Survival in Speculative Markets*

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Abstract

In this paper, I consider an exchange economy with complete markets where agents have heterogeneous beliefs and, possibly, preferences, and investigate the Market Selection Hypothesis that speculation rewards the agent with the most accurate beliefs. First, on the methodological level, I derive the relative consumption dynamics as a function of agents' effective discount factors, related to consumption decisions across time, and agents' effective beliefs, related to consumption decisions across states. Sufficient conditions for agents' survival, either in isolation or in a group, depend on the relative size of effective discount factors and on the relative accuracy of effective beliefs. Then, I show that in economies where agents maximize an Epstein-Zin utility the Market Selection Hypothesis fails: there exist parametrizations where the agent with correct beliefs vanishes and parametrizations where beliefs heterogeneity persists in the long run. Results are robust to local changes of beliefs, risk preferences, and the aggregate endowment process. These failures are shown not to occur when agents' Epstein-Zin utility has a subjective expected utility representation due to an interdependence of effective discount factors and effective beliefs.

Keywords: Heterogeneous Beliefs; Speculation; Market Selection Hypothesis; Asset Pricing; Optimal Growth Portfolio; Epstein-Zin Utility.
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1 Introduction

The dominant academic view of financial markets is that they facilitate hedging and risk diversification. A complementary view is that trade also occurs due to investors' disagreement about the return distribution of assets. Indeed, also in standard models of financial markets such as Lucas' model or the CAPM, agents' beliefs heterogeneity makes them willing to hold risky positions different from those they would have held under pure hedging. These positions are speculative, in that they include a bet on future realizations of assets' fundamentals.

Although speculative motives are certainly present, a widespread position of financial economists is that speculation can be only a transient characteristic of markets. The Market Selection Hypothesis (MSH) of Friedman (1953) applied to financial markets presumes that investors with accurate beliefs can earn high returns by taking positions against investors with inaccurate beliefs. These speculative positions should thus allow accurate traders to dominate and bring asset prices at the fundamental value implied by their beliefs. Indeed, when markets are complete, each agent can freely trade to allocate her future consumption on the paths which she believes more likely. In equilibrium, everything else being equal, the agent with the most accurate beliefs assigns the highest likelihood to paths that are actually realized and, thus, should hold everything in the long run. In bounded economies with time-separable Subjective Expected Utility (SEU) maximizers, the argument is rigorously established by Sandroni (2000) and Blume and Easley (2006). However, despite the importance of the result, the exact nature of the role of risk and time preferences in its validity is still unclear. Moreover, a number of contributions show that outside the SEU framework with bounded aggregate endowment the MSH may fail.[1]

In this paper, I investigate the effect of speculation on agents' relative consumption dynamics by identifying the separate roles of optimal consumption decisions across time and states. The contribution is twofold. Firstly, on the methodological side, I derive the relative consumption dynamics as a function of agents' effective discount factors, related to consumption decisions across time, and agents' effective beliefs, related to consumption decisions across states. Sufficient conditions for agents' survival, either in isolation or in a group, depend on the relative size of a survival index that takes into account the size of the effective discount factor and the accuracy of effective beliefs. As the intuition above suggests, the agents with the most correct beliefs are still those who, assigning the highest likelihood to paths that are actually realized, do hold everything in the long run. However, the beliefs that matter are the effective ones, and those are endogenous, in that, depending on risk preferences, they also incorporate equilibrium prices.

Secondly, and related to the MSH, I show that when, for at least one agent, consumption decisions across time and states are not interdependent, e.g. when there is at least an agent who maximizes a recursive Epstein-Zin utility, the MSH fails: there exist economies where agents with heterogeneous beliefs survive in the long run, or others where the agent with correct beliefs vanishes on a set of paths with full measure, even when risk and time preferences are homogeneous. Path dependency, with an agent dominating on some paths

[1] With complete markets, failures have been shown to emerge in models with exogenous saving, unbounded endowment and heterogeneous preferences, ambiguity aversion, recursive preferences, collateral constraints, and endogenous beliefs; I refer to the literature review section for details.

and another on others is also a possible long-run outcome. Results are robust to local changes of beliefs, risk preferences, and the aggregate endowment process. These failures are shown not to occur when the Epstein-Zin utility has a subjective expected utility representation, due to the interdependence of consumption decisions across time (effective discount factors) and states (effective beliefs).

After having introduced the general set-up (Section 2), the methodological contribution is developed in Section 3 where I focus on 2-agent economies. The full analysis of I-agent economies proceeds along the same lines but, being slightly heavier on the formal side, is postponed to Section 5.

In Section 3, I show that the dynamics of consumption shares depends on two key quantities: the ratio between the current value of next period consumption and current consumption, which accounts for optimality across time, and the ratio between the current value of next period consumption in each state and the current value of next period total consumption, which accounts for optimality across states. For an SEU maximizer with log Bernoulli utility, the first ratio (time dimension) is equal to her discount factor while the second ratio (states dimension) is equal to her beliefs. More generally, these ratios, which can be derived from the Euler equations, depend also on equilibrium state prices and time/risk preferences. Yet, they can be interpreted as an effective discount factor (for the time dimension) and effective beliefs (for the states dimension). In fact, they lead to the same consumption decision of an SEU-log agent who uses them, respectively, as discount factor and beliefs.

Sufficient conditions for an agent to survive, dominate, or vanish can be written in terms of the log of effective discount factors and accuracy of effective beliefs. I name the sum of these two terms the generalized survival index. The size of the effective discount factor matters for survival because it is related to saving and depends, other than on the discount factor, on the intertemporal elasticity of substitution (IES) and on the perceived mispricing. A unitary IES implies that perceived mispricing has no effect on saving, since income and substitution effect compensate each other. Larger (smaller) IES encourages (depresses) saving in the presence of perceived mispricing. The larger the mispricing, the larger the effect.

The accuracy of effective beliefs matters as well because it is related to the agent portfolio expected log-returns and is shown to depend both on the accuracy of beliefs and on an endogenous component that I name Non-Log-Optimality (NLO) term. Unless for agents with unitary relative risk aversion (RRA), who hold log-optimal portfolios, the NLO term is endogenous (it depends, through risk preferences, on state prices) and measures the relative accuracy of effective beliefs and beliefs. An agent with correct beliefs and non-unitary RRA has always a negative NLO term, because effective beliefs are less accurate than beliefs. An agent with non-correct beliefs may have instead a positive (negative) NLO term when beliefs and preferences imply effective beliefs that are closer (further away) to the truth than beliefs. A typical case of a positive NLO term is when optimism and risk aversion compensate each-others. Failures of the MSH depend on the interplay of this NLO contribution with the belief dependent component of the effective discount factor.

Examples of failure of the MSH are presented in Section 4, where I consider specific parametrizations of Epstein-Zin recursive utility. The examples provided are aimed to shed light on the separate role of effective discount factors (decision across time, saving) and effective beliefs (decision across states, portfolios) about MSH failures.

A parametrization which is particularly useful to clarify why the MSH can fail has all agents with unitary IES and equal discount factors, so that effective discount factors are homogeneous, but with different effective beliefs. In economies where all agents have RRA higher than 1 and there is aggregate risk, optimism may compensate risk aversion and the agent with non-correct beliefs may dominate due a high NLO term (Section 4.1). However, if optimism is too strong the outcome is long-run heterogeneity because each agent has a high NLO term, and also more accurate effective beliefs, at the state prices determined by the other agent. I provide examples of long-run heterogeneity in 2-agent economies in Section 4.2. The intuition for 2-agent economies works also for I-agent economies as shown in the examples of Appendix A.3.

Path dependency occurs instead when both agents have an RRA lower than 1 and the non correct agent is a pessimist, because each agent typically has a negative NLO term, and also less accurate effective beliefs, at the prices determined by the other agent. I provide examples of path dependency in 2-agent economies in Section 4.3.

Finally, in Section 4.4, I consider examples where all agents maximize an SEU-CRRA utility. As established by the incumbent literature, provided preferences are homogeneous, the agent with correct beliefs dominates. My contribution is to show that the result holds because, the IES coefficient being the inverse of the RRA coefficient, the endogeneity of effective beliefs (portfolios) and of effective discount factors (saving) compensate each other and what matters is only the accuracy of beliefs and the size of discount factors. In these economies, it is enough to slightly change the preferences of one agent, so that her RRA is not equal to the inverse of her IES, to obtain MSH failures such as long-run heterogeneity.

Section 5 extends the methodological contributions to I-agent economies. Appendix A presents further examples showing how failures are robust to local changes of beliefs, risk preferences, and the aggregate endowment process. Appendices B-D collect the proofs. In Appendix E, I compute effective discount factors and beliefs for general time-separable utility and, possibly, a non-linear probability weighting function. In the next section, I discuss the relation between my results and the incumbent literature.

1.1 Related Literature

Investors speculate when they take long and/or short positions that they would not have otherwise taken had they agreed on the underlying state process.[2] A number of contributions investigate the short-term effect of speculation on asset prices and the volume of trade (see e.g Varian, 1985, 1989; Harris and Raviv, 1993; Kandel and Pearson, 1995) or the relation between speculation and financial innovations (see e.g. Zapatero, 1998; Brock et al., 2009; Simsek, 2013). Here, I am instead interested in whether speculation can also have long-run consequences. My conclusion is that, also in the idealized framework of general equilibrium with complete markets, unless all agents' optimal decisions across

[2]The term speculation refers also to the purchase of an asset for the purpose of re-selling it at a higher price to those who will value it more; see Harrison and Kreps (1978) for a formal model. See also Morris (1996) and Scheinkman and Xiong (2003).

time and states are interdependent, speculation may be a persistent feature of financial markets. The result could help to explain stock market anomalies as recently suggested in Anderson et al. (2005), Hong and Stein (2007), Cogley and Sargent (2009), Yu (2011), Bhamra and Uppal (2014), Hong and Sraer (2016), Baker et al. (2016), and Bottazzi et al. (2018b).

The relation between speculation and the MSH for financial economies has received increasing attention since the works of DeLong et al. (1990, 1991) and Blume and Easley (1992). DeLong et al. (1991) find that noise traders, i.e. traders with inaccurate beliefs, might survive or even dominate against rational traders by bearing more risk. The analysis is based on a partial equilibrium model. Blume and Easley (1992) study the same issue in a temporary equilibrium model and find that, controlling for the saving rate, when the trader with the most accurate beliefs purchases a log-optimal portfolio, she gains all the wealth in the long run and brings asset prices to reflect her beliefs. However, when the trader with the most accurate beliefs does not use the log-optimal rule, Blume and Easley are able to derive conditions for this trader to vanish.[3]

Subsequent work by Sandroni (2000) and by Blume and Easley (2006) extend the analysis to general equilibrium models with endogenous saving. Under the assumptions of markets' completeness, bounded aggregate endowment, and SEU maximization with time-separable payoff, the MSH holds: provided that all agents discount future utility at the same rate, only the trader with the most accurate beliefs dominates. Results are established by comparing a survival index that depends solely on the size of discount factors and beliefs accuracy.

A related contribution is Yan (2008), where the MSH is investigated in a continuous-time economy where the aggregate endowment follows a Brownian motion and agents have CRRA preferences. Agents agree on the volatility of the aggregate endowment process but disagree on its drift. The findings by Sandroni (2000) and Blume and Easley (2006) are confirmed: provided preferences are homogeneous the accurate trader dominates.[4,5]

By finding failures of the MSH even in a general equilibrium framework, my results provide rational foundations for the findings of the earlier studies by DeLong et al. (1991) and Blume and Easley (1992). Moreover, with respect to the general equilibrium literature, I show how the use of effective discount factors and effective beliefs allows the Euler equations to be rewritten as consumption dynamics, somehow similarly to the original approach in Blume and Easley (1992) for agents' wealth dynamics. Effective discount factors and effective beliefs, which can be defined also for groups of agents, disentangle the role of consumption decisions across time and states toward survival.

Other influential works investigate the MSH in economies without intermediate con-

[3]The result provides a support for the growth-optimal portfolio derived by Kelly (1956) in equilibrium models. Blume and Easley (1992) work in an i.i.d. economy with Arrow securities. The studies surveyed in Evstigneev et al. (2009) propose a generalization of the Kelly rule for more complicated asset structures.

[4]In general, the survival index takes also into account the IES/RRA coefficient. If the aggregate endowment drift is positive, agents with a low RRA (high IES) save more and have a survival advantage.

[5]Other studies by Mailath and Sandroni (2003), Sandroni (2005), Jouini and Napp (2006, 2007), Cvitanić et al. (2012), Muraviev (2013), Bhamra and Uppal (2014), and Massari (2015, 2016) consider related cases. The main conclusion still holds, the only possible failure of the MSH is the vanishing of the accurate trader.

sumption; see in particular Kogan et al. (2006, 2017) and Cvitanić and Malamud (2010, 2011). Results in the two types of economies – with or without intermediate consumption – are known to differ due to the effect of saving in the former. Having clarified that decisions across time are not only important per se, but also for the fact that they might or might not compensate the endogenous NLO term of effective beliefs accuracy, my work reconciles the findings of the two literatures regarding the role of decisions across states.

In analyzing the MSH in Epstein-Zin economies, this paper is also related to Borovička (2019), who investigates the MSH in continuous-time exchange economies with two agents having homogeneous Epstein-Zin preferences. My results, in particular that MSH failures are possible and robust to changes in beliefs and risk preferences, confirm his findings. On the methodological side the two papers are, however, quite different. Borovička solves directly the central planner problem and characterizes agents' (general equilibrium) optimal policies exploiting the partial equilibrium limit of one agent being alone in the economy.[6] I study long-run survival in economies with possibly more than two agents and provide sufficient conditions that can be used beyond the Epstein-Zin utility case. The generalized survival index and its decomposition in terms of the size of the effective discount factor and accuracy of effective beliefs holds in general.

Another related contribution is Easley and Yang (2015) where long-run outcomes are computed in a 2-agent economy with an Epstein-Zin investor and a loss-averse investor. Consistently with my results, it is shown that the loss-averse investor vanishes because her portfolio is further away from the log-optimal one. Moreover, the loss-averse agent can survive and dominate only when she saves more than the Epstein-Zin investor.

Within the market selection literature, other studies find long-run beliefs heterogeneity. Beker and Chattopadhyay (2010) and Cogley et al. (2013) focus on 2-agent economies with incomplete markets. Beker and Espino (2011) highlights the importance of learning. Cao (2018) studies an economy where markets are endogenously incomplete due to portfolio and collateral constraints. Guerdjikova and Sciubba (2015) study economies where investors are ambiguity averse. Bottazzi and Dindo (2014) and Bottazzi et al. (2018a) extend the temporary equilibrium analysis of Blume and Easley (1992) to general asset structures, short-lived and long-lived respectively, and possibly incomplete markets. Dindo and Massari (2017) present a model with collective learning where long-run heterogeneity emerges because agents use equilibrium prices to update their beliefs, making them endogenous.

2 The Economy

In this section, I introduce a rather general model of an exchange economy with complete markets and heterogeneous agents and provide the definition of survival, dominance, and vanishing.

Time begins at date $t = 0$ and is indexed by $t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. $\mathcal{S} = \{1, 2, \ldots, S\}$ is the set of states of the world that can occur at each date $t > 0$ and $s_t \in \mathcal{S}$ denotes the state

[6] This strategy allows Borovička to derive market-selection outcomes for a larger region of Epstein-Zin utility parameters than I do here. Whether this approach is also possible in discrete-time economies and in economies with more than two agents is still an open issue.

realized in t. $\Sigma = \times_{t=0}^{\infty}\mathcal{S}$ is the set of paths $\sigma = (s_1, s_2 \dots)$ and $\sigma_t = (s_0, s_1, \dots, s_t) \in \Sigma_t$ is the history till period t (a node in the tree representation). $C(\sigma_t)$ is the cylinder set with base σ_t, $C(\sigma_t) = \{\sigma \in \Sigma | \, \sigma = (\sigma_t, \dots)\}$ and $\mathcal{F}_t$ the σ-algebra generated by the cylinders, $\mathcal{F}_t = \sigma\left(\{C(\sigma_t) \, \forall \, \sigma_t \in \Sigma_t\}\right)$. $\mathcal{F}$ is the σ-algebra generated by the union of $\mathcal{F}$, $\mathcal{F} = \sigma\left(\cup_t \mathcal{F}_t\right)$. By construction $(\mathcal{F}_t)_{t\in\mathbb{N}_0}$ is a filtration. P is a probability measure on $(\Sigma, (\mathcal{F}_t))$ and $(\Sigma, (\mathcal{F}_t), \mathrm{P})$ is the probability triple underlying the economy. Given P, $\mathrm{P}(\sigma_t)$ is the probability of the cylinder with base σ_t and $\mathrm{P}(\sigma_{t+\tau}|\sigma_t)$ is the conditional probability, defined in the usual way, provided $\sigma_{t+\tau}$ has past σ_t till t. To shorten the notation, I denote $\mathrm{P}_{s_{t+1},t}(\sigma_t) = \mathrm{P}(\sigma_{t+1}|\sigma_t)$. All stochastic processes are adapted to the filtration $(\mathcal{F}_t)$. The dependence on paths and histories may not be explicit.

The economy contains I agents and a single consumption good in each node. The consumption good in $t = 0$ is the numéraire. For each path σ, each trader $i \in \mathcal{I} = \{1, 2, \dots, I\}$ is endowed with a consumption stream $(e_t^i(\sigma))_{(t,\sigma)\in\mathbb{N}_0\times\Sigma}$. The aggregate endowment is $(e_t(\sigma))_{(t,\sigma)}$ and $g_{s_{t+1},t}(\sigma_t) = e_{t+1}(\sigma_{t+1})/e_t(\sigma_t)$ is the endowment growth rate between σ_t and $\sigma_{t+1} = (\sigma_t, s_{t+1})$.

In date $t = 0$, each agent i chooses the level of consumption in each path in order to maximize a given utility of her present and future consumption. Agent i utility is evaluated using subjective beliefs Q^i, a probability measure on $(\Sigma, (\mathcal{F}_t))$.

Agents may transfer their endowment across time and states by trading all possible contingent commodities in time zero or by trading assets in a complete market.[7] The date-0 price of consumption in node σ_t is $q(\sigma_t)$ and, assuming positive prices,

$$q_{s_{t+1},t}(\sigma_t) = \frac{q(\sigma_{t+1})}{q(\sigma_t)}$$

is the price of one unit of consumption in node $\sigma_{t+1} = (\sigma_t, s_{t+1})$ relatively to the price of one unit of consumption in node σ_t. The date-t one-period-ahead discount factor is $\delta_t = \sum_{s\in\mathcal{S}} q_{s,t}$. $r_t = \frac{1}{\delta_t}$ is the corresponding one-period interest rate. The vector of risk neutral probabilities (normalized state prices) in (t, σ) is $\mathrm{Q}_t = r_t q_t \in \Delta_+^S$.

For the moment, we do not pose any restriction on beliefs, preferences, or on the aggregate endowment process. However, we assume that there exists a competitive equilibrium with strictly positive equilibrium consumption c_t^i for all (t, σ) and for all i and strictly positive date $t = 0$ state prices $(q(\sigma_t))_{(t,\sigma)}$.[8]

We shall be interested in agents' consumption dynamics. Since aggregate consumption can be unbounded or converge to zero, we focus on the relative consumption $\phi_t = \frac{c_t^i}{\sum_{j\in\mathcal{I}} c_t^j}$. Following the incumbent literature, we define:

Definition 1 (Survival, dominance, and vanishing). *Agent i survives on σ when* $\limsup_{t\to\infty} \phi_t^i(\sigma) > 0$, *she vanishes on σ when* $\lim_{t\to\infty} \phi_t^i(\sigma) = 0$, *she dominates on σ when* $\lim_{t\to\infty} \phi_t^i(\sigma) = 1$.

[7]Other than assuming that the market is complete, I do not further specify its structure. Depending on the chosen asset structure, relevant assumptions on the asset positions should be taken to guarantee the existence of an equilibrium; see Appendix C.3.

[8]We shall come back to the issue of the existence of competitive equilibria for the examples of Section 4 in Appendix C.3.

Next, we show how the relative consumption process can be studied in terms of agents' consumption decisions across time and states.

3 Relative Consumption Dynamics

Our objective is to study the relative consumption process and to provide conditions, as well as an underlying intuition, for an agent to survive, dominate, or vanish. In this section, I concentrate on 2-agent economies and prepare the ground for the key examples of MSH failure in Section 4. The analysis of I-agent economies is postponed to Section 5.

I start by characterizing an agent's consumption process in terms of her effective discount factor, related to consumption decisions across time, and her effective beliefs, related to consumption decisions across states. An analogy with log-economies provides the intuition behind the terminology. Then, in Section 3.1, I define an agent generalized survival index, a function of effective discount factors of effective beliefs' accuracy, and show how it can be used to find the drift of the relative consumption process. Section 3.2 provides the economic interpretation of the generalized survival index in terms of saving and portfolio expected log-return. Finally, in Section 3.3, I use the generalized survival index to establish sufficient conditions for survival, dominance, and vanishing.

The analysis of the relative consumption process is particularly simple when $I = 2$ and both agents $i = 1, 2$ are Subjective Expected Utility (SEU) maximizers with Bernoulli utility $u(c_t^i) = \log(c_t^i)$. Market selection outcomes of these economies are known, at least since the seminal work of Blume and Easley (1992). We start our investigation by recalling this case because it is instructive for the subsequent generalization to other preferences, here and in Section 4, and to more than 2 agents, in Section 5.

Name β_t^i agent i discount factor in date t. From agent i Euler equations in the generic node σ_t, i.e.

$$\frac{\beta_t^i \, \mathrm{Q}_{s_{t+1},t}^i \, u'(c_{t+1}^i)}{u'(c_t^i)} = \delta_t \, \mathrm{Q}_{s_{t+1},t} \quad \text{for all } s_{t+1} \in \mathcal{S},$$

we can derive her consumption dynamics as

$$c_{t+1}^i(\sigma_t, s_{t+1}) = \frac{\beta_t^i}{\delta_t} \frac{\mathrm{Q}_{s_{t+1},t}^i}{\mathrm{Q}_{s_{t+1},t}} c_t^i \quad \text{for all } s_{t+1} \in \mathcal{S}. \tag{1}$$

Using the above for both agents $i = 1, 2$, the relative consumption dynamics reads

$$\frac{c_{t+1}^1}{c_{t+1}^2} = \frac{\beta_t^1}{\beta_t^2} \frac{\mathrm{Q}_{s_{t+1},t}^1}{\mathrm{Q}_{s_{t+1},t}^2} \frac{c_t^1}{c_t^2} \quad \text{for all } s_{t+1} \in \mathcal{S}.$$

Defining $z_t = \log \frac{c_t^1}{c_t^2} = \log \frac{\phi_t^1}{1-\phi_t^1}$, so that $z_t \to \infty \Leftrightarrow \phi_t^1 \to 1$, the former gives

$$z_{t+1} = z_t + \epsilon_{t+1} \tag{2}$$

with

$$\epsilon_{t+1}(\sigma_t, s_{t+1}) = \log \frac{\beta_t^1}{\beta_t^2} + \log \frac{\mathrm{Q}_{s_{t+1},t}^1}{\mathrm{Q}_{s_{t+1},t}^2} \quad \text{for all } s_{t+1} \in \mathcal{S}. \tag{3}$$

The expected relative consumption of agent 1 increases (decreases) if the conditional drift of the process $(z_t)_{(t,\sigma)}$ is positive (negative). The latter is

$$\mathrm{E}_{\mathrm{P}}[\epsilon_{t+1}|\,\mathcal{F}_t] = \log \beta_t^1 - \log \beta_t^2 + \left(I_{\mathrm{P}_t}(\mathrm{Q}_t^2) - I_{\mathrm{P}_t}(\mathrm{Q}_t^1)\right) \,,$$

where

$$I_{\mathrm{P}}(\mathrm{Q}) = \sum_{s\in\mathcal{S}} \mathrm{P}_s \log \frac{\mathrm{P}_s}{\mathrm{Q}_s} \,,$$

is the relative entropy of Q with respect to P (also Kullback-Leibler divergence).[9]

An agent's relative consumption increases or decreases depending on the size of her discount factor and on the accuracy of her conditional beliefs. The result is rather intuitive. In an intertemporal model, both consumption decisions across time and across states matter for survival. The date-t discount factor matters because, under log-optimality, it determines the ratio between the present value of next period consumption and current consumption:

$$\beta_t^i = \delta_t \frac{\sum_{s\in\mathcal{S}} \mathrm{Q}_{s,t}\; c_{t+1}^i(\sigma_t, s)}{c_t^i}. \tag{4}$$

The agent with the highest discount factor puts the largest weight on next period consumption, thus increasing her relative consumption.

The date-t conditional beliefs of state s_{t+1} matters because under, log-optimality, it determines the present value of next period consumption in s_{t+1} relative to the present value of consumption in all states:

$$\mathrm{Q}^i_{s_{t+1},t} = \frac{\mathrm{Q}_{s_{t+1},t}\; c_{t+1}^i(\sigma_t, s_{t+1})}{\sum_{s'\in\mathcal{S}} \mathrm{Q}_{s',t}\; c_{t+1}^i(\sigma_t, s')} \,, \quad \text{for all } s_{t+1} \in \mathcal{S}\,. \tag{5}$$

Beliefs are thus related to the portfolio decision of agent i and $I_{\mathrm{P}_t}(\mathrm{Q}_t)$ is a contribution to the expected log-return of agent i's portfolio (see Section 3.2 for a precise derivation). The more accurate the beliefs, the smaller $I_{\mathrm{P}_t}(\mathrm{Q}_t)$, and the larger the associated portfolio expected return. In relative terms, the agent with the most accurate beliefs has the largest expected log-return and thus a positive contribution toward relative consumption growth.[10]

The same analysis of the relative consumption process can be generalized beyond the SEU-log case by characterizing agents' consumption decisions across time as effective discount factors, a generalization of (4), and consumption decisions across states as effective beliefs, a generalization of (5).

Definition 2 (Effective discount factors and beliefs). *If agent i equilibrium consumption is a strictly positive adapted process, then for all (t, σ) her effective discount factor $\delta_t^i > 0$ is*

$$\delta_t^i := \frac{\delta_t \sum_{s\in\mathcal{S}} \mathrm{Q}_{s,t}\; c_{t+1}^i(\sigma_t, s)}{c_t^i} \,, \tag{6}$$

[9] The relative entropy is uniquely minimized at Q = P, strictly convex, and, although not a distance, is used as a measure of beliefs accuracy: the larger it is, the further away the distribution Q is from P.

[10] It is the expected log-return, rather than the expected return, that matters because the consumption process (1) is multiplicative rather than additive.

and her effective beliefs $\alpha_t^i \in \Delta_+^S$ *are*

$$\alpha^i_{s_{t+1},t} := \frac{\mathrm{Q}_{s_{t+1},t}\; c^i_{t+1}(\sigma_t, s_{t+1})}{\sum_{s'\in\mathcal{S}} \mathrm{Q}_{s',t}\; c^i_{t+1}(\sigma_t, s')}\,, \quad \textit{for all } s_{t+1} \in \mathcal{S}\,. \tag{7}$$

Trivially, when agent i is of SEU-log type, the optimality condition (1) implies $\delta_t^i = \beta_t^i$, the agent discount factor, and $\alpha_t^i = \mathrm{Q}_t^i$, the agent belief. More generally, effective discount factors and beliefs (δ_t^i, α_t^i) can be derived from the date-t Euler equations of the utility maximization problem of agent i and shall depend, other than on an agent's characteristics (beliefs, discount factor, time and risk preference), also on endogenous market variables such as contemporaneous state prices (δ_t, Q_t). In this case, agent i optimal consumption decisions between t and $t+1$ are the same as those taken by a SEU-log agent having a discount factor $\beta_t^i = \delta_t^i$ and beliefs $\mathrm{Q}_t^i = \alpha_t^i$, justifying the name of effective discount factor and effective beliefs. Using (δ_t^i, α_t^i) the log-economy consumption dynamics in (1) generalizes to

$$c^i_{t+1}(\sigma_t, s_{t+1}) = \frac{\delta_t^i}{\delta_t}\frac{\alpha^i_{s_{t+1},t}}{\mathrm{Q}_{s_{t+1},t}} c_t^i \quad \text{for all } s_{t+1} \in \mathcal{S}. \tag{8}$$

In Section 4, we shall derive effective discount factors and beliefs from the Euler equations of special cases of Epstein-Zin recursive utility (including the SEU-CRRA case). Agent i's decision across time shall depend on her discount factor and beliefs as well as on the market discount factor and normalized state prices, $\delta_t^i = \delta^i(\beta_t^i, \mathrm{Q}_t^i, \delta_t, \mathrm{Q}_t)$. The decision across states shall depend on beliefs and normalized state prices $\alpha_t^i = \alpha^i(\mathrm{Q}_t^i, \mathrm{Q}_t)$.[11]

3.1 Generalized Survival Index

As for the SEU-log case, the consumption dynamics in (8) can be used to study the relative consumption ratio $z_t = \log \frac{c_t^1}{c_t^2}$. Repeating here the analysis of the drift of (z_t) reveals that agent i's relative consumption increases or decreases, in expectation, depending on the sign of

$$\mathrm{E}_{\mathrm{P}}[\epsilon_{t+1} | \mathcal{F}_t] = \log \delta_t^1 - \log \delta_t^2 + \left(I_{\mathrm{P}_t}(\alpha_t^2) - I_{\mathrm{P}_t}(\alpha_t^1)\right), \tag{9}$$

that is, on the relative size of effective discount factors and on the relative accuracy of effective beliefs.

To characterize the sign of the conditional drift we define agent (conditional) *generalized survival index.*

Definition 3 (Generalized survival index). *Given agent i's effective discount factor* δ_t^i *and effective beliefs* α_t^i*, her (conditional) generalized survival index in node* σ_t *is*

$$k_t^i := \log \delta_t^i - I_{\mathrm{P}_t}(\alpha_t^i)\,. \tag{10}$$

[11]Writing the consumption dynamics in terms of effective discount factors and beliefs, as in (8), can be extended to cases where they are a function of market and individual variables (including consumption) up to date t; see Appendix E for the general SEU case and for a case with non-linear probability weighting function. Effective discount factors and beliefs might also be derived in models without inter-temporal maximization from exogenously given saving and portfolio rules, in the spirit of temporary equilibrium models à la Blume and Easley (1992); see Appendix B.

Using (9) and (10) the conditional drift of the relative consumption process (z_t) is

$$\mathrm{E_P}[\epsilon_{t+1} | \mathcal{F}_t] = k_t^1 - k_t^2 \,. \tag{11}$$

If agent i has a higher survival index than agent j, the conditional drift of the relative consumption process is in its favor and, in expectation, her relative consumption increases.

The proposed survival index (10) has three advantages over those defined by the incumbent literature. First, it shows the separate effects of consumption decisions across time, encoded in δ_t^i, and consumption decision across states, encoded in α_t^i. Second, it is agent-specific in that, even though it may depend on market prices (δ_t, Q_t), it does not directly depend on the characteristics of other agents or of the economy. For example, for given market prices, the generalized survival index of a SEU-CRRA agent is invariant to whether she is trading in a market with another SEU-CRRA agent or in a market with an Epstein-Zin agent. It is also invariant to whether the economy is bounded or unbounded.[12] Third, it can also be defined for groups, rather than only for individuals, as we shall see in Section 5.

The generalized survival index has a particularly simple form when agent i is of the SEU-log type. Given that $\delta_t^i = \beta_t^i$ and $\alpha_t^i = \mathrm{Q}_t^i$, we have $k_t^i = \log \beta_t^i - I_{P_t}(\mathrm{Q}_t^i)$. More generally, the date-t generalized survival index depends, through effective discount factors and beliefs, also on market equilibrium prices (δ_t, Q_t). Before stating sufficient conditions for survival and dominance in terms of survival indexes, the next section provides a deconstruction to aid their interpretation.

3.2 Interpreting the generalized survival index

Comparing agents' generalized survival indexes in node σ_t amounts to the ability to establish which agent has a larger expected conditional consumption log-growth rate in that node. Both effective discount factors, and thus decisions across time, and effective beliefs, decisions across states, determine the outcome. In this section, I provide an economic interpretation for the separate role of each component of the generalized survival index.

The importance of consumption decisions across time is evident from the log of effective discount factors δ_t^i and has already been discussed. It is intuitive that if in node σ_t agent i has a larger effective discount factor than agent j then, at the equilibrium interest rates, it postpones consumption more than all the other agents. Thus, the growth of its relative consumption from node σ_t to all states of the next date has a positive contribution.

The importance of consumption decisions across states is encoded in the relative entropy, and thus accuracy, of effective beliefs. In our context it measures the expected log-return of the portfolio whose payoff is next period consumption. The return of agent i portfolio with consumption decision between date t and $t+1$ if state s occurs is

$$r_{s,t}^i = \frac{c_{t+1}^i(\sigma_t, s)}{\delta_t \sum_{s' \in \mathcal{S}} \mathrm{Q}_{s',t}\, c_{t+1}^i(\sigma_t, s')} \,.$$

[12]Example $A1$ in Appendix A.1 clarifies this point. In bounded economies with time separable preferences that satisfy the Inada condition and fixed beliefs, the incumbent literature defines agent i survival index as $\log \beta^i - I_P(\mathrm{Q}^i)$, see Blume and Easley (2006) for details. The survival index changes when the economy is unbounded; see Yan (2008). See also Muraviev (2013) for other preferences.

The latter can be rewritten using agent i effective belief as

$$r^i_{s,t} = \frac{1}{\delta_t}\frac{\alpha^i_{s,t}}{\mathrm{Q}_{s,t}}\,.$$

Agent i expected log-return in date t is thus

$$\mathrm{E}_{\mathrm{P}}[\log r^i_t | \mathcal{F}_t] = \log r_t + \underbrace{I_P(\mathrm{Q}_t) - I_P(\alpha^i_t)}_{\mu^i_t}\,. \tag{12}$$

The component μ^i_t is the expected log-return in excess of the log risk-free rate and can thus be named as agent i *growth premium* in node σ_t. The accuracy of effective beliefs, $I_P(\alpha^i_t)$, matters for survival because it measures the size of the growth premium: the more accurate effective beliefs, the higher the growth premium.

To separate the effect of beliefs and preferences on the growth premium, it is instructive to decompose it as

$$\mu^i_t = \underbrace{[I_P(\mathrm{Q}_t) - I_P(\mathrm{Q}^i_t)]}_{\text{Log-optimality term}} + \underbrace{[I_P(\mathrm{Q}^i_t) - I_P(\alpha^i_t)]}_{\text{Non-Log-Optimality term}}\,. \tag{13}$$

The first part is the growth premium of an SEU-log agent with beliefs Q^i_t and is thus named the Log-optimality term. Its size depends on beliefs accuracy, correct beliefs leading to the highest possible Log-optimality term. The second part is a Non-Log-Optimality (NLO) term that takes into account the difference between the accuracy of beliefs Q^i and of effective beliefs α^i_t. It captures the effect of risk preferences. It is zero when the agent risk preferences are as with log utility, since in this case effective beliefs α^i and beliefs Q^i coincide. Otherwise, the NLO contribution measures whether agent i is better off or worse off, in terms of expected log-returns and, thus, survival, by using a non-log optimal portfolio rather than the log-optimal portfolio derived under her beliefs. If agent i beliefs are correct, the NLO term is negative, since i would have been better off, as far as her growth premium is concerned, under log-optimality. Effective beliefs are less accurate than beliefs. However, when the beliefs of agent i are not correct, effective beliefs could be more accurate than beliefs, leading to a positive NLO contribution. NLO terms measure the compensation between risk preferences and beliefs in relation to expected consumption growth.

Using growth premia, the difference of survival indexes, and thus the drift (11) of the relative consumption process can be written as

$$k^1_t - k^2_t = \log \delta^1_t - \log \delta^2_t + \mu^1_t - \mu^2_t. \tag{14}$$

where

$$\mu^1_t - \mu^2_t = \underbrace{\{I_P(Q^2_t) - I_P(Q^1_t)\}}_{\text{difference of agents' beliefs accuracy}} + \underbrace{\{[I_P(Q^1_t) - I_P(\alpha^1_t)] - [I_P(Q^2_t) - I_P(\alpha^2_t)]\}}_{\text{difference of agents' NLO terms}}. \tag{15}$$

Other than on the comparison of effective discount factors, the expected value of the relative growth depends both on the relative accuracy of beliefs and on the relative size of NLO terms.

3.3 Sufficient conditions for survival in 2-agent economies

Sufficient conditions for survival and dominance can be given in terms of differences of generalized survival indexes. In this paper, I provide two sets of conditions, both are rather intuitive. In this section, I present both sets of conditions when $I = 2$, in a rather informal way. These conditions shall be used in the examples of Section 4. The formal analysis is postponed to Section 5, where we consider sufficient conditions for general I-agent economies.

Given the link between the relative size of generalized survival indexes and the conditional drift of the relative consumption process (z_t), the first set of conditions is straightforward:

$$k^1_t > k^2_t \quad \text{for all} \quad (t,\sigma) \quad \Rightarrow \quad \text{agent 1 dominates P-a.s.,} \tag{16}$$

$$k^1_t < k^2_t \quad \text{for all} \quad (t,\sigma) \quad \Rightarrow \quad \text{agent 1 vanishes P-a.s..} \tag{17}$$

If the generalized survival index of agent i is always the highest, then, in expectation, agent i gains relative consumption. A low-of-large-number argument then implies that agent i relative consumption does converge to 1 P-a.s..[13] Note that even if an agent has the highest survival index for all possible market conditions, and thus dominates, the relative importance of consumption decision across time and states can change over time. We shall encounter such cases in the next section.

The former sufficient conditions are too coarse. Generalized survival index are endogenous and it can well happen that their relative ordering depends on market conditions. In such cases it is enough to appraise the relative size of survival indexes at the market conditions set by either agent as representative agent. Name $k^i|_j$ the generalized survival index of agent i computed at the market conditions when agent j consumes all the endowment and is the representative agent. We can establish the following:

$$k^1|_1 > k^2|_1 \quad \text{and} \quad k^1|_2 > k^2|_2 \quad \Rightarrow \quad \text{agent 1 dominates P-a.s.;} \tag{18}$$

$$k^1|_1 < k^2|_1 \quad \text{and} \quad k^1|_2 < k^2|_2 \quad \Rightarrow \quad \text{agent 1 vanishes P-a.s.;} \tag{19}$$

$$k^1|_1 < k^2|_1 \quad \text{and} \quad k^1|_2 > k^2|_2 \quad \Rightarrow \quad \text{both agents survive P-a.s.;} \tag{20}$$

$$k^1|_1 > k^2|_1 \quad \text{and} \quad k^1|_2 < k^2|_2 \quad \Rightarrow \quad \text{P-a.s. either agent 1 or 2 dominates.} \tag{21}$$

The first two conditions state when agent 1 dominates or vanishes. The conditions are stronger than the corresponding ones in (16-17) in that in order to dominate agent i must have the highest survival index only for those market conditions when either i or j is the representative agent.

Conditions (20-21) lead to other long-run outcomes. When agent i has the highest index at the market conditions set by j, then she cannot vanish. If both agents fulfill this requirement the long-run outcome is long-run heterogeneity. If instead agent i has the highest index at the market conditions set by herself, then she may dominate. If both agents fulfill this requirement, the long-run outcome is dominance of an agent and path-dependency. For almost all path, depending on the initial endowment, either agent 1 or agent 2 dominates.[14]

[13] Conditions (16-17) are based on Theorem 1 in Section 5; refer to the proof of the theorem for details.

[14] This is the approach followed, e.g., by Borovička (2019) for 2-agent economies and by Bottazzi and

4 Survival in Epstein-Zin Economies

We shall apply the sufficient conditions established in the previous section to certain parametrization of economies with 2 agents having Epstein-Zin preferences. We have two objectives. First, we show that outside the SEU framework there exists a variety of long-run outcomes: dominance of the trader with non-correct beliefs (even when discount factors and inter-temporal preferences are homogeneous), persistent heterogeneity, and path dependency. Notably, all these outcomes are possible even if the most accurate agent knows the truth. Second, we shed light on the separate role of effective discount factors, and thus decisions across time, and effective beliefs, and thus decisions across states, towards survival.

We shall restrict the analysis to economies where the aggregate endowment follows an i.i.d. process, beliefs are fixed, and Epstein-Zin utilities are such that we can explicit date-t effective discount factors and beliefs as a function of date-t equilibrium prices (δ_t, Q_t). Let us start from the aggregate endowment process.

Assumption 1 (**Aggregate endowment process).** *For all* (t, σ)*, the aggregate endowment process is such that* $g_{s_{t+1},t} = g_s$ *for all* $s \in \mathcal{S}$*, for a given vector* $g \in (-1, +\infty)^S$*, and* $\mathrm{P}(C(\sigma_t)) = \prod_{\tau=1}^{t} \rho_{s_\tau}$ *for a given measure* $\rho = (\rho_1, \ldots, \rho_S)$ *on* $(\mathcal{S}, 2^{\mathcal{S}})$*.*

With an abuse of notation, we shall use P to denote both the measure on $(\Sigma, (\mathcal{F}_t))$ and on $(\mathcal{S}, 2^{\mathcal{S}})$ so that, under Ass. 1, $\mathrm{P}(C(\sigma_t)) = \prod_{\tau=1}^{t} \mathrm{P}_{s_\tau}$. Next, we shall assume that all agents believe states to be i.i.d., that beliefs are absolutely continuous with respect to each other and the truth, and that beliefs are heterogeneous.

Assumption 2 (**Agent beliefs).** *For all agents* $i \in \mathcal{I}$*, beliefs on* $(\Sigma, (\mathcal{F}_t))$ *are generated by constant beliefs* Q^i *on* $(\mathcal{S}, 2^{\mathcal{S}})$ *with* $\mathrm{Q}^i_s > 0 \Leftrightarrow \mathrm{P}_s > 0$ *for all* $s \in \mathcal{S}$*. Moreover,* $\mathrm{Q}^i \neq \mathrm{Q}^j$ *for all* i *and* j *in* $\mathcal{I}$*.*

As with Assumption 1, the i.i.d. part of Assumption 2 is not key for the results as long as agents' disagreement persists in the long run.[15] The absolute continuity assumption is instead needed for the existence of an equilibrium; see Appendix C.3.

Regarding preferences, we shall assume that each agent $i \in \mathcal{I}$ has Epstein-Zin preferences and maximizes a recursive utility of the type

$$U^i_t = \left((1-\beta^i) c_t^{1-\rho^i} + \beta^i \left(\mathrm{E}_{\mathrm{Q}^i}[(U^i_{t+1})^{1-\gamma^i}] \right)^{\frac{1-\rho^i}{1-\gamma^i}} \right)^{\frac{1}{1-\rho^i}}, \; t \in \mathbb{N}_0, \tag{22}$$

where $\beta^i \in (0,1)$ is the discount factor, $\gamma^i \in (0, \infty)$ is the coefficient of Relative Risk Aversion (RRA), $\rho^i \in (0, \infty)$ is the inverse of the coefficient of Inter-temporal Elasticity of Substitution (IES) on a deterministic consumption path. The utility is defined also for

Dindo (2014) and Bottazzi et al. (2018a) for I-agent economies. See Theorems 2 and 3 in Section 5 for details.

[15]Technically, assuming an i.i.d. process and fixed beliefs implies that conditional expectations, and thus generalized survival indexes k^i_t as in (10), do not depend on s_t, widening the range of applicability of the sufficient conditions in Section 3.3 (see also the discussion at the end of Appendix A.3).

$\gamma^i = 1$ and $\rho^i = 1$ by taking the appropriate limits; see also Epstein and Zin (1989). If $\rho^i = \gamma^i$, then agent i utility is of the SEU-CRRA type.

In order to derive effective discount factors and beliefs, we shall restrict ourselves to the following parametrizations.

Assumption 3 (Agent preferences). *Each agent $i = 1, 2$ is of SEU-CRRA type, $\rho^i = \gamma^i$, or has unitary IES, $\rho^i = 1$.*

A well-studied case is when all agents are of the SEU-CRRA type (see Blume and Easley, 2006; Yan, 2008, for the bounded economy and unbounded economy case, respectively). In these economies one agent dominates.

Under Assumptions 1-3, effective beliefs and effective discount factors are as follows.

Proposition 1 (Effective discount factors and beliefs with Epstein-Zin utility). *Under Assumptions 1-3, for all (t, σ) agent i effective discount factors and effective beliefs are given by time independent functions of prices (δ_t, Q_t). In particular, both when $\rho^i = \gamma^i$ and when $\rho^i = 1$ effective beliefs are*

$$\alpha^i_{s,t} = \alpha^i_s(\mathrm{Q}_t) = \frac{(\mathrm{Q}^i_s)^{\frac{1}{\gamma^i}} (\mathrm{Q}_{s,t})^{1-\frac{1}{\gamma^i}}}{\sum_{s' \in S} (\mathrm{Q}^i_{s'})^{\frac{1}{\gamma^i}} (\mathrm{Q}_{s',t})^{1-\frac{1}{\gamma^i}}}\,, \quad s = 1, \ldots, S\,. \tag{23}$$

When $\rho^i = \gamma^i$ the effective discount factor is

$$\delta^i_t = \delta^i(\delta_t, \mathrm{Q}_t) = (\beta^i)^{\frac{1}{\gamma^i}} (\delta_t)^{1-\frac{1}{\gamma^i}} \left(\sum_{s' \in S} (\mathrm{Q}^i_{s'})^{\frac{1}{\gamma^i}} (\mathrm{Q}_{s',t})^{1-\frac{1}{\gamma^i}} \right)\,, \tag{24}$$

whereas when $\rho^i = 1$ the effective discount factor is $\delta^i_t = \beta^i$.[16]

Agent i effective beliefs are a geometric average of her beliefs Q^i and risk neutral probabilities Q, with weights $1/\gamma^i$ and $1 - 1/\gamma^i$ respectively. A very risk averse agent, γ^i very large, wants to bear little risk and thus puts a high weight on risk neutral probabilities, yielding a portfolio of next period consumption close to the risk-free one achieved when $\alpha^i_t = \mathrm{Q}_t$. An agent with unitary RRA has effective beliefs equal to beliefs, $\alpha^i_t = \mathrm{Q}^i$, as the SEU-log type. An agent close to risk neutrality, $\gamma^i \approx 0$, is indifferent when $\mathrm{Q}_t \approx \mathrm{Q}^i_t$, but otherwise moves consumption to those states with the highest ratio $\mathrm{Q}^i_{s,t} \,/\, \mathrm{Q}_{s,t}$. The left panel of Figure 1 gives a representation of three such types of effective beliefs in an economy with two states, $S = 2$, as a function of state 1 risk neutral price $\mathrm{Q}_1 = \mathrm{Q}$.
Beliefs and risk neutral probabilities have an effect also on effective discount factors, provided the IES coefficient is not one. The effective discount factor of a CRRA agent, and thus saving, depends on the product of two terms. The first is the geometric average of agent specific discount factor with the economy discount factor. The second is the sum of the geometric averages of beliefs and risk neutral probabilities, and it is equal to the

[16]Under Assumptions 2-3, each agent discount factor and beliefs are constant. For this reason we have omitted β^i and Q^i from the argument of effective discount factors and beliefs, otherwise $\delta^i_t = \delta^i(\beta^i_t, \mathrm{Q}^i_t, \delta_t, \mathrm{Q}_t)$ and $\alpha^i_t = \alpha^i(\mathrm{Q}^i_t, \mathrm{Q}_t)$.

normalization factor of effective beliefs. All these averages have weights $1/\gamma^i$ on the individual component and $1-1/\gamma^i$ on the market component. A high (low) RRA coefficient γ^i, and thus a low (high) IES coefficient $1/\gamma^i$, implies that the disagreement of agent i with respect to risk neutral probabilities decreases (increases) her effective discount factor. Disagreement, and thus the perceived excess earnings due to contingent consumption mispricing, let the agent decrease (increase) her saving depending on her IES.[17] The right panel of Figure 1 shows the component of agent i effective discount factor in σ_t that depends on risk neutral probabilities and beliefs:

$$\Delta^i(\mathrm{Q}_t) := \sum_{s'\in S} (\mathrm{Q}^i_{s'})^{\frac{1}{\gamma^i}} (\mathrm{Q}_{s',t})^{1-\frac{1}{\gamma^i}} . \tag{25}$$

A unitary IES coefficient implies $\Delta^i(\mathrm{Q}_t) = 1$. In accordance with the argument outlined above, higher IES (lower IES) implies $\Delta^i(\mathrm{Q}_t) > 1$ ($\Delta^i(\mathrm{Q}_t) < 1$) when risk neutral prices are far from agent i beliefs. Importantly, $\Delta^i(\mathrm{Q}_t)$ is also the normalization factor of agent i effective beliefs, providing an important interdependence between effective beliefs and effective discount factors.

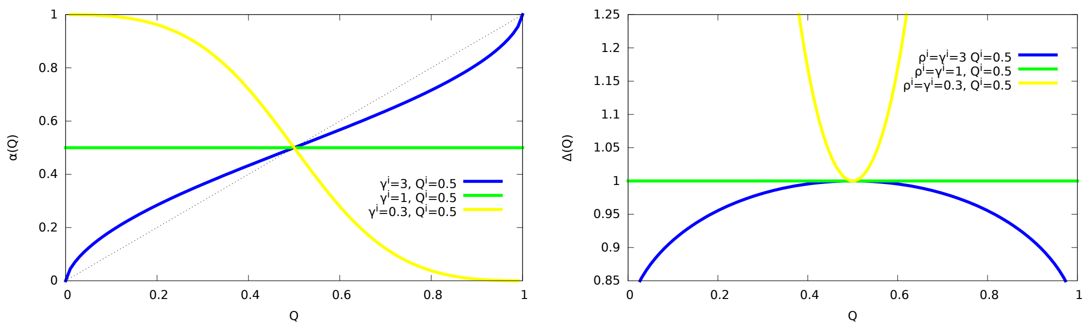

Figure 1: Left panel: agent i effective beliefs $\alpha^i(\mathrm{Q})$ as in (23) for different values of RRA γ^i when $\mathrm{Q}^i_1 = \mathrm{Q} = 0.5$. Right panel: beliefs dependent component of effective discount factors, $\Delta^i(\mathrm{Q})$ in (25), for different values of IES $1/\rho^i = 1/\gamma^i$ when $\mathrm{Q}^i = 0.5$.

Next, we apply the sufficient conditions sketched in Section 3.3 to investigate the long-run outcomes in a 2-agent 2-state economy under Assumptions 1-3. In all examples, agent 1 has correct beliefs, agent 2 has non-correct beliefs, agents have the same discount factors and are of unitary IES type. In most cases there is aggregate risk, $g_1 > g_2$, and we name the agent with highest beliefs for state 1 an optimist.

In Section 4.1, we consider an economy where agent 1, despite having correct beliefs and the same time and risk preferences of agent 2, vanishes P-a.s.. In Section 4.2, we provide an example where the outcome is long-run heterogeneity. In Section 4.3, we provide an example whereby, for almost all path, either agent 1 or agent 2 dominates, depending on the initial endowment.

In all examples above, if we make agents of SEU-CRRA type, by changing their IES parameter to match the inverse of the RRA, then the agent with correct beliefs dominates. In Section 4.4, I shall shed light on the origin of this dominance.

[17] This is the same as the "saving channel" discussed in Borovička (2019).

More examples and a robustness analysis in Appendix A generalize all these failures of the MSH, encompassing also the case of no aggregate risk.[18]

4.1 Vanishing of the correct agent

Example 1 We consider an economy with 2 states with $g^1 = 2g^2 = 2$, and 2 agents with unitary IES, the same RRA $\gamma^1 = \gamma^2 = 2$, and homogeneous discount factor β. Agent 1 has correct beliefs $\mathrm{Q}^1 = \mathrm{P} = 0.5$ while agent 2 is a mild optimist with $\mathrm{Q}^2 = 0.7$.[19] As stated in Proposition 1, a unitary IES coefficient and homogeneous discount factors imply that agents use the same effective discount factors, $\delta_t^1 = \delta_t^2 = \beta$. Heterogeneity of state-1 beliefs implies instead heterogeneous effective beliefs, as shown in the left panel of Figure 2 together with agents' exogenous beliefs. Long-run survival depends only on the relative accuracy of effective beliefs, that is, on the relative size of growth premia.

Assigning a higher probability to the state with highest growth implies that agent-2 effective belief lies above agent-1 effective belief (left panel of Figure 2). The accuracy of effective beliefs depends on equilibrium prices. The latter are found by imposing the marking clearing condition $\sum_{i=1,2} c_t^i = e_t$, leading to the implicit equation

$$\frac{\mathrm{Q}_t\, g^1}{\mathrm{Q}_t\, g^1 + (1-\mathrm{Q}_t)g^2} = \alpha^1(\mathrm{Q}_t)\phi_t^1 + \alpha^2(\mathrm{Q}_t)\phi_t^2. \tag{26}$$

The left-hand site is the normalized value in date t of the endowment in state $s_{t+1} = 1$ (normalized supply). The right hand site is the normalized value in date t of the demand for consumption in $s_{t+1} = 1$. Note that latter is a convex combination of effective beliefs with weights given by consumption shares.[20] Equilibrium risk neutral probabilities can be appraised graphically. In particular, risk neutral prices are found at the intersection of the normalized supply (black line, l.h.s. of (26)) with a convex combination of agent effective beliefs (r.h.s. of (26)). The intersection of agent i effective belief α^i in (23) with the normalized supply determines instead the equilibrium risk neutral probability in an economy where agent i is the representative agent, $\mathrm{Q}\,|_i$ (the two vertical lines in the plot). Having twice as much growth in the second state implies $\mathrm{Q}\,|_i < \mathrm{Q}^i$ for both agents $i = 1, 2$, in accordance with the risk aversion assumption.

For each equilibrium Q, the accuracy of effective beliefs, $I_\mathrm{P}(\alpha_t^i(\mathrm{Q}))$, can be directly visualized using the Euclidean distance between the effective beliefs and the truth P. In fact, when $\mathrm{P} = 0.5$ the relative entropy $I_\mathrm{P}(\mathrm{Q})$ is symmetric around its minimum $\mathrm{Q} = \mathrm{P}$, $I_\mathrm{P}(\mathrm{Q}) = I_\mathrm{P}(1-\mathrm{Q})$ and $I_\mathrm{P}(\alpha^i(\mathrm{Q})) \gtreqless I_\mathrm{P}(\alpha^j(\mathrm{Q}))$ if and only if $|\alpha^i(\mathrm{Q}) - 0.5| \gtreqless |\alpha^j(\mathrm{Q}) - 0.5|$.

As is evident from the plot, despite having less accurate beliefs, agent 2 has more accurate effective beliefs for all $\mathrm{Q} \in [\mathrm{Q}\,|_1, \mathrm{Q}\,|_2]$. The results is due to the NLO terms introduced in Section (13). Agent 2 optimism and high risk aversion (as compared to the

[18]In Appendix C.3 we provide conditions for the existence of an interior competitive general equilibrium for all the examples considered in this section. The applicability of Theorems 1-3 rests on the specific assumption for the relative consumption process (z_t) found in the same Appendix.

[19]Since there are only two states, we specify only beliefs and risk neutral probabilities of the first state.

[20]For a quick intuition of this result consider the analogy with a log-economy, provided effective beliefs are interpreted as beliefs. The step-by-step derivation is in Appendix C.1.

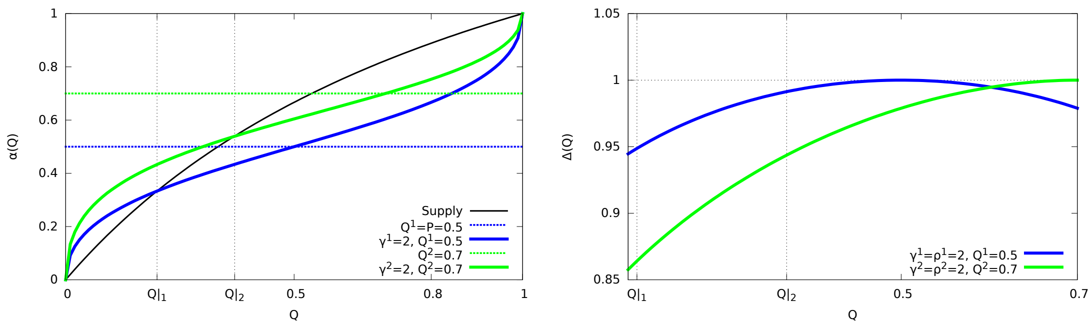

Figure 2: Left panel: Effective beliefs in a 2-agent 2-state unitary-IES economy with risk aversion $\gamma = 2$ and slight optimism. Right panel: beliefs dependent components of the effective discount factors of the corresponding CRRA economy with $\rho = \gamma = 2$.

$\gamma = 1$ case) compensate each-other and lead to a high NLO term. The opposite occurs for agent 1. Having correct beliefs, agent 1 has a negative NLO term because $\gamma^1 \neq 1$. The difference of NLO terms is larger than the inaccuracy of beliefs, resulting in a larger growth premium, and thus survival index, for agent 2 than for agent 1:

$$k_t^2 = -I_P(\alpha^2(\mathrm{Q}_t)) > k_t^1 = -I_P(\alpha^1(\mathrm{Q}_t)) \quad \text{for all} \quad (t, \sigma).$$

The sufficient condition (17) implies that the correct agent vanishes.[21]

The example can be only partially extended to economies without aggregate risk, $g_1 = g_2$. In fact, in such economies fair pricing implies $\mathrm{Q}\,|_1 = \alpha^i(\mathrm{Q}\,|_1) \Leftrightarrow \mathrm{Q}\,|_1 = \mathrm{Q}^1 = P$, so that agent 1 has the highest survival index premium at the prices where she is the representative agent:

$$k^1|_1 = -I_P(\alpha^1(\mathrm{Q}\,|_1)) = 0 > k^2|_1 = -I_P(\alpha^2(\mathrm{Q}\,|_1))\,.$$

Agent 1 cannot vanish P-a.s.. for all initial endowments. Under no-aggregate risk, vanishing of the correct agent can still occur but, but depending on the path, only for specific initial endowments, as we shall show at the end of Section 4.3.

4.2 Long-run heterogeneity

Example 2 Let us consider the same economy as in the previous section, 2-state with $g_1 = 2g_2 = 2$ and two agents with unitary IES coefficient and $\gamma^1 = \gamma^2 = \gamma = 2$. While agent 1 still has correct beliefs, $\mathrm{Q}^1 = P = 0.5$, agent 2 is more optimistic than before and assigns probability $\mathrm{Q}^2 = 0.8$ to the high growth state.

Heterogeneity of beliefs implies heterogeneity of effective beliefs, as shown in the left panel of Figure 3 together with agents' exogenous beliefs. Long-run survival depends only on the relative accuracy of effective beliefs, that is, on the relative size of growth premia.

[21]The example is close in spirit to Theorem 5.4 of Blume and Easley (1992). Figure 7 in Appendix A.2 shows that the result is robust to changing both agents beliefs, unless agent 2 becomes too optimistic (as discussed in the next example).

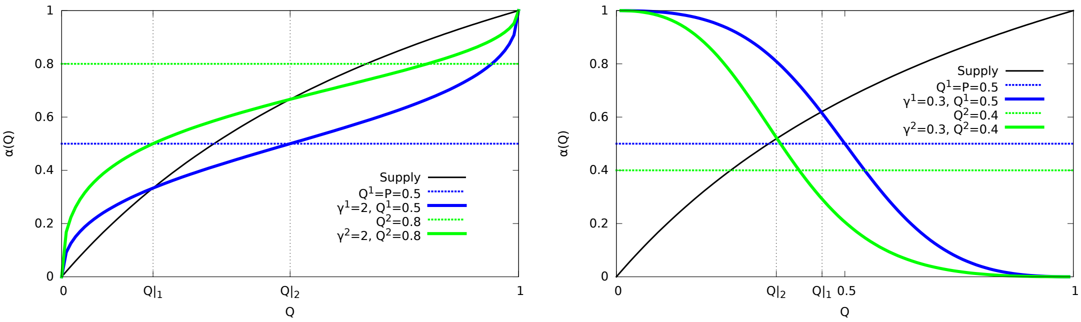

Figure 3: Effective beliefs as a function of state 1 risk neutral probability in a 2-agent 2-state economy. Left panel: low RRA. Right panel: high RRA.

For equilibrium prices close to $\mathrm{Q}\,|_1$, the interplay of beliefs and preferences is such that agent 2, despite having less accurate beliefs, has more accurate effective beliefs and thus a higher growth premium than the correct agent. The role of the NLO term is decisive, as in the previous example.

The example is built so that there are also risk neutral probabilities where agent 1 has more accurate effective beliefs. Agent 2's optimism is too strong and, close to "her" risk neutral probability $\mathrm{Q}\,|_2$, the NLO term (still positive) does not compensate for the inaccuracy of her beliefs. At these prices agent 1 NLO term is close to zero (the left plot of Fig. 3 shows that $\alpha^1(\mathrm{Q}\,|_2) \approx \mathrm{Q}^1 = \mathrm{P}$) and agent 1 has nearly correct effective beliefs.

We have established that each agent has more accurate effective beliefs, and thus a higher growth premium, at the risk neutral probability set by the other agent:

$$I_{\mathrm{P}}(\alpha^2(\mathrm{Q}\,|_1)) < I_{\mathrm{P}}(\alpha^1(\mathrm{Q}\,|_1)) \quad \text{and} \quad I_{\mathrm{P}}(\alpha^1(\mathrm{Q}\,|_2)) < I_{\mathrm{P}}(\alpha^2(\mathrm{Q}\,|_2)).$$

The latter and homogeneity of effective discount factors, $\delta_t^1 = \delta_t^2 = \beta$, lead to

$$k^1|_1 < k^2|_1 \quad \text{and} \quad k^1|_2 > k^2|_2.$$

The sufficient condition (20) implies that both agents survive P-a.s.. Equilibrium consumption shares and risk neutral probabilities keep fluctuating, as shown in the two panels of Figure 4. Moreover, although agents have the same effective discount factor, the presence of aggregate risk and the endogenous fluctuations of risk neutral probabilities cause equilibrium discount rates to fluctuate as well (higher time series in the right panel).[22]

The key elements of the example are the fact that agent 2 is both an optimist and has $\gamma^2 > 1$, so that beliefs and risk aversion compensate each other when agent 1 is the representative agent. The result is robust to small changes of beliefs and preferences. Note, however, that the long-run outcome would change if agent 2's optimism were milder (leading to vanishing on the correct agent, as in the previous example) or if agent 2 were a pessimist (leading to dominance of the correct agent). Figure 7 in Appendix A.2 shows how these findings are robust to a change of both agents' beliefs.

[22] Although agents' effective beliefs keep changing, the fact that both agents survive and have the same effective discount factor implies that they have the same average accuracy (and thus average growth premium); see Corollary 1.

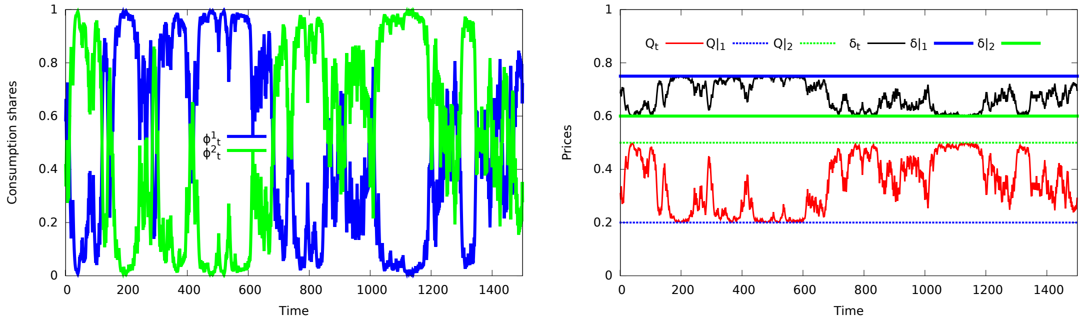

Figure 4: Market dynamics in an economy with homogeneous effective discount factors and heterogeneous effective beliefs as in the left panel of Fig 3. Left panel: relative consumption shares dynamics when the initial consumption share distribution is $\phi_0 = (0.5, 0.5)$. Right panel: risk neutral probabilities and discount rates dynamics ($\beta = 0.9$).

The example cannot be extended to economies without aggregate endowment, unless we assume that the most accurate agent does not the truth.[23] In fact, if the correct agent knows the truth and there is no aggregate risk, fair pricing implies that the correct agent has correct effective beliefs when she dominates, $\mathrm{Q}\,|_1 = \alpha^1(\mathrm{Q}\,|_1) \Leftrightarrow \mathrm{Q}\,|_1 = \mathrm{Q}^1 = \mathrm{P}$, so that agent 2 cannot have more accurate effective beliefs when agent 1 dominates. We come back to possible MSH failures with no aggregate risk case in Appendix A.2.

4.3 Path dependency

Example 3 The right panel of Figure 3 considers an economy as in the previous examples, with the important difference that agents have a low risk aversion, $\gamma^1 = \gamma^2 = \gamma = 0.3$. Agent 1 has correct beliefs, while agent 2 is pessimistic and has belief $\mathrm{Q}^2 = 0.4$.

When Q_t is close to $\mathrm{Q}\,|_1 > \mathrm{Q}\,|_2$, agent 2 considers consumption in state 1 as too expensive and, due to her low risk aversion, has a particularly low effective belief $\alpha^2(\mathrm{Q}_t)$. Agent 2's NLO term is negative and, together with the inaccuracy of her beliefs, they are lower than the NLO term of the correct agent (itself also negative).

When Q_t is close to $\mathrm{Q}\,|_2 < \mathrm{Q}\,|_1$, it is agent 1 who, facing a low price for consumption in state 1 and having low risk aversion, invests heavily in that state making $\alpha^1(\mathrm{Q}_t)$ particularly high and leading to a negative NLO tern. Moreover, in this case agent 2 NLO term is positive, since pessimism and low risk aversion $\gamma^2 < 1$, compensate each other in presence of aggregate risk. As a result, agent 2 has the largest growth premium.

I have established that each agent has the most accurate effective beliefs (highest growth premium) at the risk neutral probabilities that would prevail if she were the representative agent:

$$I_{\mathrm{P}}(\alpha^1(\mathrm{Q}\,|_1)) < I_{\mathrm{P}}(\alpha^2(\mathrm{Q}\,|_1)) \quad \text{and} \quad I_{\mathrm{P}}(\alpha^2(\mathrm{Q}\,|_2)) < I_{\mathrm{P}}(\alpha^1(\mathrm{Q}\,|_2)).$$

[23] Alternatively, we should also assume that she knows the truth but computes expected values using a probability weighting function, see the second part of Appendix E.

The latter and homogeneity of effective discount factors, $\delta_t^1 = \delta_t^2 = \beta$, lead to

$$k^1|_1 > k^2|_1 \quad \text{and} \quad k^2|_2 > k^1|_2.$$

The sufficient condition (21) implies that, depending on the path σ, either agent can dominate.

The key elements of the example are the fact that agent 2 is pessimistic and has $\gamma^2 < 1$, so that beliefs and risk aversion compensate each other when she is the representative agent. The result is robust to small changes of beliefs and preferences. Note, however, that the long-run outcome would change if agent 2's pessimism were milder (leading to vanishing of the correct agent), or if agent 2 were a mild optimist (leading to dominance of the correct agent). Figure 7 in Appendix A.2 shows how these findings are robust to a change of both agents' beliefs. Path dependency can also occur under no aggregate risk, see Figure 8 in the same appendix.

4.4 SEU-CRRA case

In all the examples just considered, the long-run outcome is at odds with the findings of the incumbent literature for exchange economies with SEU-CRRA agents where, provided preferences and discount factors are homogeneous, the agent with correct beliefs dominates. To shed light on the differences between the two settings, in what follows I modify the examples above along one dimension only: both agents' IES coefficients change from being unitary to the SEU-CRRA case where $\rho^1 = \gamma^1$ and $\rho^2 = \gamma^2$.

Example 4 Let us consider the example in Section 4.1 first. The analysis of effective beliefs is unchanged; they are such that the correct agent has the lowest growth premium for all possible equilibrium prices Q. In the SEU-CRRA setting, the latter might not be enough for agent 2 to dominate as the overall long-run outcomes depend also on effective discount factors, whose belief dependent components $\Delta^1(\mathrm{Q})$ and $\Delta^2(\mathrm{Q})$ are shown in the right panel of Figure 2. For prices $\mathrm{Q} \in [\mathrm{Q}\,|_1, \mathrm{Q}\,|_2]$, agent 1 has a larger effective discount factor (right panel of the same figure). In fact, while the speculative saving channel pushes saving down for both agents, for both IES $0.5 < 1$, the effect is stronger for agent 2 as equilibrium prices are further away from her beliefs than from those of agent 2.

While consumption decisions across states are "in favor" of agent 2, consumption decisions across time are "in favor" of agent 1. A precise examination of their difference is needed to determine long-run outcomes. The contribution of effective beliefs to the difference of generalized survival index is

$$I_{\mathrm{P}}(\alpha_t^2) - I_{\mathrm{P}}(\alpha_t^1) = \left(\frac{1}{\gamma}\right)\left(I_P(Q^2) - I_P(Q^1)\right) + \log\frac{\Delta^2(\mathrm{Q}_t)}{\Delta^1(\mathrm{Q}_t)}\,,$$

where the endogenous part comes from the difference of NLO terms. The contribution of effective discount factors is instead

$$\log\delta_t^1 - \log\delta_t^2 = \log\frac{\Delta^1(\mathrm{Q}_t)}{\Delta^2(\mathrm{Q}_t)}\,.$$

As a result

$$k_t^1 - k_t^2 = \left(\frac{1}{\gamma}\right)\left(I_{\mathrm{P}}(\mathrm{Q}^2) - I_{\mathrm{P}}(\mathrm{Q}^1)\right) = \big|_{\mathrm{Q}^1=\mathrm{P}} \frac{I_{\mathrm{P}}(\mathrm{Q}^2)}{\gamma} > 0 \quad \text{for all} \quad (t,\sigma)\,. \tag{27}$$

The sufficient condition (16) implies that the agent with correct beliefs dominates P-a.s.. Agent 1's less accurate effective beliefs are more than compensated by a higher effective discount factor, leading to dominance.[24] I have just established that in the example of Figure 2, dominance occurs through saving.

Dominance of the correct agent occurs also in the economies of Sections 4.2 and 4.3 if we set $\rho^i = \gamma^i$. However, the relative importance of decisions across time and states toward survival depends on the spcific economy. In the economy of Example-2 (3), the saving channel is crucial only at the prices where agent 1 (2) dominates.

5 Groups' Consumption Dynamics

In this section, we generalize the analysis to economies where $I > 2$ and provide the sufficient conditions for survival that we have informally anticipated in Section 3.3 for the 2-agent economy. The analysis proceeds in parallel to that of Section 3 through the definition of group effective discount factors, effective beliefs, and generalized survival indexes.

Consider a subset $\mathcal{G} \subset \mathcal{I}$ with G traders and define, for all (t,σ), the group total consumption share as $\Phi_t^{\mathcal{G}} = \sum_{i\in\mathcal{G}} \phi_t^i$ and agent i relative consumption within the group as $\phi_t^{\mathcal{G},i} = \frac{\phi_t^i}{\Phi_t^{\mathcal{G}}}$, so that $\phi_t^{\mathcal{G}} \in \Delta_+^G$. By definition, when $\mathcal{G} = \mathcal{I}$, $\Phi_t^{\mathcal{I}} = 1$ and $\phi_t^{\mathcal{I},i} = \phi_t^i$ with $\phi_t^{\mathcal{I}} = \phi_t \in \Delta_+^I$. We shall be interested in stating when a group dominates, survives, or vanishes. The terms are to be understood as in Definition 1, provided we substitute ϕ_t^i with $\phi_t^{\mathcal{G}}$.

By aggregating group consumption in nodes σ_t and σ_{t+1}, aggregate effective discount factors and effective beliefs of group $\mathcal{G}$ can be expressed as weighted average of individual effective discount factors and beliefs. Extending the definition of Equation 6, the aggregate effective discount factor is

$$\delta_t^{\mathcal{G}} := \frac{\sum_{i\in\mathcal{G}} \delta_t \sum_{s\in\mathcal{S}} \mathrm{Q}_{s,t}\; c_{t+1}^i(\sigma_t, s)}{\sum_{i\in\mathcal{G}} c_t^i} = \sum_{i\in\mathcal{G}} \delta_t^i \phi_t^{\mathcal{G},i}\,. \tag{28}$$

From Equation 7, the aggregate effective beliefs is instead

$$\alpha_{s_{t+1},t}^{\mathcal{G}} := \frac{\sum_{i\in\mathcal{G}} \mathrm{Q}_{s_{t+1},t}\; c_{t+1}^i(\sigma_t, s_{t+1})}{\sum_{i\in\mathcal{G}} \sum_{s'\in\mathcal{S}} \mathrm{Q}_{s',t}\; c_{t+1}^i(\sigma_t, s')} = \sum_{i\in\mathcal{G}} \alpha_t^i \varphi_t^{\mathcal{G},i} \quad \text{for all} \quad s_{t+1} \in \mathcal{S}, \tag{29}$$

where[25]

$$\varphi_t^{\mathcal{G},i} = \frac{\delta_t^i \phi_t^{\mathcal{G},i}}{\delta_t^{\mathcal{G}}}\,. \tag{30}$$

[24] Defining $k_{BE}^i = \log\beta^i - I_P(Q^i)$ in the spirit of Blume and Easley (2006), we can also write $k_t^1 - k_t^2 = \frac{1}{\gamma}\left(k_{BE}^1 - k_{BE}^2\right)$. Although the difference has the same sign, $k_{BE}^i \neq k_t^i$. Only the latter is agent specific in that it does not change when we modify agent $j \neq i$ preferences; see also example $A1$ in Appending A.1.

[25] By construction also $\varphi_t \in \Delta_+^G$.

Note how each agent contributes to the group effective discount factor proportionally to her relative consumption within the group, $\phi^{\mathcal{G},i}$. For the group effective beliefs, instead, each agent contributes proportionally to the consumption she decides to transfer to the next date, $\varphi^{\mathcal{G},i}$.

Defining $z_t^{\mathcal{G}} = \log \frac{\Phi_t^{\mathcal{G}}}{1-\Phi_t^{\mathcal{G}}}$, the evolution of $(z_t^{\mathcal{G}})$ is governed by

$$z_{t+1}^{\mathcal{G}} = z_t^{\mathcal{G}} + \epsilon_{t+1}^{\mathcal{G}} \,, \tag{31}$$

where, denoting $-\mathcal{G} = \mathcal{I} \setminus \mathcal{G}$,

$$\epsilon_{t+1}^{\mathcal{G}}(\sigma_{t+1}) = \log \frac{\delta_t^{\mathcal{G}}}{\delta_t^{-\mathcal{G}}} + \log \frac{\alpha_{s_{t+1},t}^{\mathcal{G}}}{\alpha_{s_{t+1},t}^{-\mathcal{G}}} \quad \text{for each } s_{t+1} \in \mathcal{S}. \tag{32}$$

As for 2-agent economies, the sign of conditional drift depends on generalized survival indexes. Given a group $\mathcal{G}$ effective discount factor $\delta_t^{\mathcal{G}}$ and effective beliefs $\alpha_t^{\mathcal{G}}$, we can define group $\mathcal{G}$ generalized survival index in node σ_t as

$$k_t^{\mathcal{G}} := \log \delta_t^{\mathcal{G}} - I_{\mathrm{P}_t}(\alpha_t^{\mathcal{G}}) \,. \tag{33}$$

From (32) and (33) the conditional drift of the log relative consumption process is[26]

$$\mathrm{E}_{\mathrm{P}}[\epsilon_{t+1}^{\mathcal{G}} | \, \mathcal{F}_t] = k_t^{\mathcal{G}} - k_t^{-\mathcal{G}} \,.$$

If group $\mathcal{G}$ has a higher survival index than group $-\mathcal{G}$, the conditional drift of the relative consumption process is in its favor and thus group $\mathcal{G}$ gains, in expectation, consumption.

Sufficient conditions for groups' survival are derived following two approaches. The first uses the average value of generalized survival indexes. Although this method is of limited applicability, due to the endogeneity of survival indexes, it is instructive because it provides conditions on the averages of generalized survival indexes when both agents survive. The other method amounts to comparing a group generalized survival indexes in the limit when the other group has unitary relative consumption.

Using the first approach, we can state sufficient conditions in terms of generalized survival indexes time averages. The proof relies on the Strong Law of Large Numbers for uncorrelated martingales.[27]

Theorem 1 (**High average generalized indexes imply dominance**). *Consider an economy with $\mathcal{I}$ agents where a competitive equilibrium exists and is interior and, given a*

[26] Given an initial equilibrium consumption distribution, the process $(z_t^{\mathcal{G}})$ is a real process adapted with respect to $\mathcal{F}_t$, the information filtration generated by the states of the world, see Appendix C.1 for details. Typically the realization of $z_t^{\mathcal{G}}$ depends on all the states of the history σ_t. This long-range dependence structure might be maintained even if we condition on recent realizations of the process such as $z_{t-1}^{\mathcal{G}}$. It is so because the information filtration generated by the process is often coarser than $(\mathcal{F}_t)$.

[27] This approach is the same used by Sandroni (2000). Theorem 1 generalizes Proposition 3 in Sandroni (2000) along three-dimensions: dominance can be characterized also at the group level; the economy can be unbounded; agents may not be SEU maximizers.

group $\mathcal{G} \subset \mathcal{I}$, the process $(z_t^{\mathcal{G}})_{(t,\sigma)}$ in (31) has bounded increments.[28] *If*

$$\text{Prob}\left\{\lim_{T\to\infty}\sum_{t=1}^{T}\frac{k_t^{\mathcal{G}}-k_t^{-\mathcal{G}}}{T}>0\right\}=1\,,$$

then group $\mathcal{G}$ dominates P-a.s.. If instead

$$\text{Prob}\left\{\lim_{T\to\infty}\sum_{t=1}^{T}\frac{k_t^{\mathcal{G}}-k_t^{-\mathcal{G}}}{T}<0\right\}=1\,,$$

then group $\mathcal{G}$ vanishes P-a.s..

The theorem states that a sufficient condition for a group to dominate on almost all paths is to have the largest average of generalized survival index on almost all paths. Having the lowest average generalized survival index is instead sufficient for vanishing. As we have seen in the examples of the previous section, two problems arise when applying the theorem. First, the relative order of survival indexes may depend on equilibrium prices, not allowing a direct comparison of their averages. Second, the theorem does not provide sufficient conditions for survival, but for dominance and vanishing only. Assume, for example, that there exists an economy where it can be established that two groups have the same average survival index. It could be so for two reasons, either because dominance of one group is too slow (for example, when the partial sum of differences of survival indexes diverges at a slower rate than T) or, rather, because both groups are doing equally well and both survive. In fact, the following corollary can be established.

Corollary 1 (Survival implies equality of average generalized indexes). *Under the assumptions of Theorem 1, if both groups $\mathcal{G}$ and $-\mathcal{G}$ survive P-a.s., then their average generalized survival indexes are equal P-a.s..*

The Corollary implies that if all agents have the same effective discount factors, then surviving agents must have effective beliefs that are, on average, equally accurate. The long-run heterogeneity example of Section 4.2 is such a case.

An alternative approach is to establish sufficient conditions for survival by evaluating generalized survival indexes in the limit of one group consuming all the endowment.[29] For the generic subset $\mathcal{G}$ of $\mathcal{I}$, given the relative consumption $\Phi^{\mathcal{G}} \in [0,1]$ and relative consumption distributions $\phi^{\mathcal{G}} \in \Delta^G$ and $\phi^{-\mathcal{G}} \in \Delta^{-G}$, denote

$$\delta^{\mathcal{G}}(\phi^{\mathcal{G}},\phi^{-\mathcal{G}},\Phi^{\mathcal{G}}) \quad \text{and} \quad \alpha^{\mathcal{G}}(\phi^{\mathcal{G}},\phi^{-\mathcal{G}},\Phi^{\mathcal{G}})$$

the group $\mathcal{G}$ effective discount factors and beliefs, respectively, when market equilibrium normalized prices and discount rates are determined by group distributions $\phi^{\mathcal{G}}$ and $\phi^{-\mathcal{G}}$ with relative size $\Phi^{\mathcal{G}}$. When group $\mathcal{G}$ has all the relative consumption we write

$$\delta^{\mathcal{G}}|_{\mathcal{G}}=\delta^{\mathcal{G}}(\phi^{\mathcal{G}},\phi^{-\mathcal{G}},\Phi^{\mathcal{G}}=1) \quad \text{and} \quad \alpha^{\mathcal{G}}|_{\mathcal{G}}=\alpha^{\mathcal{G}}(\phi^{\mathcal{G}},\phi^{-\mathcal{G}},\Phi^{\mathcal{G}}=1)\,.$$

[28] The theorem requires bounded increments in the sense that with full probability the realized innovation $\epsilon_t^{\mathcal{G}}$ should be bounded from above and from below. This is typically the case in our examples of Section 4 because agents' speculative positions are bounded. See also Appendix C.

[29] This is the approach followed e.g. by Borovička (2019) for 2-agent economies and by Bottazzi and Dindo (2014) and Bottazzi et al. (2018a) for I-agent economies.

$k^{\mathcal{G}}|_{\mathcal{G}}$ is the corresponding generalized survival index. Similarly $\delta^{\mathcal{G}}|_{-\mathcal{G}}$ and $\alpha^{\mathcal{G}}|_{-\mathcal{G}}$ are the rules used by group $\mathcal{G}$ when $\Phi^{\mathcal{G}} = 0$. The following theorem gives sufficient conditions for survival of group $\mathcal{G}$ in terms of the sign of $k^{\mathcal{G}}|_{-\mathcal{G}} - k^{-\mathcal{G}}|_{-\mathcal{G}}$.

Theorem 2 (Sufficient conditions for a group survival). *Under the assumptions of Theorem 1, assume further that the process $(z_t^{\mathcal{G}})_{t\in\mathbb{N}_0}$ conditional drift $\mathrm{E}_{\mathrm{P}}[\epsilon_{t+1}^{\mathcal{G}}|z_t^{\mathcal{G}} = z\,, \mathcal{F}_t]$ is continuous in z. If*

$$\min_{\phi^{\mathcal{G}}\in\Delta^{G},\phi^{-\mathcal{G}}\in\Delta^{-G}} \left\{k^{\mathcal{G}}|_{-\mathcal{G}} - k^{-\mathcal{G}}|_{-\mathcal{G}}\right\} > 0\,,$$

then group $\mathcal{G}$ survives P-a.s..

For a group of agents to survive it is sufficient to have a larger conditional survival index only at the state prices set by the other group, and for all possible consumption distributions within both groups. The result is rather intuitive; a group of agents survives if, under all market conditions established by the other agents, its aggregate consumption decisions across time and states lead to a higher expected growth than those of the other agents.

Finally, using "limit" generalized survival indexes it is also possible to establish when a group dominates.

Theorem 3 (Sufficient conditions for a group dominance). *Under the assumption of Theorem 2, assume further that the process $(z_t^{\mathcal{G}})_{t\in\mathbb{N}_0}$ has finite positive and negative increments.*[30] *If*

$$\min_{\phi^{\mathcal{G}}\in\Delta^{G},\phi^{-\mathcal{G}}\in\Delta^{-G}} \left\{k^{\mathcal{G}}|_{-\mathcal{G}} - k^{-\mathcal{G}}|_{-\mathcal{G}}\right\} > 0 \quad \text{and} \quad \min_{\phi^{\mathcal{G}}\in\Delta^{G},\phi^{-\mathcal{G}}\in\Delta^{-G}} \left\{k^{\mathcal{G}}|_{\mathcal{G}} - k^{-\mathcal{G}}|_{\mathcal{G}}\right\} > 0\,,$$

then group $\mathcal{G}$ dominates and group $-\mathcal{G}$ vanishes P-a.s.; if

$$\max_{\phi^{\mathcal{G}}\in\Delta^{G},\phi^{-\mathcal{G}}\in\Delta^{-G}} \left\{k^{\mathcal{G}}|_{-\mathcal{G}} - k^{-\mathcal{G}}|_{-\mathcal{G}}\right\} < 0 \quad \text{and} \quad \min_{\phi^{\mathcal{G}}\in\Delta^{G},\phi^{-\mathcal{G}}\in\Delta^{-G}} \left\{k^{\mathcal{G}}|_{\mathcal{G}} - k^{\mathcal{G}}|_{-\mathcal{G}}\right\} > 0\,,$$

then P-a.s. either group $\mathcal{G}$ or group $-\mathcal{G}$ dominates, depending on the initial endowment.

The sufficient conditions of Theorem 3 are of broader applicability than those of Theorem 1 because they require that a group has a larger generalized survival conditions than that of the other agents in the two limit cases where it has, in relative terms, all the endowment or it has none. If, given a partition, each group has the largest survival index when it has all the endowment, then either group can dominate.

Both Theorems 2 and 3 rely on applications of the martingale convergence theorem as used in Bottazzi and Dindo (2014) and Bottazzi and Dindo (2015). In economies with more than two agents, they provide only weak sufficient conditions as, even when one group has null relative consumption, the sign of the survival index difference is likely to depend on the exact consumption distribution within both groups. The same happens in economies where the endowment growth process, and/or agents beliefs, are not i.i.d..

[30]The process has finite positive and negative increments when with positive probability $\varepsilon > 0$ it increases by at least ε and it decreases with at least ε. See Appendix D.

Here, even if there are only two agents, equilibrium prices computed under the assumption that an agent, or a group of agents, consumes the aggregate endowment, become a random variable. Survival indexes should thus be compared for all the relevant possible histories of the process. Only when inequalities hold in all these cases, are they sufficient to characterize long-run outcomes.

In 2-agent i.i.d. economies, such as those considered in the examples of Section 4, provided agents' "limit" survival indexes are not identical, one of the four sign combinations given in (18-21) applies and sufficient conditions become tight. We provide examples of 3-agent economies and discuss the weakening of sufficient conditions in the examples of Appendix A.3.

6 Conclusion

This paper explains why, in dynamic stochastic exchange economies where agents have heterogeneous beliefs, speculation may not support the Market Selection Hypothesis that the trader with the most accurate beliefs dominates in the long run.

The result is established by characterizing long-run outcomes of agents' relative consumption dynamics in terms of effective discount factors and effective beliefs. The accuracy of the latter is shown to determine the size of expected log-returns, or growth premium, and can be decomposed in the accuracy of beliefs and in a Non-Log-Optimality term. Sufficient conditions for an agent (or group) survival depend on the comparison of a survival index, a function of effective discount factors and of effective beliefs accuracy, at the market conditions that prevail when the consumption share of that agent (or group) is negligible.

In the special case of SEU with Bernoulli log utility, effective beliefs and beliefs coincide so that, provided discount factors are equal, speculative positions favor the agent with the most accurate beliefs. However, outside the log utility framework, the growth premium determined by the accuracy of effective beliefs depends both on the accuracy of beliefs and on the comparison of an agent consumption decision across states with the corresponding log optimal choice. This last term is named the NLO term and is responsible for a rather rich set of long-run outcomes. In an Epstein-Zin economy where all agents have the same effective discount factor, we provide examples where all agents survive, leading to heterogeneity of beliefs also in the long run, or where the agent with the most accurate beliefs vanishes in a positive measure set (also full). Results are robust to local changes of beliefs, risk preferences, and the aggregate endowment process. SEU-CRRA economies are instead special because, due to interdependence of inter-temporal and risk preferences, the response to beliefs heterogeneity incorporated in effective discount factors and in the NLO term compensate each other. Also in SEU-CRRA economies, however, the relative importance for long-run survival of consumption decisions across time and states is shown to be different for different choices of agents' preferences and beliefs.

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A Further examples

This appendix offers further examples. Example $A.1$ in Appendix A.1 shows that in a 2-agent economy it is enough that one agent is not of SEU-CRRA type, having $\rho = 1$ and $\gamma \neq 1$, to obtain long-run heterogeneity even if the SEU-CRRA agent knows the truth. Appendix A.2 contains a robustness analysis for the examples of Section 4 and $A.1$. Appendix A.3 applies the survival sufficient conditions of Section 5 to 3-agent economies.

A.1 A slight departure from a SEU-CRRA economy

In what follows I show that it is enough that one agent is not a SEU-CRRA maximizer to obtain long-run beliefs heterogeneity. In example $A1$, I start from the benchmark case of an economy with two SEU-CRRA agents, aggregate risk, and homogeneous preferences, where it is known that the agent with accurate beliefs dominates, and I modify the saving behavior of the agent with non correct beliefs to achieve long-run heterogeneity. In particular, I assume that the non-correct agent has the same RRA as in the SEU-CRRA case but a unitary IES. The example is a contribution to the MSH literature in that changing saving behaviour was known to let the correct agent to disappear but not to create long-run heterogeneity.[31]

Example A1 Consider a 2-agent 2-state economy with $g_1 = 2g_2 = 2$ and two CRRA agents with $\rho^1 = \gamma^1 = \rho^2 = \gamma^2 = 2$. Agent 1 has correct beliefs, $\mathrm{Q}^1 = P = 0.5$, while agent 2 is a pessimist and assigns a lower probability to state 1, $\mathrm{Q}^2 = 0.2$. Agents' effective beliefs are shown in the left panel of Figure 5. Regarding effective discount factors, homogeneity of preferences and discount factors implies that the only component that matters for survival is $\Delta^i(\mathrm{Q}_t)$ as defined in (25). Both $\Delta^1(\mathrm{Q})$ and $\Delta^2(\mathrm{Q})$ are shown in the right panel of Figure 5.

As we have shown in the main text (Section 4.4), the computation of generalized survival indexes confirms the finding of the incumbent literature that the correct agent dominates. In fact,

$$k_t^1 > k_t^2 \quad \text{for all} \quad (t, \sigma) \, .$$

However, an inspection of the size of discount factors and on the accuracy of effective beliefs shows that the role of effective beliefs, and thus decisions across states, is crucial. For all equilibrium risk neutral probabilities in $[\mathrm{Q}\,|_2, \mathrm{Q}\,|_1]$, the correct agent has a lower effective discount factor but more accurate effective beliefs. The latter term dominates and thus she dominates P-a.s..

How does the long-run outcome change when agents have the same effective beliefs but different effective discount factors? In particular, assume that the agent with correct beliefs is still of SEU-CRRA type but the agent with non correct beliefs if of unitary IES type with unchanged RRA $\gamma^2 = 2$. The generalized survival index of the correct agent does not change. In particular, the effective beliefs comparison still favours the agent

[31] In bounded economies, vanishing of the correct agent occurs through a differentiated saving when her discount factor is small enough; see Sandroni (2000) and Blume and Easley (2006). In unbounded economies, the IES parameter also plays a role, see Yan (2008).

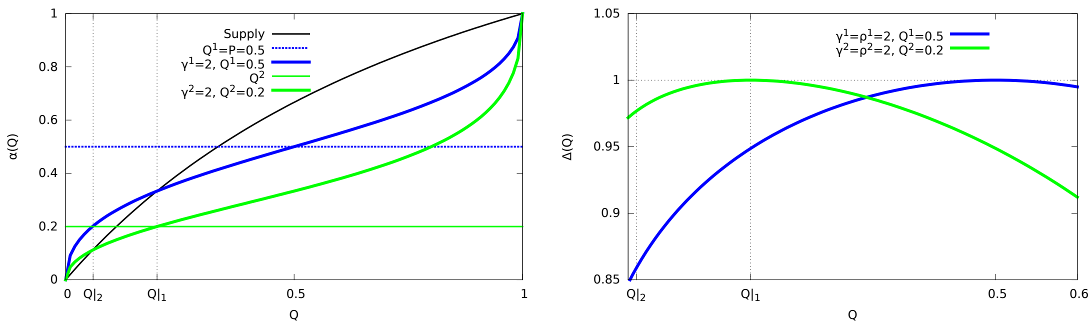

Figure 5: effective beliefs and discount factors in a 2-agent 2-state CRRA economy with high risk aversion and pessimism. Left panel: effective beliefs. Right panel: Beliefs dependent components of effective discount factors.

with correct beliefs (effective beliefs are unchanged). However, the agent with non correct beliefs, by having a higher IES than before ($1 > 0.5$) saves more than when she is of SEU-CRRA type and thus much more than the agent with correct beliefs. In particular, it is enough that the contribution of the effective discount factor overturns the contribution of effective beliefs at the equilibrium prices imposed by the correct agent when alone in the market, to guarantee that the non correct agent survives. If

$$I_P(\alpha^1(\mathrm{Q}\,|_1)) < I_P(\alpha^2(\mathrm{Q}\,|_1)) \quad \text{and} \quad \log \frac{\delta^1(\delta|_1, \mathrm{Q}\,|_1)}{\delta^2(\delta|_1, \mathrm{Q}\,|_1)} < I_P(\alpha^1(\mathrm{Q}\,|_1)) - I_P(\alpha^2(\mathrm{Q}\,|_1)),$$

then $k^2|_1 > k^1|_1$ and, by Theorem 2, agent 2 survives. At the chosen parameters this is the case. Moreover, it can also be computed that at the prices determined by agent 2, the effective beliefs relative accuracy is still stronger than the effective discount factor log ratio, implying $k^1|_2 - k^2|_2 > 0$. Theorem 2 guarantees that also agent 1 survives so that the outcome is long-run heterogeneity. The left panel of Figure 6 shows the relative consumption share dynamics for a realization of σ. The right panel shows the corresponding equilibrium discount factors and risk neutral probabilities. Both prices fluctuate in between the corresponding values set by each agent when alone. State 1 realizations push agent 1 consumption share up, decrease equilibrium discount rates, and increase state 1 risk neutral probabilities. Finally, long-run heterogeneity implies that the average accuracy of both agents effective beliefs is the same, P-a.s. (Corollary 1).

A.2 Robustness Analysis

Next, we consider a wide range of parametrizations of a 2-agent 2-state economy, including all the examples considered so far. For all these economies we evaluate agents' generalized survival indexes and use them to establish long run outcomes by means of the sufficient conditions of Theorems 2-3. The purpose is to show that dominance of the trader with less accurate beliefs, long-run heterogeneity, and path dependency are robust to local changes of beliefs, preferences, and the aggregate endowment.

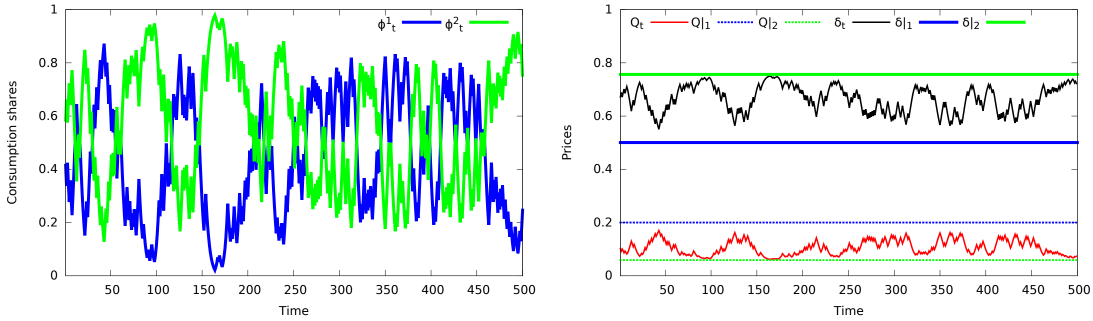

Figure 6: Market dynamics in a 2-agent 2-state economy with effective beliefs as in the left panel of Fig 5. Agent 1 is of CRRA type while agent 2 is of unitary IES type. Left panel: relative consumption shares dynamics when the initial consumption share distribution is $\phi_0 = (0.5, 0.5)$. Right panel: risk neutral probabilities and discount rates dynamics ($\beta = 0.8$).

In Figure 7, we consider 2-state 2-agent economies with $g^1 = 2g^2 = 2$, where agents have homogeneous RRA γ, unitary IES coefficient, and heterogeneous beliefs. The probability that agent 1 (2) assigns to state 1 is on the horizontal (vertical) axis. The truth is still $\mathrm{P} = 0.5$. The left panel shows long-run outcomes when the risk aversion is $\gamma = 2$. Each agent dominates (red area for agent 1, pink area for agent 2) when her effective beliefs are more accurate both at the equilibrium prices set by the other agent and at his equilibrium prices. Since risk aversion and optimism counterbalance each others, and bring effective beliefs close to the truth (high NLO term), each agent dominates when her beliefs are above the beliefs of the other agent (as we have seen in the case of Figure 2), provided optimism is not too extreme. In the latter case it could happen that it is the pessimist who dominates or that the long-run outcome is long run heterogeneity (see also the left panel of Figure 3).

In the right panel long-run outcomes are shown for a lower risk aversion, $\gamma = 0.3$. Low risk aversion leads to inaccurate effective beliefs for equilibrium prices far from an agent beliefs (negative NLO term) so that the most common outcome is path dependency. Dominance of either agent is still possible but heterogeneity is never a long-run outcome.

In Figure 8 we repeat the same analysis in the case of no aggregate risk. Fair pricing, a consequence of no aggregate risk, implies $\alpha^i(\mathrm{Q}\,|_1) = \mathrm{Q}\,|_1 \Leftrightarrow \mathrm{Q}\,|_1 = \mathrm{Q}^1$ so that when agent 1 has correct beliefs she has also the most accurate (correct!) beliefs when she is the representative agent, and cannot vanish P-a.s.. Long-run heterogeneity is also not a possible outcome. Path dependency can still be a long-run outcome and this is the case with a low RRA (right panel). When the most accurate agent is not correct there appears the same variety of long-run outcomes as with aggregate risk.

In the next figure, Figure 9, we consider other preferences parametrizations of the same economy. In particular, we fix agent 1 beliefs to the truth, $\mathrm{Q}^1 = \mathrm{P}$, and his RRA coefficient to $\gamma^1 = 2$. In the left panel, agent 1 has unitary IES. In the right panel, agent 1 is of SEU-CRRA type with $\rho^1 = \gamma^1 = 2$. Agent 2 is always of unitary IES type and might have different beliefs (horizontal axis) and different CRRA coefficients (vertical axis).

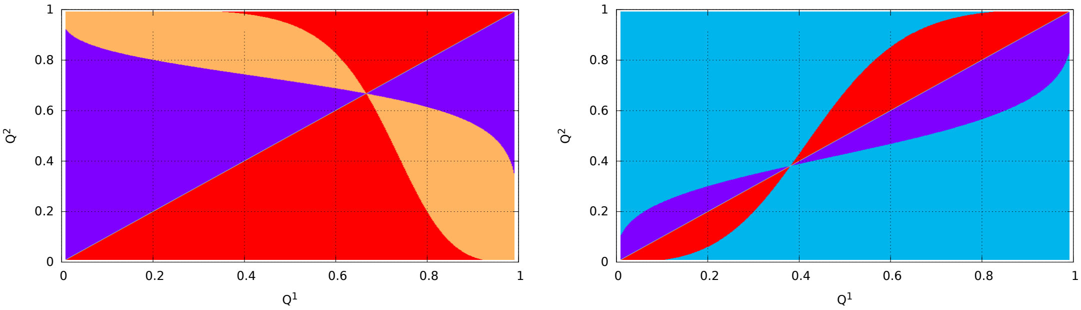

Figure 7: Long-run outcomes in a 2-agent 2-state economy with $g_1 = 2g_2 = 2$ (aggregate risk), $\mathrm{P} = 0.5$, $\rho^1 = \rho^2 = 1$, $\gamma^1 = \gamma^2 = \gamma$. To each color there corresponds a different long run outcome. In the red area the first agent dominates, in the pink area the second agent dominates, in the orange area both agents survive, in the light blue area either agent dominates depending on the selected path. Left panel: high risk aversion, $\gamma = 2 > 1$. Right panel: low risk aversion, $\gamma = 0.3 < 1$.

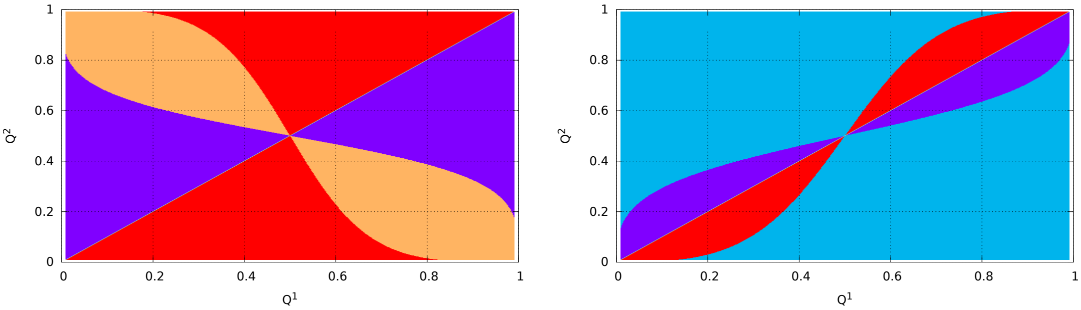

Figure 8: Long-run outcomes in a 2-agent 2-state economy with $g_1 = g_2$ (no aggregate risk), $\mathrm{P} = 0.5$, $\rho^1 = \rho^2 = 1$, $\gamma^1 = \gamma^2 = \gamma$. To each color there corresponds a different long run outcome. In the red area the first agent dominates, in the pink area the second agent dominates, in the orange area both agents survive, in the light blue area either agent dominates depending on the selected path. Left panel: high risk aversion, $\gamma = 2 > 1$. Right panel: low risk aversion, $\gamma = 0.3 < 1$.

Also in this case dominance of the inaccurate trader (pink area), heterogeneity (orange area), and path dependency (light blue area) are robust to local changes in beliefs and preferences.

A.3 Survival in 3-agent i.i.d. economies

In this appendix, we consider i.i.d. economies where agents are still using specific parametrizations of Epstein-Zin utility (Ass. 1-3) and explore the consequences of having more than two agents. The sufficient conditions of Theorems 1-3 depend on the comparison of aggregate effective beliefs and effective discount factors, and thus also on relative consumption

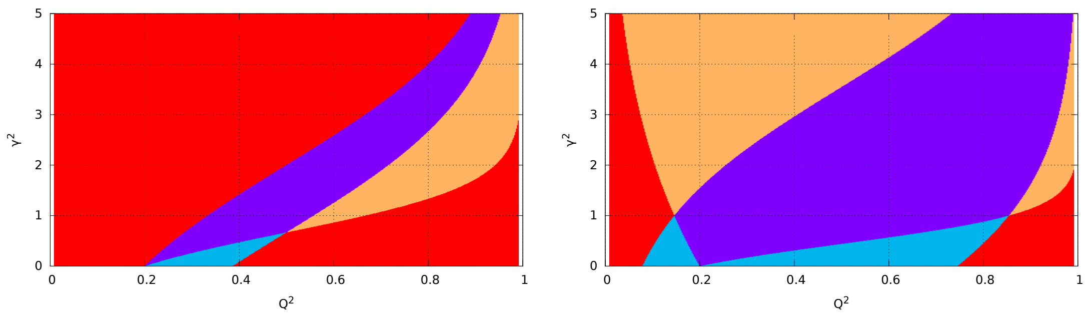

Figure 9: Long-run outcomes in a 2-agent 2-state economy with $g_1 = 2g_2 = 2$, $\mathrm{P} = 0.5$, $\mathrm{Q}^1 = \mathrm{P}$. To each color there corresponds a different long run outcome. In the read area the first agent dominates, in the pink area the second agent dominates, in the orange area both agents survive, in the light blue area either agent dominates depending on the selected path. Left panel: $\gamma^1 = 2$, $\rho^1 = \rho^2 = 1$. Right panel: $\gamma^1 = \rho^1 = 2$, $\rho^2 = 1$.

within groups. Given that the main change in applicability of sufficient conditions is already evident when $I = 3$, we stick to the latter.

A.3.1 2-state 3-agent economies

Example A2 First, I consider economies with no aggregate risk, $g^1 = g^2$, and 2-state, so that there is the advantage that effective beliefs accuracy can be evaluated graphically. All agents have unitary IES and the same discount factor, and thus homogeneous effective discount factors. Sufficient conditions for survival depend only on the accuracy of (possibly aggregate) effective beliefs. In Figure 10, I present two examples of such 3-agent economies. Let us consider group $\mathcal{G} = \{3\}$ against group $-\mathcal{G} = \{1, 2\}$. In the left panel, for all prices determined by $\{1, 2\}$, $\mathrm{Q} \in [\mathrm{Q}\,|_1, \mathrm{Q}\,|_2]$, the effective belief of 3 is more accurate than all possible aggregate effective beliefs of group $\{1, 2\}$, all convex combinations of $\alpha^1(\mathrm{Q})$ and $\alpha^2(\mathrm{Q})$. By Theorem 2, agent 3 survives. Moreover, convexity of the relative entropy and the fact that both 1 and 2 are always more accurate than agent 3 when she has all consumption, imply

$$\max_{\phi^1 \in [0,1]} \left\{ I_{\mathrm{P}}(\phi^1 \alpha^1(\mathrm{Q}\,|_3) + (1 - \phi^1)\alpha^2(\mathrm{Q}\,|_3)) - I_{\mathrm{P}}(\mathrm{Q}\,|_3) \right\} < 0\,,$$

where we have used $\alpha^3(\mathrm{Q}^3) = \mathrm{Q}^3 = \mathrm{Q}\,|_3$. By Theorem 2, group $\{1, 2\}$ survives as well.

Although we have established that neither agent 3 nor group $\{1, 2\}$ dominates, our sufficient conditions are too weak to establish whether all three agents survive. In fact, the within group $\{1, 2\}$ consumption dynamics depends on the relative consumption of agent 3. When agent 3 has most of consumption, agent 2 has more accurate effective beliefs and thus higher growth premia. The opposite occurs when agent 3 relative consumption is very low. If, for example, agent 3 were out of the market then agent 2 would dominate against agent 1.[32] As a result, there is no other partition of this 3-agent economy that

[32] A similar extinction reversal example is built in Cvitanić and Malamud (2010).

leads to decisive results. Agent 2 cannot be said to survive, vanish or dominate, because his effective beliefs accuracy relative to the aggregate beliefs of group $\{1,3\}$ depends on the within group consumption distribution. The same consideration holds when comparing 1 with group $\{2,3\}$.[33] The situation is different if we make agent 3 less risk adverse, as in the right panel of Figure 10. Here, agent 3 vanishes.

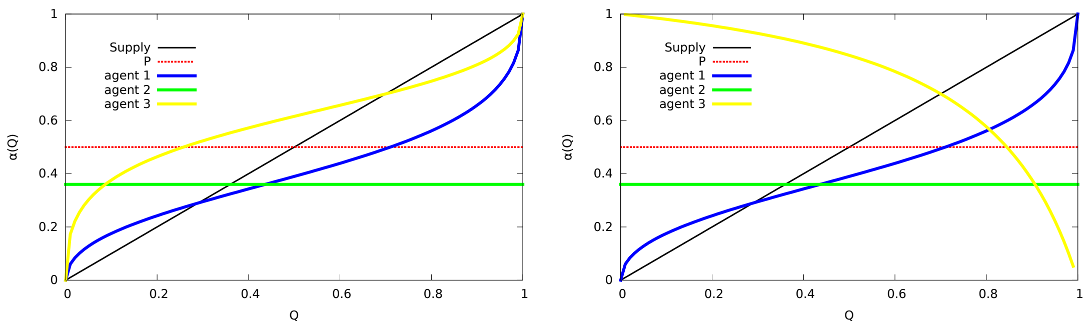

Figure 10: Effective beliefs in a 3-agent 2-state economy without aggregate risk. Left panel: both agent 3 and group $\{1,2\}$ survive. Right panel: agent 3 vanishes.

The same weakening of sufficient conditions of Section 3.3 applies also to non i.i.d. economies. When the growth rate g follows a generic process, or when beliefs are not fixed, equilibrium prices computed under the assumption that an agent, or a group of agents, consumes the aggregate endowment become a random variable. Growth premia should thus be compared for all the relevant possible histories of the process. Only when inequalities hold in all these cases, they are sufficient to characterize long-run outcomes.

A.3.2 3-state 3-agent economies

Example A3 We consider an economy with three agents and three states where sufficient conditions can be used to establish that all agents survive. All agents are of unitary IES type and, to simplify the computations and minimize the number of conditions we need to check, we shall consider homogeneous risk aversion γ and discount factor β, no aggregate risk, $\mathrm{Q}^1 = (0.4, 0.2, 0.4)$, $\mathrm{Q}^2 = (0.2, 0.4, 0.4)$, $\mathrm{Q}^3 = (0.4, 0.4, 0.2)$, and truth $\mathrm{P} = (1/3, 1/3, 1/3)$. Each agent i is correct about the relative likelihood of two states but underestimates the likelihood of the remaining state. Since sufficient conditions for survival and dominance do not rely on equality but on inequality, and all our functions are continuous, similar dynamics can be observed for slightly different parametrizations that capture the same idea. The main ingredients of the example is that no agent is correct, the truth lies in the convex hull generated by all agents beliefs, and agents are of unitary IES type and have a high enough risk aversion. Building on these features the example

[33] In order to establish whether agent 1 or 2 survive one should compare averages of effective beliefs accuracy using the invariant within group consumption distribution (having shown that it exists) when the group has all the consumption. We leave this type of results to future work.

can be extended to achieve long-run heterogeneity in I-agent I-state economies.[34]

We begin with the partition of $\mathcal{I}$ given by $\mathcal{G} = \{1, 2\}$ and $-\mathcal{G} = \{3\}$. Let us start from survival conditions for $\{1, 2\}$. According to Theorem 2 if

$$\min_{\phi^{\{1,2\}} \in \Delta^2} \left\{ k^{\{1,2\}}|_3 - k^3|_3 \right\} > 0\,,$$

then $\{1, 2\}$ survives. Given that all agents have the same effective discount factors β the latter can be written as

$$\max_{\phi^1 \in [0,1]} \left\{ I_{\mathrm{P}}(\phi^1 \alpha^1(\mathrm{Q}^3) + (1 - \phi^1)\alpha^2(\mathrm{Q}^3)) - I_{\mathrm{P}}(\mathrm{Q}^3) \right\} < 0\,, \tag{34}$$

where we have used that in absence of aggregate risk $\alpha^3(\mathrm{Q}^3) = \mathrm{Q}^3 = \mathrm{Q}\,|_3$. No matter the relative consumption within the group composed by the first two agents, their effective beliefs (as a group), should be more accurate than agent 3 beliefs. By the convexity of the relative entropy, it is enough that at the risk neutral probabilities implied by agent 3 each agent has more accurate effective beliefs. The latter holds provided each agent is enough risk averse. More formally, for $\gamma > 1$

$$I_{\mathrm{P}}(\alpha^i(\mathrm{Q}^3)) - I_{\mathrm{P}}(\mathrm{Q}^3) = \log \Delta^i(\mathrm{Q}^3) + \frac{1}{\gamma} \underbrace{\left(I_{\mathrm{P}}(\mathrm{Q}^i) - I_{\mathrm{P}}(\mathrm{Q}^3)\right)}_{=0} < 0 \quad \text{for } i = 1, 2\,.$$

Turning to the survival of agent 3, the sufficient condition for survival written only in terms of effective beliefs is

$$\min_{\phi^1 \in [0,1]} \left\{ I_{\mathrm{P}}(\phi^1 \alpha^1(\mathrm{Q}\,|_{\{1,2\}}) + (1 - \phi^1)\alpha^2(\mathrm{Q}\,|_{\{1,2\}})) - I_{\mathrm{P}}(\alpha^3(\mathrm{Q}\,|_{\{1,2\}})) \right\} > 0\,,$$

where $(\phi^1, 1 - \phi^1)$, the consumption distribution within group $\{1, 2\}$, determines also the equilibrium risk neutral probability $\mathrm{Q}\,|_{\{1,2\}}$ where effective beliefs are evaluated. We check the condition using a numerical procedure. The upper-left panel of Figure 11 shows the relative accuracy of effective beliefs as a function of the consumption share ϕ^1 for two different values of γ. When risk aversion γ is high enough, e.g. $\gamma = 2$, for all possible relative consumption within group $\{1, 2\}$, the effective belief of agent 3 is more accurate of the effective belief of group $\{1, 2\}$ leading to agent 3 survival. The consumption dynamics in the lower-right panel shows that, indeed, both agent 3 and group $\{1, 2\}$ survive. By symmetry, we could repeat the same reasoning for a different partition thus showing that all agents survive. The dynamics of risk neutral probabilities is in the upper-right panel. Differently than in 2-agent economies, and due to high risk aversion, prices may exceed the most optimist agent evaluation of the state.

The bottom-left panel of Figure 11 shows a path of consumption shares in a similar economy where risk aversion has been lowered to $\gamma = 1.5$. The outcome is still long-run

[34]In an economy with aggregate risk and high risk aversion, the same type of example can be established even if one agent knows the truth. The key element is that the price that would prevail if all agents knew the truth and had unitary RRA lies in the convex hull generated by the equilibrium prices set by each agent when she dominates.

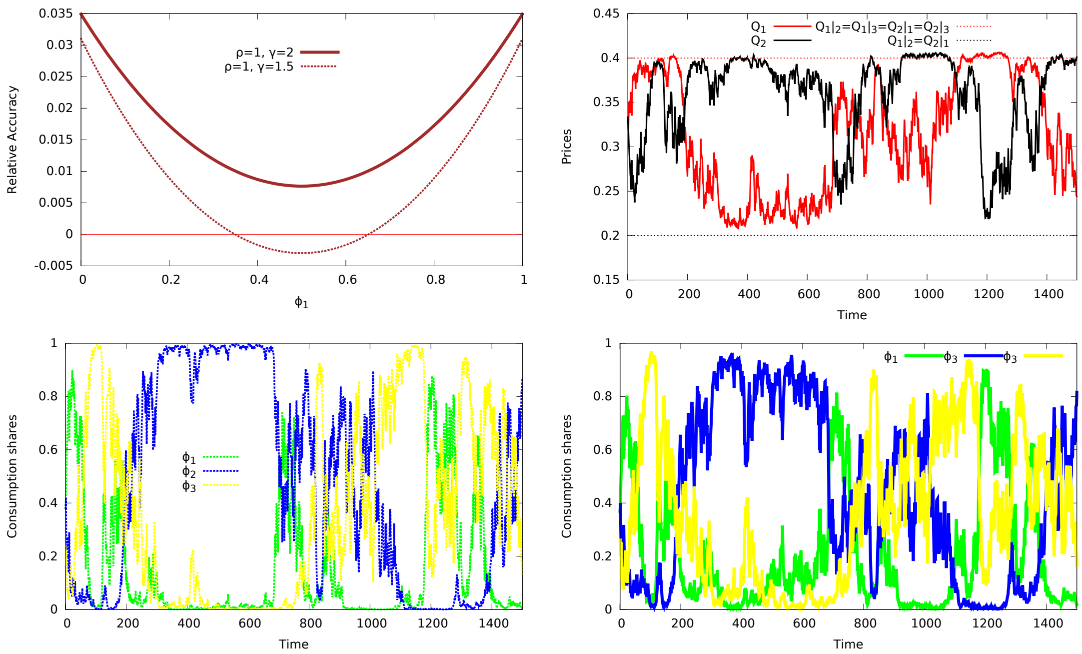

Figure 11: 3-agent, 3-state economy. All agents have homogeneous unitary ρ and homogeneous γ (either 1.5 bottom-left panel or 2 right panels). The truth is $\mathrm{P} = (1/3, 1/3, /1/3)$ and beliefs are $\mathrm{Q}^1 = (0.4, 0.2, 0.4)$, $\mathrm{Q}^2 = (0.2, 0.4, 0.4)$, $\mathrm{Q}^3 = (0.4, 0.4, 0.2)$. Top-left panel: difference of generalized survival indexes as a function of the first two agents consumption distribution. Bottom-left panel: wealth shares when $\gamma = 1.5$ and initial consumption share distribution $\phi_0 = (1/3, 1/3, /1/3)$. Bottom-right panel: wealth shares when $\gamma = 1.5$ and initial consumption share distribution $\phi_0 = (1/3, 1/3, /1/3)$. Top-right panel: risk neutral probabilities corresponding to the bottom-right panel.

heterogeneity. However, sufficient conditions for dominance of Theorem 2 cannot be applied. In fact, the dotted line of the upper-left panel shows that there could be within group consumption distributions such that agent 3 has a less effective accurate beliefs (and thus smaller growth premium) than group $\{1, 2\}$. When $\gamma = 1.5$, our sufficient conditions are too weak to imply that all agents survive.

B Wealth dynamics

In this appendix, we provide a relation between an agent effective discount factor and effective beliefs and her saving and portfolio decisions. The purpose is both to clarify the relation of the present work with the seminal contribution of Blume and Easley (1992) and to provide useful results toward the characterization of effective discount factors and beliefs under Epstein-Zin utility.

We shall define saving and portfolio decisions in terms of agents total wealth (human and financial). The latter can be recovered using effective beliefs and discount factors defined in (6-7) as follows. First, agent i total wealth in σ_t, w^i_t, is the sum of the discounted value of her present and future consumption

$$w^i_t := c^i_t + \sum_{T>0,\, \sigma_{t+T}} q_{\sigma_{t+T},t}\, c^i_{t+T}\,,$$

where σ_{t+T} takes values in $\Sigma_{t+T}(\sigma_t)$, the subset of Σ_{t+T} whose elements have a common history σ_t. Then, iterating the consumption dynamics (8) in the expression for w^i_t above, we find

$$w^i_t = c^i_t \left(1 + \delta^i_t + \delta^i_t \left(\sum_{T>0,\, \sigma_{t+T}} \prod_{\tau=1}^{T} \delta^i_{t+\tau} \alpha^i_{s_{t+\tau},t+\tau-1} \right) \right) .$$

Defining agent i's saving decision in date t as $\bar{\delta}^i_t$, such that $c^i_t = w^i_t(1-\bar{\delta}^i_t)$, from the equation above we find

$$\bar{\delta}^i_t = \delta^i_t \frac{1 + \sum_{T>0,\, \sigma_{t+T}} \prod_{\tau=1}^{T} \delta^i_{t+\tau} \alpha^i_{s_{t+\tau},t+\tau-1}}{1 + \delta^i_t + \delta^i_t \left(\sum_{T>0,\, \sigma_{t+T}} \prod_{\tau=1}^{T} \delta^i_{t+\tau} \alpha^i_{s_{t+\tau},t+\tau-1} \right)} . \tag{35}$$

When an interior equilibrium is well defined $\bar{\delta}^i_t \in (0,1)$.

Having the saving decision $\bar{\delta}^i_t$ and defining $\bar{\alpha}^i_t$ as agent i's portfolio decision in node σ_t, that is, how to allocate saved total wealth $\bar{\delta}^i_t w^i_t$ among future states, the wealth dynamics reads

$$w^i_{t+1}(\sigma_t, s_{t+1}) = \frac{\bar{\delta}^i_t}{\delta_t} \frac{\bar{\alpha}^i_{s_{t+1},t}}{\mathrm{Q}_{s_{t+1},t}} w^i_t \quad \text{for all} \quad s \in \mathcal{S}\,. \tag{36}$$

By definition $\bar{\alpha}^i_t \in \Delta^s_+$. From the above and from the consumption dynamics (8) we can also write

$$\bar{\alpha}^i_{s,t} = \alpha^i_{s,t} \frac{\delta^i_t}{\bar{\delta}^i_t} \frac{1 - \bar{\delta}^i_t}{1 - \bar{\delta}^i_{t+1}(\sigma_t, s)} \tag{37}$$

or, equivalently, using (35)

$$\bar{\alpha}^i_{s,t} = \alpha^i_{s,t} \frac{1 + \delta^i_{t+1}(\sigma_t,s) + \delta^i_{t+1}(\sigma_t,s) \left(\sum_{T>0,\, \sigma_{t+1+T}} \prod_{\tau=1}^{T} \delta^i_{t+1+\tau} \alpha^i_{s_{t+1+\tau},t+\tau} \right)}{1 + \sum_{T>0,\, \sigma_{t+T}} \prod_{\tau=1}^{T} \delta^i_{t+\tau} \alpha^i_{s_{t+\tau},t+\tau-1}} .$$

Although intertemporal and one-period decisions differ, there exists a limit under which they coincide. The following result characterizes when it is the case.

Lemma 1. *If agent i's effective discount factors are history-independent (deterministic),*

$$\delta^i_t(\sigma_t) = \delta^i_t(\sigma'_t) \quad \textit{for all} \quad t, \sigma_t, \sigma'_t,$$

then

$$\bar{\alpha}^i_t = \alpha^i_t\,.$$

If, moreover, effective discount factors are time-independent,

$$\delta^i_t = \delta^i_{t+1} = \delta^i \quad \textit{for all} \quad t,$$

then

$$\bar{\delta}^i_t = \delta^i \,.$$

Proof. If effective discount factors are history-independent, i.e. deterministic, then by (35) also $\bar{\delta}^i_t$ is deterministic and there exists a constant K such that

$$\frac{\delta^i_t(\sigma_t)}{\bar{\delta}^i_t(\sigma_t)} \frac{1-\bar{\delta}^i_t(\sigma_t)}{1-\bar{\delta}^i_{t+1}(\sigma_t,s)} = K \quad \text{for all } s \in \mathcal{S}\,.$$

The latter, (37), and the fact that both $\alpha^i \in \Delta^S_+$ and $\bar{\alpha}^i \in \Delta^S_+$ imply $K=1$ and prove the first part of the Lemma. If effective discount factors are also constant and equal to δ^i, then the sum of the geometric series of compounded rates in (35) can be computed leading to $\bar{\delta}^i_t = \delta^i$. □

A typical example is that of a SEU maximizer with Bernoulli log-utility: $\delta^i_t = \beta^i$ implies that portfolio rules are equal to beliefs, $\bar{\alpha}^i_t = \alpha^i_t = \mathrm{Q}^i_t$ and $\bar{\delta}^i_t = \delta^i_t = \beta$. As we shall prove in the next section another case is that of Epstein-Zin recursive preferences with unitary IES coefficient, where $\delta^i_t = \beta^i$ implies $\bar{\alpha}^i_t = \alpha^i_t$ and $\bar{\delta}^i_t = \delta^i_t = \beta$. In this case the consumption dynamics (8) and the wealth dynamics (36) are equivalent. In more general cases, the dynamics of wealth is much more complicated due to the dependence of the saving $\bar{\delta}_t$ and portfolio $\bar{\alpha}_t$ on the future stream of effective discount factors and beliefs.

C Proofs of Section 4

In this Appendix, we concentrate on economies where agents maximize a recursive Epstein-Zin utility as given in (22). In Appendix C.1, we derive the dynamics of relative consumption and prices as a function of agents' effective discount factors and beliefs. Then, in Appendix C.2, we derive agent i's effective beliefs and discount factors when $\rho^i = 1$ or when $\rho^i = \gamma^i$ (SEU-CRRA case). Finally, in Appendix C.3, we show that in these economies a competitive equilibrium exists and is such that the relative consumption process (z_t) satisfies all the properties needed for the applicability of the sufficient conditions of Section 5.

C.1 Market Dynamics

In this section we start from the consumption dynamics derived in Section 3,

$$c^i_{t+1} = \frac{\delta^i_t}{\delta_t} \frac{\alpha^i_{s_{t+1},t}}{\mathrm{Q}_{s_{t+1},t}} c^i_t \quad \text{for all } i \in \mathcal{I} \quad \text{for all } (t,\sigma), \tag{38}$$

and, by imposing market clearing conditions, we show how to generate the process for $z^{\mathcal{G}}_t = \log \frac{c^{\mathcal{G}}_t}{c^{-\mathcal{G}}_t}$ for any group of agents $\mathcal{G} \subset \mathcal{I}$. We shall assume that all agents i have price dependent effective discount factors and beliefs, $\delta^i_t = \delta^i(\delta_t, \mathrm{Q}_t)$ and $\alpha^i_t = \alpha^i(\mathrm{Q}_t)$ respectively.

Having defined the relative consumption as $\phi^i_t = \frac{c^i_t}{\sum_{j\in\mathcal{I}} c^j_t}$, from (38) we write the relative consumption dynamics in terms of effective beliefs and discount factor as

$$\phi^i_{t+1} = \frac{\delta^i(\delta_t, \mathrm{Q}_t)}{\delta_t} \frac{\alpha^i_{s_{t+1}}(\mathrm{Q}_t)}{\mathrm{Q}_{s_{t+1},t}\, g_{s_{t+1},t}} \phi^i_t \quad \text{for all } i \in \mathcal{I}, \quad \text{for all } (t,\sigma). \tag{39}$$

Equilibrium discount rates and normalized state prices in σ_t can now be computed by imposing the market clearing conditions $\sum_{i\in\mathcal{I}} \phi^i_t = 1$ for all (t, σ).

Aggregating (39) over states and agents we find market equilibrium prices (δ_t, Q_t) as a solution of

$$\begin{cases} \delta_t &= \dfrac{\sum_{i\in\mathcal{I}} \delta^i(\delta_t, \mathrm{Q}_t)\phi^i_t}{\sum_{s'\in\mathcal{S}} \mathrm{Q}_{s',t}\, g_{s',t}}, \\ \dfrac{\mathrm{Q}_{s,t}\, g_{s,t}}{\sum_{s'\in\mathcal{S}} \mathrm{Q}_{s',t}\, g_{s',t}} &= \displaystyle\sum_{i\in\mathcal{I}} \alpha^i_s(\mathrm{Q}_t) \frac{\delta^i(\delta_t, \mathrm{Q}_t)\phi^i_t}{\sum_{j\in\mathcal{I}} \delta^j(\delta_t, \mathrm{Q}_t)\phi^j_t} \quad \text{for all} \quad s \in \mathcal{S}\,. \end{cases} \tag{40}$$

The equilibrium discount rate δ_t, first equation, is the average of effective discount factors, weighted by relative consumption and normalized by the discounted growth rate, as in the SEU-log case. Risk neutral probabilities Q_t are instead equal to the average of effective beliefs, where agent i's impact depends, other than on her relative consumption share, also on the ratio between her effective discount factor and the aggregate effective discount factors. This is also in parallel to what happens in the SEU-log case.

The system of equations (39) and (40) and a specification of effective discount factors and effective beliefs as function of prices for all $i \in \mathcal{I}$ and (t, σ) is equivalent to agents' first order conditions and to market clearing conditions; thus it characterizes agents' relative consumption and state prices on a competitive equilibrium path. An equilibrium allocation and supporting prices can be computed iteratively if: *i*) we know that a competitive equilibrium exists and is interior; *ii*) for all (t, σ) the effective discount factors and effective beliefs of all agents can be recovered from state prices known in σ_t; *iii*) the equilibrium initial relative consumption distribution ϕ_0 is known. This is how the consumption process of each agents has been derived in Figures 4, 6, and 11.

i) and *ii*) do hold when agents' effective discount factors and effective beliefs come from the maximization of specific parametrizations of an Epstein-Zin recursive utility, as it is shown in Appendix C.3 and C.2, respectively. Regarding *iii*), note that long-run properties can be characterized even when it does not hold. In fact, provided long-run outcomes of (39) and (40) are identified for every initial consumption distribution, also equilibrium long-run outcomes are characterized.

From the agents' relative consumption process, (ϕ^i_t) for all $i \in \mathcal{I}$, one can then derive the process $(z^{\mathcal{G}}_t)$.

C.2 Proof of Proposition 1

The proof of Proposition 1 follows from Proposition 2 and Corollary 2. Corollary 3 finds market prices when i is the representative agent, $(\delta|_i, \mathrm{Q}\,|_i)$.

Proposition 2. *Consider an agent $i \in \mathcal{I}$ who maximizes an Epstein-Zin recursive utility* (22) *with discount factor β^i, RRA γ^i, IES ρ^i, and beliefs Q^i. If in equilibrium agent i effective discount factors are history-independent, or if $\gamma^i = \rho^i$, then for all (t, σ), it holds*

$$\delta_t^i = \left(\beta^i\right)^{\frac{1}{\rho^i}} \left(\delta_t\right)^{1-\frac{1}{\rho^i}} \left(\sum_{s' \in S} (\mathrm{Q}_{s'}^i)^{\frac{1}{\gamma^i}} \left(\mathrm{Q}_{s',t}\right)^{1-\frac{1}{\gamma^i}} \right)^{\frac{\gamma^i}{\rho^i} \frac{1-\rho^i}{1-\gamma^i}} , \tag{41}$$

$$\alpha_{s,t}^i = \frac{(\mathrm{Q}_s^i)^{\frac{1}{\gamma^i}} \left(\mathrm{Q}_{s,t}\right)^{1-\frac{1}{\gamma^i}}}{\sum_{s' \in S} (\mathrm{Q}_{s'}^i)^{\frac{1}{\gamma^i}} \left(\mathrm{Q}_{s',t}\right)^{1-\frac{1}{\gamma^i}}} , \quad \textit{for all} \quad s \in \mathcal{S} . \tag{42}$$

Proof. Following Epstein and Zin (1991), the Euler equation that characterizes equilibrium allocation and prices of an agent with Epstein-Zin recursive utility as (22) is

$$\frac{\mathrm{Q}_s^i}{q_{s,t}} (\beta^i)^{\frac{1-\gamma^i}{1-\rho^i}} \left(\frac{\delta_t^i \alpha_{s,t}^i}{q_{s,t}} \right)^{-\rho^i \frac{1-\gamma^i}{1-\rho^i}} \left(\frac{\bar{\alpha}_{s,t}^i}{q_{s,t}} \right)^{\frac{\rho^i - \gamma^i}{1-\rho^i}} = 1 \quad \text{for all} \quad s,\, t,\, \sigma_t , \tag{43}$$

where we have used that $\frac{\bar{\alpha}_{s,t}^i}{q_{s,t}}$ is the return of agent i portfolio.

If effective discount factors are history-independent, then Lemma 1 applies so that $\bar{\alpha}_t^i$ and α_t^i coincide. In the CRRA case, $\gamma^i = \rho^i$, so that the full portfolio $\bar{\alpha}_t^i$ does not enters in the first order condition. In both cases, solving (43) in terms of α_t^i and δ_t^i proves the result. □

The corollary applies the previous proposition when agent i has unitary IES parameter.

Corollary 2. *If agent i maximizes an Epstein-Zin recursive utility with unitary IES coefficient $\rho^i = 1$, then for all (t, σ) $\bar{\delta}_t^i = \delta_t^i = \beta^i$ and $\bar{\alpha}_t^i = \alpha_t^i = \alpha^i(\mathrm{Q}_t)$ as in* (42).

Proof. Other than from direct substitution of $\rho^i = 1$ in (41), the result can be established starting from the Euler equation of the recursive formulation limit, see Epstein and Zin (1991). □

Note that the corollary applies also when beliefs and growth rates are not i.i.d..

Finally, we find equilibrium prices in an i.i.d. economy with a representative agent who has Epstein-Zin recursive utility.

Corollary 3. *In an economy where Assumptions 1-2 hold, if agent i is the representative agent and maximizes an Epstein-Zin recursive utility as in* (22) *with beliefs Q^i, discount factor β^i, RRA coefficient γ^i, and IES parameter ρ^i, then for all t and all σ_t*

$$\bar{\alpha}_{s,t}^i = \alpha_{s,t}^i = \frac{\mathrm{Q}_s^i \, g_s^{1-\gamma^i}}{\sum_{s \in \mathcal{S}} \mathrm{Q}_{s'}^i \, g_{s'}^{1-\gamma^i}} \quad \textit{for all } s , \quad \textit{and} \tag{44}$$

$$\bar{\delta}_t^i = \delta_t^i = \delta_t \left(\sum_s \mathrm{Q}_s \, g_s \right) , \tag{45}$$

where

$$
\mathrm{Q}_{s,t} = \frac{\mathrm{Q}^i_s \, g_s^{-\gamma^i}}{\sum_{s'\in\mathcal{S}} \mathrm{Q}^i_{s'} \, g_{s'}^{-\gamma^i}} \quad \textit{for all } s\in\mathcal{S}\,, \quad \textit{and} \tag{46}
$$

$$
\delta_t = \beta^i \left(\sum_{s\in\mathcal{S}} \mathrm{Q}^i_s \, g_s^{-\gamma^i}\right)\left(\sum_{s\in\mathcal{S}} \mathrm{Q}^i_s \, g_s^{1-\gamma^i}\right)^{\frac{\gamma^i-\rho^i}{1-\gamma^i}} . \tag{47}
$$

Proof. In a representative agent economy, stationarity of portfolios, saving, and equilibrium prices follows from Assumptions 1-2 and Lemma 1. Their actual expression can be found by imposing market clearing conditions (40) given the effective discount factors and beliefs as in (41-42). □

C.3 Properties of the process $(z^{\mathcal{G}}_t)_{t\in\mathbb{N}_0}$

In this section we shall show that for all the examples considered in Section 4 and in Appendix A, the process $(z^{\mathcal{G}}_t)$ is *i*) well defined and *ii*) suited for the application of Theorems 1-3. *i*) follows from the fact that for each example there exists an interior competitive equilibrium, as we shall show in Proposition 3. *ii*) follows from the fact that, in each example, the process of relative consumption is bounded and has finite positive and negative increments as shown in Proposition 4.

Proposition 3. *For all examples considered in Section 4 and Appendix A there exists an interior competitive equilibrium under time-*0 *trading.*

Proof. Given an economy with a set $\mathcal{I}$ of Epstein-Zin agents, consumption paths $(c^i_t)_{(t,\sigma)}$ for all i, normalized states prices $(\mathrm{Q}_t)_{(t,\sigma)}$, and market discount rates $(\delta_t)_{(t,\sigma)}$ generated by (39-40) with effective beliefs and discount factors as in (41-42) are an equilibrium of the exchange economy for a given initial allocation $\{\phi^i_0 \text{ for all } i\in\mathcal{I}\}$ provided that an interior equilibrium is shown to exist. The latter requires that agents' value function are finite, so that recursive preferences are well defined and Euler equations are sufficient, see also Epstein and Zin (1989) and Ma (1993). We shall show that this is the case for all the examples considered in Section 4 and in Appendix A.

Under time-0 trading the existence of an equilibrium follows from Peleg and Yaari (1970), provided the recursive formulation of utility gives a well define utility over consumption streams and provided strict desirability holds. Both require finiteness of the value functions. Since Epstein-Zin preferences are dynamically consistent, as long as markets are (dynamically) complete and an equilibrium exists, time-0 trading and sequential trading or Arrow securities achieve the same equilibrium allocations. Depending on the chosen asset structure, different assumptions are necessary to guarantee the existence of an equilibrium: under date $t = 0$ trading no bankruptcy is allowed, under sequential trading of Arrow securities no bankruptcy and no Ponzi schemes are allowed, see also Araujo and Sandroni (1999).

When an equilibrium exists, it must be interior: agents consumption is positive on all paths σ due to the fact that consumption in t and expected value of date $t+1$ utility are evaluated via a CES aggregator with a finite elasticity of substitution equal to $1/\rho$. As a result, it is never optimal to have zero consumption.

Regarding the finiteness of agents' value functions, a sufficient condition is that each agent value function is finite when he consumes all the aggregate endowment along the paths of maximal and minimal growth, that is, assuming that there is no uncertainty in the economy. To see why, name s^+ the state of maximal growth and s^- the state of minimal growth. Agent i utility on the path $\{e_t^s\} = \{e_0, g_s e_0, g_s^2 e_0, \ldots\}$, with s either s^+ or s^-, can be easily computed from (22) as

$$U_0^i = \left((1-\beta)\sum_{t=0}^{\infty} e_0 \left(\beta^i g_s^{1-\rho^i}\right)^t\right)^{\frac{1}{1-\rho^i}} .$$

The time zero utility is finite provided that

$$\beta^i g_s^{1-\rho^i} < 1 \quad s = s^+, s^- . \tag{48}$$

Note also that since the path of maximum and minimum growth are certain, and no agent consumes all the aggregate endowment, then for all σ

$$(e_t^{s-})_t \leq (c_t^i)_t \leq (e_t^{s+})_t$$

(inequalities for sequences are to be meant component by component). Adding that preferences are monotone, the latter implies

$$U_0^i((c_t^i)_{(t,\sigma)}) \in (-\infty, \infty)$$

for all the feasible allocations $(c_t^i)_{(t,\sigma)}$, provided that agent i preferences satisfy the bound (48).

The argument is concluded by checking that for each agent $i \in \mathcal{I}$ discount factors β^i and IES coefficients ρ^i are such that both inequalities (48) hold. In all the examples of Section 4 and Appendix A, we have $g_{s^+} \leq 2$ and $g_{s^-} = 1$ so that the inequality is always satisfied for s^-. The inequality for s^+ is non problematic in Examples 1 and $A.1$ since $\rho \geq 1$. Example 4 consider cases with $\rho \geq 1$ as well as cases with $\rho < 1$. When $\rho < 1$ we need to assume that β is small enough. A value of $\beta = 0.5$ suffices for our purposes.

Note that we have not excluded the possibility that multiple equilibria exist. As long as each equilibrium obeys (39-40), the market selection outcome derived from our sufficient condition hold. □

Proposition 4. *For all examples considered in Section 4 and in Appendix A, the log-relative consumption process $z_t^{\mathcal{G}}$ has continuous drift and finite increments, that is, there exists a $B > 0$ such that*

$$|z_{t+1}^{\mathcal{G}} - z_t^{\mathcal{G}}| < B \quad \mathrm{P}-\textit{almost surely}.$$

Moreover in the Examples with unitary IES the process has finite positive and negative increments, that is, there exists a $\eta > 0$ such that

$$\mathrm{Prob}\{z_{t+1}^{\mathcal{G}} - z_t^{\mathcal{G}} > \eta | \Im_t\} > \eta \quad \textit{and} \quad \mathrm{Prob}\{z_{t+1}^{\mathcal{G}} - z_t^{\mathcal{G}} < -\eta | \Im_t\} > \eta . \tag{49}$$

Proof. For all examples continuity of the drift of (z_t) follows from the continuity of the effective discount factors and effective beliefs in market prices and from the fact that, through the implicit function theorem, market prices in (40) are themselves a continuous function of (z_t).

We divide the proofs in three parts depending on the specific examples.

Unitary IES In Examples 1, 2, 3, $A.2$, $A.3$ and in those of Figures 7, 8, 10 (left panel), agents have unitary IES. The proof that the process has finite positive and negative increments is as follows.

The process $z_t^{\mathcal{G}}$ has innovation

$$\epsilon_{s,t+1}^{\mathcal{G}} = \log \frac{\alpha_s^{\mathcal{G}}(\mathrm{Q}_t, \phi_t^{\mathcal{G}})}{\alpha_s^{-\mathcal{G}}(\mathrm{Q}_t, \phi_t^{-\mathcal{G}})} \quad \text{in} \, (\sigma_t, s) \, ,$$

where $\alpha^{\mathcal{G}}$ and $\alpha^{-\mathcal{G}}$ are convex combinations of individual effective beliefs as in (23) with weights $\phi_t^{\mathcal{G}}$ and $\phi_t^{-\mathcal{G}}$, respectively, and Q_t is the market clearing effective belief in σ_t. Smoothness of effective beliefs in Q_t imply that each market clearing effective probability is continuous in the weights $\phi = (\phi^{\mathcal{G}}, \phi^{-\mathcal{G}})$. Thus, applying the Weierstrass theorem, for each s there exists a maximum innovation given by

$$\epsilon_s = \max_{\phi \in \Delta^I} \left\{ \left| \log \frac{\alpha_s^{\mathcal{G}}(\mathrm{Q}(\phi), \phi^{\mathcal{G}})}{\alpha_s^{-\mathcal{G}}(\mathrm{Q}(\phi), \phi^{-\mathcal{G}})} \right| \right\} .$$

Choosing $B > \max\{\epsilon_s \, , s \in \mathcal{S}\}$ suffices for proving that the process has finite increments

Turning to the existence of η such that (49) holds, note first that by construction $\alpha^{\mathcal{G}}(\mathrm{Q}_t) \neq \alpha^{-\mathcal{G}}(\mathrm{Q}_t)$ for all (t, σ). The latter plus the the fact that in equilibrium there are no arbitrages (or equivalently that aggregate effective beliefs are in the simplex Δ^S) imply that for all $\phi \in \Delta^I$ there exists at least an s and an s' such that

$$\epsilon_s^{\mathcal{G}}(\phi) > 0 \quad \text{and} \quad \epsilon_{s'}^{\mathcal{G}}(\phi) < 0 \, .$$

Name $\epsilon^+(\phi)$ the upper envelope of all the functions $\{\epsilon_s^{\mathcal{G}}(\phi) \, s \in \mathcal{S}\}$ and $\epsilon^-(\phi)$ its lower envelop. By construction the two functions are continuous in ϕ and, Δ^I being compact, they have a maximum and a minimum. Moreover, since by the non arbitrage argument $\epsilon^+(\phi) > 0$ and $\epsilon^-(\phi) < 0$, the minimum of $\epsilon^+(\phi)$ is positive, $\epsilon^{+-} > 0$, and the maximum of $\epsilon^-(\phi)$ is negative, $\epsilon^{-+} < 0$. Choosing

$$\eta = \min\{\epsilon^{+-}, |\epsilon^{-+}|, \mathrm{P}_s \; s \in \mathcal{S}\}$$

finishes the proof.

CRRA In Example 4 there are two agents, so that $\mathcal{G} = \{1\}$, and all are of SEU-CRRA type. The proof that increments are finite is as follows.

Using CRRA effective discount factors (24) and beliefs (23), for every s define

$$f_s(\mathrm{Q}_s) = \epsilon_{s,t+1} = \log \frac{(\beta^1\,\mathrm{Q}_s^1)^{\frac{1}{\gamma^1}}}{(\beta^2\,\mathrm{Q}_s^2)^{\frac{1}{\gamma^2}}}\,\mathrm{Q}_s^{\frac{1}{\gamma^2}-\frac{1}{\gamma^1}},$$

where $\epsilon_{s,t+1}$ is the increment of the process $z_t = \log \frac{\phi_t^1}{\phi_t^2}$ when $s_{t+1} = s$. Since $\beta^1 = \beta^2 = \beta$ and $\gamma^1 = \gamma^2 = \gamma$ we have

$$f_s(\mathrm{Q}_s) = \frac{1}{\gamma}\log\left(\frac{\mathrm{Q}_s^1}{\mathrm{Q}_s^2}\right).$$

To prove that increments are finite, we need to show that there exists a B such that

$$|f_s(\mathrm{Q}_s)| < B \quad \text{for every } s \in \mathcal{S}.$$

Given that $f_s(\mathrm{Q}_s)$ is a constant, choosing $B > \max\{|f_s(\mathrm{Q}_s)|,\, s \in \mathcal{S}\}$ proves the result.

CRRA and unitary IES In Example $A.1$ there are two agents, one is of SEU-CRRA type while the other has unitary IES, the proof the increments are finite is as follows.

Name $\mathrm{Q}\,|_i$ and $\delta|_i$ the normalized state price vector and market discount rate set by agent i when he consumes all the aggregate endowment. In this 2-agent 2-state economy state prices are in the interior of the set $\Omega_\mathrm{Q} = \times_{s\in\mathcal{S}}\,[\min\{\mathrm{Q}_s\,|_1, \mathrm{Q}_s\,|_2\}, \max\{\mathrm{Q}_s\,|_1, \mathrm{Q}_s\,|_2\}]$ and market discount rate in the interior of $\Omega_\delta = [\min\{\delta|_1, \delta|_2\}, \max\{\delta|_1, \delta|_2\}]$. Given the smoothness of both agents' effective discount factors (24) and beliefs (23), for each s there exists a maximum innovation of the process $z_t = \log \frac{\phi_t^1}{\phi_t^2}$ given by

$$\epsilon_s = \max\left\{\left|\log \frac{\delta^i(\delta, q)\alpha_s^i(q)}{\delta^j(\delta, q)\alpha_s^j(q)}\right| \text{for } q \in \Omega_q\,, \delta \in \Omega_\delta\right\}.$$

Choosing $B > \max\{\epsilon_s\,, s \in \mathcal{S}\}$ suffices for the requirement. □

D Proofs of Section 5

D.1 Proof of Theorem 1 and of Corollary 1

Proof. Proposition 1 is a direct application of the Law of Large Numbers for uncorrelated martingales, see also Proposition 1 in Sandroni (2000) for a similar application.

Consider the additive process $z_t^{\mathcal{G}}$ with innovation $\epsilon_t^{\mathcal{G}}$. By assumption the process $\{Z_t\}$ with

$$Z_t = \epsilon_t^{\mathcal{G}} - \mathrm{E}[\epsilon_t^{\mathcal{G}}|\mathcal{F}_{t-1}]$$

is a uncorrelated martingale with bounded variance so that, by the SLLN for uncorrelated martingales,

$$\lim_{T\to\infty} \frac{\sum_{t=1}^T Z_t}{T} = 0 \quad \text{P-almost surely}\,.$$

By assumption

$$\lim_{T\to\infty} \frac{\sum_{t=1}^{T} \mathrm{E}[\epsilon_t^{\mathcal{G}}|\mathcal{F}_{t-1}]}{T} > 0 \quad \text{P-almost surely}\,.$$

The latter implies $\sum_{t=1}^{T} \epsilon_t^{\mathcal{G}} = +\infty$ so that $\lim_{T\to\infty} z_T^{\mathcal{G}} = +\infty$ too.

Given the proof of the Proposition, the Corollary can be proved by contradiction. □

D.2 Proof of Theorems 2-3

Proof. Given a filtered probability space $(\mathrm{P}, \Sigma, \Im)$ and a real adapted process $\{x_t\}$ defined on $(\Sigma, \{\Im_t\})$, Bottazzi and Dindo (2015) exploit the Martingale Convergence Theorem as in Lamperti (1960) to prove the two following Theorems.

Theorem 4. *Consider a finite increments process x_t with $|x_{t+1} - x_t| < B$ P-a.s.. If there exist $M > B$ and $\epsilon > 0$ such that, P-a.s., $\mathrm{E}[x_{t+1}|x_t = x, \Im_t] < x - \epsilon$ for all $x > M$ and $\mathrm{E}[x_{t+1}|x_t = x, \Im_t] > x + \epsilon$ for all $x < -M$, then there exists a real interval $L = (a, b)$ such that for any t it is $\mathrm{Prob}\{x_{t'} \in L \text{ for some } t' > t\} = 1$.*

Theorem 5. *Consider a finite increments process x_t with $|x_{t+1} - x_t| < B$ P-a.s.. and such that for all t $\mathrm{Prob}\{x_{t+1} - x_t > \eta|\Im_t\} > \eta$ for some $\eta > 0$. If there exist $M > B$ and $\epsilon > 0$ such that, P-a.s., $\mathrm{E}[x_{t+1}|x_t = x, \Im_t] > x + \epsilon$ for all $x > M$ and $\mathrm{E}[x_{t+1}|x_t = y, \Im_t] > x + \epsilon$ for all $x < -M$, then $\mathrm{Prob}\{\lim_{t\to\infty} x_t = +\infty\} = 1$.*

The next theorem is instead a consequence of Theorem 4.3 in Bottazzi and Dindo (2014) and of the fact the process x has finite positive and negative increments.

Theorem 6. *Consider a finite increments process x_t with $|x_{t+1} - x_t| < B$ P-a.s.. and such that for all t $\mathrm{Prob}\{x_{t+1} - x_t > \eta|\Im_t\} > \eta$ and $\mathrm{Prob}\{x_{t+1} - x_t < -\eta|\Im_t\} > \eta$ for some $\eta > 0$. If there exist $M > B$ and $\epsilon > 0$ such that, P-a.s., $\mathrm{E}[x_{t+1}|x_t = x, \Im_t] > x + \epsilon$ for all $x > M$ and $\mathrm{E}[x_{t+1}|x_t = y, \Im_t] < x - \epsilon$ for all $x < -M$, then for P-almost all path $\sigma \in \Sigma$ there exists two sets of initial conditions, $U^+(\sigma)$ and $U^-(\sigma)$ with $U^+(\sigma) \cup \eta^-(\sigma) = \mathbb{R}$, such that $\lim_{t\to\infty} x_t(\sigma) = +\infty$ if $x_0 \in U^+(\sigma)$ and $\lim_{t\to\infty} x_t(\sigma) = -\infty$ if $x_0 \in U^-(\sigma)$.*

Theorems 2-3 are a direct application of the three theorems above with $x_t = z_t^{\mathcal{G}}$. In fact, by the permanence of sign theorem, continuity of the conditional drift in z guarantees that the sign of the drift of the process in the limit of one group dominating is equal to the the sign of the drift in a properly chosen neighborhood around it. □

E Beyond the SEU-CRRA case

Sandroni (2000) and Blume and Easley (2006) show that discount factors and beliefs determine long-run survival for all economies where preferences are represented by a time-separable SEU with Bernoulli utility $u(c)$ satisfying $u'(c) > 0$, $u''(c) < 0$, and $\lim_{c\to 0} u'(c) = +\infty$, provided that the aggregate endowment is bounded from above and from below. In an i.i.d. economy such as the one studied in Section 4, the agent with the highest value of the survival index $k_{BE}^i = \log \beta^i - I_{\mathrm{P}}(\mathrm{Q}^i)$ dominates.

In this Appendix, we compute effective discount factor and beliefs for such general time-separable SEU. For each agent i, name $f_i(c^i) = 1/(u^i(c^i))'$. The properties of u imply that f_i is non-negative, strictly increasing, with well defined inverse $f_i^{-1}(\cdot)$, and $f_i(0) = 0$. Solving the Euler equations leads to the following effective discount factor and belief:

$$
\begin{aligned}
\delta_t^i &= \frac{\delta_t}{c_t^i} \sum_{s' \in \mathcal{S}} f_i^{-1}\left(\frac{\beta^i \, \mathrm{Q}^i_{s',t}}{\delta_t \, \mathrm{Q}_{s',t}} f_i\left(c_t^i\right) \right) \mathrm{Q}_{s',t}\,, \\
\alpha^i_{s,t} &= \frac{f_i^{-1}\left(\frac{\beta^i \, \mathrm{Q}^i_{s,t}}{\delta_t \, \mathrm{Q}_{s,t}} f_i\left(c_t^i\right) \right) \mathrm{Q}_{s,t}}{\sum_{s'} f_i^{-1}\left(\frac{\beta^i \, \mathrm{Q}^i_{s',t}}{\delta_t \, \mathrm{Q}_{s',t}} f_i\left(c_t^i\right) \right) \mathrm{Q}_{s',t}}\,, \quad \text{for all} \quad s \in \mathcal{S}\,.
\end{aligned}
\tag{50}
$$

As in the SEU-CRRA case, both the effective discount factor and effective beliefs depends, other than on beliefs, on risk neutral probabilities and are thus endogenous. If, instead, the survival index computed by the incumbent literature depends only on the exogenously given discount factor β^i and beliefs Q^i, then it must be that the endogenous component of the effective discount factor and beliefs compensate each other, at least in the limit when one agent is the representative one.

The same would happen if we considered an utility with nonlinear probability weighting function such as

$$
U^i(c) = \sum_{t \geq 0, \sigma_t} \beta^t v^i(\mathrm{Q}^i(\sigma_t)) u^i(c_t^i),
$$

where $v^i(x) = x^{\eta^i}$, $\eta^i > 0$, to preserve time-consistency. For example, when $f_i(c) = c^{\gamma^i}$ we obtain the same effective belief and effective discount factor of the SEU-CRRA case in (41-42) with RRA γ^i but where discount factor and beliefs are replaced, respectively, by

$$
\begin{aligned}
\tilde{\beta}^i &= \beta \textstyle\sum_{s' \in \mathcal{S}} v^i(\mathrm{Q}^i_{s'}), \\
\tilde{\mathrm{Q}}^i_s &= \frac{v^i(\mathrm{Q}^i_s)}{\sum_{s' \in \mathcal{S}} v^i(\mathrm{Q}^i_{s'})}, \quad \text{for all} \quad s \in \mathcal{S}\,.
\end{aligned}
$$

The economy has thus the same possible long-run outcomes of a SEU-CRRA economy, dominance of a unique trader unless survival indexes are equal, but the survival index takes into account modified beliefs and discount factors. For example, in i.i.d. economies with bounded endowment, or with homogeneous γ, the agent with the highest value of $\log \tilde{\beta}^i - I_{\mathrm{P}}(\tilde{\mathrm{Q}}^i)$ dominates P-a.s.. Correct beliefs are not necessarily selected for but, generically, a unique agent dominates. If all agents have the same $\tilde{\beta}^i$, the dominating agent is the one with most accurate modified beliefs $\tilde{\mathrm{Q}}^i$.