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Weak Dependence of CRRA on Standard Deviation in the Case of Truncated Normal Distribution of Returns

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ABSTRACT: This paper analyzes the dependence of the Certainty Equivalent Return of a Constant Relative Risk Aversion, *CER[CRRA],* on the Standard Deviation of the Return with the hypothesis of a Truncated Normal distribution of returns and for some level of Relative Risk Aversion *(RRA)* parameter. The paper compares this dependence with those detected by an Annualized Geometrical Return *(AGR)* and by *CER* of the Quadratic Utility Function, *CER[Q].* The behavior of *CER[CRRA]* is more similar to *AGR* than *CER[Q]* and only for a higher value of *RRA* is it possible to find substantial differences, even if in this case we find values of Standard Deviation that have discontinuity points for the concavity.

Using a ranking criteria equal to the one introduced by Morningstar for a set of Funds, the paper shows that, in a wide range for monthly Standard Deviation and Mean of the Returns, the ranking done by *CER[CRRA]* is similar to the one induced by *AGR*, and that a *CER[Q]* has essentially different behavior.

It will be shown that Morningstar ranking may be considered a particular case of the *CER[CRRA]* and thus all the considerations can be applied to the well-known Morningstar Rating methodology.

An application is made to Italian Pension Funds.

JEL Classification Numbers: G11, G14, G24

Key Words: Constant Relative Risk Aversion, Certainty Equivalent Return, Standard Deviation, Quadratic Utility Function, Morningstar, Italian Pension Funds.

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1. Introduction

A considerable amount of literature has been devoted to the way to measure the performances of both investment and pension funds. While great attention has been paid to the aspect of the fairness of the measures to rank the superior performances of the funds, less attention has been given to the sensitivity of these measures to the risk.

The relevant paper by Ingersoll et al. (2007) shows the conditions under which a manipulation-proof measure exists and what its characterizing properties are. This measure was called Manipulation-Proof Performance Measure (MPPM). The article examines seven popular measures, four based on ratios: Sharpe (1966), Sortino and van der Meer (1991), Leland (1999), and Sortino et al. (1999)), and three based on regression intercepts: the CAPM alpha, Treynor and Mazuy (1966), and Henriksson and Merton (1981)), concluding that it is not difficult to game these measures meaningfully.

Ingersoll et al. (2007) claim that the existence of a set of performance measures which are sufficiently manipulation-proof for practical use is important. They say that *"this is particularly relevant in the presence of transactions costs, which may offset whatever ''performance'' gains a manager might hope to generate from trading for the purpose of manipulating the measure".*
They propose the following specific form of MPPM:

$$\widehat{\Theta} = \frac{1}{(1-\rho)\Delta t} ln\left(\frac{1}{T}\sum_{t=1}^{T}\left[(1+r_t)/(1+r_{ft})\right]^{1-\rho}\right)$$

where r_{ft} and r_t are the per-period (not annualized) interest rate and the rate of return on the portfolio over period t. It is noticed that the popular Morningstar Risk Adjusted Rating (MRAR) (Morningstar, 2009), which was introduced in July 2002, is a transformation of the proposed MPPM.
If we consider the MRAR formula:

$$MRAR(\gamma) = \left[\frac{1}{T}\sum_{t=1}^{T}\left[(1+r_t)/(1+r_{ft})\right]^{-\gamma}\right]^{-\Delta t/\gamma} - 1$$

we can easily see that $MRAR(\gamma) = e^{\widehat{\Theta}} - 1$, where $\gamma = \rho - 1$.
So the MRAR has the advantage of being a manipulation proof performance measure and also has an empirical application in the calculation of Morningstar star rating by using[1] MRAR(2).

The performance measure MRAR is based on the utility function of constant relative risk aversion (CRRA) form and states that the level of utility is the same between the certainty equivalent geometric excess return (for a given value of γ) and the expected excess of the fund. We use the notation *CER[CRRA(γ)]* for the entire class of these measures.

In this article we show that the MPPM property is gained to the detriment of the dependence on the Standard Deviation, developing ranking criteria that principally depend on the Annualized Geometrical Return (*AGR)*. The behavior of *CER[CRRA]* is more similar to *AGR* than that of *CER* of the Quadratic Utility Function *CER[Q]*. Only for a higher value of Risk Aversion *(RRA)* parameter is it

[1] Morningstar (2009), p. 12.

possible to find substantial differences, even if in this case we find values of Standard Deviation that have discontinuity points for the concavity.
Using a ranking criteria equal to the one introduced by Morningstar for a set of Funds, the paper shows that, in a wide range for monthly Standard Deviation and Mean of the Returns, the ranking done by *CER[CRRA]* is similar to the one induced by *AGR*, and that a *CER[Q]* has a behavior essentially different.
It will be shown that Morningstar ranking may be considered a particular case of the *CER[CRRA]* and thus all the considerations can be applied to the well-known Morningstar Rating methodology.

In this paper we consider Funds without derivatives products, which cannot lose more than the initial capital, and our considerations are developed in a finite set of Funds.

Although we do not consider the use of derivatives, we nevertheless take into account the criticism of the financial literature related to the assumption of normal or lognormal distributions for portfolio returns. In fact, here we consider a normal truncated distribution characterized by the asymmetry of returns in their range of variation.

The paper is organized as follows. Section 2 introduces the *CER[CRRA(γ)]*, where $\gamma + 1$ is the RRA parameter, in the case of returns distributed as a Truncated Normal variable and develops the matters with the Differential Geometry in 3D. Section 3 explains how the Morningstar Rating can be seen as a particular case of the *CER[CRRA]* and introduces *AGR* as first reference, while Section 4 introduces *CER[Q]* as second reference. Section 5 develops a measure of dependence on the Standard Deviation, using the First Derivative of the Implicit Function (FDIF) of *CER*, showing that *CER[CRRA(γ)]* is similar to *AGR* and has changes of the concavity for high values of Standard Deviation and γ parameter. Section 6 illustrates a ranking using the Morningstar criteria, and shows the similarity with both the ranking induced by *CER[CRRA(γ)]* for low value of γ and that one induced by *AGR* in some range of Standard Deviation and Mean. On the contrary, *CER[Q]* has a different behavior with respect to *AGR*, detecting that it depends on the first two moments of the Truncated Normal distribution. Section 7 sets out the conclusions.

2. The Constant Relative Risk Aversion (CRRA) Utility Function.

Consider a general *CRRA* Utility Function:

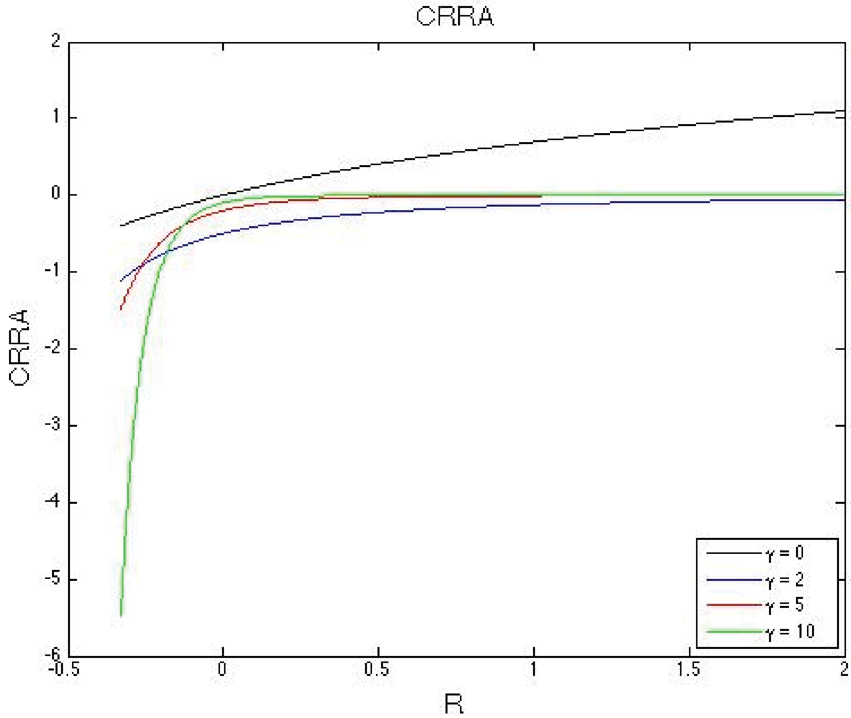
(2.1) $$CRRA(\gamma) = \begin{cases} -W^{-\gamma}/\gamma, & \gamma \neq 0 \\ logW & \gamma = 0 \end{cases}$$

where:

$$W = W_0(1 + R)$$

is *Wealth* with the initial amount W_0 and the (monthly) return R, γ is a parameter that expresses an investor's sensitivity to risk, say the degree of his risk aversion.
The following Figure 2.1 shows the behavior of the CRRA with respect to different values of the γ parameter.

Figure 2.1: Constant Relative Risk Aversion Utility Functions



$\gamma < -1$: the investor is risk loving rather than risk averse.
$\gamma = -1$: means that the degree of risk aversion is zero: The investor is indifferent between a risk-free choice and a risky choice as long as the arithmetic average expected return is the same.
$\gamma = 0$: the investor is indifferent between a risk free choice and a risky choice as long as the geometric average expected return is the same.
$\gamma > 0$: the investor is risk averse and calls a premium against his choice of a risky asset, the larger is the value of γ the greater the risk premium.
In this paper, we do not consider $\gamma < 0$.

Assuming for simplicity $W_0 = 1$, the *ARA (Absolute Risk Aversion)* and *RRA (Relative Risk Aversion)* for the CRRA we have the expressions:

$$ARA[CRRA(\gamma)] = \frac{\gamma + 1}{1 + R}, \qquad RRA[CRRA(\gamma)] = \gamma + 1$$

For the scope of Funds' rating, it is preferable to use the Certainty Equivalent Return, which does not depend on the Funds' money values and represents the riskless Return that provides the same level of utility as the variable Return to the investor.

The (monthly) Certainty Equivalent Return Utility Function with parameter γ and a generic distribution D, denoted with $CER_m[CRRA(\gamma)_D]$, by definition is:

$$1 + CER_m[CRRA(\gamma)_D] = \begin{cases} \left(E[(1+R)^{-\gamma}]\right)^{-\frac{1}{\gamma}} & \gamma > 0 \\ e^{E[ln(1+R)]} & \gamma = 0 \end{cases}$$

It is reasonable to consider the annualized values $CER[CRRA(\gamma)_D]$ of $CER_m[CRRA(\gamma)_D]$, defined by:

(2.2)
$$CER[CRRA(\gamma)_D] = \begin{cases} \{E[(1+R)^{-\gamma}]\}^{-\frac{12}{\gamma}} - 1 & \gamma > 0 \\ e^{12E[\ln(1+R)]} - 1 & \gamma = 0 \end{cases}$$

Some considerations about the range of the return R are useful.

$R = -1$ is a singularity point for the (2.1), when $\gamma > 0$; it means that $R > -1$ is a condition that we have to pose. Moreover, it is coherent to the fact that our analysis is concentrated on Funds that cannot lose more than 100% of their initial value and excludes derivative products.

To have the maximum generality, consider R as a normal random variable $R \sim N(\mu, \sigma^2)$ constrained to assume values only values in the interval $K = (k_1, k_2)$, with $-1 < k_1 < 0 < k_2 \leq \infty$ and $k_1 < \mu < k_2$. In this paper the computations are done for $k_1 = -0.99$, $k_2 = \infty$.

The density function for the Truncated Normal *(TN)* distribution, $f_{TN}(R)$, is:

$$f_{TN}(R) = \begin{cases} \dfrac{\phi\left(\frac{R-\mu}{\sigma}\right)}{\sigma \Delta\Phi_K} = \dfrac{e^{-(R-\mu)^2/2\sigma^2}}{\int_{k_1}^{k_2} e^{-(R-\mu)^2/2\sigma^2} dR} & R \in K \\ 0 & R \notin K \end{cases}$$

where:

$$\phi(\xi) = \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}, \Phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-\frac{\tau^2}{2}} d\tau$$

and

$$h_2 = \frac{k_2 - \mu}{\sigma}, \quad h_1 = \frac{k_1 - \mu}{\sigma}, \Delta\Phi_K = \Phi(h_2) - \Phi(h_1)$$

The quantity $\Delta\Phi_K$ represents the probability that $R \subset K$.

Is it possible to compute the expected value of $CRRA(\gamma)$ for $\gamma > 0$:

(2.3)
$$E[CRRA(\gamma)_{TN}](\sigma, \mu) = \frac{\int_{k_1}^{k_2} \frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^{\gamma}} dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2} dx}$$

that we consider as a function of (σ, μ)?

Using the expectation (2.3), the annual Certainty Equivalent Return $CER[CRRA(\gamma)_{TN}](\sigma, \mu)$ becomes:

(2.4) $$CER[CRRA(\gamma)_{TN}](\sigma,\mu) = \left[\int_{k_1}^{k_2} \frac{f_{TN}(x)}{(1+x)^{\gamma}} dx\right]^{-\frac{12}{\gamma}} - 1 = \left[\frac{\int_{k_1}^{k_2} \frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^{\gamma}} dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2} dx}\right]^{-\frac{12}{\gamma}} - 1$$

that for $\gamma = 0$ becomes:

(2.5) $$CER[CRRA(0)_{TN}](\sigma,\mu) = e^{12E[\ln(1+R)]} - 1 = exp\left[12\frac{\int_{k_1}^{k_2} ln(1+x)e^{-(x-\mu)^2/2\sigma^2} dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2} dx}\right] - 1$$

To graph $CER[CRRA(\gamma)_{TN}](\sigma,\mu)$ in the space $[\sigma_{TN}(\sigma,\mu), \mu_{TN}(\sigma,\mu), CER[CRRA(\gamma)_{TN}](\sigma,\mu)]$, we use the Differential Geometry and briefly we introduce the transformation between spaces.

We compute the Mean, $\mu_{TN}(\sigma,\mu)$, and the Standard Deviation, $\sigma_{TN}(\sigma,\mu)$, of the Truncated Normal variable *R*, that are functions of (σ,μ) and represent a transformation $(\sigma,\mu) \rightarrow (\sigma_{TN},\mu_{TN})$:

(2.6)
$$\sigma_{TN}(\sigma,\mu) = \sqrt{\frac{\int_{k_1}^{k_2} x^2 e^{-(x-\mu)^2/2\sigma^2} dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2} dx} - \left[\frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2/2\sigma^2} dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2} dx}\right]^2}$$

$$\mu_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2/2\sigma^2} dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2} dx}$$

The condition we have to impose is that the transformation $(\sigma,\mu) \rightarrow (\sigma_{TN},\mu_{TN})$ defined by (2.6) is bijective, that is for every point (σ,μ) there corresponds only one point (σ_{TN},μ_{TN}) and vice-versa. It is well known that a necessary condition for bijective mappings is that the determinant of the Jacobian matrix *J* must be different from zero:

(2.7) $$\det(J) = det\begin{pmatrix} \frac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\sigma} & \frac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\mu} \\ \frac{\partial\mu_{TN}(\sigma,\mu)}{\partial\sigma} & \frac{\partial\mu_{TN}(\sigma,\mu)}{\partial\mu} \end{pmatrix} \neq 0$$

Using parametric representation with the parameters (σ,μ), it will be possible to graph the surface defined by the three functions $\sigma_{TN}(\sigma,\mu), \mu_{TN}(\sigma,\mu), CER[CRRA(\gamma)_{TN}](\sigma,\mu)$ in the parametric space $[\sigma_{TN}(\sigma,\mu), \mu_{TN}(\sigma,\mu), CER[CRRA(\gamma)_{TN}](\sigma,\mu)]$ where:

x axis = $\sigma_{TN}(\sigma,\mu)$.
y axis = $\mu_{TN}(\sigma,\mu)$.
z axis = $CER[CRRA(\gamma)_{TN}](\sigma,\mu)$

The three functions $\sigma_{TN}(\sigma,\mu), \mu_{TN}(\sigma,\mu), CER[CRRA(\gamma)_{TN}](\sigma,\mu)$ depend on (σ,μ) defined in $[(\sigma_{Min},\sigma_{Max})\times(\mu_{Min},\mu_{Max})]$ in the Cartesian space (σ,μ).

Using vectorial notation, the surface is defined by the vector $\boldsymbol{r}(\sigma,\mu)$ in the space $[\sigma_{TN}(\sigma,\mu), \mu_{TN}(\sigma,\mu), CER[CRRA(\gamma)_{TN}](\sigma,\mu)]$, where $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ are the relative unit vectors:

(2.8) $$\boldsymbol{r}(\sigma,\mu) = \sigma_{TN}(\sigma,\mu)\boldsymbol{i} + \mu_{TN}(\sigma,\mu)\boldsymbol{j} + CER[CRRA(\gamma)_{TN}](\sigma,\mu)\boldsymbol{k}$$

For regularity of the surface, the Jacobian Matrix J_1:

(2.9) $$J_1 = \begin{pmatrix} \dfrac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\sigma} & \dfrac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\mu} \\ \dfrac{\partial\mu_{TN}(\sigma,\mu)}{\partial\sigma} & \dfrac{\partial\mu_{TN}(\sigma,\mu)}{\partial\mu} \\ \dfrac{\partial CER[CRRA(\gamma)_{TN}](\sigma,\mu)}{\partial\sigma} & \dfrac{\partial CER[CRRA(\gamma)_{TN}](\sigma,\mu)}{\partial\mu} \end{pmatrix}$$

must have rank two; e.g. this condition is satisfied if (2.7) is true (as is proved in Appendix A).

Now we can graph with parametric representation the $CER[CRRA(\gamma)_{TN}](\sigma,\mu)$ for $\gamma = 0, 2, 5, 10$.
We consider a range:

$$[(\sigma_{Min} \div \sigma_{Max})\times(\mu_{Min} \div \mu_{Max})] = [(0.001 \div 0.150)\times(-0.100 \div 0.100)]$$

that will be transformed by (2.6) in the range:

(2.10) $$\left[\left(\sigma_{TN\_Min} \div \sigma_{TN\_Max}\right)\times\left(\mu_{TN\_Min} \div \mu_{TN\_Max}\right)\right] = [(0.001 \div 0.150)\times(-0.100 \div 0.100)]$$

This ostensible equality between the numeric values of the ranges is due to the narrow amplitude of the ranges; if we use a wider range for $[(\sigma_{Min},\sigma_{Max})\times(\mu_{Min},\mu_{Max})]$, we reach a considerable difference between the ranges, due to the non linearity of the transformation (see Appendix A).

(2.10) is a reasonable range, taking in account that it represents a monthly value. E.g., S&P500 had a monthly volatility of 0.090 in a window of 12 months from 2006 until 2014, and 0.04 for average monthly Return in the same window.

Figures 2.2.1÷2.2.8 show the 3D representation and the Iso-utility curves for $CER[CRRA(\gamma)_{TN}]$ *for* $\gamma = 0, 2, 5, 10$.

Figure 2.2.1: 3D for $\gamma = 0$

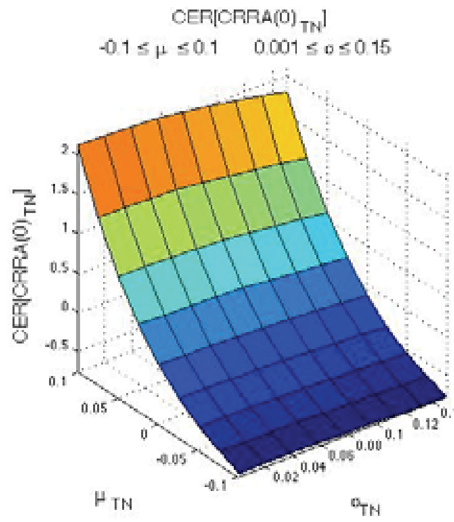


Figure 2.2.2: Iso-utility curves for $\gamma = 0$

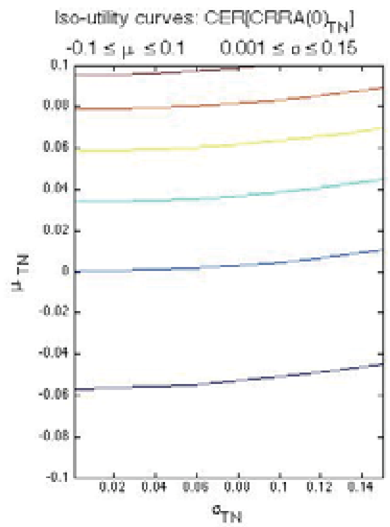


Figure 2.2.3: 3D for $\gamma = 2$

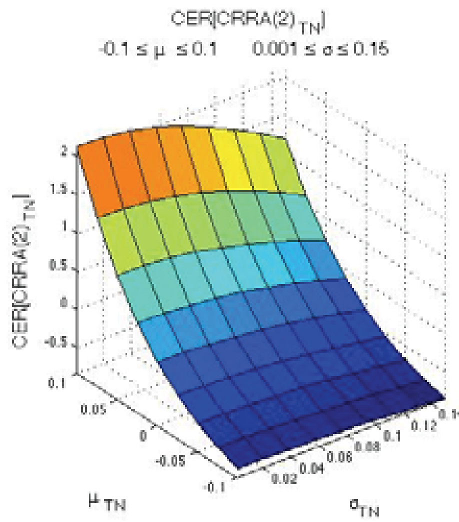


Figure 2.2.4: Iso-utility curves for $\gamma = 2$

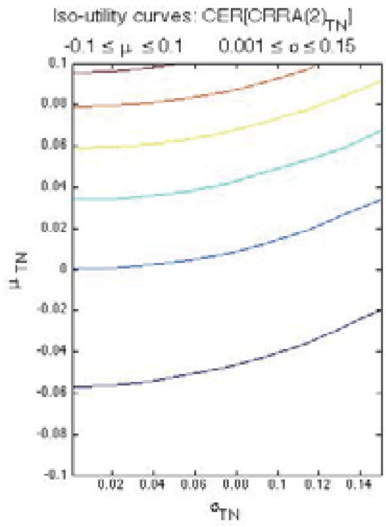
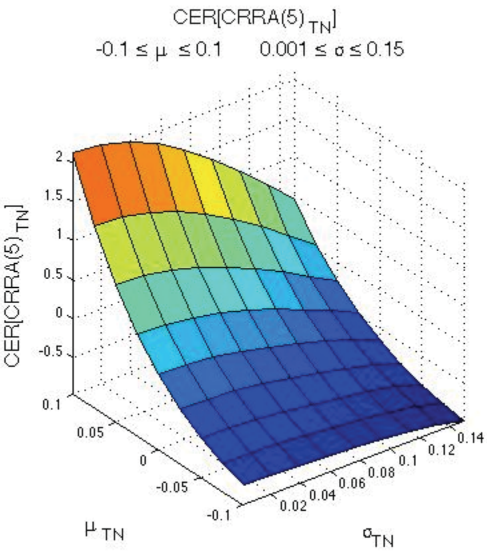


Figure 2.2.5: 3D for $\gamma = 5$ *Figure 2.2.6: Iso-utility curves for* $\gamma = 5$



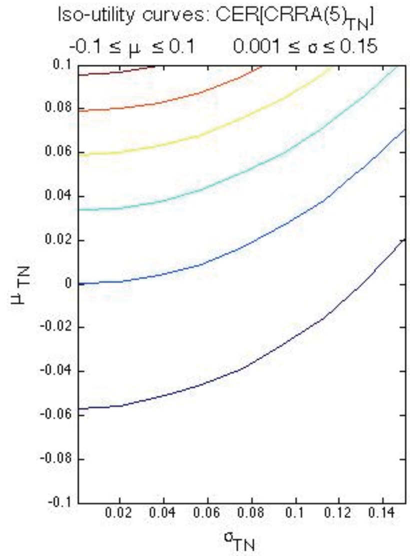
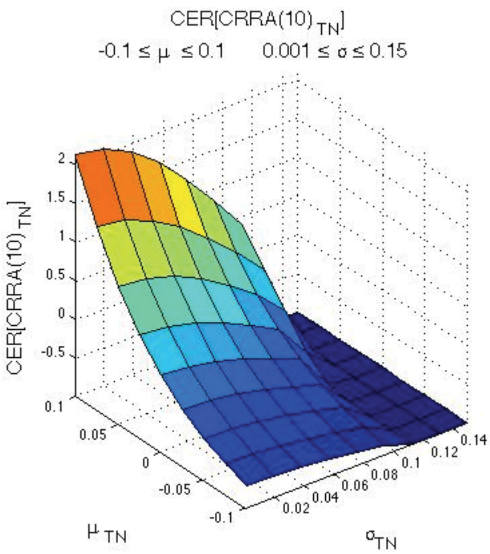
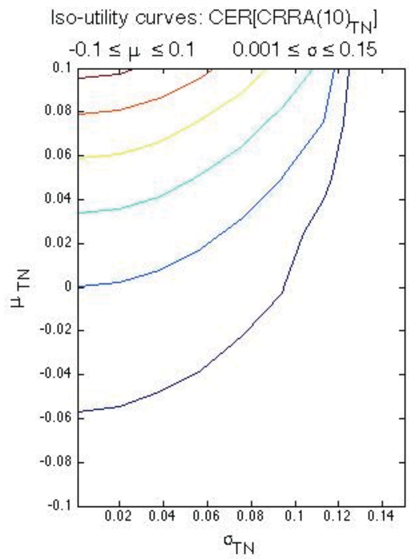


Figure 2.2.7: 3D for $\gamma = 10$ *Figure 2.2.8: Iso-utility curves for* $\gamma = 10$





We can note in Figure 2.2.7 that for σ_{TN} values approximately greater than 0.12, $CER[CRRA(10)_{TN}]$ has very flat values.

Consider now a generic pair $(\sigma_j, \mu_j) \subset [(\sigma_{Min} \div \sigma_{Max})\times(\mu_{Min} \div \mu_{Max})], j = 1, \dots, J$. Due to the bijective transformation (2.6) it corresponds to the pair $(\sigma_{TN,j}, \mu_{TN,j}) \subset [(\sigma_{TN\_Min} \div \sigma_{TN\_Max})\times(\mu_{TN\_Min} \div \mu_{TN\_Max})]$.

Every pair $(\sigma_{TN,j}, \mu_{TN,j})$ can represent a Fund, F_j, which has the (monthly) Return $\{R_{TN,j}; j = 1, \dots, K\}$ described by a Truncated Normal distribution.

We can graph every F_j in the space $[\sigma_{TN}(\sigma,\mu), \mu_{TN}(\sigma,\mu), CER[CRRA(\gamma)_{TN}](\sigma,\mu)]$ from a point with coordinates $[\sigma_{TN,j}(\sigma_j,\mu_j), \mu_{TN,j}(\sigma_j,\mu_j), CER[CRRA(\gamma)_{TN,j}](\sigma_j,\mu_j)]$, and it is possible to measure the dependency of $CER[CRRA(\gamma)_{TN,j}](\sigma_j,\mu_j)$ on $\sigma_{TN,j}(\sigma_j,\mu_j)$ (Section 5) followed by the induced ranking (Section 6).

3. The specific case of Morningstar Rating methodology.

We consider now the link between (2.2) and the Morningstar Rating.

Given a period of T months, the Morningstar Risk-Adjusted Return for a Fund F_j $\{ j = 1, \dots, J\}$ is defined as follows[2]:

$$(3.1) \qquad MRAR(\gamma)_j = \begin{cases} MRAR(2)_j = \left[\frac{1}{T}\sum_{t=1}^{T}(1 + ER_{t,j})^{-2}\right]^{\frac{-12}{2}} - 1, & \gamma = 2 \\ MRAR(0)_j = \left[\prod_{t=1}^{T}(1 + ER_{t,j})\right]^{\frac{12}{T}} - 1, & \gamma = 0 \end{cases}$$

where:

$$ER_{t,j} = \frac{1 + LR_{t,j}}{1 + Rf_t} - 1$$

is the monthly geometric excess return. $LR_{t,j}$ is the monthly return including the commissions and Rf_t is the monthly return of the risk free rate. In this paper we consider $Rf_t = 0$ and no commissions, meaning that we can consider the monthly return $R_{t,j}$ instead of the $ER_{t,j}$.

Morningstar considers the value $\gamma = 2$ consistent with the risk aversion of the typical retail customers and uses the values of $MRAR(2)_j$ to rank the Funds.

$MRAR(\gamma)_j$ are the values of the Certainty Equivalent described in (2.2), for $\gamma = 2$ and $\gamma = 0$ using the time series average of $(1 + R_{t,j})^{-2}$ and of $ln\,(1 + R_{t,j})$ as an estimate of $E\left[(1 + R_j)^{-2}\right]$ and $E[ln\,(1 + R_j)]$ computed for a generic Fund F_j.

Consider now J Funds $\{F_j\,;\, j = 1, \dots, J\}$, each of which has a sequence of T (monthly) Returns $\{R_{TN,t,j}\,;\, t = 1, \dots, T;\, j = 1, \dots, J\}$ with Truncated Normal distribution $f_{TN,j}(R)$. For every F_j, the Returns $R_{TN,t,j}$ are independent and identically distributed with Expected Return $\mu_{TN,j}$ and Standard Deviation $\sigma_{TN,j}$. It is possible to compute the sample Expected Return and the sample Standard Deviation by using:

[2] Morningstar (2009), pp. 11-12.

$$\mu_{TN,T,j} = T^{-1} \sum_{t=1}^{T} R_{TN,t,j} \quad \sigma_{TN,T,j} = \left\{ (T-1)^{-1} \sum_{t=1}^{T} \left[R_{TN,t,j} - \mu_{TN,T,j} \right]^2 \right\}^{1/2}.$$

It is well known that $lim_{T\to\infty} \mu_{TN,T,j} = \mu_{TN,j}$ and $lim_{T\to\infty} \sigma_{TN,T,j} = \sigma_{TN,j}$.

The empirical pdf is : $\quad f_{TN,T,j}(R) = \frac{1}{T}\sum_{t=1}^{T} \delta\left(R - R_{TN,t,j}\right)$

where $\delta\left(R - R_{TN,t,j}\right)$ is the Delta Dirac function centered in $R_{TN,t,j}$.

By the Glivenko-Cantelli Theorem we have:

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \delta\left(R - R_{TN,t,j}\right) = f_{TN,j}(R)$$

Consider now the (3.1), again in the hypothesis of R distributed as a Truncated Normal, for $T \to \infty$:

$$\lim_{T\to\infty} \left[\frac{1}{T} \sum_{t=1}^{T} \frac{1}{(1+R_{TN,t,j})^2} \right]^{-\frac{12}{2}} - 1 = \left[\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \frac{1}{\left(1+R_{TN,t,j}\right)^2} \right]^{-\frac{12}{2}} - 1$$

$$= \left[\int_{k_1}^{k_2} \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \delta\left(x - R_{TN,t,j}\right) \frac{1}{(1+x)^2} dx \right]^{-\frac{12}{2}} - 1$$

$$= \left[\int_{k_1}^{k_2} \frac{f_{TN,j}(x)}{(1+x)^2} dx \right]^{-\frac{12}{2}} - 1$$

that is, we find the expression (2.4) computed for $\gamma = 2$ and for the Fund F.

Thus, with the hypothesis of R distributed as a Truncated Normal, we will consider $CER\left[CRRA(2)_{TN,j}\right]\left(\sigma_j, \mu_j\right)$ as the $\lim_{T\to\infty} MRAR(2)_j$. With the same rationale, $CER\left[CRRA(0)_{TN,j}\right]\left(\sigma_j, \mu_j\right)$ is the $\lim_{T\to\infty} MRAR(0)_j$.

Note that $MRAR(0)_j$ is the Annualized Geometric Return computed for the Fund F_j, denoted by AGR_j: in this way we assume that, measuring satisfaction using $CER[CRRA(0)_{TN}]$ is the same as if we measured satisfaction based on the AGR for large T. This is coherent with the fact that maximizing the AGR is equivalent to maximizing the expected value of log-utility function, that is the (2.1) for $\gamma = 0$.

Consequently, the well-known Morningstar Risk, *MRisk*, defined by:

$$MRisk_j = MRAR(0)_j - MRAR(2)_j$$

can also be computed in the Truncated Normal hypothesis as:

$$MRisk_{TN,j}\left(\sigma_j, \mu_j\right) \equiv MRisk_{TN,j} = CER\left[CRRA(0)_{TN,j}\right]\left(\sigma_j, \mu_j\right) - CER\left[CRRA(2)_{TN,j}\right]\left(\sigma_j, \mu_j\right)$$

In the hypothesis done for the distribution of $R_{t,j}$, $CER\left[CRRA(2)_{TN,j}\right](\sigma_j, \mu_j)$ is a good representation of $MRAR(2)_j$ behavior.

4. Quadratic Utility Function

Consider the following general Quadratic Utility Function (QUF):

(4.1) $$QUF(W) \equiv QUF = a + bW - cW^2 \qquad b, c > 0$$

where W is defined as in (2.1).

If the function (4.1) has positive first derivative and negative second derivative, it represents a risk-averse person with insatiable appetite, that is:

$$QUF' = b - 2cW > 0 \Longrightarrow W < \frac{b}{2c} \equiv W_0(1+\mu_M)$$
$$QUF'' = -2c < 0 \qquad \Longrightarrow c > 0$$

(4.2) $$ARA[QUF] = -\frac{QUF''}{QUF'} = \frac{2c}{b - 2cW} > 0 \ , RRA[QUF] = \frac{2cW}{b - 2cW}$$

Without stating a hypothesis on the distribution of the (monthly) return R, we define the expected return $E[R]$, the standard deviation $SD[R] = E[(R - E[R])^2]$ and $\mu_M = max\{E[R] \,|W \leq W_0(1 + \mu_M)\}$.
$W_0(1 + \mu_M)$ is the maximum value allowed for W such that (4.1) maintain its characteristic of Risk aversion.

Proposition 4.1: *With the definition* $b = 2cW_0(1 + \mu_M)$, *the expected value of QUF in* (4.1), $E[Q(\mu_M)](SD[R], E[R])$, *is a function of both Standard Deviation* $SD[R]$ *and Expected Return* $E[R]$ *represented by a paraboloid in the space* $(SD[R], E[R], E[Q(\mu_M)](SD[R], E[R]))$ *with downward concavity, whose vertex is given by the point* $\left(0, \mu_M, E[Q(\mu_M)](0, \mu_M)\right)$. *That is:*

$$E[Q(\mu_M)](SD[R], E[R]) = QUF(W_0) + cW_0^2\mu_M^2 - cW_0^2[SD[R]^2 + (E[R] - \mu_M)^2]$$

where $QUF(W_0) = a + bW_0 - cW_0^2 = a + 2cW_0(1 + \mu_M)W_0 - cW_0^2$.

The Certainty Equivalent Return for the Quadratic Utility Function centered in μ_M, $CER[Q(\mu_M)]$, *has the following annualized expression:*

(4.3) $$CER[Q(\mu_M)](SD[R], E[R]) \equiv CER[Q(\mu_M)] = \left(1 + \mu_M - \sqrt{SD[R]^2 + (E[R] - \mu_M)^2}\right)^{12} - 1$$

$CER[Q(\mu_M)](SD[R],E[R])$ represent a squeezed cone in the space $(SD[R],E[R],E[Q(\mu_M)](SD[R],E[R])$), whose vertex is given by the point $\left(0,\mu_M,CER[Q(\mu_M)](0,\mu_M)\right)$.

Proof: Appendix B.

For the ARA and RRA computed for QUF with μ_M parameter we have:

$$(4.4) \qquad ARA[Q(\mu_M)] = \frac{1}{W_0(\mu_M - E[R])}\ , \qquad RRA[Q(\mu_M)] = \frac{1+E[R]}{\mu_M - E[R]}$$

This expression implies that (absolute and relative) risk aversion increases with increments of expected return.
The economic literature claims this aspect of the monotone increasing implied by the quadratic utility function is unrealistic. In spite of this problem, QUF plays an important role in portfolio analysis because it is perfectly consistent with the mean-variance analysis[3]. Furthermore, the quadratic function is very useful because it can be seen, according to the Taylor expansion, as the second-order approximation of any utility function[4].

Until now, the distribution for Return involved in $CER[Q(\mu_M)]$ has not been defined. To compare its behavior with the $CER[CRRA(\gamma)_{TN}]$, we also apply the (4.3) to a Return with a Truncated Normal distribution denoted with $CER[Q(\mu_M)_{TN}]$.

For $CER[Q(\mu_M)_{TN}]$ we choose $\mu_M = 0.15$ greater than $\mu_{TN\_Max} = 0.10$, see (2.10), due to the fact that for μ_M=μ_{TN_Max} the First Derivative of the Implicit Function, defined in Section 5, becomes infinite and that for $\mu_M < \mu_{TN\_Max}$ the concavity of the (4.3) has not the risk averse properties.

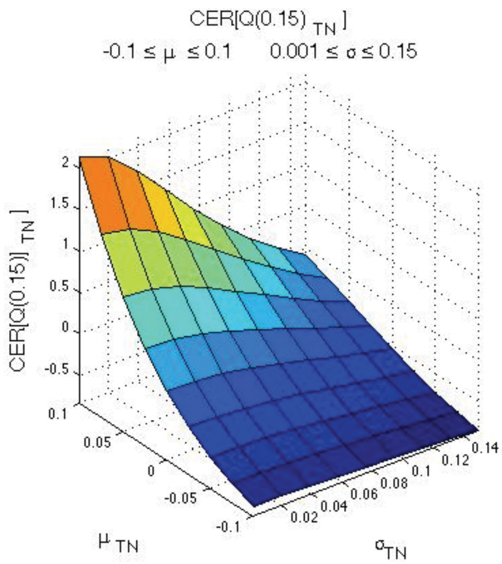
Figure 4.1.1 shows the three dimensional graphs representing the $CER[Q(\mu_M)_{TN}]$.and Figure 4.1.2 shows the iso-utility curves that we can obtain from the equality $CER[Q(\mu_M)_{TN}] = K$, where K is a constant.

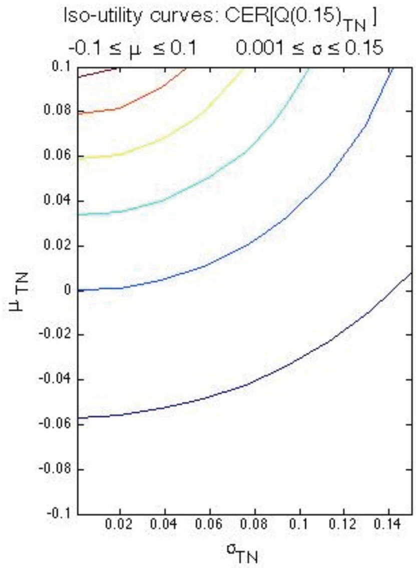
---

[3] Very often investors take their decisions on the basis of both Expected Returns and Standard Deviations parameters.

[4] For example, Quadratic Utility function provides an excellent approximation of Logarithmic Utility function, which exhibits decreasing absolute and constant relative risk aversion. Using historical security return data, Pulley (1983) shows that the approximations are very good, and in many cases the optimal portfolios, computed with the maximization of the expected logarithmic utility, are virtually identical compared with those obtained maximizing appropriate mean-variance formulations.

Figure 4.1.1: 3D $CER[Q(\mu_M)_{TN}]$ *for* $\mu_M = 0.15$ *Figure 4.1.2: Iso-utility curves for* $\mu_M = 0.15$





Again, the $CER[Q(\mu_M)_{TN}]$ is computed for a set of Funds F_j, and consequently its value is $CER[Q(\mu_M)_{TN,j}](\sigma_j, \mu_j)$.

5. A measure of the dependence on σ_{TN}: First Derivative of the Implicit Function.

As stated above, the first aim of this paper is to show that $CER[CRRA(\gamma)_{TN}]$ has a lack of dependence on σ_{TN}. For this purpose, we compare the dependence with that of the $CER[CRRA(0)_{TN}]$. It will be observed that this dependence is very similar for low values of γ. To better underline this point, we also compare the dependence of the $CER[CRRA(0)_{TN}]$ with $CER[Q(\mu_M)_{TN}]$, noticing the different result.
As an analytic measure of the dependence by σ_{TN}, we compute the First Derivative of the Implicit Function (FDIF) μ_{TN} defined by the level curve of the $CER[CRRA(\gamma)](\sigma, \mu)$ in the space $[\sigma_{TN}(\sigma, \mu), \mu_{TN}(\sigma, \mu), CER[CRRA(\gamma)_{TN}](\sigma, \mu)]$. The FDIF has a parametric representation with the parameters (σ, μ), and is drawn in the space mentioned above.

(5.1) $$\frac{d\mu_{TN}\{CER[CRRA(\gamma)]_{TN}\}(\sigma, \mu)}{d\sigma_{TN}} = -\frac{\dfrac{\partial CER[CRRA(\gamma)]_{TN}}{\partial \sigma_{TN}}}{\dfrac{\partial CER[CRRA(\gamma)]_{TN}}{\partial \mu_{TN}}}$$

In Appendix C the generic expression for the FDIF is computed, through the transformation $(\sigma, \mu) \rightarrow (\sigma_{TN}, \mu_{TN})$.
In Appendix D it is shown that (5.1) is equal to:

$$(5.2)\quad \frac{d\mu_{TN}\{CER[CRRA(\gamma)_{TN}]\}(\sigma,\mu)}{d\sigma_{TN}} = \frac{\frac{\partial\mu_{TN}}{\partial\sigma}\frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\mu} - \frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\sigma}\frac{\partial\mu_{TN}}{\partial\mu}}{\frac{\partial\sigma_{TN}}{\partial\sigma}\frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\mu} - \frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\sigma}\frac{\partial\sigma_{TN}}{\partial\mu}}$$

and that it is also possible to find all the partial derivatives involved in (5.2).
In Appendix E we find all the elements to compute:

$$(5.3)\quad \frac{d\mu_{TN}\{CER[CRRA(0)_{TN}]\}(\sigma,\mu)}{d\sigma_{TN}} = \frac{\frac{\partial\mu_{TN}}{\partial\sigma}\frac{\partial CER[CRRA(0)_{TN}]}{\partial\mu} - \frac{\partial CER[CRRA(0)_{TN}]}{\partial\sigma}\frac{\partial\mu_{TN}}{\partial\mu}}{\frac{\partial\sigma_{TN}}{\partial\sigma}\frac{\partial CER[CRRA(0)_{TN}]}{\partial\mu} - \frac{\partial CER[CRRA(0)_{TN}]}{\partial\sigma}\frac{\partial\sigma_{TN}}{\partial\mu}}$$

Finally, in Appendix F:

$$(5.4)\quad \frac{d\mu_{TN}\{CER[Q(\mu_M)_{TN}]\}(\sigma,\mu)}{d\sigma_{TN}} = \frac{\frac{\partial\mu_{TN}}{\partial\sigma}\frac{\partial CER[Q(\mu_M)_{TN}]}{\partial\mu} - \frac{\partial CER[Q(\mu_M)_{TN}]}{\partial\sigma}\frac{\partial\mu_{TN}}{\partial\mu}}{\frac{\partial\sigma_{TN}}{\partial\sigma}\frac{\partial CER[Q(\mu_M)_{TN}]}{\partial\mu} - \frac{\partial CER[Q(\mu_M)_{TN}]}{\partial\sigma}\frac{\partial\sigma_{TN}}{\partial\mu}}$$

The expressions above allow to graphs the FDIF defined by $CER[CRRA(\gamma)_{TN}]$:

Figure 5.1.1: FDIF for $\gamma = 0$

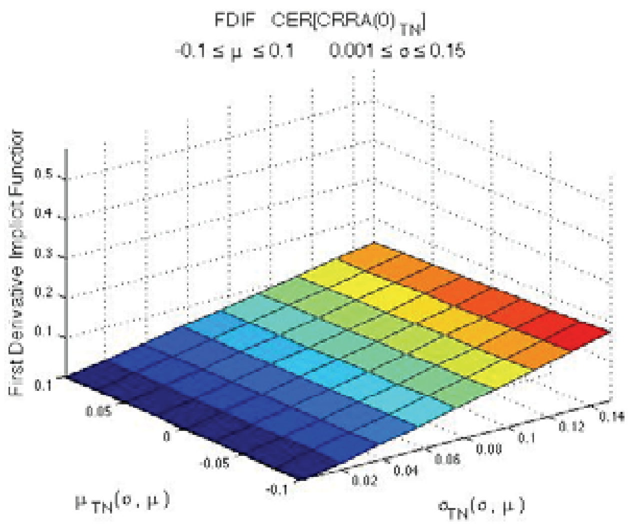


Figure 5.1.2: FDIF for $\gamma = 2$

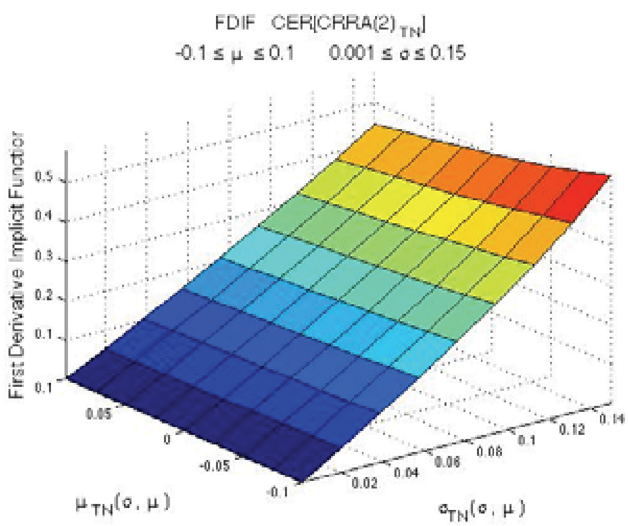
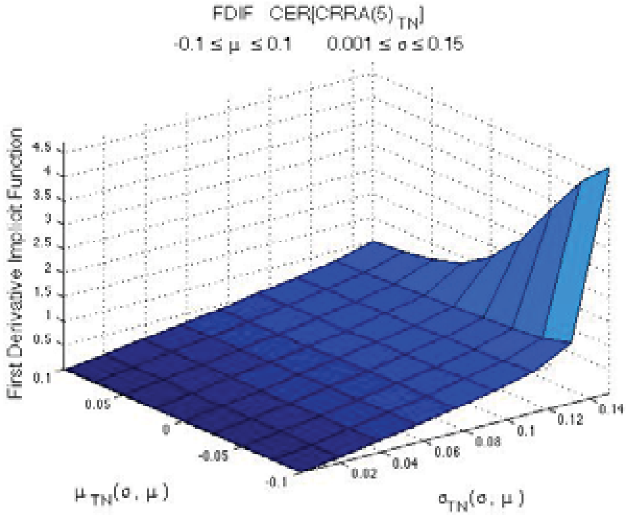
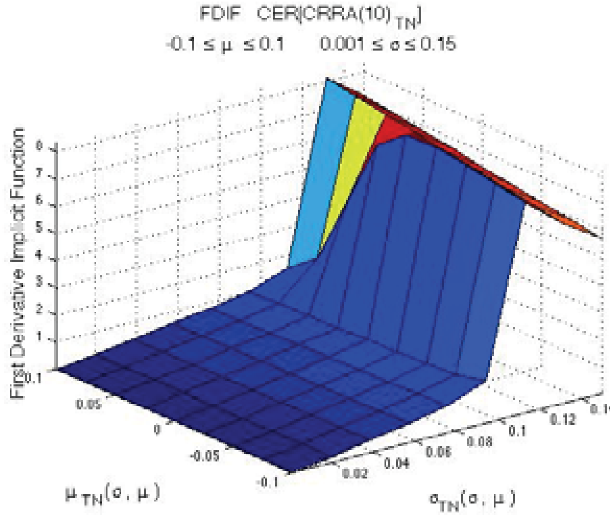


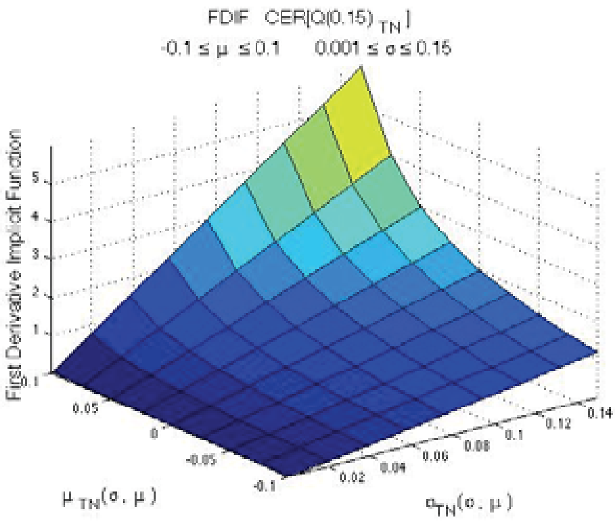
Figure 5.1.3: FDIF for $\gamma = 5$ *Figure 5.1.4: FDIF for $\gamma = 10$*





and by $CER[Q(0.15)_{TN}]$:

Figure 5.2: FDIF defined by $CER[Q(0.15)_{TN}]$



Note that all four Figures 5.1 have quite a linear dependence, obviously until different values of σ_{TN}; this will be quantified in the following Tables 5.1 ÷5.4.
Finally, as a measure of the dependence by σ_{TN}, we define:

$$\Delta\%[CC(\gamma,0)](\sigma,\mu) = \frac{(5.2)-(5.3)}{(5.3)}$$

(5.5)

$$\Delta\%[QC(\mu_M,0)](\sigma,\mu) = \frac{(5.4)-(5.3)}{(5.3)}$$

that are drawn below.

Figure 5.3.1: $\Delta\%[CC(2,0)]$

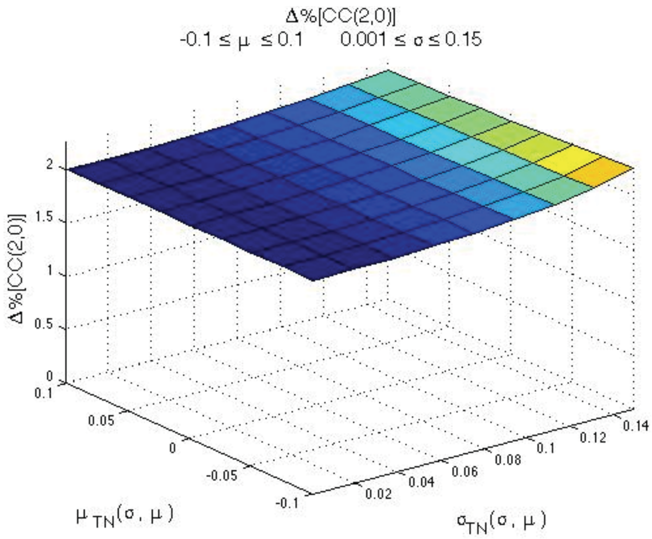


Figure 5.3.2: $\Delta\%[CC(5\,,0)]$

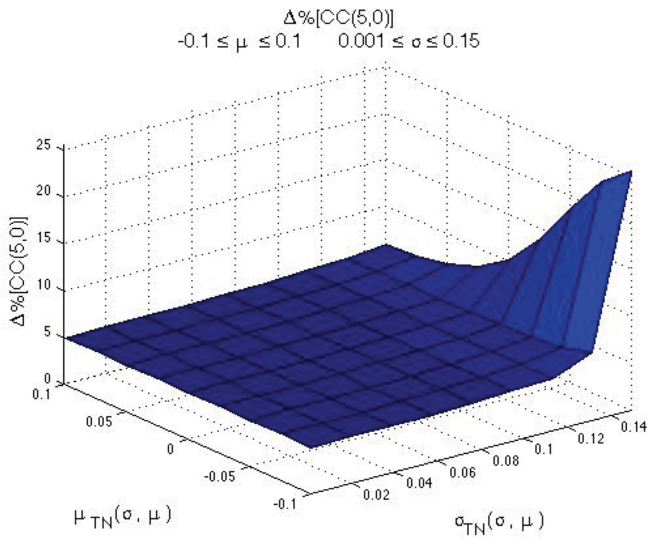


Figure 5.3.3: $\Delta\%[CC(10,0)]$

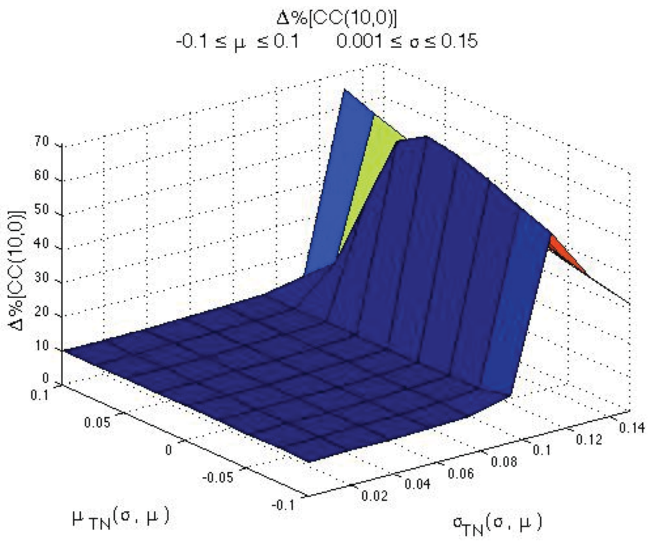
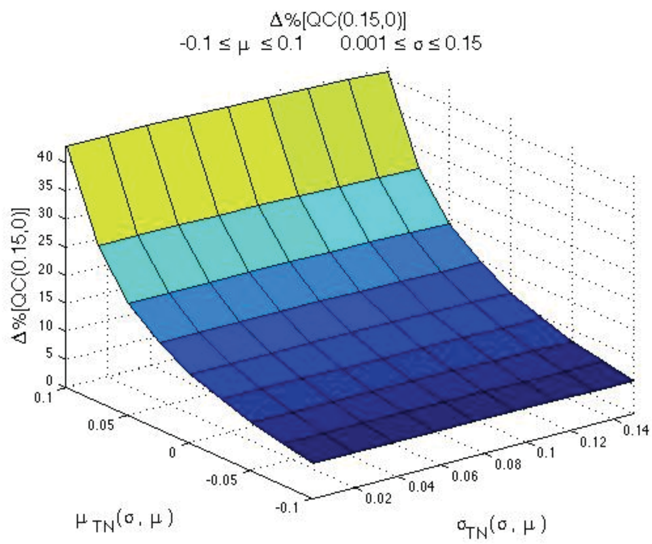


Figure 5.3.4: $\Delta\%[QC(0.15,0)]$



We divide the range (2.10) in 81 cross points, for each of which we compute the values of (5.5) that it is possible to see in the following tables; the values for σ_{TN} are indicated in the last row and the values for μ_{TN} are indicated in the first column.

Table 5.1: $\Delta\%[CC(2,0)]\sigma_{TN}, \mu_{TN}$

μ_{TN}	Δ%[CC(2,0)]								
0.100	2.000	2.003	2.011	2.025	2.046	2.075	2.115	2.172	2.260
0.075	2.000	2.003	2.010	2.024	2.043	2.070	2.108	2.159	2.237
0.050	2.000	2.003	2.010	2.022	2.041	2.066	2.101	2.148	2.217
0.025	2.000	2.002	2.009	2.021	2.039	2.063	2.095	2.139	2.200
0.000	2.000	2.002	2.009	2.020	2.037	2.059	2.089	2.130	2.186
-0.025	2.000	2.002	2.008	2.019	2.035	2.056	2.084	2.122	2.173
-0.050	2.000	2.002	2.008	2.018	2.033	2.053	2.080	2.115	2.162
-0.075	2.000	2.002	2.008	2.017	2.031	2.050	2.075	2.108	2.152
-0.100	2.000	2.002	2.007	2.017	2.030	2.048	2.072	2.103	2.143
σ_{TN}	0.001	0.020	0.038	0.057	0.076	0.094	0.113	0.131	0.150

Table 5.2: $\Delta\%[CC(5,0)](\sigma_{TN}, \mu_{TN})$

μ_{TN}	Δ%[CC(5,0)]								
0.100	5.000	5.014	5.056	5.129	5.241	5.408	5.667	7.228	25.594
0.075	5.000	5.014	5.053	5.121	5.226	5.381	5.615	6.425	22.939
0.050	5.000	5.013	5.050	5.115	5.213	5.357	5.570	6.062	18.572
0.025	5.000	5.012	5.047	5.108	5.201	5.335	5.531	5.886	13.833
0.000	5.000	5.012	5.045	5.103	5.190	5.315	5.496	5.786	10.176
-0.025	5.000	5.011	5.043	5.098	5.180	5.297	5.465	5.718	7.972
-0.050	5.000	5.011	5.041	5.093	5.170	5.281	5.437	5.665	6.826
-0.075	5.000	5.010	5.039	5.088	5.162	5.266	5.411	5.620	6.270
-0.100	5.000	5.010	5.037	5.084	5.154	5.252	5.388	5.581	5.998
σ_{TN}	0.001	0.020	0.038	0.057	0.076	0.094	0.113	0.131	0.150

Table 5.3: $\Delta\%[CC(10,0)](\sigma_{TN}, \mu_{TN})$

μ_{TN}	Δ%[CC(10,0)]								
0.100	10.000	10.053	10.209	10.494	10.969	13.805	58.843	42.247	31.411
0.075	10.000	10.050	10.197	10.465	10.904	11.900	62.393	44.880	33.436
0.050	10.000	10.047	10.187	10.438	10.846	11.548	65.970	47.585	35.516
0.025	10.000	10.045	10.177	10.413	10.794	11.413	69.301	50.362	37.651
0.000	10.000	10.043	10.168	10.391	10.747	11.313	71.037	53.211	39.840
-0.025	10.000	10.041	10.159	10.370	10.704	11.227	65.503	56.131	42.083
-0.050	10.000	10.039	10.151	10.351	10.665	11.151	43.478	59.123	44.382
-0.075	10.000	10.037	10.144	10.334	10.629	11.081	21.226	62.183	46.736
-0.100	10.000	10.035	10.138	10.317	10.597	11.019	13.539	65.303	49.145
σ_{TN}	0.001	0.020	0.038	0.057	0.076	0.094	0.113	0.131	0.150

Table 5.4: $\Delta\%[QC(\mu_M, 0)](\sigma_{TN}, \mu_{TN})$

μTN	Δ%[QC(0,15,0)]								
0.100	6.200	6.193	6.174	6.142	6.096	6.037	5.962	5.870	5.757
0.075	7.222	7.215	7.194	7.159	7.110	7.046	6.966	6.867	6.747
0.050	8.500	8.492	8.469	8.431	8.378	8.308	8.220	8.112	7.982
0.025	10.143	10.134	10.108	10.066	10.007	9.929	9.832	9.713	9.570
0.000	12.333	12.323	12.294	12.246	12.179	12.090	11.980	11.846	11.684
-0.025	15.400	15.388	15.354	15.298	15.219	15.116	14.987	14.831	14.644
-0.050	20.000	19.985	19.944	19.876	19.779	19.654	19.497	19.308	19.081
-0.075	27.667	27.648	27.594	27.505	27.380	27.216	27.013	26.767	26.474
-0.100	43.000	42.972	42.893	42.763	42.579	42.341	42.044	41.685	41.258
σTN	0.001	0.020	0.038	0.057	0.076	0.094	0.113	0.131	0.150

Looking at Table 5.1, where $\gamma = 2$, we see that $\Delta\%[CC(2,0)]$ has high but flat values for all the values of μ_{TN} and for a wide range of σ_{TN}, until $\sigma_{TN} = 0.131$. Obviously we expect high values, but it is unexpected that they remain flat for a wide range. E.g., for the row corresponding to $\mu_{TN} = -0.100$, the values change from 2.000 for $\sigma_{TN} = 0.001$ to 2.103 for $\sigma_{TN} = 0.131$.
It means that, in the region where the difference is so flat, the dependence by σ_{TN} is almost equal between $CER[CRRA(2)_{TN}]$ and $CER[CRRA(0)_{TN}]$, and thus their capability to carry out a ranking is influenced almost in the same manner by σ_{TN}.

In short, using $CER[CRRA(2)_{TN}]$ for ranking, we will have similar results as if we use $CER[CRRA(0)_{TN}]$.
The above considerations lead to the conclusion that also the ranking done by Morningstar suffers from the same drawbacks.

For $\gamma = 5$ the dependence by σ_{TN} is greater, but it remains quite constant even if in a narrower range than for $\gamma = 2$. Note that there is an increase in the value and obviously in the shape as can be seen in Figure 5.3.2 for $\sigma_{TN} \approx 0.12$.

For $\gamma = 10$ the increase is more evident, but after $\sigma_{TN} \approx 0.12$ the values of $\Delta\%[CC(10,0)]$ decrease when σ_{TN} increases. This appears as an anomaly that can be detected also by looking at Figures 2.2.7 and 5.1.4. This change of concavity indicates that, for a value of σ_{TN} greater than 0.12, the sensitivity to σ_{TN} does not change.
It may be a source of misunderstandings that this value exists for σ_{TN}; it seems a cut-off value. In order to explore this aspect, we consider the values of $CER[CRRA(10)_{TN}]$ for $\sigma_{TN} > 0.150$:

Figure 5.4.1: $CER[CRRA(10)_{TN}]$

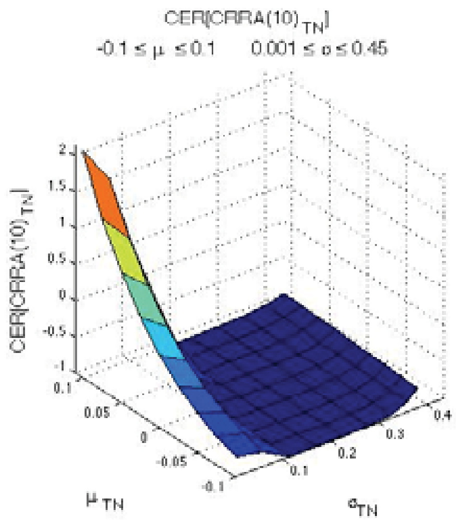
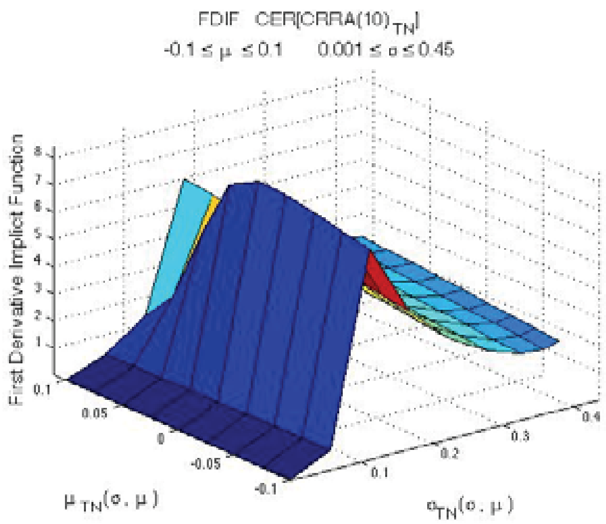


Figure 5.4.2: FDIF $CER[CRRA(10)_{TN}]$



In this case $CER[CRRA(10)_{TN}]$ becomes flat, and its FDIF decreases while σ_{TN} increases: after the value $\sigma_{TN} \approx 0.12$, the sensitivity to σ_{TN} remains constant.

For the $CER[Q(\mu_M)_{TN}]$ case, $\Delta\%[QC(\mu_M, 0)]$ is shown to be a continuous value variable, meaning that this Utility Function, even though it depends on the first two moments, has a gradual dependence on the σ_{TN} without steps.

6. The Ranking's Dependence on Standard Deviation

It is possible to compare the risk-adjusted Funds using CER for all cases mentioned above. An alternative to utility theory is simply to select the Fund that has the highest AGR; it is well known that maximizing the AGR is equivalent to maximizing the expected value of log-utility function, that is, the (2.1) for $\gamma = 0$.
We divide the range (2.10):

$$\left[\left(\sigma_{TN\_Min} \div \sigma_{TN\_Max}\right)\times\left(\mu_{TN\_Min} \div \mu_{TN\_Max}\right)\right] = [(0.001 \div 0.1500)\times(-0.100 \div 0.100)]$$

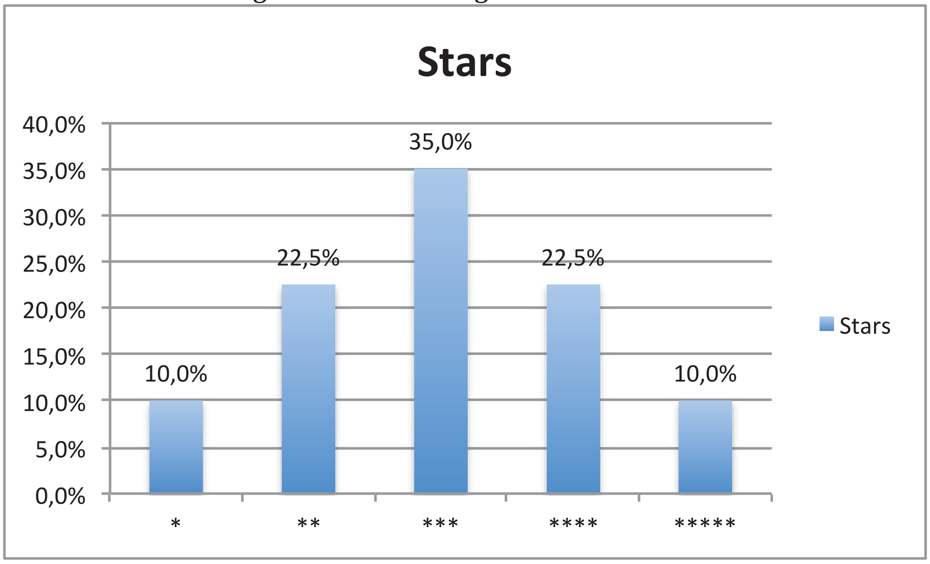
into 9 grids:

$$\left[\left(\sigma_{TN\_Min(i)} \div \sigma_{TN\_Max(i)}\right)\times\left(\mu_{TN\_Min(i)} \div \mu_{TN\_Max(i)}\right)\right], i \subset [1, \dots 9]$$

and each of the grids contains 169 (=13*13) equally spaced cross points; every point represents a Fund, with its values of $\left(\sigma_{TN,j}, \mu_{TN,j}\right)$ for which will be computed the values of $CER\left[CRRA(\gamma)_{TN,j}\right]$ and $CER\left[Q(\mu_M)_{TN,j}\right]$.

We use the same ranking criteria used by Morningstar. The Funds are scored by the following bell curve:

Figure 6.1: Morningstar's bell curve



The Funds are sorted using the values of the $CER[CRRA(\gamma)_{TN}]$ or $CER[Q(\mu_M)_{TN}]$ and the rating is assigned as follows:

5 stars to the first 10.0%,
4 stars to the next 22.5%
3 stars to the next 35.0%
2 stars to the next 22.5%
1 star to the last 10.0%.

We compare a ranking based on $CER[CRRA(\gamma)_{TN}]$, for $\gamma = 2, 5, 10$ and based on $CER[Q(\mu_M)_{TN}]$ with a ranking induced by $CER[CRRA(0)_{TN}]$. The lack of dependence on σ_{TN} for lower γ suggests that the ranking may be quite similar, at least in the region pointed out in Section 5 where $CER[CRRA(\gamma)_{TN}]$ has a lack of dependence on σ_{TN}.

So we have simulated the behavior and the capability to supply a Rating by $CER[CRRA(\gamma)_{TN}]$ and $CER[Q(\mu_M)_{TN}]$ for every grid separately, to point out the grids where a ranking done using $CER[CRRA(\gamma)_{TN}]$ may be equivalent to that of $CER[CRRA(0)_{TN}]$ and if there is a difference with respect to the ranking done with $CER[Q(\mu_M)_{TN}]$.
We count how many changes of Rating we have for every grid, in absolute and in percentage values.
For example, in the first row of the following table, we have for grid 2 $[(0.052 \div 0.099)\times(-0.100 \div -0.037)]$ the value 16 in column Δ Rating: this means that only 16 of the 169 Funds, for the grid 2 have a different rank from the ones defined by $CER[CRRA(0)_{TN}]$, and we have 8 Funds with a difference of +1 star and 8 with -1 star.
For grid 1 we have 0: this means that, if we have all the 169 Funds concentrated in this grid, it is equivalent to ranking with $CER[CRRA(2)_{TN}]$ or with $CER[CRRA(0)_{TN}]$.

Table 6.1: Number of Funds that have changed Rating for $\gamma = 2$

grid	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ Rating
1	0.001	0.048	-0.100	-0.037	0	0	0	169	0	0	0	0
2	0.052	0.099	-0.100	-0.037	0	0	8	153	8	0	0	16
3	0.103	0.150	-0.100	-0.037	0	0	19	131	19	0	0	38
4	0.001	0.048	-0.032	0.032	0	0	0	169	0	0	0	0
5	0.052	0.099	-0.032	0.032	0	0	8	153	8	0	0	16
6	0.103	0.150	-0.032	0.032	0	0	17	135	17	0	0	34
7	0.001	0.048	0.037	0.100	0	0	0	169	0	0	0	0
8	0.052	0.099	0.037	0.100	0	0	7	155	7	0	0	14
9	0.103	0.150	0.037	0.100	0	0	17	135	17	0	0	34
Total	0.001	0.150	-0.100	0.100	0	0	76	1369	76	0	0	152

Table 6.2: Number of Funds that have changed Rating for $\gamma = 5$

grid	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ Rating
1	0.001	0.048	-0.100	-0.037	0	0	3	163	3	0	0	6
2	0.052	0.099	-0.100	-0.037	0	0	23	123	23	0	0	46
3	0.103	0.150	-0.100	-0.037	0	7	32	86	42	2	0	83
4	0.001	0.048	-0.032	0.032	0	0	2	165	2	0	0	4
5	0.052	0.099	-0.032	0.032	0	0	21	127	21	0	0	42
6	0.103	0.150	-0.032	0.032	0	1	36	95	36	1	0	74
7	0.001	0.048	0.037	0.100	0	0	2	165	2	0	0	4
8	0.052	0.099	0.037	0.100	0	0	19	131	19	0	0	38
9	0.103	0.150	0.037	0.100	0	1	33	100	35	0	0	69
Total	0.001	0.150	-0.100	0.100	0	9	171	1155	183	3	0	366

Table 6.3: Number of Funds that have changed Rating for $\gamma = 10$

grid	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ Rating
1	0.001	0.048	-0.100	-0.037	0	0	13	143	13	0	0	26
2	0.052	0.099	-0.100	-0.037	0	3	35	91	39	1	0	78
3	0.103	0.150	-0.100	-0.037	4	19	38	48	38	16	6	121
4	0.001	0.048	-0.032	0.032	0	0	12	145	12	0	0	24
5	0.052	0.099	-0.032	0.032	0	1	36	94	38	0	0	75
6	0.103	0.150	-0.032	0.032	4	20	35	49	38	20	3	120
7	0.001	0.048	0.037	0.100	0	0	9	151	9	0	0	18
8	0.052	0.099	0.037	0.100	0	1	35	96	37	0	0	73
9	0.103	0.150	0.037	0.100	5	19	33	49	43	17	3	120
Total	0.001	0.150	-0.100	0.100	13	63	246	866	267	54	12	655

Table 6.4: Number of Funds that have changed Rating for $\mu_M = 0.15$

grid	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ Rating
1	0.001	0.048	-0.100	-0.037	0	0	1	167	1	0	0	2
2	0.052	0.099	-0.100	-0.037	0	0	16	137	16	0	0	32
3	0.103	0.150	-0.100	-0.037	0	0	25	119	25	0	0	50
4	0.001	0.048	-0.032	0.032	0	0	4	161	4	0	0	8
5	0.052	0.099	-0.032	0.032	0	0	22	125	22	0	0	44
6	0.103	0.150	-0.032	0.032	0	1	35	96	37	0	0	73
7	0.001	0.048	0.037	0.100	0	0	11	147	11	0	0	22
8	0.052	0.099	0.037	0.100	0	3	34	92	40	0	0	77
9	0.103	0.150	0.037	0.100	0	9	38	71	46	5	0	98
Total	0.001	0.150	-0.100	0.100	0	13	186	1115	202	5	0	406

The following 4 tables replicate the previous in percentages.
For example, in the following table we have for grid 2 the value 0.095: this means that only 9.5% of the Funds in grid 2 [[0.052÷ 0.099) X (-0.100 ÷ -0.037)] have different ranks from the one defined by $CER[CRRA(0)_{TN}]$, and we have 4.7% Funds with a difference of +1 star and 4.7% with -1 star.

Table 6.5: % of Funds that have changed Rating for $\gamma = 2$

grid	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ% Rating
1	0.001	0.048	-0.100	-0.037	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000
2	0.052	0.099	-0.100	-0.037	0.000	0.000	0.047	0.905	0.047	0.000	0.000	0.095
3	0.103	0.150	-0.100	-0.037	0.000	0.000	0.112	0.775	0.112	0.000	0.000	0.225
4	0.001	0.048	-0.032	0.032	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000
5	0.052	0.099	-0.032	0.032	0.000	0.000	0.047	0.905	0.047	0.000	0.000	0.095
6	0.103	0.150	-0.032	0.032	0.000	0.000	0.101	0.799	0.101	0.000	0.000	0.201
7	0.001	0.048	0.037	0.100	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000
8	0.052	0.099	0.037	0.100	0.000	0.000	0.041	0.917	0.041	0.000	0.000	0.083
9	0.103	0.150	0.037	0.100	0.000	0.000	0.101	0.799	0.101	0.000	0.000	0.201
Total	0.001	0.150	-0.100	0.100	0.000	0.000	0.050	0.900	0.050	0.000	0.000	0.100

Table 6.6: % of Funds that have changed Rating for $\gamma = 5$

grid	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ% Rating
1	0.001	0.048	-0.100	-0.037	0.000	0.000	0.018	0.964	0.018	0.000	0.000	0.036
2	0.052	0.099	-0.100	-0.037	0.000	0.000	0.136	0.728	0.136	0.000	0.000	0.272
3	0.103	0.150	-0.100	-0.037	0.000	0.041	0.189	0.509	0.249	0.012	0.000	0.491
4	0.001	0.048	-0.032	0.032	0.000	0.000	0.012	0.976	0.012	0.000	0.000	0.024
5	0.052	0.099	-0.032	0.032	0.000	0.000	0.124	0.751	0.124	0.000	0.000	0.249
6	0.103	0.150	-0.032	0.032	0.000	0.006	0.213	0.562	0.213	0.006	0.000	0.438
7	0.001	0.048	0.037	0.100	0.000	0.000	0.012	0.976	0.012	0.000	0.000	0.024
8	0.052	0.099	0.037	0.100	0.000	0.000	0.112	0.775	0.112	0.000	0.000	0.225
9	0.103	0.150	0.037	0.100	0.000	0.006	0.195	0.592	0.207	0.000	0.000	0.408
Total	0.001	0.150	-0.100	0.100	0.000	0.006	0.112	0.759	0.120	0.002	0.000	0.241

Table 6.7: % of Funds that have changed Rating for $\gamma = 10$

grid	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ% Rating
1	0.001	0.048	-0.100	-0.037	0.000	0.000	0.077	0.846	0.077	0.000	0.000	0.154
2	0.052	0.099	-0.100	-0.037	0.000	0.018	0.207	0.538	0.231	0.006	0.000	0.462
3	0.103	0.150	-0.100	-0.037	0.024	0.112	0.225	0.284	0.225	0.095	0.036	0.716
4	0.001	0.048	-0.032	0.032	0.000	0.000	0.071	0.858	0.071	0.000	0.000	0.142
5	0.052	0.099	-0.032	0.032	0.000	0.006	0.213	0.556	0.225	0.000	0.000	0.444
6	0.103	0.150	-0.032	0.032	0.024	0.118	0.207	0.290	0.225	0.118	0.018	0.710
7	0.001	0.048	0.037	0.100	0.000	0.000	0.053	0.893	0.053	0.000	0.000	0.107
8	0.052	0.099	0.037	0.100	0.000	0.006	0.207	0.568	0.219	0.000	0.000	0.432
9	0.103	0.150	0.037	0.100	0.030	0.112	0.195	0.290	0.254	0.101	0.018	0.710
Total	0.001	0.150	-0.100	0.100	0.009	0.041	0.162	0.569	0.176	0.036	0.008	0.431

Table 6.8: % of Funds that have changed Rating for $\mu_M = 0.15$

grid	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ% Rating
1	0.001	0.048	-0.100	-0.037	0.000	0.000	0.006	0.988	0.006	0.000	0.000	0.012
2	0.052	0.099	-0.100	-0.037	0.000	0.000	0.095	0.811	0.095	0.000	0.000	0.189
3	0.103	0.150	-0.100	-0.037	0.000	0.000	0.148	0.704	0.148	0.000	0.000	0.296
4	0.001	0.048	-0.032	0.032	0.000	0.000	0.024	0.953	0.024	0.000	0.000	0.047
5	0.052	0.099	-0.032	0.032	0.000	0.000	0.130	0.740	0.130	0.000	0.000	0.260
6	0.103	0.150	-0.032	0.032	0.000	0.006	0.207	0.568	0.219	0.000	0.000	0.432
7	0.001	0.048	0.037	0.100	0.000	0.000	0.065	0.870	0.065	0.000	0.000	0.130
8	0.052	0.099	0.037	0.100	0.000	0.018	0.201	0.544	0.237	0.000	0.000	0.456
9	0.103	0.150	0.037	0.100	0.000	0.053	0.225	0.420	0.272	0.030	0.000	0.580
Total	0.001	0.150	-0.100	0.100	0.000	0.009	0.122	0.733	0.133	0.003	0.000	0.267

The following Figures represent more intuitively Tables 6.5÷6.8; every grid is represented as a parallelepiped, whose height is Δ% Rating that is visible in the last column of the Tables.

Total % of change of Rating for Grid:

Figure 6.2.1: $\gamma = 2$

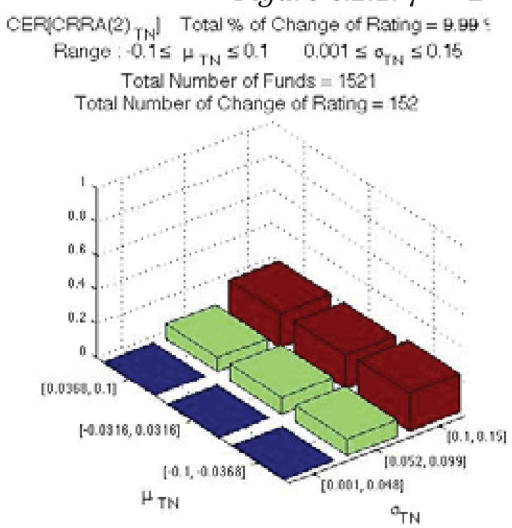


Figure 6.2.2: $\gamma = 5$

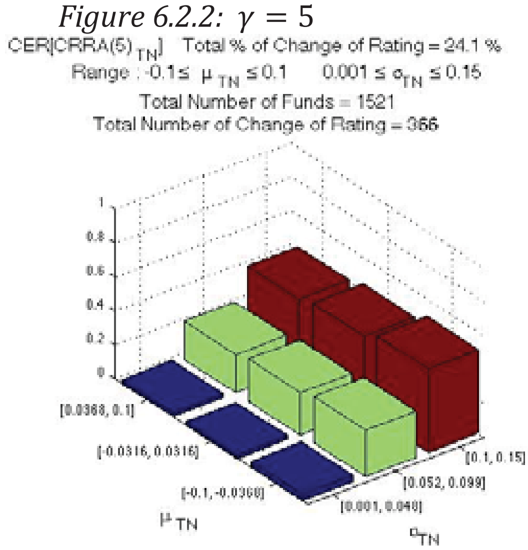


Figure 6.2.3: $\gamma = 10$

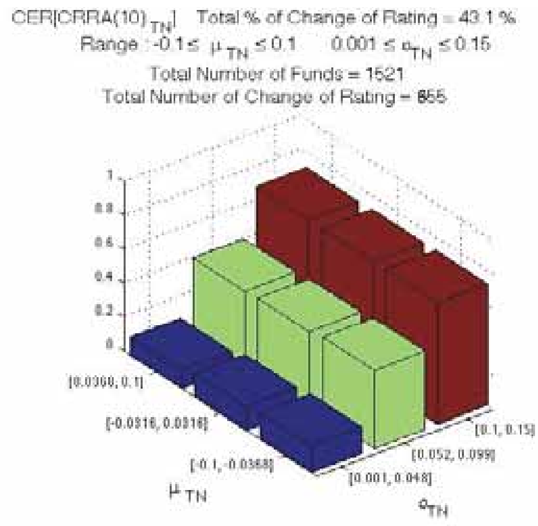
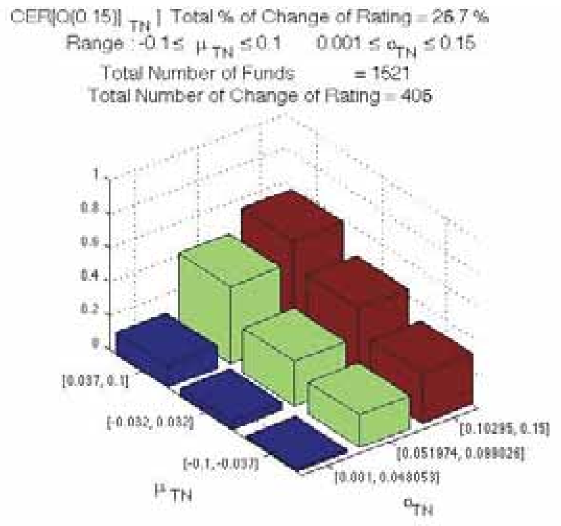


Figure 6.2.4: $\mu_M = 0.15$



We can conclude that there are grids where the ranking of $CER[CRRA(\gamma)_{TN}]$ is equal to the ranking of $CER[CRRA(0)_{TN}]$. Especially for $\gamma = 2$, Figure 6.2.1 and Table 6.5, for $0.001 \leq \sigma_{TN} \leq 0.048$, grids 1, 4, 7 , show zero changes of rating.
Also for $\gamma = 5$ in the same grids we have at most 3.6% change of Rating, which is a very low level.
For $\gamma = 10$ we have a different situation, in the same grids the percentages of change of rating lie between 10.7% and 15.4% (Table 6.7). For the grids with greater σ_{TN} the sensitivity increases greatly and the percentages of change of rating increase to 71.6% in grid 3 of Table 6.7.

The situation slightly improves if we consider a rating done with 1521 (=13*13*3*3) funds distributed in only one grid in a range (2.10).

Table 6.9: Number of Funds that have changed Rating in one grid

	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ Rating
ϒ = 2	0.001	0.150	-0.100	0.100	0	0	97	1327	97	0	0	194
ϒ = 5	0.001	0.150	-0.100	0.100	0	1	224	1070	226	0	0	451
ϒ = 10	0.001	0.150	-0.100	0.100	37	116	203	658	466	41	0	863
μ_M = 0,15	0.001	0.150	-0.100	0.100	0	13	215	1052	241	0	0	469

Table 6.10: % of Funds that have changed Rating in one grid

	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ% Rating
ϒ = 2	0.001	0.150	-0.100	0.100	0.000	0.000	0.064	0.872	0.064	0.000	0.000	0.128
ϒ = 5	0.001	0.150	-0.100	0.100	0.000	0.001	0.147	0.703	0.149	0.000	0.000	0.297
ϒ = 10	0.001	0.150	-0.100	0.100	0.024	0.076	0.133	0.433	0.306	0.027	0.000	0.567
μ_M = 0,15	0.001	0.150	-0.100	0.100	0.000	0.009	0.141	0.692	0.158	0.000	0.000	0.308

Figure 6.3.1: $\gamma = 2$

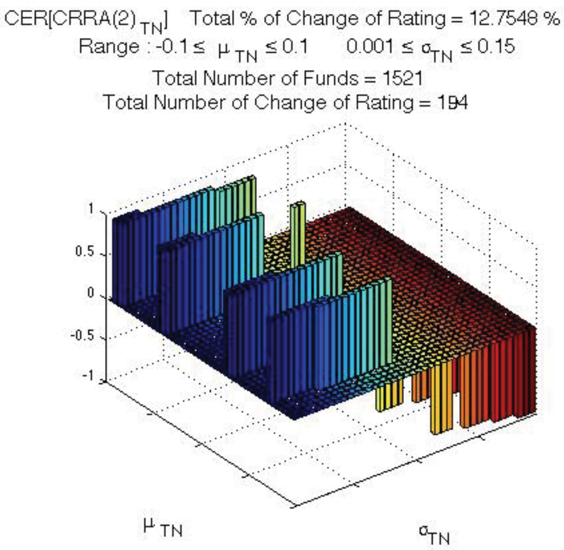


Figure 6.3.2: $\gamma = 5$

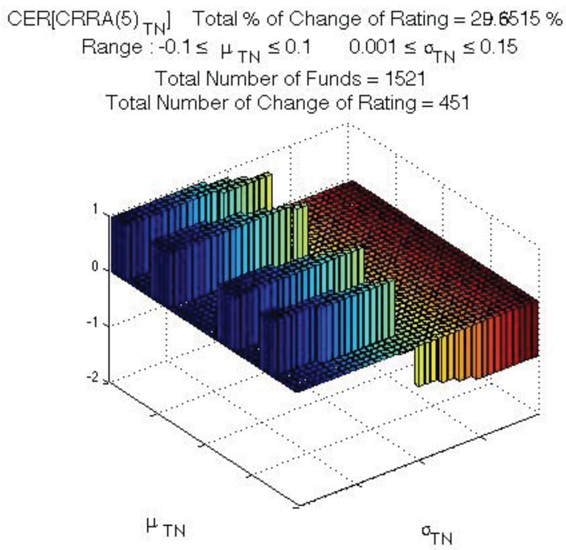


Figure 6.3.3: $\gamma = 10$

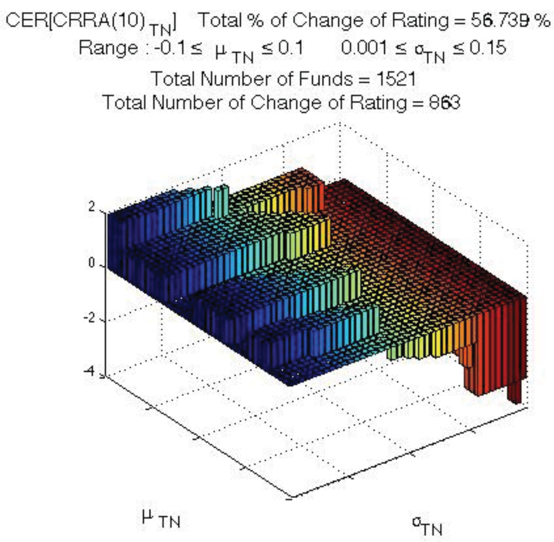
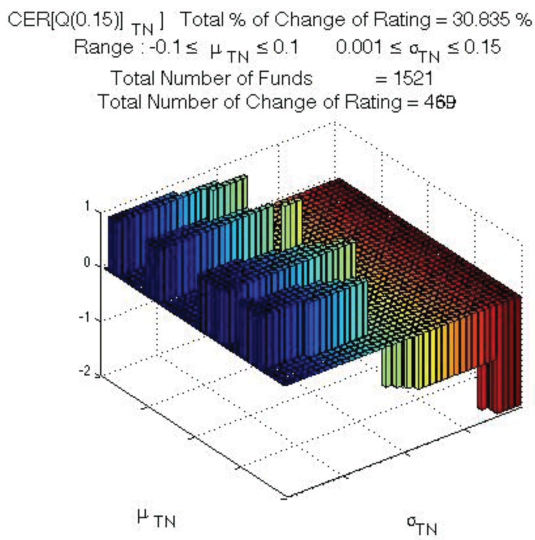


Figure 6.3.4: $\mu_M = 0.15$



If we increase the number of Funds, to approximate the continuous behavior of the change of rating, the situation does not change as can be seen in the following tables.

Table 6.11: % of Funds that have changed Rating in one grid for $\gamma = 2$

Funds' N.	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ% Rating
1521	0.001	0.150	-0.100	0.100	0.000	0.000	0.064	0.872	0.064	0.000	0.000	0.128
2401	0.001	0.150	-0.100	0.100	0.000	0.000	0.060	0.880	0.060	0.000	0.000	0.120
3481	0.001	0.150	-0.100	0.100	0.000	0.000	0.060	0.879	0.060	0.000	0.000	0.121
4761	0.001	0.150	-0.100	0.100	0.000	0.000	0.060	0.880	0.060	0.000	0.000	0.120
6241	0.001	0.150	-0.100	0.100	0.000	0.000	0.061	0.879	0.061	0.000	0.000	0.121

Table 6.12: % of Funds that have changed Rating in one grid for $\gamma = 5$

Funds' N.	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ% Rating
1521	0.001	0.150	-0.100	0.100	0.000	0.001	0.147	0.703	0.149	0.000	0.000	0.297
2401	0.001	0.150	-0.100	0.100	0.000	0.000	0.144	0.711	0.145	0.000	0.000	0.289
3481	0.001	0.150	-0.100	0.100	0.000	0.001	0.146	0.706	0.147	0.000	0.000	0.294
4761	0.001	0.150	-0.100	0.100	0.000	0.000	0.144	0.711	0.145	0.000	0.000	0.289
6241	0.001	0.150	-0.100	0.100	0.000	0.000	0.145	0.708	0.146	0.000	0.000	0.292

Table 6.13: % of Funds that have changed Rating in one grid for $\gamma = 10$

Funds' N.	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ% Rating
1521	0.001	0.150	-0.100	0.100	0.024	0.076	0.133	0.433	0.306	0.027	0.000	0.567
2401	0.001	0.150	-0.100	0.100	0.023	0.076	0.132	0.441	0.299	0.028	0.000	0.559
3481	0.001	0.150	-0.100	0.100	0.023	0.076	0.133	0.441	0.299	0.028	0.000	0.559
4761	0.001	0.150	-0.100	0.100	0.024	0.075	0.132	0.443	0.298	0.028	0.000	0.557
6241	0.001	0.150	-0.100	0.100	0.024	0.075	0.132	0.443	0.299	0.027	0.000	0.557

Table 6.14: % of Funds that have changed Rating in one grid for $\mu_M = 0.15$

Funds' N.	σ_{TN_Min}	σ_{TN_Max}	μ_{TN_Min}	μ_{TN_Max}	-3	-2	-1	0	1	2	3	Δ% Rating
1521	0.001	0.150	-0.100	0.100	0.000	0.009	0.141	0.692	0.158	0.000	0.000	0.308
2401	0.001	0.150	-0.100	0.100	0.000	0.007	0.143	0.693	0.157	0.000	0.000	0.307
3481	0.001	0.150	-0.100	0.100	0.000	0.007	0.144	0.692	0.157	0.000	0.000	0.308
4761	0.001	0.150	-0.100	0.100	0.000	0.007	0.142	0.695	0.156	0.000	0.000	0.305
6241	0.001	0.150	-0.100	0.100	0.000	0.007	0.142	0.694	0.156	0.000	0.000	0.306

We conclude that *CRRA*, even if it is one of the most used utility functions, weakly depends on Standard Deviation, σ_{TN}, for low levels of γ. Due to the fact that every risk definition is linked with Standard Deviation, *CRRA* seems inadequate to rank Funds at least for low values of γ and surely for $\gamma = 2$.

Moreover, for $\gamma = 10$, CRRA has a cut-off value for σ_{TN}, as we have pointed out at the end of Section 5; this cut-off decreases when γ increases.

7. Conclusions

This paper analyzes the behaviour of the *CER[CRRA]*, with a comparison between *AGR* and *CER[Q]*. The dependence on Standard Deviation is analytically detected by the FDIF, which is possible to calculate using a parametric representation of the *CER* of the involved utility function. It is shown that the dependence on Standard Deviation is weak for low levels of γ and for higher levels of γ it is possible to find anomalous behavior, in the sense that the concavity of the *CER[CRRA]* changes sign and the dependence on Standard Deviation decreases when Standard Deviation increases. The behaviour of the *CER[CRRA]* is similar to the one detected by *AGR*, and it is easy to see that the *CER[Q]*, even if the *QUF* depends only on the first two moments, has a behaviour more regular and, especially, dependent on Standard Deviation. We chose the analytic approach, with the assumption that the distribution of the Return is a Truncated Normal; this is motivated by the wide generality of the Normal and the consideration that the Return cannot be lower than -1.

As a particular case, we analyse the behavior of the Morningstar ranking methodology, showing that its dependence on the Risk is irrelevant[5] and that the ranking done is similar to the ranking done using *AGR*.

Even if Standard Deviation is not a coherent measure of risk, it is a basis for every consideration about the Risk, and is relevant to measuring the dependence by Standard Deviation for an instrument, *CER[CRRA]*, which is a candidate for measuring the Funds using any definition of risk[6].

Considerations that are based on Standard Deviation or on any definition of risk ultimately involve the capability of doing a ranking, and it is reasonable to expect that a ranking done using CER[CRRA] should be different from one done using AGR. It has been demonstrated that this difference is weak for low values of γ and becomes relevant only for high levels of γ. In addition, increasing the number of Funds to simulate a continuous domain maintains the results unchanged. On the contrary, CER[Q] induces a ranking that is different by the one induced using AGR.
Following these considerations, it seems that *CER[CRRA]* has a reduced capability to rank the Funds.

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[5] Our results give a methodological support to other studies that have examined the role of risk measurements in the Morningstar rating under various aspects. For example, Lisi and Caporin (2012) show that ratings obtained with the setting of Morningstar are very similar to those obtained by assuming that the investor is risk-indifferent.

[6] In any case, the behavior does not change if the measure of risk is the Expected Shartfall, as shown in Corradin and Sartore (2016).

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Appendix A. Transformation $[\boldsymbol{\sigma},\boldsymbol{\mu}] \to [\boldsymbol{\sigma_T}(\boldsymbol{\sigma},\boldsymbol{\mu}),\boldsymbol{\mu_T}(\boldsymbol{\sigma},\boldsymbol{\mu})]$**.**

By the definitions of Standard Deviation and the Mean, for the Truncated Normal variable we have:

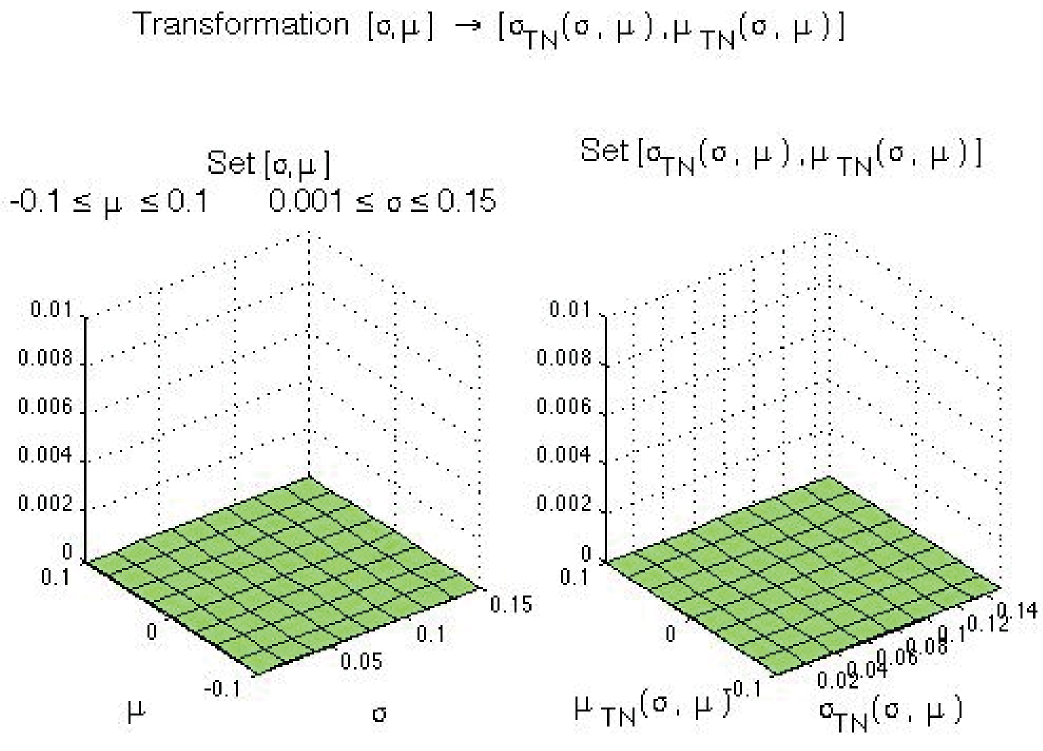
$$\sigma_{TN}(\sigma,\mu) = \sqrt{\frac{\int_{k_1}^{k_2} x^2 e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx} - \left[\frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}\right]^2}$$

(A.1)

$$\mu_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}$$

The (A.1) transforms the set $[\sigma,\mu]$ in the set $[\sigma_{TN}(\sigma,\mu),\mu_{TN}(\sigma,\mu)]$ as it is possible to see in the following Figure A.1:

Figure A.1: Transformation $[\sigma,\mu] \to [\sigma_{TN}(\sigma,\mu),\mu_{TN}(\sigma,\mu)]$



This ostensible equality between the numeric values of the ranges is due to the narrow amplitude of the ranges; if we use a wider range for $[(\sigma_{Min},\sigma_{Max})\times(\mu_{Min},\mu_{Max})]$, then we do not reach this equality between the ranges, due to the non linearity of the transformation (Figures A.3, A.4).
With the definitions:

$$\tau = \frac{x-\mu}{\sigma}, h_2 = \frac{k_2-\mu}{\sigma}, \quad h_1 = \frac{k_1-\mu}{\sigma},$$

$$I1 = \sigma\int_{h_1}^{h_2} e^{-\tau^2/2}\, d\tau, \qquad I2 = \int_{h_1}^{h_2} \tau e^{-\tau^2/2}\, d\tau \qquad I3 = \int_{h_1}^{h_2} \tau^2 e^{-\tau^2/2}\, d\tau$$

$$I4 = \sigma\int_{h_1}^{h_2} (\mu+\sigma\tau) e^{-\tau^2/2}\, d\tau, \quad I5 = \int_{h_1}^{h_2} (\mu+\sigma\tau)\tau e^{-\tau^2/2}\, d\tau, \quad I6 = \int_{h_1}^{h_2} (\mu+\sigma\tau)\tau^2 e^{-\tau^2/2}\, d\tau,$$

$$I7 = \sigma\int_{h_1}^{h_2} (\mu+\sigma\tau)^2 e^{-\tau^2/2}\, d\tau, I8 = \int_{h_1}^{h_2} (\mu+\sigma\tau)^2\tau e^{-\tau^2/2}\, d\tau, I9 = \int_{h_1}^{h_2} (\mu+\sigma\tau)^2\tau^2 e^{-\tau^2/2}\, d\tau,$$

we compute the following partial derivatives:

(A.2)

$$\frac{\partial\mu_{TN}(\sigma,\mu)}{\partial\sigma} = \frac{I1I6 - I3I4}{I1^2}, \qquad \frac{\partial\mu_{TN}(\sigma,\mu)}{\partial\mu} = \frac{I1I5 - I2I4}{I1^2}$$

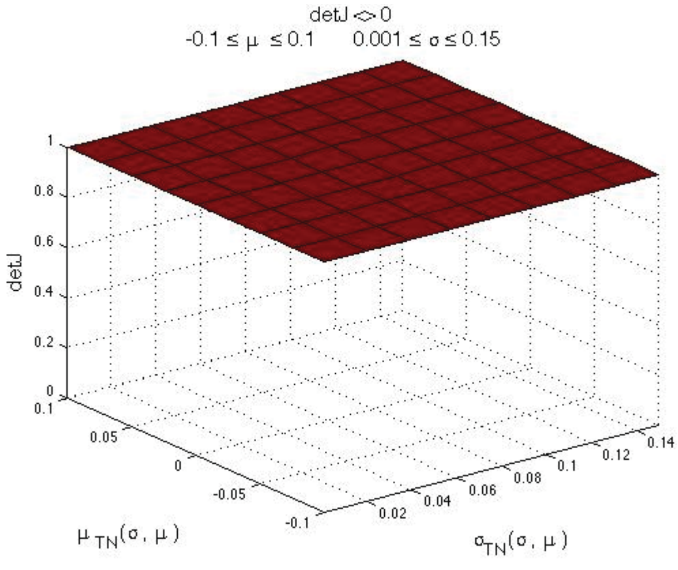
$$\frac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\sigma} = \frac{1}{2\,\sigma_{TN}(\sigma,\mu)}\left[\frac{I1I9 - I3I7}{I1^2} - 2\,\mu_{TN}(\sigma,\mu)\frac{\partial\mu_T(\sigma,\mu)}{\partial\sigma}\right]$$

$$\frac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\mu} = \frac{1}{2\,\sigma_{TN}(\sigma,\mu)}\left[\frac{I1I8 - I2I7}{I1^2} - 2\,\mu_{TN}(\sigma,\mu)\frac{\partial\mu_T(\sigma,\mu)}{\partial\mu}\right]$$

Now it is possible to verify (2.7), here reported for brevity:

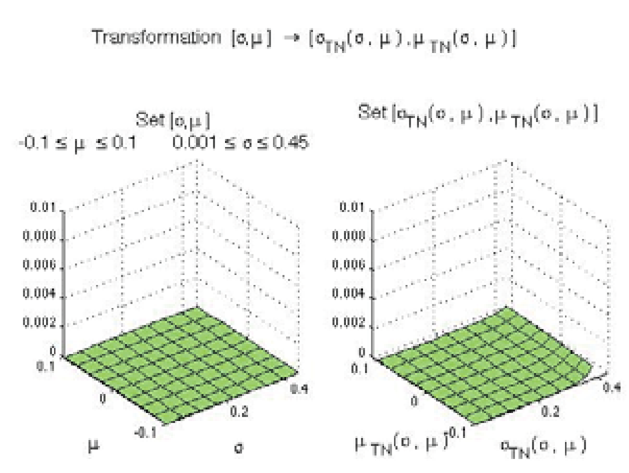
$$detJ = det\begin{pmatrix} \frac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\sigma} & \frac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\mu} \\ \frac{\partial\mu_{TN}(\sigma,\mu)}{\partial\sigma} & \frac{\partial\mu_{TN}(\sigma,\mu)}{\partial\mu} \end{pmatrix} \neq 0$$

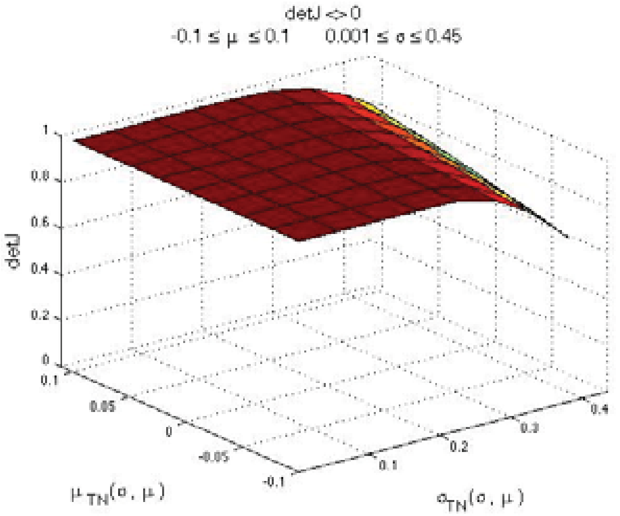
Figure A.2: $detJ \neq 0$



For wider range, we can note the non linearity of the transformation:

Figure A.3: Transformation *Figure A.4: $detJ \neq 0$*





Appendix B. Proof of Proposition 4.1

Proposition 4.1: *With the definition $b = 2cW_0(1+\mu_M)$, the expected value of QUF in (4.1), $E[Q(\mu_M)](SD[R], E[R])$, is a function of Standard Deviation $SD[R]$ and Expected Return $E[R]$ represented by a paraboloid in the space $(SD[R], E[R], E[Q(\mu_M)](SD[R], E[R]))$ with downward concavity, whose vertex is given by the point $\left(0, \mu_M, E[Q(\mu_M)](0, \mu_M)\right)$. That is:*

$$E[Q(\mu_M)](SD[R], E[R]) = QUF(W_0) + cW_0^2\mu_M^2 - cW_0^2[SD[R]^2 + (E[R]-\mu_M)^2]$$

where $QUF(W_0) = a + bW_0 - cW_0^2 = a + 2cW_0(1+\mu_M)W_0 - cW_0^2$.

The Certainty Equivalent Return for the Quadratic Utility Function centered in μ_M, $CER[Q(\mu_M)]$, has the following annualized expressions:

(4.3) $$CER[Q(\mu_M)](SD[R], E[R]) \equiv CER[Q(\mu_M)] = \left(1 + \mu_M - \sqrt{SD[R]^2 + (E[R]-\mu_M)^2}\right)^{12} - 1$$

$CER[Q(\mu_M)](SD[R], E[R])$ represents a squeezed cone in the space $(SD[R], E[R], E[Q(\mu_M)](SD[R], E[R]))$, whose vertex is given by the point $\left(0, \mu_M, CER[Q(\mu_M)](0, \mu_M)\right)$.

Proof: Consider the expected value of the Quadratic Utility Function(4.1):

$$E[Q(\mu_M)] = E[a + bW - cW^2]$$

$$= E[a + bW_0(1 + R) - cW_0^2(1 + R)^2]$$

$$= a + bW_0(1 + E[R]) - cW_0^2(1 + 2E[R] + E[R^2])$$

$$= a + bW_0 + bW_0E[R] - cW_0^2 - 2cW_0^2E[R] - cW_0^2(SD[R]^2 + E[R]^2)$$

$$= QUF(W_0) + W_0E[R](b - 2cW_0) - cW_0^2(SD[R]^2 + E[R]^2)$$

Substituting the parameter b with its expression, we have:

$$E[Q(\mu_M)] = QUF(W_0) + W_0E[R](2cW_0 + 2c\mu_M W_0 - 2cW_0) - cW_0^2(SD[R]^2 + E[R]^2)$$

$$= QUF(W_0) + 2cW_0^2E[R]\mu_M - cW_0^2(SD[R]^2 + E[R]^2)$$

Adding and subtracting the same quantity $cW_0^2\mu_M^2$and considering the $E[Q(\mu_M)]$ as a function of $SD[R]$ and $E[R]$ we obtain:

$$E[Q(\mu_M)](SD[R], E[R]) = QUF(W_0) + cW_0^2\mu_M^2 - cW_0^2[SD[R]^2 + (E[R] - \mu_M)^2]$$

$$(B.1) \qquad = QUF(W_0) + cW_0^2\{\mu_M^2 - cW_0^2[SD[R]^2 + (E[R] - \mu_M)^2]\}$$

The expression (B.1) represents a paraboloid in the space $(SD[R], E[R], E[Q(\mu_M)](SD[R], E[R]))$ with downward concavity, whose vertex is the point $\left(0, \mu_M, E[Q(\mu_M)](0, \mu_M)\right)$.

By the definition of the Certainty Equivalent Return:

$$QUF\left(W_0(1 + CER_m[Q(\mu_M)])\right) = E[Q(\mu_M)](SD[R], E[R])$$

where $CER_m[Q(\mu_M)]$ means (monthly) Certainty Equivalent Return for the Quadratic Utility Function with center in μ_M.

We have:

$$a + 2cW_0(1 + \mu_M)(1 + CER_m[Q(\mu_M)]) - cW_0{}^2(1 + CER_m[Q(\mu_M)])^2 =$$

$$= QUF(W_0) + cW_0^2\{\mu_M^2 - [\sigma^2 + (\mu - \mu_M)^2]\}$$

and solving for $CER_m[Q(\mu_M)]$ the result is:

$$CER_m[Q(\mu_M)](SD[R], E[R]) \equiv CER_m[Q(\mu_M)] = \mu_M - \sqrt{SD[R]^2 + (E[R] - \mu_M)^2}$$

because the other solution, with positive sign in front of the square root, has no economic sense.

The annualized expression is:

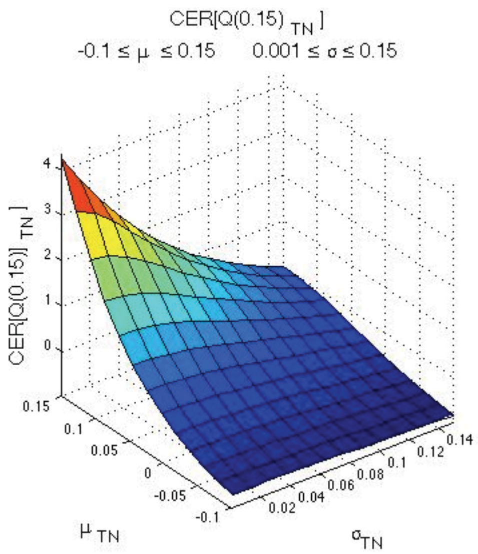
$$(B.2) \quad CER[Q(\mu_M)](SD[R], E[R]) \equiv CER[Q(\mu_M)] = \left(1 + \mu_M - \sqrt{SD[R]^2 + (E[R] - \mu_M)^2}\right)^{12} - 1$$

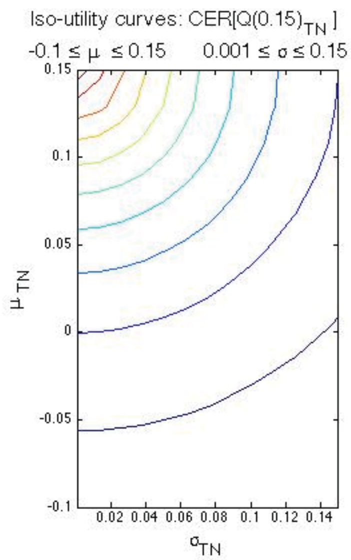
$CER[Q(\mu_M)](SD[R], E[R])$ represents a rotating squeezed cone in the space $(SD[R], E[R], E[Q(\mu_M)](SD[R], E[R]))$ with downward concavity, whose vertex is given by the point $(0, \mu_M, CER[Q(\mu_M)](0, \mu_M))$.

Until now it has not been defined a distribution for Return involved in $CER[Q(\mu_M)]$; to compare its behavior with the $CER[CRRA(\gamma)_{TN}]$, we apply also the (4.3) to a Return with a Truncated Normal distribution and will call it $CER[Q(\mu_M)_{TN}]$.

The Figure B.1 shows three dimensional graphs representing the $CER[Q(\mu_M)_{TN}]$. The figure is defined on the half-plane (σ_{TN}, μ_{TN}) with $\sigma_{TN} > 0$ and $\mu_{TN} < \mu_M$, and only in this plane it maintains the concavity coherent with its Risk aversion characteristic.
The Figure B.2 shows the iso-utility curves that can be obtained from the equality $CER[Q(\mu_M)_{TN}] = K$, where K is a constant; we have a sheaf of circumferences on the plane (σ_{TN}, μ_{TN}) with centre $(0, \mu_M)$.
The parameter μ_M is sufficient to identify $CER[Q(\mu_M)_{TN}]$ and $RRA[Q(\mu_M)_{TN}]$.

Figure B.1: 3D $CER[Q(\mu_M)_{TN}] =$ *for* $\mu_M = 0.15$ *Figure B.2: Iso-utility curves for* $\mu_M = 0.15$





Appendix C
Consider (2.6) here reported for brevity, defining the Standard Deviation $\sigma_{TN}(\sigma, \mu)$ and the Mean $\mu_{TN}(\sigma, \mu)$ of the Truncated Normal variable are:

$$\sigma_{TN}(\sigma, \mu) = \sqrt{\frac{\int_{k_1}^{k_2} x^2 e^{-(x-\mu)^2 / 2\sigma^2} dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2 / 2\sigma^2} dx} - \left[\frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2 / 2\sigma^2} dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2 / 2\sigma^2} dx}\right]^2}$$

(2.6)

$$\mu_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2/2\sigma^2}\, dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\, dx}$$

We know by the Appendix A that the transformation $(\sigma,\mu) \rightarrow (\sigma_{TN}, \mu_{TN})$ defined by (2.6) is bijective, it means that for every point (σ,μ) correspond only one point (σ_{TN}, μ_{TN}) and vice versa.

This means that the determinant of the Jacobian matrix J must be different from zero:

(2.7) $$detJ = det\begin{pmatrix} \frac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\sigma} & \frac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\mu} \\ \frac{\partial\mu_{TN}(\sigma,\mu)}{\partial\sigma} & \frac{\partial\mu_{TN}(\sigma,\mu)}{\partial\mu} \end{pmatrix} \neq 0$$

Consider the following parametric representation of a surface:

x axis = $\sigma_{TN}(\sigma,\mu)$.
y axis = $\mu_{TN}(\sigma,\mu)$.
z axis = *Generic Function* = $\psi(\sigma,\mu)$.

The three functions $\sigma_{TN}(\sigma,\mu), \mu_{TN}(\sigma,\mu), \psi(\sigma,\mu)$ are defined in the Cartesian subspace $[(\sigma_{Min}, \sigma_{Max})\times(\mu_{Min}, \mu_{Max})]$ of the space (σ,μ).

Using the vector notation, the surface is defined by the vector $\boldsymbol{r}(\sigma,\mu)$ in the space $[\sigma_{TN}(\sigma,\mu), \mu_{TN}(\sigma,\mu), \psi(\sigma,\mu)]$, with:

$(C.1)$ $$\boldsymbol{r}(\sigma,\mu) = \sigma_{TN}(\sigma,\mu)\boldsymbol{i} + \mu_{TN}(\sigma,\mu)\boldsymbol{j} + \psi(\sigma,\mu)\boldsymbol{k}$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the relative unit vectors.

For regularity of the surface, the Jacobian Matrix J_1:

$(C.2)$ $$J_1 = \begin{pmatrix} \frac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\sigma} & \frac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\mu} \\ \frac{\partial\mu_{TN}(\sigma,\mu)}{\partial\sigma} & \frac{\partial\mu_{TN}(\sigma,\mu)}{\partial\mu} \\ \frac{\partial\psi(\sigma,\mu)}{\partial\sigma} & \frac{\partial\psi(\sigma,\mu)}{\partial\mu} \end{pmatrix}$$

must have rank two; e.g. this condition is satisfied if (2.7) is true.
The condition (2.7), saying that the transformation $(\sigma,\mu) \rightarrow (\sigma_{TN}, \mu_{TN})$ is bijective, means that the inverse transformation $\sigma(\sigma_{TN}, \mu_{TN})$, $\mu(\sigma_{TN}, \mu_{TN})$ exists locally.
It is possible to write:

$$\psi(\sigma,\mu) = \psi\big(\sigma(\sigma_{TN}, \mu_{TN})\,, \mu(\sigma_{TN}, \mu_{TN})\big)$$

Computing the partial derivatives:

$$(C.3)\quad \begin{aligned} \frac{\partial\psi\big(\sigma(\sigma_{TN},\mu_{TN})\,,\mu(\sigma_{TN},\mu_{TN})\big)}{\partial\sigma_{TN}} &= \frac{\partial\psi}{\partial\sigma}\frac{\partial\sigma}{\partial\sigma_{TN}}+\frac{\partial\psi}{\partial\mu}\frac{\partial\mu}{\partial\sigma_{TN}} \\ \frac{\partial\psi\big(\sigma(\sigma_{TN},\mu_{TN})\,,\mu(\sigma_{TN},\mu_{TN})\big)}{\partial\mu_{TN}} &= \frac{\partial\psi}{\partial\sigma}\frac{\partial\sigma}{\partial\mu_{TN}}+\frac{\partial\psi}{\partial\mu}\frac{\partial\mu}{\partial\mu_{TN}} \end{aligned}$$

By the Theorem of the Inverse Function:

$$\begin{pmatrix} \frac{\partial\sigma(\sigma_{TN},\mu_{TN})}{\partial\sigma_{TN}} & \frac{\partial\sigma(\sigma_{TN},\mu_{TN})}{\partial\mu_{TN}} \\ \frac{\partial\mu(\sigma_{TN},\mu_{TN})}{\partial\sigma_{TN}} & \frac{\partial\mu(\sigma_{TN},\mu_{TN})}{\partial\mu_{TN}} \end{pmatrix} = \begin{pmatrix} \frac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\sigma} & \frac{\partial\sigma_{TN}(\sigma,\mu)}{\partial\mu} \\ \frac{\partial\mu_{TN}(\sigma,\mu)}{\partial\sigma} & \frac{\partial\mu_{TN}(\sigma,\mu)}{\partial\mu} \end{pmatrix}^{-1}$$

that has a solution for (2.7), we can write:

$$\begin{pmatrix} \frac{\partial\sigma(\sigma_{TN},\mu_{TN})}{\partial\sigma_{TN}} & \frac{\partial\sigma(\sigma_{TN},\mu_{TN})}{\partial\mu_{TN}} \\ \frac{\partial\mu(\sigma_{TN},\mu_{TN})}{\partial\sigma_{TN}} & \frac{\partial\mu(\sigma_{TN},\mu_{TN})}{\partial\mu_{TN}} \end{pmatrix} = \begin{pmatrix} \frac{1}{detJ}\frac{\partial\mu_{TN}}{\partial\mu} & -\frac{1}{detJ}\frac{\partial\sigma_{TN}}{\partial\mu} \\ -\frac{1}{detJ}\frac{\partial\mu_{TN}}{\partial\sigma} & \frac{1}{detJ}\frac{\partial\sigma_{TN}}{\partial\sigma} \end{pmatrix}$$

and substituting in (C.3):

$$\frac{\partial\psi}{\partial\sigma_{TN}} = \frac{1}{detJ}\frac{\partial\psi}{\partial\sigma}\frac{\partial\mu_{TN}}{\partial\mu} - \frac{1}{detJ}\frac{\partial\psi}{\partial\mu}\frac{\partial\mu_{TN}}{\partial\sigma} \quad \Rightarrow -detJ\frac{\partial\psi}{\partial\sigma_{TN}} = \frac{\partial\mu_{TN}}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial\mu_{TN}}{\partial\mu}$$

$$\frac{\partial\psi}{\partial\mu_{TN}} = -\frac{1}{detJ}\frac{\partial\psi}{\partial\sigma}\frac{\partial\sigma_{TN}}{\partial\mu} + \frac{1}{detJ}\frac{\partial\psi}{\partial\mu}\frac{\partial\sigma_{TN}}{\partial\sigma} \quad \Rightarrow \quad detJ\frac{\partial\psi}{\partial\mu_{TN}} = \frac{\partial\sigma_{TN}}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial\sigma_{TN}}{\partial\mu}$$

$$\begin{aligned} \frac{\partial\psi}{\partial\sigma_{TN}} &= -\frac{1}{detJ}\left(\frac{\partial\mu_{TN}}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial\mu_{TN}}{\partial\mu}\right) \\ \frac{\partial\psi}{\partial\mu_{TN}} &= \frac{1}{detJ}\left(\frac{\partial\sigma_{TN}}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial\sigma_{TN}}{\partial\mu}\right) \end{aligned}$$

Finally, the First Derivative of the Implicit Function in the space $[\sigma_{TN}(\sigma,\mu),\mu_{TN}(\sigma,\mu),\psi(\sigma,\mu)]$ is:

$$(C.4)\quad \frac{d\mu_{TN}\{\psi\}(\sigma,\mu)}{d\sigma_{TN}} = -\frac{\frac{\partial\psi}{\partial\sigma_{TN}}}{\frac{\partial\psi}{\partial\mu_{TN}}} = \frac{\frac{\partial\mu_{TN}}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial\mu_{TN}}{\partial\mu}}{\frac{\partial\sigma_{TN}}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial\sigma_{TN}}{\partial\mu}}$$

This is a generic expression, that will be used substituting instead of $\psi(\sigma,\mu)$ the expressions for $CER[CRRA(\gamma)_{TN}]$ and $CER[Q(\mu_M)_{TN}]$ that is computed in Appendix D, E and F.

The expression (C.4) can be seen in the geometrical way.

The orthogonal unit vector of the surfaces is done by:

$$\boldsymbol{N}(\sigma,\mu) \equiv \boldsymbol{N} = \frac{\dfrac{\partial \boldsymbol{r}(\sigma,\mu)}{\partial \sigma} \times \dfrac{\partial \boldsymbol{r}(\sigma,\mu)}{\partial \mu}}{\left\| \dfrac{\partial \boldsymbol{r}(\sigma,\mu)}{\partial \sigma} \times \dfrac{\partial \boldsymbol{r}(\sigma,\mu)}{\partial \mu} \right\|}$$

where:

$$(C.5) \qquad \frac{\partial \boldsymbol{r}(\sigma,\mu)}{\partial \sigma} \times \frac{\partial \boldsymbol{r}(\sigma,\mu)}{\partial \mu} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial \sigma_{TN}(\sigma,\mu)}{\partial \sigma} & \dfrac{\partial \mu_{TN}(\sigma,\mu)}{\partial \sigma} & \dfrac{\partial \psi(\sigma,\mu)}{\partial \sigma} \\ \dfrac{\partial \sigma_{TN}(\sigma,\mu)}{\partial \mu} & \dfrac{\partial \mu_{TN}(\sigma,\mu)}{\partial \mu} & \dfrac{\partial \psi(\sigma,\mu)}{\partial \mu} \end{vmatrix}$$

Developing (C.5) and omitting the dependence on (σ,μ) we can write:

$$\frac{\partial \boldsymbol{r}}{\partial \sigma} \times \frac{\partial \boldsymbol{r}}{\partial \mu} = \left[\frac{\partial \mu_{TN}}{\partial \sigma}\frac{\partial \psi}{\partial \mu} - \frac{\partial \psi}{\partial \sigma}\frac{\partial \mu_{TN}}{\partial \mu}\right]\boldsymbol{i} - \left[\frac{\partial \sigma_{TN}}{\partial \sigma}\frac{\partial \psi}{\partial \mu} - \frac{\partial \psi}{\partial \sigma}\frac{\partial \sigma_{TN}}{\partial \mu}\right]\boldsymbol{j} + \left[\frac{\partial \sigma_{TN}}{\partial \sigma}\frac{\partial \mu_{TN}}{\partial \mu} - \frac{\partial \mu_{TN}}{\partial \sigma}\frac{\partial \sigma_{TN}}{\partial \mu}\right]\boldsymbol{k}$$

The tangent vector T is derived by N with a counterclockwise rotation and, due to the rotation, the values of the component along the σ_{TN} and μ_{TN} axes exchange their absolute values between them; furthermore, the component along μ_{TN} axis changes also its sign; thus the projection of T on the plane (σ_{TN}, μ_{TN}) defines an angle whose tangent is done by:

$$\frac{\dfrac{\partial \mu_{TN}}{\partial \sigma}\dfrac{\partial \psi}{\partial \mu} - \dfrac{\partial \psi}{\partial \sigma}\dfrac{\partial \mu_{TN}}{\partial \mu}}{\dfrac{\partial \sigma_{TN}}{\partial \sigma}\dfrac{\partial \psi}{\partial \mu} - \dfrac{\partial \psi}{\partial \sigma}\dfrac{\partial \sigma_{TN}}{\partial \mu}}$$

that is (C.4).

Appendix D. First Derivative of the Implicit Function for $CER[CRRA(\gamma)_{TN}]$.

Consider the expression (2.4), here reported for brevity:

$$CER[CRRA(\gamma)_{TN}](\sigma,\mu) \equiv CER[CRRA(\gamma)_{TN}] = \left[\frac{\int_{k_1}^{k_2} \dfrac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^{\gamma}}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}\right]^{-\frac{12}{\gamma}} - 1$$

As first step, we compute the first derivatives respect σ, μ of the integrals in the numerator and denominator:

$$\frac{\partial}{\partial\sigma}\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx=\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx$$

$$\frac{\partial}{\partial\mu}\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx=\int_{k_1}^{k_2}\frac{(x-\mu)}{\sigma^2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx$$

$$\frac{\partial}{\partial\sigma}\int_{k_1}^{k_2}e^{-(\xi-\mu)^2/2\sigma^2}dx=\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}e^{-(\xi-\mu)^2/2\sigma^2}dx$$

$$\frac{\partial}{\partial\mu}\int_{k_1}^{k_2}e^{-(\xi-\mu)^2/2\sigma^2}dx=\int_{k_1}^{k_2}\frac{(x-\mu)}{\sigma^2}e^{-(\xi-\mu)^2/2\sigma^2}dx$$

and it follows that:

$$\frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\sigma}=-\frac{12}{\gamma}\left[\frac{\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx}{\int_{k_1}^{k_2}e^{-(x-\mu)^2/2\sigma^2}dx}\right]^{-\frac{12+\gamma}{\gamma}}\frac{\partial}{\partial\sigma}\left[\frac{\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx}{\int_{k_1}^{k_2}e^{-(x-\mu)^2/2\sigma^2}dx}\right]$$

$$=-\frac{12}{\gamma}\left[\frac{\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx}{\int_{k_1}^{k_2}e^{-(x-\mu)^2/2\sigma^2}dx}\right]^{-\frac{12+\gamma}{\gamma}} *$$

$$\left\{\frac{\left[\int_{k_1}^{k_2}e^{-(x-\mu)^2/2\sigma^2}dx\right]\left[\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx\right]-\left[\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx\right]\left[\int_{k_1}^{k_2}\frac{(x-\mu)^2}{\sigma^3}e^{-(x-\mu)^2/2\sigma^2}dx\right]}{\left[\int_{k_1}^{k_2}e^{-(x-\mu)^2/2\sigma^2}dx\right]^2}\right\}$$

$$\frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\mu}=-\frac{12}{\gamma}\left[\frac{\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx}{\int_{k_1}^{k_2}e^{-(x-\mu)^2/2\sigma^2}dx}\right]^{-\frac{12+\gamma}{\gamma}}\frac{\partial}{\partial\mu}\left[\frac{\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx}{\int_{k_1}^{k_2}e^{-(x-\mu)^2/2\sigma^2}dx}\right]=$$

$$= -\frac{12}{\gamma}\left[\frac{\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx}{\int_{k_1}^{k_2}e^{-(x-\mu)^2/2\sigma^2}dx}\right]^{-\frac{12+\gamma}{\gamma}} *$$

$$\left\{\frac{\left[\int_{k_1}^{k_2}e^{-(x-\mu)^2/2\sigma^2}dx\right]\left[\int_{k_1}^{k_2}\frac{(x-\mu)}{\sigma^2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx\right]-\left[\int_{k_1}^{k_2}\frac{e^{-(x-\mu)^2/2\sigma^2}}{(1+x)^\gamma}dx\right]\left[\int_{k_1}^{k_2}\frac{(x-\mu)}{\sigma^2}e^{-(x-\mu)^2/2\sigma^2}dx\right]}{\left[\int_{k_1}^{k_2}e^{-(x-\mu)^2/2\sigma^2}dx\right]^2}\right\}$$

with the definitions:

$$\tau = \frac{x-\mu}{\sigma}, \qquad I10 = \sigma\int_{h_1}^{h_2}\frac{e^{-\tau^2/2}}{(1+\mu+\sigma\tau)^\gamma}d\tau,$$

$$I11 = \int_{h_1}^{h_2}\tau\frac{e^{-\tau^2/2}}{(1+\mu+\sigma\tau)^\gamma}d\tau, \quad I12 = \int_{h_1}^{h_2}\tau^2\frac{e^{-\tau^2/2}}{(1+\mu+\sigma\tau)^\gamma}d\tau$$

and using the definitions already done in Appendix A we have:

$$\frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\sigma} = -\frac{12}{\gamma}\left(\frac{I10}{I1}\right)^{-\frac{12+\gamma}{\gamma}}\left(\frac{I1I12 - I10I3}{I1^2}\right)$$

$(D.1)$

$$\frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\mu} = -\frac{12}{\gamma}\left(\frac{I10}{I1}\right)^{-\frac{12+\gamma}{\gamma}}\left(\frac{I1I11 - I10I2}{I1^2}\right)$$

Specifying $\mu_{TN}\{CER[CRRA(\gamma)_{TN}]\}$ as the Implicit Function defined by $CER[CRRA(\gamma)_{TN}]$, by (C.4) we have:

$$(D.2) \quad \frac{d\mu_{TN}\{CER[CRRA(\gamma)_{TN}]\}(\sigma,\mu)}{d\sigma_{TN}} = -\frac{\frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\sigma_T}}{\frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\mu_T}}$$

$$= \frac{\frac{\partial\mu_{TN}}{\partial\sigma}\frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\mu} - \frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\sigma}\frac{\partial\mu_{TN}}{\partial\mu}}{\frac{\partial\sigma_{TN}}{\partial\sigma}\frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\mu} - \frac{\partial CER[CRRA(\gamma)_{TN}]}{\partial\sigma}\frac{\partial\sigma_{TN}}{\partial\mu}}$$

It is possible to compute (D.2) substituting the partial derivatives with its expressions computed in (A.2) and (D.1).

Appendix E. First derivative of the Implicit Function for $CER[CRRA(0)_{TN}]$.

Consider the expression (2.5) here reported for brevity:

$$CER[CRRA(0)_{TN}] = e^{12E[\ln(1+R)]} - 1 = exp\left[12\frac{\int_{k_1}^{k_2} ln(1+x)e^{-(x-\mu)^2/2\sigma^2}dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}dx}\right] - 1$$

with the definitions:

$$\tau = \frac{x-\mu}{\sigma}, \qquad I13 = \sigma\int_{h_1}^{h_2} ln\,(1+\mu+\sigma\tau)e^{-\tau^2/2}\,d\tau,$$

$$I14 = \int_{h_1}^{h_2} \tau ln\,(1+\mu+\sigma\tau)e^{-\tau^2/2}\,d\tau, \quad I15 = \int_{h_1}^{h_2} \tau^2 ln\,(1+\mu+\sigma\tau)e^{-\tau^2/2}\,d\tau$$

and using the same procedure and notations of the Appendix D we have:

(E.1)

$$\frac{\partial CER[CRRA(0)_{TN}]}{\partial\sigma} = 12\left(\frac{I1I15 - I13I3}{I1^2}\right)exp\left(12\frac{I13}{I1}\right)$$

$$\frac{\partial CER[CRRA(0)_{TN}]}{\partial\mu} = 12\left(\frac{I1I14 - I13I2}{I1^2}\right)exp\left(12\frac{I13}{I1}\right)$$

Specifying $\mu_{TN}\{CER[CRRA(0)_{TN}]\}$ as the Implicit Function defined by $CER[CRRA(0)_{TN}]$, by (C.4) we have:

$$(E.2)\quad \frac{d\mu_{TN}\{CER[CRRA(0)_{TN}]\}(\sigma,\mu)}{d\sigma_{TN}} = -\frac{\frac{\partial CER[CRRA(0)_{TN}]}{\partial\sigma_{TN}}}{\frac{\partial CER[CRRA(0)_{TN}]}{\partial\mu_{TN}}}$$

$$= \frac{\frac{\partial\mu_{TN}}{\partial\sigma}\frac{\partial CER[CRRA(0)_{TN}]}{\partial\mu} - \frac{\partial CER[CRRA(0)_{TN}]}{\partial\sigma}\frac{\partial\mu_{TN}}{\partial\mu}}{\frac{\partial\sigma_{TN}}{\partial\sigma}\frac{\partial CER[CRRA(0)_{TN}]}{\partial\mu} - \frac{\partial CER[CRRA(0)_{TN}]}{\partial\sigma}\frac{\partial\sigma_{TN}}{\partial\mu}}$$

It is possible to compute (E.2) substituting the partial derivatives with its expressions computed in (A.2) and (E.1).

Appendix F . First derivative of the Implicit Function for $CER[Q(\mu_M)_{TN}]$.

Consider the expression (4.3) here reported for brevity and applied to a Truncated Normal case:

$$CER[Q(\mu_M)_{TN}](\sigma,\mu) = \left(1 + \mu_M - \sqrt{\sigma_{TN}^2 + (\mu_{TN} - \mu_M)^2}\right)^{12} - 1$$

We have:

$$\frac{\partial CER[Q(\mu_M)_{TN}]}{\partial \sigma} = -\frac{12\left[1 + \mu_M - \sqrt{\sigma_{TN}^2 + (\mu_{TN} - \mu_M)^2}\right]^{11}}{2\sqrt{\sigma_{TN}^2 + (\mu_{TN} - \mu_M)^2}} * 2\left[\sigma_{TG}\frac{\partial \sigma_{TN}}{\partial \sigma} + (\mu_{TN} - \mu_M)\frac{\partial \mu_{TN}}{\partial \sigma}\right]$$

(F.1)

$$\frac{\partial CER[Q(\mu_M)_{TN}]}{\partial \mu} = -\frac{12\left[1 + \mu_M - \sqrt{\sigma_{TN}^2 + (\mu_{TG} - \mu_M)^2}\right]^{11}}{2\sqrt{\sigma_{TN}^2 + (\mu_{TN} - \mu_M)^2}} * 2\left[\sigma_{TN}\frac{\partial \sigma_{TN}}{\partial \mu} + (\mu_{TN} - \mu_M)\frac{\partial \mu_{TN}}{\partial \mu}\right]$$

Specifying $\mu_{TN}\{CER[Q(\mu_M)_{TN}]\}$ as the Implicit Function defined by $CER[Q(\mu_M)_{TN}]$, by (C.4) we have:

$$(F.2) \quad \frac{d\mu_{TN}\{CER[Q(\mu_M)_{TN}]\}(\sigma,\mu)}{d\sigma_{TN}} = -\frac{\frac{\partial CER[Q(\mu_M)_{TN}]}{\partial \sigma_{TN}}}{\frac{\partial CER[Q(\mu_M)_{TN}]}{\partial \mu_{TN}}}$$

$$= \frac{\frac{\partial \mu_{TN}}{\partial \sigma}\frac{\partial CER[Q(\mu_M)_{TN}]}{\partial \mu} - \frac{\partial CER[Q(\mu_M)_{TN}]}{\partial \sigma}\frac{\partial \mu_{TN}}{\partial \mu}}{\frac{\partial \sigma_{TN}}{\partial \sigma}\frac{\partial CER[Q(\mu_M)_{TN}]}{\partial \mu} - \frac{\partial CER[Q(\mu_M)_{TN}]}{\partial \sigma}\frac{\partial \sigma_{TN}}{\partial \mu}}$$

and with (A.2) and (F.1) it is possible to compute (F.2).

Appendix G. The case of the Italian Pension Funds.

The opportunity to study the $CRRA(\gamma)$ utility Function arose with the aim to rank the Italian Pension Funds. Obviously the Italian Pension Fund have returns that cannot be lower than 1.

We applied the Morningstar approach(3.1) and this produced a ranking not so far from that one induced by the AGR (that Morningstar call MRAR(0)). Then, we compared this result with the one supplied by $CER[Q(\mu_M)_S]$ where S means applied to the samples. We think that it would be difficult to explain to the investors that the evaluation of their Funds is not linked to their Standard Deviation.

Consider J Funds $\{F_j\ ;\ j=1,\dots,J\}$ each with a sequence of T (monthly) Returns $\{F_j\ ;\{R_{t,j}\ ;\ j=1,\dots,J;\ t=1,\dots T\}\ j=1,\dots,J\}$; their distribution is unknown and we rank the Funds with $CER[Q(\mu_M)_S]$, with Expected Return and the Standard Deviation:

$$E[R]_{T,j}=T^{-1}\sum_{t=1}^{T}R_{t,j}\ ,\ SD[R]_{T,j}=\left\{(T-1)^{-1}\sum_{t=1}^{T}\left[R_{t,j}-E[R]_{T,j}\right]^2\right\}^{1/2}.$$

We indicate with μ_M the maximum value of the sequence of the Expected, i.e. $\mu_M\geq max\{E[R]_{T,j}, j=1\dots J\}$ and we chose $b=2cW_0(1+\mu_M)$. The rationale of this latter choice is that the maximum of $CER[Q(\mu_M)_S]$ will be reached at the point $\left(0,\mu_M,CER[Q(\ \mu_M)_S](0,\mu_M)\right)$ on the space $(SD[R],E[R],CER[Q(\mu_M)_S](SD[R],E[R])\ )$. It means that, within the set of Funds, the investor has the maximum satisfaction in the following ideal state: Standard Deviation = 0 and maximum value of the Expected Return among all that are available. We chose $\mu_M=0.081$, a value that guarantee that a Fund with positive $E[R]_{T,j}$ have always a rating higher than one with a negative yield (Corradin and Sartore, 2016).

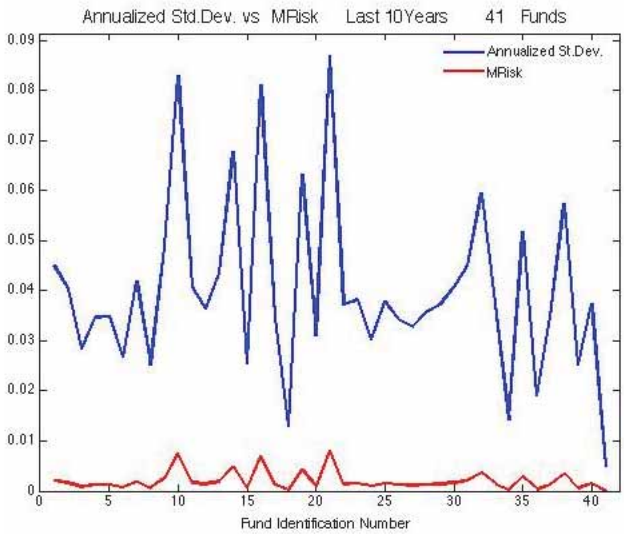
We computed a ranking with $CER[Q(0.081)_S]$ and *MRAR(2).* The discrepancies between them could be shown also in the field of Morningstar applications. Lisi and Caporin (2012), with an empirical analysis, put in evidence that the Morningstar rating system is mainly influenced by profitability, and only marginally by risk. Here, we give a new application with the computation of the two mentioned approaches, $CER[Q(0.081)_S]$ and *MRAR(2)*, performed on the actual sample given by the monthly return values of the Italian Pension Funds. The sample of 41 Funds, collects all the Funds with not less than 10 years of activity from July 2004 until June 2014.

The Morningstar Risk (MRisk) is computed with the relation:

$$MRisk=MRAR(0)-MRAR(2)$$

In this case the comparison is done with the Annualized Standard Deviation $=\sqrt{12}SD[R]_{T,j}$

Figure G.1: Annualized Standard Deviation vs MRisk

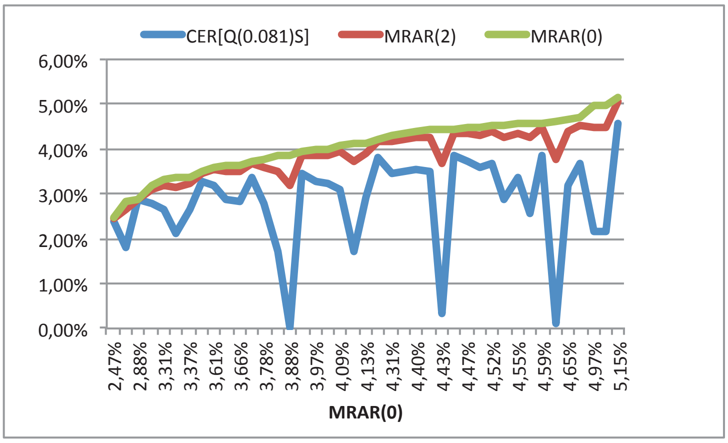


We can see that there is a large discrepancy in the measurement scale even though the behavior looks very similar, in fact the correlation coefficient between the two measures is 0.96322.

MRisk is largely lower with respect to the Standard Deviation measure.

We can compare graphically the $CER[Q(0.081)_S]$, *MRAR(2)* and *MRAR(0)* versus *MRAR(0)*, *MRisk*, Standard Deviation.

Figure G.2: Comparison $CER[Q(0.081)_S]$ *, MRAR(2) and MRAR(0) vs MRAR(0)*



Figures G.2 shows big downturns for $CER[Q(0.081)_S]$ when MRAR(2) has very weak downturns at the same points. This different behavior is attributable to the different sensitivity of the two measures regarding the Standard Deviation.

It is graphically evident that MRAR(2) Rating has a smoother behavior and almost the same increasing trend of MRAR(0).

Similarly, we can see the following graphs:

Figure G.3: Comparison $CER[Q(0.081)_S]$, *MRAR(2) and MRAR(0) vs MRisk*

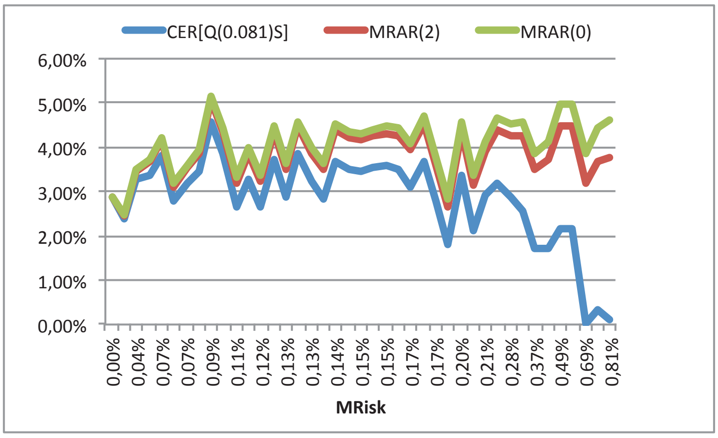
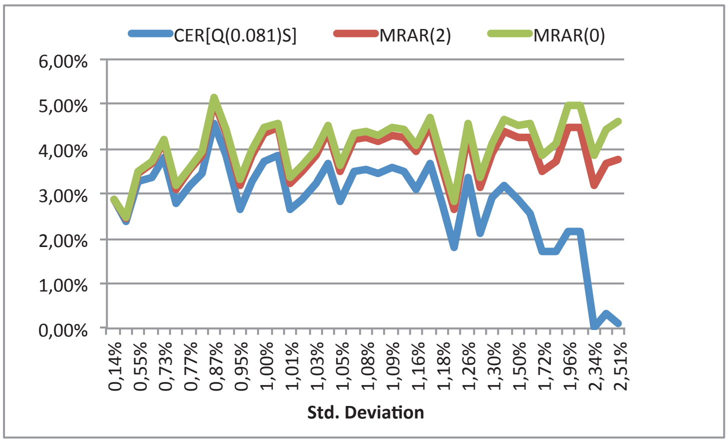


Figure G.4: Comparison $CER[Q(0.081)_S]$*MRAR(2) and MRAR(0) vs Standard Deviation*



Again, both the Figures G.3 and G.4 show that for high values of Standard Deviation or *MRisk*, *MRAR(2)* is clearly less sensitive comparing with $CER[Q(0.081)_S]$. Furthermore, there is graphical evidence that the *MRAR(2)* does not decrease for high levels of Standard Deviations or *MRisk*, and this confirm the consideration done in Section 5.

This different behavior translates directly into the rating scale. In Figures G.5, G.6 and G.7 we compare the rating of the Italian Pension Funds induced by the $CER[Q(0.081)_S]$, *MRAR(2)* and *MRAR(0)* as a function of *MRAR(0)* , *MRisk* and Standard Deviation respectively.

Figure G.5: Comparison Rating $CER[Q(\mu_M)_S]$, MRAR(2) and MRAR(0) vs MRAR(0)

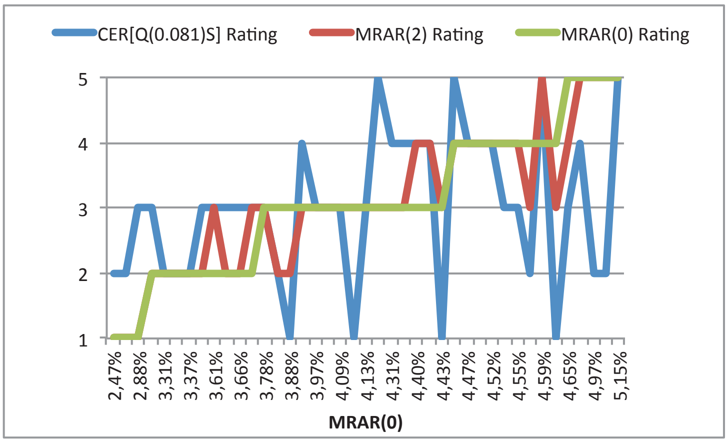


Figure G.6: Comparison Rating $CER[Q(0.081)_S]$, MRAR(2) and MRAR(0) vs MRisk

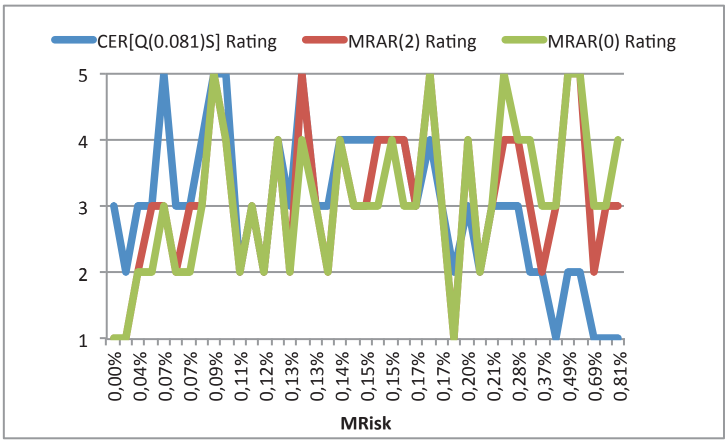
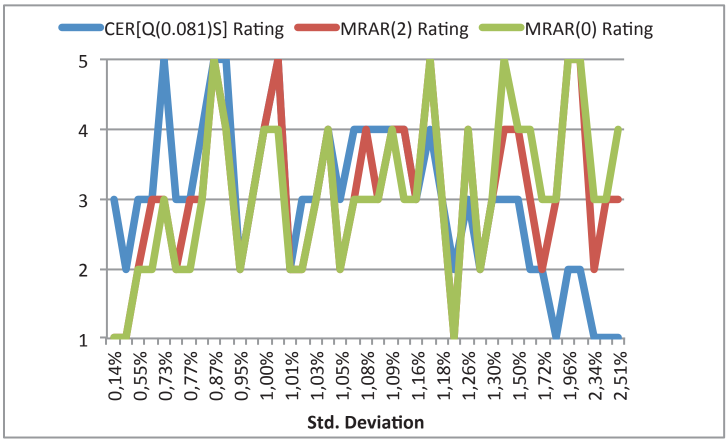


Figure G.7: Comparison Rating$CER[Q(0.081)_S]$, MRAR(2) and MRAR(0) vs Standard Deviation



Also in the Figures G.5, G.6, G.7 there is graphical evidence that the *MRAR(2) Rating* does not decrease for high levels of Standard Deviation or *MRisk* and seems to have a behavior similar to *MRAR(0) Rating*.
Beyond the graphical evidence, we can look at the correlation coefficients between the different measures.

Table G.1: Correlations

Std. Deviation	MRisk	CER[Q(0.081)S	MRAR(2)	MRAR(0)	CER[Q(0.081)S Rating	MRAR(2) Rating	MRAR(0) Rating
1.00000	0.96328	-0.68220	0.23650	0.48952	-0.53636	0.24051	0.45739
	1.00000	-0.79404	0.09928	0.37176	-0.61770	0.11856	0.34633
		1.00000	0.52572	0.26882	0.89602	0.46714	0.24727
			1.00000	0.96065	0.59657	0.93118	0.89283
				1.00000	0.38414	0.90183	0.92963
					1.00000	0.54000	0.34000
						1.00000	0.90000
							1.00000

In Table G.1 we can notice the strong dependence of *MRAR(2)* and *MRAR(2)* Rating to the *MRAR(0)*, along with their scarce sensitivity to the risk. On the contrary, $CER[Q(0.081)_S]$ and $CER[Q(0.081)_S]$ Rating exhibit negative correlation coefficients with both Standard Deviation and MRisk, as expected, instead of positive values of MRAR(2) and MRAR(2) Rating.

Another proof that we have done is to compute the Rating using MRAR(0) and to measure the differences between the Rating done with MRAR(2) and $CER[Q(0.081)_S]$. Also this matter proves that the Ranking done with MRAR(2) has scarce sensibility to the risk, indeed it depends on the *AGR*.

Indeed, in our job we have analyzed 41 Funds with more than 10 years samples; 23 changing of Ranking between *MRAR(2)* and $CER[Q(0.081)_S]$ and only 10 between *MRAR(2)* and *MRAR(0)*. This further justify the impression that the Ranking done by *MRAR(2)* has a weak dependence on the Standard Deviation.