prevails; for intermediate values of network effects, the two scenarios coexist.*

More precisely, the following proposition holds true.

Proposition 4.3. *Suppose* $J_A = J_B =: J$ *and* $4\tau\sigma = 1$*; define two threshold levels for network externality as follows:*

$$J^l(\sigma) = \frac{(8-\sqrt{14})\,\sigma}{24}; \quad J^h(\sigma) = \frac{\sigma}{2}.$$

Then, three possibilities apply:

1. *if* $J \le J^l(\sigma)$*,* (σ,σ) *is the unique subgame perfect Nash equilibrium;*
2. *if* $J \ge J^h(\sigma)$*,* $(0,0)$ *is the unique subgame perfect Nash equilibrium;*
3. *if* $J^l(\sigma) < J < J^h(\sigma)$*,* $(0,0)$ *and* (σ,σ) *are both subgame perfect Nash equilibria; nevertheless,* (σ,σ) *is Pareto efficient.*

Proposition 4.3 shows that, even in case of symmetric firms, all the three possibilities identified in Remark 4.1 may apply. In particular, when network externalities are small, the two firms locate at the extremes of the linear interval. This finding is consistent with previous studies: under weak network effects, firms choose a position yielding a market with maximal product differentiation (see d'Aspremont et al. (1979)). Indeed, the closer the firms are on the consumer spectrum, the higher is the price competition; therefore, competitors maximize horizontal differentiation, lowering price competition, and this results in a segmentation of the market. It is not difficult to see that, in this case,

$$p_A^* = \frac{12\sigma^2\tau - J_A - 2J_B}{3}; \quad p_B^* = \frac{12\sigma^2\tau - 2J_A - J_B}{3}, \tag{23}$$

and

$$q^* = \frac{12\sigma^2\tau - J_A - 2J_B}{3(8\sigma^2\tau - J_A - J_B)}. \tag{24}$$

		Firm B	
		0	σ
Firm A	0	$\pi_A(0,0),\ \pi_B(0,0)$	$\pi_A(0,\sigma),\ \pi_B(0,\sigma)$
	σ	$\pi_A(\sigma,0),\ \pi_B(\sigma,0)$	$\pi_A(\sigma,\sigma),\ \pi_B(\sigma,\sigma)$

Table 2: Normal form location game.

Firms increase their prices when the sensitivity to cognitive distance, τ, is high and/or when the length of the interval of consumers' heterogeneity, σ, is high. Note, moreover, that this scenario is consistent with the simplified setting discussed in Section 3, where firms were located at the extremes of the consumers' spectrum.

Under the second situation of Proposition 4.3 (Remark 4.1), the firms choose strategically to converge at the center of the interval (no differentiation). Note that they also *choose* a setting where SNE are in place:[23] only one firm will act as the monopolist, exerting a price given by

$$p_A^* = \frac{2J_A + J_B}{3} \quad \text{or} \quad p_B^* = \frac{J_A + 2J_B}{3}. \tag{25}$$

The expected market share uniquely depends on the network effect levels:

$$\mathbb{E}(Q) = 1 - \theta^* = \frac{1}{3} + \frac{1}{3}\left(\frac{J_A}{J_A + J_B}\right); \quad \mathbb{E}(1-Q) = \frac{1}{3} + \frac{1}{3}\left(\frac{J_B}{J_A + J_B}\right). \tag{26}$$

In particular, for a fixed value of J_B, $\mathbb{E}(Q)$ increases in J_A from the value of 1/3 to the maximum value of 2/3. In the third situation of Proposition 4.3, for intermediate values of the network externality, the two equilibria coexist although (σ, σ) is always Pareto dominant.

We now show, by means of a numerical example, the behavior of the model when $J_A > J_B$. Since only the extreme equilibria can be potential outcomes, we only consider the four basic strategies $(0,0)$, $(0,\sigma)$, $(\sigma,0)$ and (σ,σ). Table 2 summarizes a *skeleton version* of the location game.

We choose the following values of the parameters. We fix $\sigma = 2$ and $\tau = 1/8$ (so that $4\sigma\tau = 1$). Concerning social recognition, we fix $J_B = 0.3$ and consider three values of J_A (see Table 3), to match the three situations of Remark 4.1.

[23]The algebraic condition separating the weak and strong regimes (the sign of $c_A + c_B$) depends now on (α, β). Therefore, when firms are in charge to set their location, the emerging regime is endogenous as well.

Scenario 1	Scenario 2	Scenario 3
$(0.4, 0.3)$	$(1.4, 0.3)$	$(0.8, 0.3)$

Table 3: Values of J_A and J_B for the three scenarios.

We run the model and we observe results by filling the normal-form game matrix as described in Table 2.

Scenario 1		*Firm B*	
		0	σ
Firm A	0	0.1921, 0.1587	0.5342, 0.1675
	σ	0.1921, 0.4923	**0.8418**, **0.8084**

It is clear that under Scenario 1, the strategy (σ, σ) strictly dominates any other strategy and is the unique Nash equilibrium. Let us say that this is the *benchmark situation* where low (or zero) network externality is in place. In this region firms compete based on product differentiation and each of them serves a niche on the market.

Scenario 2		*Firm B*	
		0	σ
Firm A	0	**0.6281**, **0.2614**	0.9000, 0.000
	σ	0.0926, 0.0593	0.7729, 0.4063

Under the second scenario, the no differentiation equilibrium prevails. Here, only one firm survives on the market, so horizontal differentiation is lost and firms prefer to serve to the median consumer. Recall that the survival of the firm is probabilistic: firms locate at the center in order to increase their chances to survive and monopolize the market. This is in line with the Hotelling (1929) model and with the *median voter election model* (see Enelow and Hinich (1984)) in which two political parties converge to the median point of the voters interval in order to increase their chances to win the competition. Note, finally, that the no differentiation equilibrium can be regarded as a *trap* in the sense that (σ, σ) would be Pareto dominant, in terms of expected returns. However, being in region $\mathcal{M}_\theta$, only one firm will survive. Therefore, in case of survival, the realized payoff will be higher than the expected payoff. For instance, for what firm A is concerned, realized profits will be $p_A^* \cdot 1 = 1.0333$, much higher than the expected payoff 0.6281. Finally, from the consumers perspective, prices are lower under the no differentiation case compared to the maximal differentiation case. Therefore, they benefit from lower prices.[24]

Scenario 3		Firm B	
		0	σ
Firm A	0	**0.3646, 0.1980**	0.5444, 0.0444
	σ	0.1494, 0.3160	**0.8107, 0.6441**

Under the third scenario, the two Nash equilibria coexist, although (σ, σ) is Pareto dominant. As before, the no differentiation equilibrium can be interpreted as a trap. When firms coordinate on the suboptimal equilibrium, resulting in a forced monopoly, both of them loose profits on average.

Concerning the dependence of market shares in the social recognition parameter, Figure 10 shows the market shares of the two firms at the equilibrium (here, $\sigma = 1$, $\tau = 1$ and $J_B = 2$). First of all, notice that for these parameters, firm B cannot result to be the monopolist (unless multiple equilibria are present). As already noticed in Section 3, under WNE q^* increases with J_A; however, it never reaches the monopolistic level, since, when $J_A \geq 2$, the two firms change their location strategy and position themselves at the center. Recall that under this scenario, $\lim_{J_A \to \infty} \mathbb{E}^*(1 - Q) = \theta^* = 1/3$. Therefore, no matter of the values of the parameters, the probability for firm A to be out of the market remains sensibly high even when J_A becomes huge.

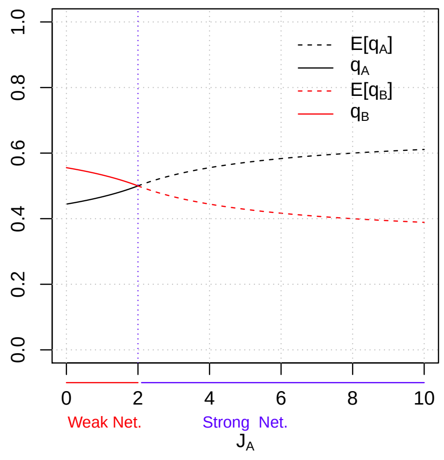

Figure 10: Optimal market shares varying J_A when $\tau = \sigma = 1$ and $J_B = 2$.

[24]We will see later that the situation changes for huge values of J_A: in this case, prices increase and consumers are worse off compared to the maximal differentiation scenario.

Finally, Figure 11 shows how optimal prices vary with J_A. Under WNE and *maximal differentiation*, prices decrease with J_A: an increase in network effects signals a fiercer market competition. When, $J_A \geq 2$, firms switch to *strong network effects* and *no differentiation*: price competition does not hold any more. Under this scenario, there is a positive probability for each firm to monopolize the market, and product differentiation dissipates; firms' profits only depend on the strength of their social recognition parameters, as we can see by looking at (25). Although both firms experiment an increase in prices, firm A is able to set a higher price due to its higher social recognition.

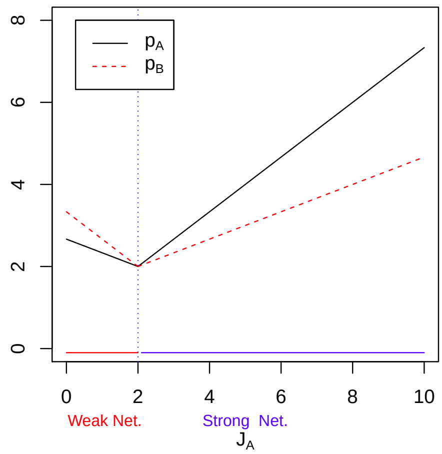

Figure 11: Optimal prices varying J_A ($\tau = \sigma = 1$ and $J_B = 2$).

5 Conclusions

We have studied a duopoly model where firms choose a location on the consumers' preference interval and compete on prices, whereas a large population of heterogeneous potential adopters strategically chooses one of the two available goods by means of a consumer choice game. The peculiarity of our approach is the presence of firm-specific network effects and the formalization of a three-stage game where the two competitors strategically choose both locations and prices and potential adopters play a version of a consumer choice game accordingly.

By assuming that consumers' tastes are uniformly distributed, we are able to fully characterize the emergent equilibria in locations, prices and market shares. Our analysis confirms previous studies in showing that, under weak network effects, competition increases when network externality increases, thus, abating prices; moreover, under this scenario, *maximal differentiation* emerges and each firm serves a niche of the market. However, a result never detected in the field of market competition emerges: under strong network effects firms converge to serve the *median consumer*. With this respect, our duopoly competition resembles the idea of political parties converging to a *median positioning*, thus disregarding ideologies or political niches. The two competitors end up meeting at the center of the interval; eventually, only one succeeds to monopolize the market, while the other is out. Both firms maintain a positive probability to win the competition, although the brand leader has always a higher chance to succeed.

We also detect situations in which multiple equilibria coexist. In this case, the uncertainty related to the possibility of *adverse coordination of consumers*, makes the brand leader partially loose its market power. Indeed, the specter of a *disastrous equilibrium*, where the brand leader has a zero-market share, gets positive probability. Apart from the mathematically interesting discontinuity in the pricing strategies and profit functions, the appearance of multiple equilibria relates our duopoly model to the literature of social choices. Indeed, the bad equilibrium resembles the *low-level equilibria* (or social traps) widely discussed in the literature in social dynamics. As a matter of fact, for intermediate values of social recognition, the market may end up in a Pareto inefficient *no differentiation trap* where both competitors are worse off compared to the maximal differentiation regime. However, in our case, it is not correct to speak about *socially suboptimal equilibria*: under this scenario, consumers take advantage of lower prices on the market. This effect vanishes when social recognition exceeds a certain level; at some point, the prices for the two products turn out to be even higher than in the monopoly region, thus causing worse market conditions for consumers. The variety of results in terms of differentiation in locations is much richer compared to what conjectured in previous studies. In Grilo et al. (2001), in a model similar to ours, without firm-specific network externalities, the authors conjecture that firms would never converge to the no differentiation scenario. We have proven, instead, that no differentiation and even coexistence of the two opposite situations can be actually detected.

A final methodological comment. As already stressed, our game consists in three successive stages where locations, prices and, finally, market shares are set. Under this perspective, it can be viewed by different angles: Hotelling location models, Bertrand competition on prices and consumer choice models.

The possibility of merging under a unifying framework three research fields pertaining to game theory, industrial organization and economic theory is, to our opinion, a notable byproduct of this research.

A Proofs

Proof of Proposition 2.1

To prove existence of a Nash equilibrium in the n-player consumer game, it is more convenient to look at *optimal thresholds*, $(t_i^{th})_{i=1,\dots,n} \in \overline{\mathbb{R}}^n$, rather than at the vector of binary actions. Indeed, (2) suggests a one-to-one relationship between ω_i and t_i^{th}. The idea is to show that the *best response map* is continuous over a convex and compact domain and thus it admits at least one fixed point. Let consider t^*_{-i} be the $(n-1)$-dimensional vector formed by the optimal thresholds excluding agent i.

Note that q_i, the expected utility of agent i about the market share, is expressed as follows:

$$q^i = \frac{1}{n}\,\mathbb{E}^i\left[\sum_{j=1}^{n} \mathbb{I}_{\{\omega_j = A\}}\right],$$

where $\mathbb{E}^i[\cdot]$ denotes the expectation taken with respect to the joint distribution of the vector of types $(t_j)_{j\neq i}$.[25] Therefore, t_i^{th} can be written as

$$t_i^{th} = \frac{1}{Z}\cdot\Big[-p_A + p_B - J_B + \tau[\beta^2-\alpha^2] + \frac{J_A+J_B}{n}\,\mathbb{I}_{\{t_i<t_i^{th}\}}\,\frac{J_A+J_B}{n}\sum_{j\neq i}\mathbb{I}_{\{t_j<t_j^{th}\}}\Big], \tag{27}$$

where $Z = 2\tau(\alpha+\beta)$. Consider two different vectors $t^{*,1}_{-i}$ and $t^{*,2}_{-i}$. As said, we want to show that t_i^{th} is continuous in t^*_{-i}. Equation (27) suggests that

$$\left|t_i^{th}(t^{*,1}_{-i}) - t_i^{th}(t^{*,2}_{-i})\right| \le c\,\mathbb{P}\Big(\bigcup_{j\neq i}\min\{t^{*,1}_{-i}(j), t^{*,2}_{-i}(j)\} \le t_j \le \max\{t^{*,1}_{-i}(j), t^{*,2}_{-i}(j)\}\Big).$$

By continuity of the distribution of the types, the r.h.s. of the previous inequality goes to zero as $t^{*,1}_{-i} \to t^{*,2}_{-i}$. This ensures continuity of the best response map w.r.t the players' thresholds. Given that the best response map continuous on $\overline{\mathbb{R}}^n$,

[25] We could equivalently use $\frac{1}{n-1}\sum_{j\neq i}\mathbb{I}_{\{\omega_j=A\}}$, in place of q^i. Indeed, when n is large, the marginal contribution of the choice of agent i becomes negligible and the two aggregate statistics are indistinguishable.

it admits at least one fixed point and so at least one Nash equilibrium in pure strategies exists.

Concerning the limit as $n \to \infty$ we need to introduce an aggregate variable. Consider equation (2), aggregate on the number of agents and divide by n to obtain

$$\tilde{q}^n := \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{w_i = A\}} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{t_i < t_i^{th}\}}. \tag{28}$$

The sequence $(\tilde{q}^n)_n$, taking values on $[0, 1]$, is tight. Therefore, all subsequences almost surely converge to a limit $q \in [0, 1]$. It remains to show that the limits of such sequences are the solutions to the fixed point equation $F(t_m(q)) = q$. This at least at an heuristic level, follows directly from (28). As said, the l.h.s. converges to q. Concerning the r.h.s. it basically represents the *empirical distribution* of the n-dimensional system; it can be formally proved that, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{t_i < t_i^{th}\}} = F(t_m).$$

Therefore, for $n \to \infty$, we obtain $q = F(t_m(q))$.[26]

Proof of Proposition 3.1

It is not difficult to verify that the possible solutions to (4), expressed as functions of p_A and p_B, verify the following scheme:

i) if $p_B - p_A \le -c_B$, $q = 0$ is a solution to (4);

ii) if $-c_B < p_B - p_A \le c_A$, $q = \frac{p_B - p_A + c_B}{c_A + c_B}$ is a solution to (4);

iii) if $p_B - p_A > c_A$, $q = 1$ is a solution to (4).

As a consequence, both (10) and (11) immediately follow. In particular, under SNE, the intervals where $q = 0$ and $q = 1$ are solutions to (4) overlap, giving rise to multiplicity of solutions.

Proof of Corollary 3.1

Under WNE, it is easy to see that $\gamma_A \ge 1$ translates into $J_A \ge \sigma$. More difficult is the situation under SNE. According to the discussion following Lemma 3.1, we

[26] A formal proof of this limit is out of the scope of this paper. Details can be found in Dai Pra et al. (2013).

know that under SNE, the proper duopoly situation cannot be reached and that θ, as described in (9), represents now the *probability of firm B to monopolize the market.* In this respect, $1-\theta$ signals the probability of firm A to monopolize the market. Since we are now dealing with the *ex-ante* conditions, we set prices at zero, so that $1-\theta = c_A/(c_A+c_B)$. Therefore, in this case, vertical differentiation amounts in this probability to be equal to one. This is obtained when $J_B = \sigma$. Since, by assumption, J_A is larger than J_B, we obtain again that $J_A \geq \sigma$ is the necessary condition to have vertical differentiation in the model.

Proof of Proposition 3.2

Since the demands are linear and decreasing in their own prices, a Nash equilibrium in pure strategies exists. Assuming that the competitor's price is fixed, with a slight abuse of notation, we write, respectively,

$$\pi_A(p_A) = p_A\left(\frac{p_B+c_B}{c_A+c_B}\right) - \frac{p_A^2}{c_A+c_B}; \tag{29}$$

$$\pi_B(p_B) = p_B\left(\frac{p_A+c_A}{c_A+c_B}\right) - \frac{p_B^2}{c_A+c_B}. \tag{30}$$

Therefore, there are two values for which π_A (π_B resp.) are zero:

$$\begin{aligned} \pi_A(p_A) = 0 \Leftrightarrow p_A = 0 \text{ or } p_A^0 = p_B + c_B; \\ \pi_B(p_B) = 0 \Leftrightarrow p_B = 0 \text{ or } p_B^0 = p_A + c_A. \end{aligned} \tag{31}$$

From (10), we infer that, when $p_A = p_A^0$, then $q = 0$. Therefore, in this case, firm B becomes a monopolist. Conversely, when $p_B = p_B^0$, the market share is $q = 1$, letting firm A be the monopolist. More in details, the *FOCs* read

$$\begin{cases} \dfrac{d\pi_A}{dp_A} = q + p_A\dfrac{dq}{dp_A} = 0 \\ \dfrac{d\pi_B}{dp_B} = 1 - q - p_B\dfrac{dq}{dp_B} = 0 \end{cases} \tag{32}$$

Replacing equation (10) on (32), we obtain

$$p_A = \frac{p_B - J_B + \sigma}{2}; \quad p_B = \frac{p_A - J_A + \sigma}{2}.$$

Hence, critical prices and market share are

$$p_A^* = \frac{3\sigma - J_A - 2J_B}{3}, \quad p_B^* = \frac{3\sigma - 2J_A - J_B}{3}, \tag{33}$$

$$q^* = \frac{3\sigma - J_A - 2J_B}{3(2\sigma - J_A - J_B)}. \tag{34}$$

Thanks to (7), we immediately obtain (14). Moreover, according to equation (34), we obtain the (algebraic) behavior of q^*:

$$q^* < 0 \Leftrightarrow 3\sigma < J_A + 2J_B; \tag{35}$$

$$0 \leq q^* \leq 1 \Leftrightarrow 3\sigma \geq J_A + 2J_B \text{ and } 3\sigma \geq 2J_A + J_B; \tag{36}$$

$$1 < q^* \Leftrightarrow 3\sigma < 2J_A + J_B. \tag{37}$$

When $3\sigma < 2J_A + J_B$ (corresponding to region M), the critical price p_A^* suggested by the *FOC* is not admissible, because $q^* > 1$ is not feasible. Furthermore, region M is the only region where $p_B^* < 0$. Indeed, p_B^* turns out to be negative as soon as $3\sigma < 2J_A + J_B$. Since a negative price (which would correspond to negative profits) is not acceptable, the best strategy for firm B is the *boundary solution*, i.e., to choose $p_B = 0$ (or, equivalently, to be out of the market). Conversely, firm A chooses its most profitable feasible price: since $p_A^* < p_A^M$ is not feasible and $p_A > p_A^M$ is not optimal, the most profitable choice for A is the monopolist price p_A^M. Figure 12 represents π_A under the scenario depicted, respectively, by equation (36) (left panel) and (37) (right panel).[27] The non-shaded areas correspond to admissible regions. Concluding, when $3\sigma < J_B + 2J_A$ ($3\sigma < J_A + 2J_B$), firm B

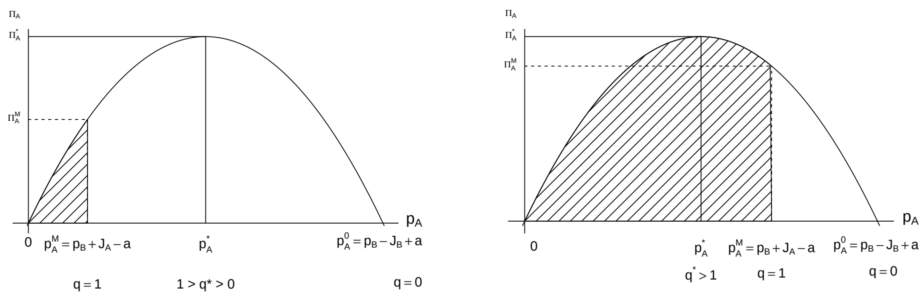

Figure 12: π_A under two different scenarios: when equation (36) holds (left panel) or when (37) holds (right panel). Unfeasible regions are shaded.

(A) steps out of market, while the competitor sets the price p_A^M (p_B^M):

$$\begin{array}{llll} p_A^* = p_A^M = J_A - \sigma & \text{and} \quad p_B^* = 0 & \text{when} & 3\sigma \leq J_B + 2J_A; \\ p_B^* = p_B^M = J_B - \sigma & \text{and} \quad p_A^* = 0 & \text{when} & 3\sigma \leq J_A + 2J_B. \end{array} \tag{38}$$

Consequently, when equation (38) holds, either $q = 1$ or $q = 0$ and the market becomes a monopoly.

[27] With the values of the parameters we are using, the scenario as in equation (35), where firm B becomes the monopolist, is not admissible.

Proof of Lemma 3.1

Assuming a monotonically increasing cumulative distribution function $F(\cdot)$, standard arguments[28] show that an equilibrium point is locally stable if and only if

$$\frac{dF(t_m(q))}{dq} < 1.$$

In our model, the stable equilibria are $q = \{0, 1\}$; in fact, $dF(t_m(q))/dq = 0$, as soon as $q = \{0, 1\}$. Conversely,

$$\frac{dF(t_m(q))}{dq} = \frac{J_A + J_B}{2\sigma} > 1 \quad \text{when} \quad 0 < q < 1.$$

Proof of Proposition 3.3

Although conceptually different, the algebraic form of the maximization problem looks very similar to the one of Proposition 3.2. In this case, the *FOCs* imply

$$p_A = \frac{p_B + J_A - \sigma}{2}; \quad p_B = \frac{p_A + J_B - \sigma}{2}.$$

Hence,

$$p_A^* = \frac{-(2c_A + c_B)}{3}; \quad p_B^* = \frac{-(c_A + 2c_B)}{3}. \tag{39}$$

Finally, by replacing $p_B^* - p_A^* = J_B - J_A/3$ in the intermediate equilibrium of equation (11), we obtain

$$\theta^* = \frac{1}{3} + \frac{c_B}{3(c_A + c_B)}. \tag{40}$$

Thanks to equation (40), we identify two scenarios: in the first one, q^* can be either 1 or 0 with positive probability (corresponding to the case where $0 < \theta^* < 1$); in the second scenario, just one of the two border solutions is admissible. In this latter case, either firm A monopolizes the market, which happens when $E^*[Q] = 1 \Leftrightarrow 1 - \theta^* \geq 1 \Leftrightarrow \theta^* \leq 0$, or firm B monopolizes the market, when $1 - E^*[Q] = 1 \Leftrightarrow \theta^* \geq 1$. Specifically,

$$q^* = 1 \Leftrightarrow c_A + 2c_B \geq 0,$$
$$q^* = 0 \Leftrightarrow 2c_A + c_B \geq 0.$$

Otherwise, $q \in \{0; 1\}$ behaves as a Bernoullian r.v. with parameter $(1 - \theta^*)$.

Similarly as before, when $c_A + 2c_B \geq 0$ ($2c_A + c_B \geq 0$), firm A (B) becomes the monopolist. Moreover,

$$\begin{array}{llll} p_A^* = p_A^M & = J_A - a & \text{and} \quad p_B = 0 & \text{when} \quad c_A + 2c_B \geq 0, \\ p_B^* = p_B^M & = J_B - a & \text{and} \quad p_A = 0 & \text{when} \quad 2c_A + c_B \geq 0. \end{array} \tag{41}$$

[28] See, for instance, Granovetter and Soong (1983) for an application in the context of threshold models.

Proof of Proposition 4.1

To derive all the possible solutions to (4), define

$$\theta = \frac{p_B - p_A + c_B}{c_A + c_B}. \tag{42}$$

Eventually,

i) if $p_B - p_A \leq -c_B$, $q = 0$ is a solution to (4);

ii) if $-c_B < p_B - p_A \leq c_A$, $q = \frac{p_B - p_A + c_B}{c_A + c_B}$ is a solution to (4);

iii) if $p_B - p_A > c_A$, $q = 1$ is a solution to (4).

As a consequence, the self-consistent Nash equilibria of the infinite-player consumer game are as follows:

- under WNE ($c_A + c_B > 0$), q is such that

$$q = \begin{cases} 0 & \text{if} \quad p_B - p_A \leq -c_B \\ \theta & \text{if} \quad -c_B < p_B - p_A \leq c_A \\ 1 & \text{if} \quad p_B - p_A > c_A \end{cases} \tag{43}$$

- under SNE ($c_A + c_B < 0$), q is such that

$$q = \begin{cases} 0 & \text{if} \quad p_B - p_A \leq -c_B \\ \{0; \theta; 1\} & \text{if} \quad -c_B < p_B - p_A \leq c_A \\ 1 & \text{if} \quad p_B - p_A > c_A \end{cases} \tag{44}$$

We now discuss subgame perfect equilibria in prices and market shares; as before, it is convenient to separate the two regimes for network externality.

<u>Weak network effects:</u>

Under WNE, (21) ensures that $c_A > -c_B$. Therefore, the equilibrium level q for the market share is unique. Concerning the supply side of the market, similarly as before, profits are given by

$$\pi_A(p_A) = p_A \left(\frac{p_B + c_B}{c_A + c_B} \right) - \frac{p_A^2}{c_A + c_B}; \quad \pi_B(p_B) = p_B \left(\frac{p_A + c_A}{c_A + c_B} \right) - \frac{p_B^2}{c_A + c_B}.$$

Solving the *FOCs*, we obtain the unique critical point

$$p_A^* = \frac{c_A + 2c_B}{3}; \quad p_B^* = \frac{2c_A + c_B}{3}. \tag{45}$$

Moreover, since $p_B^* - p_A^* = (J_B - J_A)/3$, q^* reads

$$q^* = \frac{c_A + 2c_B}{3(c_A + c_B)}. \tag{46}$$

On the other hand, boundary solutions are

$$\begin{aligned} \pi_A(p_A) = 0 \Leftrightarrow p_A = 0 \text{ or } p_A^M = -c_A; \\ \pi_B(p_B) = 0 \Leftrightarrow p_B = 0 \text{ or } p_B^M = -c_B. \end{aligned} \tag{47}$$

According to (46), we are able to define the regions where the solution is feasible, i.e., $0 \leq q^* \leq 1$, depending on the parameters $(J_A, J_B, \alpha, \beta, \tau, \sigma)$. We have:

$$q^* \leq 0 \Leftrightarrow c_A + 2c_B \leq 0; \tag{48}$$

$$1 \leq q^* \Leftrightarrow 2c_A + c_B \leq 0. \tag{49}$$

$$0 < q^* < 1 \Leftrightarrow \quad \text{otherwise}. \tag{50}$$

Since prices and quantities cannot be negative, firm B will "decide" to step out of market when $2c_A + c_B < 0$, while, on the meantime, the competitor will take the most profitable feasible price p_A^M. Similarly, firm A will do the same when $c_A + 2c_B < 0$ while, on the meantime, the competitor will take the most profitable feasible price p_B^M. Therefore,

- if $2c_A + c_B < 0$, $p_B^* = 0$ and $p_A^* = p_A^M = -c_A$;
- if $c_A + 2c_B < 0$, $p_A^* = 0$ and $p_B^* = p_B^M = -c_B$.

Strong network effects:

When multiple equilibria are in place (region $\mathcal{M}_\theta$), we assume that the prevailing market share Q is a random variable taking value $q = 0$ with probability θ and $q = 1$ with probability $1 - \theta$, where θ is as defined in (42). Under this assumption, firms A and B maximize their expected profits

$$\mathbb{E}(\pi_A) = p_A \cdot \left(\frac{p_A - p_B + c_A}{c_A + c_B} \right); \quad \mathbb{E}(\pi_B) = p_B \cdot \left(\frac{p_B - p_A - c_B}{c_A + c_B} \right). \tag{51}$$

Similarly as before we obtain the following prices:

$$p_A^* = \frac{-2c_A - c_B}{3}; \quad p_B^* = \frac{-c_A - 2c_B}{3}. \tag{52}$$

Substituting in (42), we derive (22). We identify two scenarios: in the first one, q^* can be either 1 or 0 with positive probability (corresponding to the case where $0 <$

$\theta^* < 1$); in the second scenario, just one of the two border solution is admissible. In this latter case, either firm A monopolizes the market, which happens when $E^*[Q] = 1 \Leftrightarrow 1 - \theta^* \geq 1 \Leftrightarrow \theta^* \leq 0$, or firm B monopolizes the market, when $1 - E^*[Q] = 1 \Leftrightarrow \theta^* \geq 1$. More in details,

$$q^* = 0 \Leftrightarrow 2c_A + c_B > 0;$$

$$q^* = 1 \Leftrightarrow c_A + 2c_B > 0.$$

Proof of Proposition 4.2

We need to introduce some notations. For any fixed value of $\beta \in [0, \sigma]$, define

$$\begin{aligned} \alpha_1(\beta) &= \frac{J_A + J_B}{4\tau\sigma} - \beta \\ \alpha_2(\beta) &= \frac{-3\sigma\tau + \sqrt{9\sigma^2\tau^2 - \tau[6\sigma\tau\beta - \tau\beta^2 - 2J_A - J_B]}}{\tau} \\ \alpha_3(\beta) &= \frac{3\sigma\tau - \sqrt{9\sigma^2\tau^2 - \tau[-6\sigma\tau\beta - \tau\beta^2 + J_A + 2J_B]}}{\tau}. \end{aligned} \tag{53}$$

It is not difficult to see that $\alpha = \alpha_1(\beta)$ represents the line separating regions of weak and strong network effects; $\alpha = \alpha_2(\beta)$ represents the curve where $q^*(\alpha, \beta) = 1$ and $\alpha = \alpha_3(\beta)$ the curve where $q^*(\alpha, \beta) = 0$.
Moreover, the curves represented by equations $\alpha = \alpha_2(\beta)$ and $\alpha = \alpha_3(\beta)$ intersect at the point (α, β) such that $\alpha_1(\beta) = \alpha_2(\beta) = \alpha_3(\beta)$. Those equations are used in the next lemma to specify conditions on α to have pairs of locations (α, β) pertaining to the different regions of duopoly and monopoly.

Lemma A.1. *Set*

$$\alpha_{min}(\beta) = \min(\alpha_2(\beta), \alpha_3(\beta)); \quad \alpha_{max}(\beta) = \max(\alpha_2(\beta), \alpha_3(\beta)).$$

Then

$$\alpha_{min}(\beta) \leq \alpha_1(\beta) \leq \alpha_{max}(\beta).$$

Moreover,

1. *If* $\alpha > \alpha_{max}(\beta)$, *then* $(\alpha, \beta) \in \mathcal{D}$;
2. *If* $\alpha_{min}(\beta) \leq \alpha \leq \alpha_{max}(\beta)$, *then* $(\alpha, \beta) \in \mathcal{M}_A \cup \mathcal{M}_B$;
3. *If* $\alpha < \alpha_{min}(\beta)$, $(\alpha, \beta) \in \mathcal{M}_\theta$.

Back to the proof of Proposition 4.2, we look for pairs (α^*, β^*), such that

$$\pi_A(\alpha^*, \beta^*) \geq \pi_A(\alpha, \beta^*), \quad \text{for all } \alpha \in [0, \sigma]\,;$$
$$\pi_B(\beta^*, \alpha^*) \geq \pi_B(\beta, \alpha^*), \quad \text{for all } \beta \in [0, \sigma]\,.$$

We first exclude pairs $(\alpha, \beta) \in \mathcal{M}_A \cup \mathcal{M}_B$. Here, one firm monopolizes the market: either $q^* = 0$ or $q^* = 1$. Regardless which of the two circumstances apply, a location pair (α, β) cannot be a Nash equilibrium; indeed, the eventual loser would play a strategy forcing the opponent to enter either in $\mathcal{D}$ or $\mathcal{M}_\theta$; in this way, it has at least a chance to make some profits.

We now consider the case in which $0 < q^* < 1$. This means that we restrict our attention to strategies located in $\mathcal{D}$. According to Proposition 4.1, subgame perfect prices and market share are given by:

$$p_A^*(\alpha, \beta) = \frac{c_A + 2c_B}{3}, \quad p_B^*(\alpha, \beta) = \frac{2c_A + c_B}{3},$$
$$q^*(\alpha, \beta) = \frac{c_A + 2c_B}{3(c_A + c_B)}.$$

Relying on these expressions, we can write profits in terms of (α, β):

$$\pi_A^*(\alpha, \beta) = \frac{(c_A + 2c_B)^2}{9(c_A + c_B)}; \quad \pi_B^*(\beta, \alpha) = \frac{(2c_A + c_B)^2}{9(c_A + c_B)}.$$

We now concentrate on π_A. On this region, the FOC for π_A reads:

$$\frac{\partial \pi_A^*}{\partial \alpha} = 4\tau q^* \left(\sigma(1 - q^*) - \frac{\alpha^*}{3} \right) = 0 \iff \alpha^* = 3\sigma(1 - q^*), \tag{54}$$

where the equivalence follows since, by assumption, $q^* > 0$.

By (54), it is easy to see that α^* is a local maximum for π_A. Although difficult to prove in closed-form, numerical analysis suggests that it is never reached in the region $\mathcal{D}$. As a consequence, the only candidates for maximality are on the border. By virtue of Lemma A.1, the domain for α to ensure $(\alpha, \beta) \in \mathcal{D}$ is $\alpha_{max}(\beta) < \alpha \leq \sigma$. Therefore, for fixed β, a maximum in this region is reached only if

$$\pi_A^*(\sigma, \beta) \geq \lim_{\alpha \to \alpha_{max}(\beta)} \pi_A^*(\alpha, \beta) \tag{55}$$

and, in this case, the maximizer is $\alpha = \sigma$. Again, it turns out to be difficult to provide an algebraic condition on parameters to satisfy (55). Numerical analysis suggests that both situations are possible. In particular, if (55) fails for each β then there is no equilibrium in region $\mathcal{D}$. Conversely, when (55) holds, then $\alpha = \sigma$ is a local maximum for π_A. Therefore, firm A plays $\alpha^* = \sigma$. For what firm B is concerned, two situations are possible: either $\beta = \sigma$ is the best response and (σ, σ) is a Nash equilibrium, or alternatively it is not possible to reach an equilibrium in $\mathcal{D}$ and we can expect to have at least one in $\mathcal{M}_\theta$.[29]

It remains to discuss the case $(\alpha,\, \beta) \in \mathcal{M}_\theta$. Thanks to Proposition 4.1, we have

$$p_A^* = \frac{-2c_A - c_B}{3}, \quad p_B^* = \frac{-c_A - 2c_B}{3}, \quad \theta^* = \frac{c_A + 2c_B}{3(c_A + c_B)},$$

where θ^* is the probability of firm B to act as the monopolist. Again, we look for a Nash equilibrium in pure strategies. Each firm maximizes its own expected profit with respect to it's location conditioned upon the (optimal) location of the competitor.

$$\mathbb{E}(\pi_A^*(\alpha, \beta)) = -\frac{(-2c_A - c_B)^2}{9(c_A + c_B)}; \;\; \mathbb{E}(\pi_B^*(\beta, \alpha)) = -\frac{(c_A + 2c_B)^2}{9(c_A + c_B)}.$$

The *FOC* for π_A reads:

$$\frac{\partial \pi_A^*}{\partial \alpha} = -\frac{4\tau(1-\theta^*)}{3} \cdot (\sigma\alpha + 3\theta^*),$$

which is evidently negative for any α. A similar result holds for π_B. Therefore, in this situation, only $(0, 0)$ is a plausible equilibrium. Summarizing, only the two symmetric and extreme solutions are candidate locations to become subgame perfect equilibria of the game.

Proof of Proposition 4.3

When $J_A = J_B =: J$, we have $c_A = c_B = 2\sigma\tau(\alpha + \beta) - J := c$. Moreover, if $4\sigma\tau = 1$, it is easy to see that $\pi_A^*(0, 0) = \pi_B^*(0, 0) = J/2$ and $\pi_A^*(\sigma, \sigma) = \pi_B^*(\sigma, \sigma) = |\sigma - J|/2$. The absolute value guarantees the positivity both under WNE and SNE. Having said that, in case $J > \sigma$, we necessarily are under the SNE regime. In this case, we know from Proposition 4.2 that (σ, σ) is strictly dominated by $(0, 0)$ and, therefore, under this circumstance, $(0, 0)$ is the only possible equilibrium. For this reason, we now concentrate on the case where $J < \sigma$, so that $(\sigma, \sigma) \in \mathcal{D}$ and $\pi_A^*(\sigma, \sigma) = (\sigma - J)/2$. Evidently, $\pi_A^*(\sigma, \sigma) > \pi_A^*(0, 0)$ for small values of J. Anyway, to guarantee that (σ, σ) is the only Nash equilibrium for the game described by Table 2, it must be $\pi_A^*(\sigma, 0) < \pi_A^*(\sigma, \sigma)$. To this aim, we compute $\pi_A^*(\sigma, 0)$. We have

$$\pi_A^*(\sigma, 0) = \frac{(5\sigma - 12J)^2}{12^2(\sigma - 2J)}. \tag{56}$$

[29] The same rationale seen for π_A applies to π_B: only $\beta = \sigma$ is a plausible best response, since no internal local maximum for π_B is feasible.

It can be proved that $\pi_A^*(\sigma, \sigma) = \pi_A^*(\sigma, 0)$ if and only if $J = \sigma \cdot (8 - \sqrt{14})/24$. Therefore, $\pi_A^*(\sigma, \sigma)$ is the unique Nash equilibrium if and only if

$$J < J^l = \frac{(8 - \sqrt{14})\sigma}{24}.$$

On the opposite, when J is large enough, the only Nash equilibrium is $(0, 0)$. Uniqueness is broken as soon as $J < J^h$ under which $\pi_A^*(0, 0) = \pi_A^*(0, \sigma)$. It is not difficult to see that the latter condition on profits requires that $(0, \sigma) \in \mathcal{M}_A$. In this case, $\pi_A^*(0, \sigma) = c + \tau\sigma^2$. It turns out that $\pi_A^*(0, \sigma) = \pi_A^*(0, 0)$ if and only if $J = J^h = \sigma/2$. Therefore, $(0, 0)$ is the unique Nash equilibrium if and only if $J > J^h = \sigma/2$, and this completes the proof.

References

Akerlof, G. (1997). Social Distance and Social Decisions. *Econometrica*, 65(5):1005–1028.

Anderson, S. P., Palma, A. D., and Thisse, J. F. (1992). *Discrete Choice Theory of Product Differentiation*. MIT Press.

Becker, G. S. (1974). A Theory of Social Interactions. *Journal of Political Economy*, 82(6):1063–1093.

Belleflamme, P. and Peitz, M. (2015). *Industrial organization: markets and strategies*. Cambridge University Press.

Brock, W. A. and Durlauf, S. N. (2001). Discrete Choice with Social Interactions. *The Review of Economic Studies*, 68(2):235–260.

Dai Pra, P., Sartori, E., and Tolotti, M. (2013). Strategic Interaction in Trend-Driven Dynamics. *Journal of Statistical Physics*, 152:724–741.

d'Aspremont, C., Gabszewicz, J., and Thisse, J. (1979). On Hotelling's "Stability in Competition". *Econometrica*, 47(5):1145–50.

Downs, A. (1957). *An Economic Theory of Democracy*. Harper. Google-Books-ID: kLEGAAAAMAAJ.

Enelow, J. M. and Hinich, M. J. (1984). *The Spatial Theory of Voting: An Introduction*. CUP Archive. Google-Books-ID: lXY6AAAAIAAJ.

Glaeser, E. L., Sacerdote, B., and Scheinkman, J. A. (1996). Crime and Social Interactions. *The Quarterly Journal of Economics*, 111(2):507–548.

Granovetter, M. (1978). Threshold Models of Collective Behavior. *American Journal of Sociology*, 83:1420–1443.

Granovetter, M. and Soong, R. (1983). Threshold models of diffusion and collective behavior. *The Journal of Mathematical Sociology*, 9(3):165–179.

Grilo, I., Shy, O., and Thisse, J. (2001). Price competition when consumer behavior is characterized by conformity or vanity. *Journal of Public Economics*, 80(3):385–408.

Hotelling, H. (1929). Stability in Competition. *The Economic Journal*, 39(153):41–57.

Kalai, E. (2004). Large Robust Games. *Econometrica*, 72(6):1631–1665.

Katz, M. and Shapiro, C. (1985). Network Externalities, Competition, and Compatibility. *American Economic Review*, 75(3):424–40.

Lamberson, P. and Page, S. E. (2018). First mover or higher quality? optimal product strategy in markets with positive feedbacks. *Journal of Economics & Management Strategy*, 27(1):40–52.

Reimann, M., Castao, R., Zaichkowsky, J., and Bechara, A. (2012). How we relate to brands: Psychological and neurophysiological insights into consumerbrand relationships. *Journal of Consumer Psychology*, 22(1):128–142.

Schelling, T. C. (1971). Dynamic models of segregation. *The Journal of Mathematical Sociology*, 1(2):143–186.

Singh, N. and Vives, X. (1984). Price and Quantity Competition in a Differentiated Duopoly. *The RAND Journal of Economics*, 15(4):546–554.

Wirl, F. (2008). Tragedy of the Commons in a Stochastic Game of a Stock Externality. *Journal of Public Economic Theory*, 10(1):99–124.

Wood, W. and Hayes, T. (2012). Social Influence on consumer decisions: Motives, modes, and consequences. *Journal of Consumer Psychology*, 22(3):324–328.