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On the relations between Markov chain lumpability and reversibility

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Abstract In the literature, the notions of lumpability and time reversibility for large Markov chains have been widely used to efficiently study the functional and non-functional properties of computer systems. In this paper we explore the relations among different definitions of lumpability (strong, exact and strict) and the notion of time-reversed Markov chain. Specifically, we prove that an exact lumping induces a strong lumping on the reversed Markov chain and a strict lumping holds both for the forward and the reversed processes. Based on these results we introduce the class of $\lambda\rho$-reversible Markov chains which combines the notions of lumping and time reversibility modulo state renaming. We show that the class of autoreversible processes, previously introduced in [23], is strictly contained in the class of $\lambda\rho$-reversible chains.

Keywords Stochastic models · time reversibility · lumpability · quantitative analysis

1 Introduction

The theory of Markov chains is the foundation of several approaches to the design and verification of computer systems. Many performance evaluation methods are based on models whose underlying stochastic processes are Markov chains (see, e.g., [11]) upon which Quality of Service (QoS) prediction methods for component-based software systems are defined. Similar considerations can be made for quantitative model checking techniques (see, e.g., [2]).

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Nevertheless, the specification of low level models in terms of Markov chains can be very time consuming and prone to errors due to the complexity of contemporary computers' hardware and software architectures. For this reason, several high level formalisms/languages have been introduced to allow for a compositional specification of complex systems while maintaining an underlying Markov chain. Examples of such formalisms are Markovian queueing networks [21], generalized stochastic Petri nets [25] and for the languages we mention the Performance Evaluation Process Algebra (PEPA) [16] and the Interactive Markov chains [15] which are quite popular in the community of model checking and performance evaluation.

Although the use of high-level specification formalisms highly simplifies the design of compositional/hierarchical quantitative models, the stochastic process underlying even a very compact model may have a number of states that makes its analysis a very difficult, if not computationally impossible, task. In order to study models with a very large state space without resorting to approximation or simulation techniques we can identify the following routes:

– *State space reduction.* According to this approach the modeller tries to reduce the state space of the underlying Markov chain by aggregating states with equivalent behaviours (according to a notion of equivalence that may vary). An interesting class of these aggregation methods that can be decided by the structural analysis of the original Markov chain is known as *lumping*. In the literature, several notions of lumping are introduced: strong and weak lumping [19], exact lumping [29], and strict lumping [6]. Interestingly, for Markovian process algebras there is a strong connection between the notion of *bisimulation* and that of *strong lumping* (see, e.g., [16]).
– *Analysis of the reversed process.* The idea of studying the behaviour of processes in the reversed time has been deeply exploited not only for the analysis of computing systems but also for the analysis of physical systems (see [33,18]). It is well-known that if a Markov chain $X(t)$ is stochastically indistinguishable from $X(\tau - t)$ for all τ and t in the time domain (reals or integers) then both the transient and the stationary analyses are very efficient and numerically stable.
– *Exact model decomposition.* If the stochastic model is defined in terms of cooperations of several components, the *product-form* theory allows one to derive the stationary performance indices by the analysis of the processes underlying the single components considered in isolation. Product-form models have been widely investigated in queueing theory [5,18], stochastic Petri nets [22,4] and Markovian process algebra [14,30]. It is worth of notice that there is a strict relation between the theory of reversed processes developed in [33,18,11], called *quasi-reversibility*, and the product-form results (ee, e.g., [9,12,10,14]).
– *Regularity of the state-space.* Some stochastic models have an underlying Markov chain whose transition/rate matrix exhibits some block regular

structures that allow for the application of a class of methods called *matrix geometrics* and *matrix analytics* [27].

The first contribution of this paper is the investigation of the relations among the various definitions of lumpability (strong, exact and strict) and the notion of reversed process. Specifically, we prove that the conditions for the exact lumpability relative to a certain aggregation of states are sufficient to obtain a strong lumpability in the reversed Markov chain with respect to the same aggregation. Moreover, if a Markov chain is strictly lumpable then also its reversed process is strictly lumpable with respect to the same partition. We also study the relations between the notion of weak-similarity on states [34, 23] and that of strict lumpability. Then, we introduce the notion of $\lambda\rho$-reversibility. Given an aggregation of states that is a strict lumping, we say that the Markov chain is $\lambda\rho$-reversible if the lumped Markov chain is reversible modulo a state renaming. We study the properties of this class of processes and show that the computation of the stationary state probability is very efficient. Moreover, we give a characterisation of this class of processes in the style of Kolmogorov's criteria for reversible chains and prove that it allows for the decision of $\lambda\rho$-reversibility without constructing the reversed process. Finally, we show that the class of autoreversible Markov chains introduced in [23] is strictly contained in the class of $\lambda\rho$-reversible processes. Our results extend the applicability of time reversibility and find applications in different contexts where the Markov chains underlying the models have strong symmetrical properties. This is the case for some queueing systems, such as that studied in Section 7.2, and also other non queueing models like the one presented in King's seminal work on cache analysis [20] (see Section 7.1). Although the conditions required by our results may seem strict, in many cases they provide an effective way for deriving exact solutions for large Markov chains as witnessed by the wide literature exploiting time reversibility for practical purposes.

1.1 Related work

The notion of exact lumpability is introduced by Schweitzer in [29] where the author shows that Takahashi's *aggregation/disaggregation* algorithm [32] converges in one step if the Markov chain satisfies certain properties. A connection between exact lumping and time reversibility is observed in [31] where the authors propose a new algorithm for the computation of the stationary probability distributions based on aggregation and disaggregation method. They observe that the algorithm provides exact results in the case of reversible chains. With respect to the latter paper, we make explicit the connection between exact lumping and reversed processes, including in our results any ergodic chain (not only time-reversible). In [33, 18] the authors study a class of Markov chains called *dynamically reversible* which is strictly included in the class of $\lambda\rho$-reversible chains. The purpose of [33, 18] is to study a class of physical systems in which the time reversed chain shows the same behaviour as the original one modulo an involution of the state names. The result has been applied to model the growth of crystal in [8]. With respect to these works we study the connection between the forward and reversed chain modulo a strict lumpability and an arbitrary renaming of states (not necessary an involution). In [14] the author extends the formulation of Kolmogorov's criteria for reversible chains to ergodic Markov chains. Although we use the result of [14] in our proofs, it is important to notice that we can decide the property of $\lambda\rho$-reversibility solely based on the analysis of the given process, without building its reversed counterpart. This is important since the definition of the reversed process has in general the same computational complexity as the computation of the stationary distribution.

This paper is an extended and improved version of [24, 23]. With respect to the previous works, this paper includes a detailed study of the relations among the notions of reversibility, lumpability and autoreversibility, contains all the proofs of the theorems and presents various applications in the context of queueing analysis and Markovian process algebras.

1.2 Structure of the paper

The paper is structured as follows. Section 2 introduces the fundamental notions and the notation. In section 3 we introduce the definitions of state similarity [34] and weak similarity [23] and study the relations with the notions of lumping. In Section 4 we prove the results on reversed processes, exact and strict lumpability of Markov chains. Section 6 shows that the definition of autoreversibility [23] is encompassed by that of $\lambda\rho$-reversibility. Then, in Section 7, we show some applications in different fields such as queueing theory and Markovian process algebra. Finally, Section 8 concludes the paper.

2 Theoretical Background

In this section we review some aspects of the theory of Markov processes which will be required in the sequel. The arguments presented hereafter apply to continuous time Markov processes with a discrete state space (CTMCs) and they can be formulated also for Discrete Time Markov Chains (DTMCs).

2.1 Preliminaries on Markov processes

Let $X(t)$ be a stochastic process taking values in a countable state space S for $t \in \mathbb{R}^+$. If $(X(t_1), X(t_2), \ldots, X(t_n))$ has the same distribution as the process $(X(t_1+\tau), X(t_2+\tau), \ldots, X(t_n+\tau))$ for all $t_1, t_2, \ldots, t_n, \tau \in \mathbb{R}^+$ then the stochastic process $X(t)$ is said to be *stationary*. The stochastic process $X(t)$ is a *Markov* process if for $t_1 < t_2 < \cdots t_n < t_{n+1}$ the joint distribution of $(X(t_1), X(t_2), \ldots, X(t_n), X(t_{n+1}))$ is such that

$$P(X(t_{n+1}) = i_{n+1} \mid X(t_1) = i_1, X(t_2) = i_2, \ldots, X(t_n) = i_n) = P(X(t_{n+1}) = i_{n+1} \mid X(t_n) = i_n).$$

In other words, for a Markov process its past evolution until the present state does not influence the conditional (on both past and present states) probability distribution of future behaviour.

A Continuous-Time Markov Chain (CTMC) is a Markov process in continuous time with a denumerable state space $\mathcal{S}$. A Markov process is *time homogeneous* if the conditional probability $P(X(t+\tau) = j \mid X(t) = i)$ does not depend upon t, and is *irreducible* if every state in S can be reached from every other state. A state in a Markov process is called *recurrent* if the probability that the process will eventually return to the same state is one. A recurrent state is called *positive-recurrent* if the expected number of steps until the process returns to it is less than infinity. A Markov process is *ergodic* if it is irreducible and all its states are positive recurrent. A process satisfying all these assumptions possesses an *equilibrium* (or *steady-state*) *distribution*, that is the unique collection of positive numbers π_k with $k \in S$ summing to unity such that:

$$\lim_{t \to \infty} P(X(t) = k \mid X(0) = i) = \pi_k \, ,$$

with $\pi_k \in \mathbb{R}^+$. The transition rate between two states i and j is denoted by q_{ij}. The infinitesimal generator matrix $\mathbf{Q}$ of a Markov process is such that the q_{ij}'s are the off-diagonal elements while the diagonal elements are formed as the negative sum of the non-diagonal elements of each row, i.e., $q_{ii} = -\sum_{\substack{h \in S \\ h \neq i}} q_{ih}$. The steady-state distribution $\boldsymbol{\pi}$ is the unique vector of positive numbers π_k with $k \in S$, summing to unit and satisfying the system of the global balance equations (GBEs):

$$\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}.$$

Any non-trivial solution of the GBE differs by a constant but only one satisfies the normalising condition $\sum_{k \in S} \pi_k = 1$.

Henceforth, we assume the ergodicity of the CTMCs that we study.

2.2 Reversibility

The analysis of an ergodic CTMC with equilibrium distribution can be greatly simplified if it satisfies the property that when the direction of time is reversed the behaviour of the process remains the same.

Given an ergodic CTMC in steady-state, $X(t)$ with $t \in \mathbb{R}^+$, we call $X(\tau - t)$ its reversed process. In the following we denote by $X^R(t)$ the reversed process of $X(t)$. It can be shown that $X^R(t)$ is also a stationary CTMC.

We say that $X(t)$ is *reversible* if it is stochastically identical to $X^R(t)$, i.e., $(X_{t_1}, \ldots, X_{t_n})$ has the same distribution as $(X_{\tau - t_1}, \ldots, X_{\tau - t_n})$ for all $t_1, t_2, \ldots, t_n, \tau \in \mathbb{R}^+$ [18, Ch. 1]. For an ergodic CTMC there exist simple necessary and sufficient conditions for reversibility expressed in terms of the equilibrium distribution $\boldsymbol{\pi}$ and the transition rates q_{ij}.

Proposition 1 *(Detailed balance equations [18]) A stationary Markov process with state space S and infinitesimal generator $\mathbf{Q}$ is* reversible *if and only if the following system of detailed balance equations are satisfied for a set of positive π_i, $i \in S$ summing to unity:*

$$\pi_i q_{ij} = \pi_j q_{ji}$$

for all states $i, j \in S$, with $i \neq j$. If such a set of π_i exists, then it is the equilibrium distribution of the reversible chain.

Clearly, a reversible CTMC $X(t)$ and its dual $X^R(t)$ have the same steady-state distribution since they are stochastically identical.

An important property of reversible CTMCs is the Kolmogorov's criterion which states that the reversibility of a process can be established directly from its transition rates. In particular the following proposition can be proved:

Proposition 2 *(Kolmogorov's criterion [18]) A stationary Markov process with state space S and infinitesimal generator $\mathbf{Q}$ is* reversible *if and only if its transition rates satisfy the following equation: for every finite sequence of states $i_1, i_2, \ldots i_n \in S$,*

$$q_{i_1 i_2} q_{i_2 i_3} \cdots q_{i_{n-1} i_n} q_{i_n i_1} = q_{i_1 i_n} q_{i_n i_{n-1}} \cdots q_{i_3 i_2} q_{i_2 i_1} . \tag{1}$$

2.3 Reversed process

The *reversed process* $X^R(t)$ of a Markov process $X(t)$ can always be defined even when $X(t)$ is not reversible. In [14] the author shows that $X^R(t)$ is a CTMC and proves that the transition rates are defined in terms of the equilibrium distribution of the process $X(t)$ as stated below.

Proposition 3 *(Reversed process transition rates [14]) Given the stationary Markov process $X(t)$ with state space S and infinitesimal generator $\mathbf{Q}$, the transition rates of the reversed process $X^R(t)$, forming its infinitesimal generator $\mathbf{Q}^R$, are defined as follows:*

$$q^R_{ji} = \frac{\pi_i}{\pi_j} q_{ij} , \tag{2}$$

where q^R_{ji} denotes the transition rate from state j to state i in the reversed process. The equilibrium distribution $\boldsymbol{\pi}$ is the same for both the forward and the reversed process.

Observe that we can replace in Equation (2) any non-trivial solution of the GBE. Roughly speaking, we can say that the knowledge of the reversed process' transition rates allows for an efficient computation of the invariant measure of the process and vice versa the latter allows for an efficient definition of the reversed process. In [14] the author generalises the Kolmogorov's criteria in order to encompass non-reversible CTMCs. Hereafter, for a given state i we denote by q_i (resp. q_i^R) the quantity $\sum_{h\in S,h\neq i} q_{ij}$ (resp. $\sum_{h\in S,h\neq i} q_{ij}^R$).

Proposition 4 *(Kolmogorov's generalised criteria [14]) For a given a stationary Markov process with state space S and infinitesimal generator $\mathbf{Q}$, $\mathbf{Q}^R = (q_{ij}^R)_{i,j\in S}$ is the infinitesimal generator of its reversed process if and only if the following conditions hold:*

1) $q_i^R = q_i$ *for every state* $i \in S$*;*
2) *for every finite sequence of states* $i_1, i_2, \ldots, i_n \in S$*,*

$$q_{i_1 i_2} q_{i_2 i_3} \cdots q_{i_{n1} i_n} q_{i_n i_1} = q^R_{i_1 i_n} q^R_{i_n i_{n-1}} q^R_{i_2 i_3} q^R_{i_2 i_1}. \tag{3}$$

Proposition 4 suggests us a proof method for verifying whether a vector $\boldsymbol{\pi}$ satisfies the GBE system $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$ for a given process $X(t)$. It consists of:

1. defining the reversed process $X^R(t)$ using Proposition 3 and assuming $\boldsymbol{\pi}$,
2. verifying the generalised Kolmogorov's criteria of Proposition 4.

If the generalised Kolmogorov's criteria are verified and $\sum_{k\in S} \pi_k = 1$ then, by uniqueness of the steady-state distribution, we can conclude that $\boldsymbol{\pi}$ is the steady-state distribution of the process.

2.4 Lumpability

In the context of performance and reliability analysis, the notion of *lumpability* provides a model simplification technique which can be used for generating an aggregated Markov process that is smaller than the original one but allows one to determine exact results for the original process.

The concept of lumpability can be formalized in terms of equivalence relations over the state space of the Markov process. Any such equivalence induces a *partition* on the state space of the Markov chain and aggregation is achieved by aggregating equivalent states into macro-states, thus reducing the overall state space. In general, when a CTMC is aggregated the resulting stochastic process will not have the Markov property. However if the partition satisfies the so-called *strong* lumpability condition [19, 1], the property is preserved and the steady-state solution of the aggregated process may be used to simplify the computation of the solution of the original one.

Let $\sim$ be an equivalence relation over the state space of a CTMC. If the original state space is $\{0, 1, \ldots, n\}$ then the aggregated state space is some $\{[i_0]_\sim, [i_1]_\sim, \ldots, [i_N]_\sim\}$, where $[i]_\sim$ denotes the set of states that are equivalent to i and $N \leq n$, ideally $N \ll n$. Hereafter, we use the following notation:

$$q_{i[k]} = \sum_{j\in[k]_\sim} q_{ij} \qquad q_{[k]i} = \sum_{j\in[k]_\sim} q_{ji}.$$

By a slight abuse of notation, if no confusion arises, we simply write $[i]$ to denote the equivalence class $[i]_\sim$ relative to the equivalence relation $\sim$.

Strong lumpability has been introduced in [19] and studied in, e.g., [6, 31].

Definition 1 (Strong Lumpability) Let $X(t)$ be a CTMC with state space $S = \{0, 1, \ldots, n\}$ and $\sim$ be an equivalence relation over S. We say that $X(t)$ is *strongly lumpable* with respect to $\sim$ (resp. $\sim$ is a *strong lumpability* for $X(t)$) if for any $[k] \neq [l]$ and $i, j \in [l]$, it holds that $q_{i[k]} = q_{j[k]}$.

Thus, an equivalence relation over the state space of a Markov process is a strong lumpability if it induces a partition into equivalence classes such that for any two states within an equivalence class their aggregated transition rates to any other class are the same. Notice that every Markov process is strongly lumpable with respect to the identity relation and also the trivial relation having only one equivalence class. In [19] the authors prove that for an equivalence relation $\sim$ over the state space of a Markov process $X(t)$, the aggregated process is a Markov process for every initial distribution if, and only if, $\sim$ is a strong lumpability for $X(t)$. Moreover, the transition rate between two aggregated states $[i]$ and $[j]$ is equal to $q_{i[j]}$.

Let $\sim$ be an equivalence relation over the state space of a Markov process $X(t)$. We denote by $\widetilde{X}(t)$ the aggregated process with respect to the specific relation $\sim$. If the relation $\sim$ is a strong lumpability then we denote by $\widetilde{\mathbf{Q}} = (\widetilde{q}_{[i][j]})_{[i],[j] \in S/\sim}$ the infinitesimal generator of $\widetilde{X}(t)$ which is defined as stated below.

Proposition 5 *(Aggregated process) Let $X(t)$ be a CTMC and $\sim$ be an equivalence relation over the state space of $X(t)$. The following statements are equivalent*

– *$\sim$ is a strong lumpability for $X(t)$;*
– *$\widetilde{X}(t)$ is a Markov process.*

Moreover if $\sim$ is a strong lumpability for $X(t)$ then for all $[i], [j] \in S/\sim$, it holds that $\widetilde{q}_{[i][j]} = q_{i[j]}$ where $\widetilde{\mathbf{Q}}$ is the infinitesimal generator of $\widetilde{X}(t)$.

A probability distribution $\boldsymbol{\pi}$ is *equiprobable* with respect to a partition of the state space S of an ergodic Markov process if for all the equivalence classes $[i] \in S/\sim$ and for all $i_1, i_2 \in [i]$, $\pi_{i_1} = \pi_{i_2}$. In [29] the notion of exact lumpability as a sufficient condition for a distribution to be equiprobable with respect to a partition is introduced.

Definition 2 (Exact Lumpability) Let $X(t)$ be a CTMC with state space $S = \{0, 1, \ldots, n\}$ and $\sim$ be an equivalence relation over S. We say that $X(t)$ is *exactly lumpable* with respect to $\sim$ (resp. $\sim$ is an exact lumpability for $X(t)$) if for any $[k], [l] \in S/\sim$ and $i, j \in [l]$, it holds that $q_{[k]i} = q_{[k]j}$.

An equivalence relation is an exact lumpability if it induces a partition on the state space such that for any two states within an equivalence class the aggregated transition rates into such states from any other class are the same. Notice that Definition 2 does not require $[k] \neq [l]$ which means that the transitions rates from a class to itself must be considered and that the definition of $q_{[k]i}$ with $i \in [k]$ takes into account the diagonal elements of the infinitesimal generator.

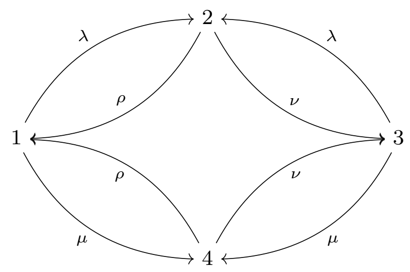


Fig. 1: A strongly, but not exactly, lumpable CTMC.

The following proposition states that any exact lumpability induces an equiprobable distribution over its partition.

Proposition 6 *(Equiprobable distribution [6]) Let $X(t)$ be a CTMC with state space $S = \{0, 1, \ldots, n\}$ and $\sim$ be an equivalence relation over S. If $X(t)$ is* exactly lumpable *with respect to $\sim$ (resp. $\sim$ is an exact lumping for $X(t)$) then for all $i_1 \sim i_2$, $\pi_{i_1} = \pi_{i_2}$.*

As for strong lumpabability, every Markov process is exactly lumpable with respect to the identity relation. However, differently from strong lumpability, the relation having only one equivalence class is in general not an exact lumpability since, in this case, the equiprobability of its equilibrium distribution would not hold.

Finally, we introduce the notion of strict lumpability as an equivalence relation over the state space of a Markov process that is a strong lumpability with an equiprobable distribution.

Definition 3 (Strict Lumpability) Let $X(t)$ be a CTMC and $\sim$ be an equivalence relation over its state space. We say that $X(t)$ is *strictly lumpable* with respect to $\sim$ if it is both strongly and exactly lumpable with respect to $\sim$ (resp. $\sim$ is a strict lumpability for $X(t)$ if, and only if, it is both a strong and an exact lumpability).

Example 1 Consider the CTMC depicted in Fig. 1 with $\rho \neq \nu$. Let $S = \{1, 2, 3, 4\}$ be its state space and $\sim$ be the equivalence relation such that $1 \sim 3$ and $2 \sim 4$, inducing the partition $S/\sim = \{\{1, 3\}, \{2, 4\}\}$. It is easy to see that $\sim$ is a strong lumpability for $X(t)$ but it is not an exact lumpability. Indeed, for instance, $q_{\{2,4\},1} \neq q_{\{2,4\},3}$ when $\rho \neq \nu$. □

Example 2 Consider the CTMC with state space $S = \{i_1, i_2, j_1, j_2, j_3\}$ depicted in Fig. 2. Let $\sim$ be the equivalence relation defined by the reflexive and transitive closure of: $i_1 \sim i_2$, $j_1 \sim j_2$ and $j_2 \sim j_3$. The state space S is partitioned into the classes: $S/\sim = \{[i], [j]\}$, where $[i] = \{i_1, i_2\}$ and

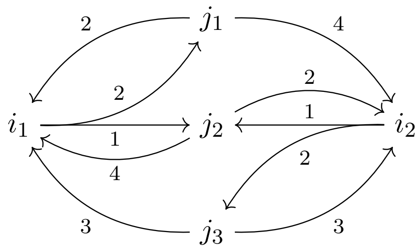


Fig. 2: A strictly lumpable CTMC.

$[j] = \{j_1, j_2, j_3\}$. Observe that

$$
\begin{array}{lll}
q_{j_1[i]} = q_{j_2[i]} = q_{j_3[i]} = 6 & q_{[i]j_1} = q_{[i]j_2} = q_{[i]j_3} = 2 & q_{[j]j_1} = q_{[j]j_2} = q_{[j]j_3} = -6 \\
q_{i_1[j]} = q_{i_2[j]} = 3 & q_{[i]i_1} = q_{[i]i_2} = -3 & q_{[j]i_1} = q_{[j]i_2} = 9
\end{array}
$$

By Definitions 1 and 2, $\sim$ is a strict lumpabability for $X(t)$. □

The next corollary follows from Propositions 5 and 6.

Corollary 1 *Let $X(t)$ be a CTMC with state space S and $\sim$ be a strict lumpability for $X(t)$. Then*

- *$\widetilde{X}(t)$ is a Markov process;*
- *$\pi_{i_1} = \pi_{i_2}$, for all $i_1, i_2 \in S$ such that $i_1 \sim i_2$;*
- *$\widetilde{q}_{[i][j]} = q_{i[j]}$, for all $[i], [j] \in S/\sim$.*

The following proposition will be used later on.

Proposition 7 *Let $X(t)$ be a CTMC with state space S and $\sim \subseteq S \times S$ be a strict lumpability for $X(t)$. Then, for each class $[i], [j] \in S/\sim$ it holds:*

$$n_i q_{i[j]} = n_j q_{[i]j} \,,$$

where n_h is the cardinality of the equivalence class $[h]$, with $h = i, j$.

Proof Let $[i] \neq [j]$. By definition of strict lumpability we can write:

$$n_i q_{i[j]} = \sum_{k\in[i]} q_{k[j]} = \sum_{k\in[i]}\sum_{h\in[j]} q_{kh} = \sum_{h\in[j]}\sum_{k\in[i]} q_{kh} = \sum_{h\in[j]} q_{[i]h} = n_j q_{[i]j} \,.$$

Consider now $i, j \in [i]$, then trivially $n_i = n_j$ and hence we have to prove that $q_{i[i]} = q_{[i]j}$. We can write $q_{[i]j}$ as:

$$q_{[i]j} = \sum_{\substack{k\in[i]\\ k\neq j}} q_{kj} + q_{jj}$$

which, by the fact that $q_{jj} = -\sum_{\substack{k\in S\\ k\neq j}} q_{jk}$, can be written as:

$$q_{[i]j} = \sum_{\substack{k\in[i]\\ k\neq j}} q_{kj} - \sum_{[\ell]\neq[i]} q_{j[\ell]} - \sum_{\substack{k\in[i]\\ k\neq j}} q_{jk} \,.$$

Since $\sim$ is a strong lumping we have $\sum_{[\ell]\neq[i]} q_{j[\ell]} = \sum_{[\ell]\neq[i]} q_{i[\ell]}$ and by the definition of exact lumping we have $q_{[i]i} = q_{[j]i}$. Hence, for any pair $i, j \in [i]$:

$$\sum_{\substack{k\in[i]\\ k\neq j}} q_{kj} - \sum_{\substack{k\in[i]\\ k\neq j}} q_{jk} = \sum_{\substack{k\in[i]\\ k\neq i}} q_{ki} - \sum_{\substack{k\in[i]\\ k\neq i}} q_{ik} = \Delta_{[i]} \,.$$

We prove that $\Delta_{[i]} = 0$:

$$\sum_{j\in[i]} \Delta_{[i]} = n_i \Delta_{[i]} = \sum_{j\in[i]} \sum_{\substack{k\in[i]\\ k\neq j}} q_{kj} - \sum_{j\in[i]} \sum_{\substack{k\in[i]\\ k\neq j}} q_{jk} = 0 \,.$$

To conclude the proof we rewrite $q_{i[i]}$ as:

$$q_{i[i]} = \sum_{\substack{k\in[i]\\ k\neq i}} q_{ik} - \sum_{[\ell]\neq[i]} q_{i[\ell]} - \sum_{\substack{k\in[i]\\ k\neq i}} q_{ik} = - \sum_{[\ell]\neq[i]} q_{i[\ell]} \,.$$

Subtracting the expressions of $q_{[i]j}$ and that of $q_{i[i]}$, and recalling that the CTMC is strongly lumpable, we obtain:

$$\sum_{\substack{k\in[i]\\ k\neq i}} q_{kj} - \sum_{\substack{k\in[i]\\ k\neq j}} q_{jk} = \Delta_{[i]} = 0 \,,$$

which concludes the proof. □

2.5 Similarity of states

In this section we introduce the notion of *similar states* proposed by Yap in [34]. We then generalize this definition and introduce the notion of *weakly similar states* that is at the basis of the novel concept of *autoreversibility* for CTMCs presented in Section 6.

Definition 4 (Similar states) Two distinct states i_1 and i_2 of a CTMC are similar if their rates to every other state agree, i.e., $q_{i_1 j} = q_{i_2 j}$ $\forall j \neq i_1, i_2$.

In [34] the author shows that the similarity relation is not transitive (and hence it is not an equivalence relation). Moreover, given a partition $S_1, \ldots, S_t$ of the chain's state space such that within the same class there are only similar states (but similar states may belong to different classes) then $S_1, \ldots, S_t$ is a strong lumping [19] for the original CTMC. In [34] the author discusses the applicability of this result to the analysis of DNA sequences.

3 Weak similarity

In this section we introduce a new notion of state similarity, named *weak similarity*, inspired by the one proposed by Yap in [34]. Specifically, we relate those states whose rates to and from any class of weakly similar states agree. We show that weak similarity is an equivalence relation and indeed it is a strict lumpability for the original CTMC. Strict lumpability plays a pivotal role in our work. The symmetry conditions required by Definition 5 allow us to prove that weak similarity is a strict lumpability in Theorem 1.

In the following, $i \to j$ denotes a transition from state i to state j and $[j]_{\sim_w}$ denotes the set of states which are weakly similar to j. Moreover, we denote by $m_{i[j]_{\sim_w}}$ the number of transitions from state i to the set of states in $[j]_{\sim_w}$ and by $m_{[j]_{\sim_w}i}$ the number of transitions from the set $[j]_{\sim_w}$ to i.

Definition 5 (Weak similarity [23]) Given a CTMC with state space S, a reflexive and symmetric relation $\sim_w \subseteq S \times S$ is a weak similarity if:

1. for every $i_1 \to j_1$ and $i_2 \to j_2$ such that $i_1 \sim_w i_2$ and $j_1 \sim_w j_2$ it holds that $q_{i_1 j_1} = q_{i_2 j_2}$;
2. for every state j and for every state i_1 and i_2 such that $i_1 \sim_w i_2$, it holds that $m_{i_1[j]_{\sim_w}} = m_{i_2[j]_{\sim_w}}$ and $m_{[j]_{\sim_w}i_1} = m_{[j]_{\sim_w}i_2}$

where for every $i, j \in S$, $m_{i[j]_{\sim_w}} = |\{i \to k : \ j \sim_w k\}|$ and $m_{[j]_{\sim_w}i} = |\{k \to i : \ j \sim_w k\}|$.

Before giving the intuition behind each of the definition items above, we prove the following proposition.

Proposition 8 *Weak similarity is an equivalence relation.*

Proof We prove that $\sim_w$ is a transitive relation. Let i_1, i_2, i_3 be three states such that $i_1 \sim_w i_2$ and $i_2 \sim_w i_3$.

In order to prove the first item of Definition 5, let j_1, j_2, j_3 be three states such that $i_1 \to j_1$, $i_2 \to j_2$ and $i_3 \to j_3$ and $j_1 \sim_w j_2$ and $j_2 \sim_w j_3$. From $i_1 \sim_w i_2$ we have $q_{i_1 j_1} = q_{i_2 j_2}$ and from $i_2 \sim_w i_3$ we have $q_{i_2 j_2} = q_{i_3 j_3}$, hence $q_{i_1 j_1} = q_{i_3 j_3}$.

To prove the second item of Definition 5 consider a state j. From $i_1 \sim_w i_2$ we have $m_{i_1[j]_{\sim_w}} = m_{i_2[j]_{\sim_w}}$ and $m_{[j]_{\sim_w}i_1} = m_{[j]_{\sim_w}i_2}$. From $i_2 \sim_w i_3$ we have $m_{i_2[j]_{\sim_w}} = m_{i_3[j]_{\sim_w}}$ and $m_{[j]_{\sim_w}i_2} = m_{[j]_{\sim_w}i_3}$. Hence $m_{i_1[j]_{\sim_w}} = m_{i_3[j]_{\sim_w}}$ and $m_{[j]_{\sim_w}i_1} = m_{[j]_{\sim_w}i_3}$. □

Informally, Condition 1 of Definition 5 asks that all the transitions from the states of an equivalence class $[i]_{\sim_w}$ to any state of an equivalence class $[j]_{\sim_w}$ have the same rate. Condition 2 asks that the number transitions from equivalent states to the states of a fixed equivalence class are the same, and the number transitions from the states of a fixed equivalence class to distinct equivalent states are the same. This case arises for instance in the aggregating technique for interleaving of identical components in Markovian process algebra as presented in [13].

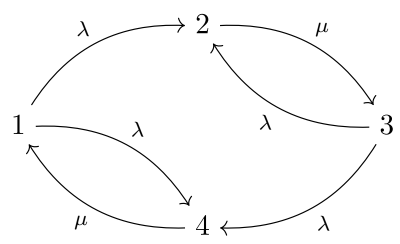


Fig. 3: A simple example of weak similarity.

Notice that, in general, Yap's notion of similarity classes does not imply weak similarity. Indeed, let us consider the example presented in [34] of a CTMC with infinitesimal generator

$$\mathbf{Q} = \begin{bmatrix} * & \alpha_2 & \alpha_3 \\ \alpha_1 & * & \alpha_3 \\ \alpha_1 & \alpha_4 & * \end{bmatrix}$$

where $*$ denotes the negative sum of the row off-diagonal entries and all α_i are distinct. Notice that according to Definition 4 state 1 is similar to state 2 and the latter is similar to state 3. However, states 1 and 3 are not similar since $\alpha_2 \neq \alpha_4$. Notice that in the case of weak similarity if $1 \sim_w 2$, then it must hold $\alpha_1 = \alpha_2$ by Condition 1 of Definition 5. In [34] the author proves that for reversible CTMCs, state similarity becomes an equivalence relation. In these cases it implies our notion of weak similarity.

Example 3 Let us consider the CTMC depicted by Fig. 3. We can easily prove that the equivalence relation $\sim_w$ defined by the symmetric and reflexive closure of $1 \sim_w 3$ and $2 \sim_w 4$ is a weak similarity. Therefore, the state space S can be partitioned into the following equivalence classes:

$$S/\sim_w = \{\{1,3\},\{2,4\}\}\,.$$

A more complex example is depicted in Fig. 4. In this case one can easily prove that the equivalence relation defined as the symmetric and reflexive closure of $1 \sim_w 3$ and $4 \sim_w 6$ is a weak similarity. As a consequence, the state space S can be partitioned into equivalence classes as follows:

$$S/\sim_w = \{\{1,3\},\{4,6\},\{2\},\{5\}\}\,.$$

Propositions 9 and Theorem 1 allow us to characterise the equilibrium distribution of an ergodic CTMC on which a weak similarity relation is defined.

Proposition 9 *Given a CTMC with state space S, if $i, j \in S$ and $i \sim_w j$ then $q_i = q_j$ where $q_h = \sum_{k \in S \smallsetminus \{h\}} q_{hk}$ is the total rate out of state h, with $h = i, j$.*

Proof The proof follows by Conditions 1 and 2 of Definition 5. □

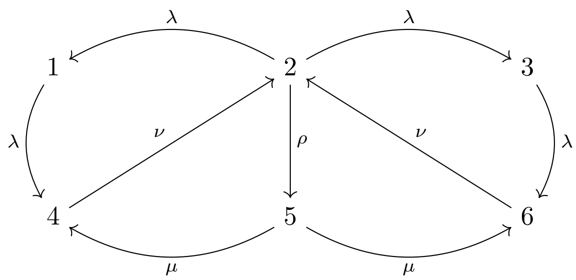


Fig. 4: A more complex example of weak similarity.

Theorem 1 shows that if $\sim_w$ is a weak similarity over the state space of a Markov process $X(t)$ then $\sim_w$ is also a strict lumpability for $X(t)$ (but the opposite is, in general, not true).

Theorem 1 *Let $X(t)$ be a CTMC with state space S and $\sim_w$ be a weak similarity over S. Then $\sim_w$ is a strict lumpability for $X(t)$.*

Proof Let $\sim_w \subseteq S \times S$ be a weak similarity and $[i]_{\sim_w}$ denote the equivalence class of all elements in S which are weakly similar to i. By Definition 5, for every $i_1 \to j_1$ and $i_2 \to j_2$ such that $i_1 \sim_w i_2$ and $j_1 \sim_w j_2$, and for every $[j]_{\sim_w}$ it holds that

1) $q_{i_1 j_1} = q_{i_2 j_2}$
2) $m_{i_1 [j]_{\sim_w}} = m_{i_2 [j]_{\sim_w}}$
3) $m_{[j]_{\sim_w} i_1} = m_{[j]_{\sim_w} i_2}$.

Hence, for any $[k]_{\sim_w} \neq [l]_{\sim_w}$ and $i, j \in [l]_{\sim_w}$, by items 1) and 2) we have

$$q_{i[k]_{\sim_w}} = q_{j[k]_{\sim_w}}.$$

Moreover, by items 1) and 3), for any $[k]_{\sim_w}, [l]_{\sim_w}$ and $i, j \in [l]_{\sim_w}$ we have

$$q_{[k]_{\sim_w} i} = q_{[k]_{\sim_w} j}$$

i.e., by Definition 3, $\sim_w$ is a strict lumpability. □

Example 4 Consider the CTMC depicted in Fig. 2 and the equivalence relation $\sim$ defined in Example 2. It is easy to see that $\sim$ is a strict lumpability but is is not a weak similarity since, for instance, $q_{j_1 i_1} \neq q_{j_1 i_2} \neq q_{j_3 i_1}$.

Another property of weak similarity that will be used in Section 4 is the following:

Proposition 10 *Given a CTMC and a weak similarity relation $\sim_w \subseteq S \times S$, for each class $[i]_{\sim_w}, [j]_{\sim_w} \in S/\sim_w$ it holds:*

$$n_i m_{i[j]_{\sim_w}} = n_j m_{[i]_{\sim_w} j} \tag{4}$$

where n_h is the cardinality of the equivalence class $[h]_{\sim_w}$, with $h = i, j$.

Proof The proof follows from Definition 5 by observing that the total number of arcs from class $[i]_{\sim_w}$ to $[j]_{\sim_w}$ can be computed using either the expression on the left-hand-side or that on the right-hand-side of Equation (4). □

4 Reversibility and Lumpability

In this section we prove the main result of our paper. The following theorem states that if an equivalence relation over the state space of a Markov process is an exact lumpability then it is a strong lumpability for the reversed process.

Theorem 2 *Let $X(t)$ be a CTMC with state space S and $X^R(t)$ its reversed process. Let $\sim$ be an exact lumpability for $X(t)$. Then $\sim$ is a strong lumpability for $X^R(t)$.*

Proof Suppose that $\sim$ is an exact lumpability for $X(t)$, i.e., for any $[k], [l]$ and $i, j \in [l]$, $q_{[k]i} = q_{[k]j}$. We prove that $\sim$ is a strong lumpability for $X^R(t)$. Indeed, by Equation (2) of Proposition 3,

$$q^R_{i[k]} = \sum_{h \in [k]} q^R_{ih} = \sum_{h \in [k]} \frac{\pi_h}{\pi_i} q_{hi} \,.$$

Since, from Proposition 6, states in the same equivalence class have the same distribution, we can write:

$$\sum_{h \in [k]} \frac{\pi_h}{\pi_i} q_{hi} = \frac{\pi_k}{\pi_i} \sum_{h \in [k]} q_{hi} = \frac{\pi_k}{\pi_i} q_{[k]i} \,.$$

Now, since from the definition of exact lumpability, for any $[k], [l]$, with $[k] \neq [l]$ and $i, j \in [l]$, $q_{[k]i} = q_{[k]j}$ and $\pi_i = \pi_j$ we have that:

$$q^R_{i[k]} = \frac{\pi_k}{\pi_i} q_{[k]i} = \frac{\pi_k}{\pi_j} q_{[k]j} = q^R_{j[k]} \,,$$

proving that $\sim$ is a strong lumpability for the reversed process $X^R(t)$. □

In general, if $\sim$ is a strong lumpability for $X(t)$ then $\sim$ is neither a strong nor an exact lumpability for $X^R(t)$.

Example 5 Let $X(t)$ be the CTMC depicted in Fig. 5 and $\sim$ be the equivalence relation defined as the symmetric and reflexive closure of $1 \sim 3$ and $2 \sim 4$. It is easy to prove that $\sim$ is a strong lumpability for $X(t)$. Let us now consider the reversed process $X^R(t)$ represented in Fig. 6 with $\delta = 2\beta + \gamma$ and $\zeta = \epsilon\gamma + 2\beta(\epsilon + \lambda)$. One can trivially prove that $\sim$ is neither a strong nor an exact lumpability for $X^R(t)$.

Theorem 3 states that an equivalence relation is a strict lumpability for a Markov process if, and only if, it is a strict lumpability for its reversed process.

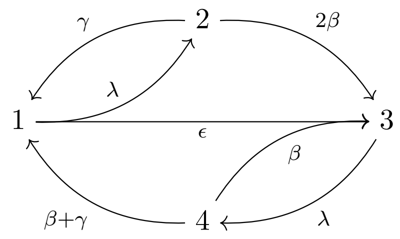


Fig. 5: A strongly but not exactly lumpable and non-reversible CTMC.

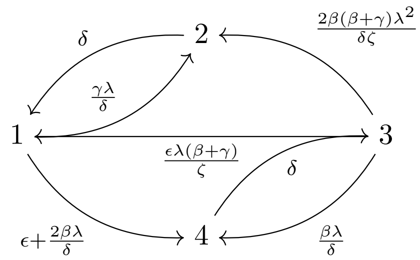


Fig. 6: Reversed process of the model in Fig. 5.

Theorem 3 *Let $X(t)$ be a CTMC with state space S and $X^R(t)$ its reversed process. An equivalence relation $\sim \subseteq S \times S$ is a strict lumpability for $X(t)$ if, and only if, $\sim$ is a strict lumpability for $X^R(t)$.*

Proof Suppose that $\sim$ is a strict lumpability for $X(t)$, i.e., for any $[k] \neq [l]$ and $i, j \in [l]$, $q_{i[k]} = q_{j[k]}$ and for any $[k], [l]$, $q_{[k]i} = q_{[k]j}$. From the fact that $\sim$ is an exact lumpability for $X(t)$, by Theorem 2, $\sim$ is a strong lumpability for $X^R(t)$. We now prove that $\sim$ is also an exact lumpability for $X^R(t)$. Indeed, by Equation (2) of Proposition 3,

$$q^R_{[k]i} = \sum_{h\in[k]} q^R_{hi} = \sum_{h\in[k]} \frac{\pi_i}{\pi_h} q_{ih} \,.$$

Since, from Proposition 6, states in the same equivalence class have the same distribution, we can write:

$$\sum_{h\in[k]} \frac{\pi_i}{\pi_h} q_{ih} = \frac{\pi_i}{\pi_k} \sum_{h\in[k]} q_{ih} = \frac{\pi_i}{\pi_k} q_{i[k]} \,.$$

Now, since $\sim$ is a strong lumpability, for any $[k] \neq [l]$ and $i, j \in [l]$, $q_{i[k]} = q_{j[k]}$ and $\pi_i = \pi_j$, we have:

$$q^R_{[k]i} = \frac{\pi_i}{\pi_k} q_{i[k]} = \frac{\pi_j}{\pi_k} q_{j[k]} = q^R_{[k]j} \,.$$

In order to prove that $\sim$ is an exact lumpability for $X^R(t)$ it remains to prove that for any $[k]$ and $i, j \in [k]$ we have $q^R_{[k]i} = q^R_{[k]j}$ that can be written as:

$$q^R_{ii} + \sum_{h\in[i],h\neq i} q^R_{hi} = q^R_{jj} + \sum_{h\in[j],h\neq j} q^R_{hj} \,. \tag{5}$$

By definition of q^R_{ii} we obtain:

$$q^R_{ii} = -\sum_{h\neq i} q^R_{ih} = -\sum_{[k],i\not\in[k]} q^R_{i[k]} - \sum_{h\in[i],h\neq i} q^R_{ih}$$

and

$$q^R_{jj} = -\sum_{h\neq j} q^R_{jh} = -\sum_{[k],j\not\in[k]} q^R_{j[k]} - \sum_{h\in[j],h\neq j} q^R_{jh} \,.$$

By substituting the definitions of q^R_{ii} and q^R_{jj} in Equation (5), we obtain:

$$\begin{aligned} &-\sum_{[k],i\not\in[k]} q^R_{i[k]} - \sum_{h\in[i],h\neq i} q^R_{ih} + \sum_{h\in[i],h\neq i} q^R_{hi} \\ &\qquad = -\sum_{[k],j\not\in[k]} q^R_{j[k]} - \sum_{h\in[j],h\neq j} q^R_{jh} + \sum_{h\in[j],h\neq j} q^R_{hj} \,. \end{aligned} \tag{6}$$

From the fact that $\sim$ is a strong lumpability for $X^R(t)$, we have $\sum_{[k],i\not\in[k]} q^R_{i[k]} = \sum_{[k],j\not\in[k]} q^R_{j[k]}$ and then Equation (6) reduces to

$$-\sum_{h\in[i],h\neq i} q^R_{ih} + \sum_{h\in[i],h\neq i} q^R_{hi} = -\sum_{h\in[j],h\neq j} q^R_{jh} + \sum_{h\in[j],h\neq j} q^R_{hj} \,. \tag{7}$$

Now observe that $\sim$ is an exact lumpability for $X(t)$ and then for any equivalence class $[k]$ and $i, j \in [k]$, $q_{[k]i} = q_{[k]j}$, i.e.,

$$q_{ii} + \sum_{h\in[i],h\neq i} q_{hi} = q_{jj} + \sum_{h\in[j],h\neq j} q_{hj}$$

and, by the fact that $\sim$ is also a strong lumpability for $X(t)$, we can write:

$$-\sum_{h\in[i],h\neq i} q_{ih} + \sum_{h\in[i],h\neq i} q_{hi} = -\sum_{h\in[j],h\neq j} q_{jh} + \sum_{h\in[j],h\neq j} q_{hj} \,. \tag{8}$$

By Proposition 3 and Equation (2), since equivalent states have the same equilibrium probability, we have that for all $h \in [l]$ $q_{lh} = q^R_{hl}$ with $l = i, j$. Hence, Equation (8) can be written as:

$$-\sum_{h\in[i],h\neq i} q^R_{hi} + \sum_{h\in[i],h\neq i} q^R_{ih} = -\sum_{h\in[j],h\neq j} q^R_{hj} + \sum_{h\in[j],h\neq j} q^R_{jh} \,,$$

which multiplying by -1 gives exactly Equation (7), proving that $\sim$ is an exact lumpability for $X^R(t)$.

The proof that if $\sim$ is a strict lumpability for $X^R(t)$ then $\sim$ is a strict lumpability for $X(t)$ is analogous. □

Given a stochastic process $X(t)$ with state space S and an equivalence relation $\sim$ over S, we denote by $\widetilde{X}(t)$ and $\widetilde{X^R}(t)$ the aggregated processes with respect to $\sim$ corresponding to $X(t)$ and $X^R(t)$, respectively.

Corollary 2 *Let $X(t)$ be a CTMC with state space S and $\sim \subseteq S \times S$ be an equivalence relation. If $\sim$ is a strict lumpability for $X(t)$ then both the aggregated processes $\widetilde{X}(t)$ and $\widetilde{X^R}(t)$ satisfy the Markov property.*

Proof The fact that $\widetilde{X}(t)$ satisfies the Markov property follows from the definition of strong lumpability, whereas $\widetilde{X^R}(t)$ is a Markov process because by Theorem 3 we know that $X^R(t)$ is strictly lumpable with respect to $\sim$. □

If $X(t)$ is a reversible CTMC then exact lumpability is a necessary and sufficient condition for strict lumpability.

Proposition 11 *Let $X(t)$ be a reversible CTMC with state space S and $\sim \subseteq S \times S$ be an equivalence relation. $\sim$ is a strict lumpability for $X(t)$ if, and only if, $\sim$ is an exact lumpability for $X(t)$.*

Proof If $\sim$ is a strict lumpability for $X(t)$ then, by definition, $\sim$ is an exact lumpability for $X(t)$. Conversely, if $\sim$ is an exact lumpability for $X(t)$, since $X(t)$ is reversible, i.e., its reversed process is stochastically identical to $X(t)$, then $\sim$ is also a strong lumpability for $X(t)$. This implies that $\sim$ is a strict lumpability for $X(t)$. □

We now investigate the relationships between $\widetilde{X^R}(t)$ and the reversed process of $\widetilde{X}(t)$, denoted by $(\widetilde{X})^R(t)$. We prove that they are stochastically identical when $X(t)$ is strictly lumpable.

Theorem 4 *Let $X(t)$ be a CTMC with state space S and $\sim \subseteq S \times S$ be a strict lumpability for $X(t)$. Then the Markov processes $\widetilde{X^R}(t)$ and $(\widetilde{X})^R(t)$ are stochastically identical.*

Proof First observe that $\widetilde{X^R}(t)$ and $(\widetilde{X})^R(t)$ have the same state space that is $S/\sim$ and by Theorem 3 $\widetilde{X^R}(t)$ is a Markov process. Moreover, they have the same equilibrium distribution, i.e., for all $[i] \in S/\sim$, $\pi_{[i]} = \sum_{h \in [i]} \pi_h$.

Now we prove that they have the same transition rates. Let $\widetilde{\mathbf{Q}^R}$ and $(\widetilde{\mathbf{Q}})^R$ be the infinitesimal generators of $\widetilde{X^R}(t)$ and $(\widetilde{X})^R(t)$, respectively. We show that for any $[i], [j] \in S/\sim$, $[i] \neq [j]$, $\widetilde{q^R}_{[i][j]} = (\widetilde{q})^R_{[i][j]}$. By definition of $\widetilde{X^R}(t)$,

$$\widetilde{q^R}_{[i][j]} = \sum_{h \in [j]} q^R_{ih} = q^R_{i[j]}$$

while, considering the equiprobability of the equilibrium probability induced by the exact lumping:

$$(\widetilde{q})^R_{[i][j]} = \frac{\pi_{[j]}}{\pi_{[i]}} q_{j[i]} = \frac{n_j}{n_i} \sum_{h \in [i]} \frac{\pi_j}{\pi_h} q_{jh} = \frac{n_j}{n_i} \sum_{h \in [i]} q^R_{hj} = \frac{n_j}{n_i} q^R_{[i]j} \, .$$

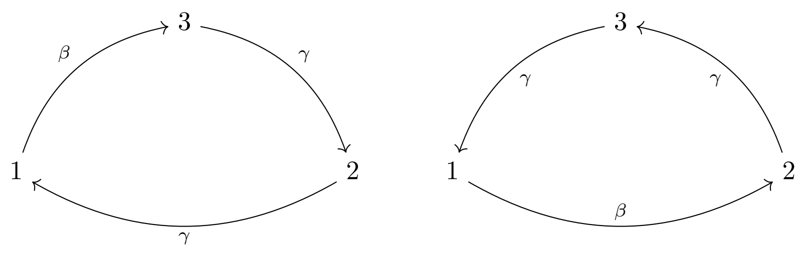


Fig. 7: A ρ-reversible CTMC $X(t)$ and its reversed process $X^R(t)$

From the fact that $\sim$ is a strict lumpability for $X^R(t)$, by Proposition 7, we have $n_j/n_i q^R_{[i]j} = q^R_{i[j]}$ and this proves that $\widetilde{q^R}_{[i][j]} = q^R_{i[j]} = (\widetilde{q})^R_{[i][j]}$. □

5 Lumpable-based reversibility

Many stochastic processes are not reversible, however the corresponding aggregated processes with respect to a lumpable relation may be reversible modulo some renaming of the state names. In this section we generalise the notion of reversibility and introduce a novel notion named *$\lambda\rho$-reversibility*.

Hereafter, a *renaming* ϱ over the state space of a Markov process is a bijection on S. For a Markov process $X(t)$ with state space S we denote by $\varrho(X)(t)$ the same process where the state names are changed according to ϱ. More formally, let $\mathbf{Q}$ and $\boldsymbol{\pi}$ be the infinitesimal generator and the equilibrium distribution of $X(t)$; $\mathbf{Q}'$ and $\boldsymbol{\pi}'$ be the infinitesimal generator and the equilibrium distribution of $\varrho(X)(t)$. It holds that for all $i, j \in S$,

$$q_{ij} = q'_{\varrho(i)\varrho(j)} \quad \text{and} \quad \pi_i = \pi'_{\varrho(i)}.$$

We first introduce the notions of ρ-reversibility and λ-reversibility. Then, they will be combined to obtained the definition of $\lambda\rho$-reversibility.

Definition 6 (ρ-reversibility) A CTMC $X(t)$ with state space S is *ρ-reversible* if there exists a renaming ϱ on S such that $X(t)$ and $\varrho(X^R)(t)$ are stochastically identical. In this case we say that $X(t)$ is ρ-reversible with respect to ϱ.

Example 6 Consider the CTMC $X(t)$ and its reversed process $X^R(t)$ depicted in Fig. 7. It is easy to see that $X(t)$ is not reversible. However, if we consider the renaming ϱ defined as: $\varrho(1) = 1$, $\varrho(2) = 3$, $\varrho(3) = 2$ then we can prove that $X(t)$ and $\varrho(X^R)(t)$ are stochastically identical, i.e., $X(t)$ is ρ-reversible.

Definition 7 (λ-reversibility) A CTMC $X(t)$ with state space S is *λ-reversible* with respect to a strict lumpability $\sim$ for $X(t)$ if $X(t)$ and $\widetilde{X^R}(t)$ are stochastically identical.

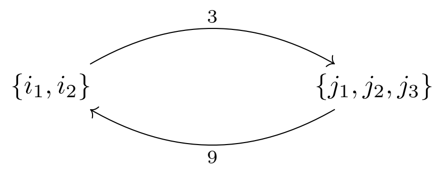


Fig. 8: The aggregated process of the CTMC in Fig. 2.

Notice that, since by Theorem 4, $\widetilde{X^R}(t)$ and $(\widetilde{X})^R(t)$ are stochastically identical, we can say that $X(t)$ is λ-reversible with respect to a strict lumpability $\sim$ over S if $\widetilde{X}(t)$ is reversible.

Example 7 Let $X(t)$ be the CTMC depicted in Fig. 2 and $\sim$ be the strict lumpability presented in Example 2. The state space S is partitioned into the following classes: $S/\sim = \{\{i_1, i_2\}, \{j_1, j_2, j_3\}\}$. The aggregated process $\widetilde{X}(t)$, depicted in Fig. 8, is reversible. Hence, $X(t)$ is λ-reversible with respect to $\sim$.

Definition 8 plays a pivotal role in the theory we develop hereafter. Based on the notion of $\lambda\rho$-reversibility, we study efficient ways of deriving the equilibrium probabilities and compare the class of $\lambda\rho$-reversible CTMCs with other classes previously introduced in the literature [18, 23].

Definition 8 ($\lambda\rho$-reversibility) A CTMC $X(t)$ with state space S is said to be $\lambda\rho$-*reversible* with respect to a strict lumpability $\sim$ for $X(t)$ and a renaming ϱ on $S/\sim$ if $\widetilde{X}(t)$ and $\varrho(\widetilde{X^R})(t)$ are stochastically identical.

It is clear that a Markov process is ρ-reversible when it is $\lambda\rho$-reversible with respect to the trivial lumpability. Moreover, λ-reversibility corresponds to $\lambda\rho$-reversibility with respect to the trivial renaming.

Analogously to what has been shown in [18] for reversible CTMCs (see Proposition 1) we prove in Proposition 12 necessary and sufficient conditions for $\lambda\rho$-reversibilty based on the existence of the solution for the system of linear equations called detailed balance equations. We denote by $\varrho[i]$ the renaming of the class $[i]$ according to ϱ.

We first introduce a Lemma which will simplify the proof of Proposition 12.

Lemma 1 *Let $X(t)$ be a CTMC with state space S and infinitesimal generator* $\mathbf{Q}$ *and ϱ be a renaming of the states such that $q_i = q_{\varrho(i)}$ for all $i \in S$. If there exists a set of positive real numbers π_i, $i \in S$, summing to unity satisfying:*

$$\pi_i q_{ij} = \pi_j q_{\varrho(j)\varrho(i)} \quad \forall i, j \in S\,, \tag{9}$$

then π_i is the unique equilibrium distribution of $X(t)$.

Proof We carry out the proof by substitution of the expression of π_i given by Equation (9) in the system of global balance equations of $X(t)$. We have:

$$\pi_i \sum_{\substack{j \in S \\ i \neq j}} q_{ij} = \sum_{\substack{j \in S \\ i \neq j}} \pi_j q_{ji}\,,$$

that divided by π_i gives:

$$\sum_{\substack{j\in S\\ i\neq j}} q_{ij} = \sum_{\substack{j\in S\\ i\neq j}} \frac{\pi_j}{\pi_i} q_{ji} = \sum_{\substack{j\in S\\ i\neq j}} q_{\varrho(i)\varrho(j)} \,.$$

Since ϱ is a bijection, this reduces to $q_i = q_{\varrho(i)}$ which is an identity by hypothesis. □

Proposition 12 *(Detailed balance equations for $\lambda\rho$-reversible processes) Let $X(t)$ be a CTMC with state space S and infinitesimal generator $\mathbf{Q}$, let $\sim$ be a strict lumpability for $X(t)$ such that the infinitesimal generator $\widetilde{\mathbf{Q}}$ of $\tilde{X}(t)$ is defined as stated in Corollary 1 and let ϱ be a renaming on $S/\sim$ such that $\widetilde{q}_{[i]} = \widetilde{q}_{\varrho[i]}$ for all $[i] \in S/\sim$. Then $X(t)$ is $\lambda\rho$-reversible with respect to $\sim$ and ϱ if and only if there exists a set of positive real numbers π_i summing to unity, with $i \in S$, such that the following system of detailed balance equations are satisfied: for all $[i],[j] \in S/\sim$ with $[i] \neq [j]$, for all $i \in [i]$, $j \in [j]$ and $j' \in \varrho[j]$:*

$$n_i \pi_i \widetilde{q}_{[i][j]} = n_j \pi_j \widetilde{q}_{\varrho[j]\varrho[i]}$$

or, equivalently,

$$n_i \pi_i q_{i[j]} = n_j \pi_j q_{j'\varrho[i]}$$

where n_h is the cardinality of the equivalence class $[h]$, with $h = i, j$. If such a solution π_i exists then it is the equilibrium distribution of $X(t)$, while $\pi_{[i]} = n_i \pi_i$ is the equilibrium distribution of $\widetilde{X}(t)$.

Proof Let $\widetilde{\mathbf{Q}}$ be the infinitesimal generator of $\tilde{X}(t)$ and $\widetilde{\mathbf{Q}}'$ be the infinitesimal generator of $\varrho(\widetilde{X^R})(t)$. We prove that a $\lambda\rho$ reversible CTMC satisfies the system of detailed balance equations. Observe that by definition of renaming and the fact that $\widetilde{X^R}(t)$ and $(\widetilde{X})^R(t)$ are stochastically identical, for $[i],[j] \in S/\sim$

$$\widetilde{q}\,'_{\varrho[j]\varrho[i]} = (\widetilde{q})^R_{[j][i]} \,. \tag{10}$$

Since $\widetilde{X}(t)$ and $\varrho(\widetilde{X^R})(t)$ are stochastically identical we have $\widetilde{q}\,'_{\varrho[j]\varrho[i]} = \widetilde{q}_{\varrho[j]\varrho[i]}$ and hence, by Equation (10), $(\widetilde{q})^R_{[j][i]} = \widetilde{q}_{\varrho[j]\varrho[i]} = q_{j'\varrho[i]}$ for $j' \in \varrho[j]$. Then, by Proposition 3, we obtain the desired result, that is $\pi_{[i]} q_{i[j]} = \pi_{[j]} q_{j'\varrho[i]}$. In order to prove that $\widetilde{q}_{[i]} = \widetilde{q}_{\varrho[i]}$ it suffices to observe that since $\widetilde{X}(t)$ and $\varrho(\widetilde{X^R})(t)$ are stochastically identical then the residence time in state $[i]$ and $\varrho[i]$ must be the same, hence $\widetilde{q}_{[i]} = \widetilde{q}^R_{\varrho[i]}$ and by Proposition 4 the residence time in the forward and the reversed processes must be the same for each state, i.e., $\widetilde{q}^R_{\varrho[i]} = \widetilde{q}_{\varrho[i]}$. Observe that $\pi_{[i]}$ is the equilibrium distribution of $\widetilde{X}(t)$ by Lemma 1.

Now we prove that the detailed balance equations and the condition $\widetilde{q}_{[i]} = \widetilde{q}_{\varrho[i]}$ imply the fact that $X(t)$ is $\lambda\rho$-reversible with respect to $\sim$ and ϱ. Observe that if there exists a set of $\pi_{[i]} = n_i \pi_i$ satisfying the detailed balance equations then by Lemma 1 this must be the equilibrium distribution of $\widetilde{X}(t)$. Moreover, by the fact that $\sim$ is a strict lumpability and Corollary 1, $\pi_i = \pi_j$ for all $i \sim j$ is the the equilibrium distribution of $X(t)$. Hence, by Proposition 3 we write:

$$\widetilde{q}^R_{\varrho[i],\varrho[j]} = \frac{\pi_{[j]}}{\pi_{[i]}} \widetilde{q}_{\varrho[j]\varrho[i]} = \frac{\pi_{[j]}}{\pi_{[i]}} q_{j'\varrho[i]} \,,$$

with $j' \in \varrho[j]$. By using the detailed balance equations, we have $q_{j'\varrho[i]} = \pi_{[i]}/\pi_{[j]} q_{i[j]}$, obtaining $\widetilde{q}^R_{\varrho[i],\varrho[j]} = q_{i[j]} = \widetilde{q}_{[i][j]}$.

□

Corollary 3 *Let $X(t)$ be a CTMC with state space S and infinitesimal generator $\mathbf{Q}$, let $\sim$ be a strict lumpability for $X(t)$, ϱ be a renaming on $S/\sim$, and $\pi_{[i]}$ be the equilibrium distribution of $\widetilde{X}(t)$. If the transition rates of $X(t)$ satisfy the following equation for all $[i],[j] \in S/\sim$ with $[i] \neq [j]$, for all $i \in [i]$, $j \in [j]$ and $j' \in \varrho[j]$:*

$$n_i \pi_i q_{i[j]} = n_j \pi_j q_{j'\varrho[i]}$$

where n_h is the cardinality of the equivalence class $[h]$ with $h = i, j$, then $X(t)$ is $\lambda\rho$-reversible with respect to $\sim$ and ϱ.

Proof Since the detailed balance equations are trivially satisfied for the steady-state distribution, we must prove $\widetilde{q}_{[i]} = \widetilde{q}_{\varrho[i]}$. By the detailed balance equation we have $q_{i'\varrho[j]} = \pi_{[j]}/\pi_{[i]} q_{j[i]}$, with $i' \in \varrho[i]$. Therefore, we can write:

$$\widetilde{q}_{\varrho[i]} = \sum_{\substack{[j]\in S/\sim \\ [j]\neq[i]}} \widetilde{q}_{\varrho[i]\varrho[j]} = \sum_{\substack{[j]\in S/\sim \\ [j]\neq[i]}} q_{i'\varrho[j]} = \sum_{\substack{[j]\in S/\sim \\ [j]\neq[i]}} \frac{\pi_{[j]}}{\pi_{[i]}} q_{j[i]} = \sum_{\substack{[j]\in S/\sim \\ [j]\neq[i]}} \frac{\pi_{[j]}}{\pi_{[i]}} \widetilde{q}_{[j][i]} \,.$$

The right-hand-side term of this equation must be equal to $\tilde{q}_{[i]}$ since by hypothesis $\pi_{[i]}$ satisfies the system of global balance equations. Theorefore, by Proposition 12, $X(t)$ is $\lambda\rho$-reversible with respect to $\sim$ and ϱ. □

By applying the Kolmogorov's criterion we obtain the following characterization of lumpable reversibility.

Proposition 13 *Let $X(t)$ be a CTMC with state space S and infinitesimal generator $\mathbf{Q}$, $\sim$ be a strict lumpability for $X(t)$ and ϱ be a renaming on $S/\sim$. $X(t)$ is $\lambda\rho$-reversible with respect to $\sim$ and ϱ if and only if its transition rates satisfy:*

- *for every $[i] \in S/\sim$ and $i' \in \varrho[i]$:*

$$\sum_{\substack{[k]\in S/\sim \\ [k]\neq[i]}} q_{i[k]} = \sum_{\substack{[k]\in S/\sim \\ [k]\neq\varrho[i]}} q_{i'[k]} \,;$$

- *for every cycle $[i_1],[i_2],\ldots[i_n] \in S/\sim$ and $i'_1 \in \varrho[i_1], i'_2 \in \varrho[i_2], \ldots i'_n \in \varrho[i_n]$,*

$$q_{i_1[i_2]} q_{i_2[i_3]} \cdots q_{i_{n-1}[i_n]} q_{i_n[i_1]} = q_{i'_1\varrho[i_n]} q_{i'_n\varrho[i_{n-1}]} \cdots q_{i'_3\varrho[i_2]} q_{i'_2\varrho[i_1]} \,.$$

Proof Let $\widetilde{\mathbf{Q}}$ be the infinitesimal generator of $\tilde{X}(t)$ and $\widetilde{\mathbf{Q}}'$ be the infinitesimal generator of $\varrho(\widetilde{X^R})(t)$.

($\Rightarrow$) First notice that, since $\sim$ is a strict lumpability for $X(t)$,

$$\sum_{\substack{[k]\in S/\sim\\ [k]\neq[i]}} q_{i[k]} = \tilde{q}_{[i]} \qquad \sum_{\substack{[k]\in S/\sim\\ [k]\neq\varrho[i]}} q_{i'[k]} = \tilde{q}_{\varrho[i]}\,,$$

with $i' \in \varrho[i]$. Moreover, $\tilde{q}_{[i]} = \tilde{q}'_{\varrho[i]}$ since $\widetilde{X^R}(t)$ and $(\widetilde{X})^R(t)$ are stochastically identical, and $\tilde{q}'_{\varrho[i]} = \tilde{q}_{\varrho[i]}$ since $\widetilde{X}(t)$ and $\varrho(\widetilde{X}^R)(t)$ are stochastically identical, i.e., $\tilde{q}_{[i]} = \tilde{q}_{\varrho[i]}$. Now consider $[i_1],[i_2],\dots[i_n] \in S/\sim$ and $i'_1 \in \varrho[i_1], i'_2 \in \varrho[i_2],\dots i'_n \in \varrho[i_n]$. By Proposition 12,

$$q_{i_1[i_2]}q_{i_2[i_3]}\cdots q_{i_{n-1}[i_n]}q_{i_n[i_1]} = \frac{\pi_{[i_2]}}{\pi_{[i_1]}}q_{i'_2\varrho[i_1]}\,\frac{\pi_{[i_3]}}{\pi_{[i_2]}}q_{i'_3\varrho[i_2]}\cdots\frac{\pi_{[i_n]}}{\pi_{[i_{n-1}]}}q_{i'_n\varrho[i_{n-1}]}\,\frac{\pi_{[i_1]}}{\pi_{[i_n]}}q_{i'_1\varrho[i_n]}$$

and by simplifying it yelds

$$q_{i_1[i_2]}q_{i_2[i_3]}\cdots q_{i_{n-1}[i_n]}q_{i_n[i_1]} = q_{i'_1\varrho[i_n]}q_{i'_n\varrho[i_{n-1}]}\cdots q_{i'_3\varrho[i_2]}q_{i'_2\varrho[i_1]}.$$

($\Leftarrow$) First observe that, since $\widetilde{X}(t)$ is irreducible, for all $[j],[k] \in S/\sim$ we can find a chain $[j] = [j_0] \to [j_1] \to \cdots \to [j_{n-1}] \to [j_n] = [k]$ (for $n \geq 1$) of one-step transitions. From the hypothesis that $\widetilde{X}(t)$ and $\varrho(\widetilde{X^R})(t)$ are stochastically identical, there is also a chain $\varrho[k] = \varrho[j_n] \to \varrho[j_{n-1}] \to \cdots \to \varrho[j_1] \to \varrho[j_0] = \varrho[j]$ with $\varrho[j_w] \in S/\sim$ for $w \in [0,\dots,n]$.

Consider an arbitrary state $[i_0] \in S/\sim$ as a reference state and $[i] \in S/\sim$. Let $[i] = [i_n] \to [i_{n-1}] \to \cdots \to [i_1] \to [i_0]$ and $\varrho[i_0] \to \varrho[i_1] \to \cdots \to \varrho[i_{n-1}] \to \varrho[i_n] = \varrho[i]$ ($n \geq 1$) be two chains of one-step transitions in $\widetilde{X}(t)$. We prove that:

$$\pi_{[i]} = C_{i_0}\prod_{k=1}^{n}\frac{q_{i'_{k-1}\varrho[i_k]}}{q_{i_k[i_{k-1}]}}\,, \tag{11}$$

where $C_{i_0} \in \mathbb{R}^+$ and $i'_k \in \varrho[i_k]$ for $k = 0,\dots,n$. We show that $\pi_{[i]}$ is well-defined. Indeed, if $[i] = [j_m] \to [j_{m-1}] \to \cdots \to [j_1] \to [j_0] = [i_0]$ ($m \geq 1$) is another chain, we can always find a chain $[i_0] = [h_0] \to [h_1] \to \cdots \to [h_{l-1}] \to [h_l] = [i]$. By hypothesis, for any $h'_k \in \varrho[h_k]$ with $k \in [0,\dots,l]$ and $j'_k \in \varrho[j_h]$ with $k \in [0,\dots,m]$ we have:

$$\prod_{k=1}^{m} q_{j_k[j_{k-1}]}\prod_{k=1}^{l} q_{h_{k-1}[h_k]} = \prod_{k=1}^{l} q_{h'_k\varrho[h_{k-1}]}\prod_{k=1}^{m} q_{j'_{k-1}\varrho[j_k]}\,. \tag{12}$$

Moreover, considering the one-step chain $[i] = [i_n] \to [i_{n-1}] \to \cdots \to [i_1] \to [i_0] = [h_0] \to [h_1] \to \cdots \to [h_{l-1}] \to [h_l] = [i]$, $h'_k \in \varrho[h'_k]$ for $k \in [0,\dots,l]$, and $i'_k \in \varrho[i'_k]$ for $k \in [0,\dots,n]$ we have:

$$\prod_{k=1}^{n} q_{i_k[i_{k-1}]}\prod_{k=1}^{l} q_{h_{k-1}[h_k]} = \prod_{k=1}^{l} q_{h'_k\varrho[h_{k-1}]}\prod_{k=1}^{n} q_{i'_{k-1}\varrho[i_k]}\,. \tag{13}$$

By Equations (12) and (13), we obtain:

$$\prod_{k=1}^{m} \frac{q_{j'_{k-1}\varrho[j_k]}}{q_{j_k[j_{k-1}]}} = \prod_{k=1}^{n} \frac{q_{i'_{k-1}\varrho[i_k]}}{q_{i_k[i_{k-1}]}} .$$

Hence:

$$\pi_{[i]} = C_{i_0} \prod_{k=1}^{n} \frac{q_{i'_{k-1}\varrho[i_k]}}{q_{i_k[i_{k-1}]}} ,$$

where $C_{i_0} \in \mathbb{R}^+$, is well-defined. In order to prove that this is the equilibrium probability of $[i] \in S/\sim$ we show that it satisfies the system of GBE for $[i]$.

$$\pi_{[i]}\tilde{q}_{[i]} = \sum_{[j]\in S/\sim} \pi_{[j]}\tilde{q}_{[j][i]} ,$$

which can be written as:

$$\tilde{q}_{[i]} = \sum_{\substack{[j]\in S/\sim\\ [j]\neq[i]}} \frac{\pi_{[j]}}{\pi_{[i]}} q_{j[i]} .$$

By Proposition 12 we have:

$$\tilde{q}_{[i]} = \sum_{\substack{[j]\in S/\sim\\ [j]\neq[i]}} \frac{q_{i'\varrho[j]}}{q_{j[i]}} q_{j[i]} = \sum_{\substack{[j]\in S/\sim\\ [j]\neq[i]}} q_{i'\varrho[j]} .$$

for $i' \in \varrho[i]$. Hence:

$$\tilde{q}_{[i]} = \sum_{\substack{[j]\in S/\sim\\ [j]\neq[i]}} q_{i'[j]} = \tilde{q}_{\varrho[i]} ,$$

which is an identity by hypothesis. Now let $[i], [j] \in S/\sim$ such that $q_{j[i]} > 0$. Then

$$\pi_{[j]} = C_{i_0} \frac{q_{i'\varrho[j]}}{q_{j[i]}} \prod_{k=1}^{n} \frac{q_{i'_{k-1}\varrho[i_k]}}{q_{i_k[i_{k-1}]}} = \pi_{[i]} \frac{q_{i'\varrho[j]}}{q_{j[i]}} ,$$

for $i' \in \varrho[i]$. Hence, by Proposition 12, $X(t)$ is $\lambda\rho$-reversible. □

The next two corollaries provide a method to compute the steady state probability of a $\lambda\rho$-reversible and a ρ-reversible, respectively, CTMC.

Corollary 4 *Let $X(t)$ be a $\lambda\rho$-reversible CTMC with respect to a strict lumpability $\sim$ and a renaming ϱ on $S/\sim$. Then for all $[i] \in S/\sim$,*

$$\pi_{[i]} = C_{i_0} \prod_{k=1}^{n} \frac{q_{i'_{k-1}\varrho[i_k]}}{q_{i_k[i_{k-1}]}} \tag{14}$$

where $i_0 \in S/\sim$ is an arbitrary reference state, $i'_k \in \varrho[i_k]$ for all $k \in [0, \ldots, n]$, $[i] = [i_n]$ and $C_{i_0} \in \mathbb{R}^+$.

Corollary 5 *Let $X(t)$ be a ρ-reversible CTMC with respect to a renaming ϱ on S. Then for all $i \in S$,*

$$\pi_i = C_{i_0} \prod_{k=1}^{n} \frac{q_{\varrho(i_{k-1})\varrho(i_k)}}{q_{i_k i_{k-1}}} \tag{15}$$

where $i_0 \in S/\sim$ is an arbitrary reference state, $i = i_n$ and $C_{i_0} \in \mathbb{R}^+$.

6 Autoreversibility

In this section we introduce the notion of *autoreversibility* for a given Markov process [23] and prove that it is a $\lambda\rho$-reversibility.

The notion of autoreversibility for a given Markov process is formalized in terms of two relations over its states: a *reversal bisimilarity* $\sim_r$ which allows us to relate "reversed" states and a *reversal equivalence relation* $\sim$ which relates states corresponding to the same class of reversed states as shown in the following example.

Example 8 (Autoreversibility) Consider the CTMC depicted by Fig. 4. Since it has a finite number of states and is irreducible it is trivially ergodic. Assume $\lambda = 2a$, $\mu = a$, $\nu = 8a$ and $\rho = 4a$, with $a \in \mathbb{R}^+$. This chain is autoreversible and the reversal bisimilarity $\sim_r$ is:

$$\{(1,5),(3,5),(2,4),(6,2),(5,1),(5,3),(4,2),(2,6)\}\,.$$

The equivalence relation $\sim$ groups together the states with the same reversed, hence we obtain the following equivalence classes: $\{1,3\}$, $\{4,6\}$, $\{2\}$, $\{5\}$ with cardinality $n_1 = n_3 = 2$, $n_4 = n_6 = 2$, $n_2 = 1$ and $n_5 = 1$. Observe that $\{2\}$ is the class of the "reversed" of $\{1,3\}$ and $\{4,6\}$ is the class of the "reversed" of $\{5\}$.

We will prove that if $X(t)$ is autoreversible then it is also $\lambda\rho$-reversible with respect to a strict lumpability $\sim$ and a renaming ϱ on $S/\sim$ such that

- the reversal equivalence relation determines the equivalent classes belonging to $S/\sim$ and
- the reversal bisimulation characterises the renaming ϱ on $S/\sim$ since it relates the states of S belonging to a class $[i] \in S/\sim$ with the states in the renaming of $[i]$, i.e., belonging to $\varrho[i] \in S/\sim$.

Hereafter, we say that the reversal bisimulation relates "reversed" states since the induced renaming ϱ establishes a relation between the states of $\widetilde{X}(t)$ and the states of $\varrho(\widetilde{X})(t)$ which coincides with the reversed of $\widetilde{X}(t)$.

The *reversal bisimulation* over the states of a CTMC is a coinductive definition, in the style of bisimulation [26], formally expressed as follows. The advantage of such a definition consists in providing both a recursive definition on the state space and a well established bisimulation based proof method.

Definition 9 (Reversal bisimulation) Consider a CTMC with state space S and infinitesimal generator $\mathbf{Q}$. A symmetric relation $\mathcal{R} \subseteq S \times S$ is a *reversal bisimulation* if

1) for every $(i, i') \in \mathcal{R}$, $q_i = q_{i'}$;
2) for every $(i_1, i'_1) \in \mathcal{R}$ and for every finite sequence of one-step transitions $i_1 \to i_2 \to \ldots \to i_{n-1} \to i_n$ there exist $i'_n \to i'_{n-1} \to \cdots \to i'_2 \to i'_1$ such that $(i_k, i'_k) \in \mathcal{R}$ for all $k \in \{1, \ldots, n\}$ and

$$q_{i_1 i_2} q_{i_2 i_3} \cdots q_{i_{n-1} i_n} q_{i_n i_1} = q_{i'_1 i'_n} q_{i'_n i'_{n-1}} \cdots q_{i'_3 i'_2} q_{i'_2 i'_1}.$$

We are interested in the relation which is the largest reversal bisimulation, formed by the union of all reversal bisimulations.

The following proposition ensures that any union of reversal bisimulations is itself a reversal bisimulation.

Proposition 14 *Consider a CTMC with state space S and infinitesimal generator $\mathbf{Q}$. Let $\mathcal{R}_1, \mathcal{R}_2 \subseteq S \times S$ be two reversal bisimulations. Then $\mathcal{R}_1 \cup \mathcal{R}_2$ is a reversal bisimulation.*

Proof Let $(i, i') \in \mathcal{R}_1 \cup \mathcal{R}_2$. Then either $(i, i') \in \mathcal{R}_1$ or $(i, i') \in \mathcal{R}_2$ and hence Conditions 1 and 2 of Definition 9 are satisfied. □

Based on the above result we can define the maximal reversal bisimulation as the union of all reversal bisimulations.

Definition 10 (Reversal bisimilarity) Given a CTMC with state space S, we denote by $\sim_r$ the maximal reversal bisimulation over S which is defined by

$$\sim_r = \bigcup \{\mathcal{R} \mid \mathcal{R} \text{ is a reversal bisimulation}\}.$$

If $\sim_r \subseteq S \times S$ is complete, i.e., for all $i \in S$ there exists $i' \in S$ such that $i \sim_r i'$, then $\sim_r$ is called *reversal bisimilarity over S*.

Notice that reversal bisimilarity $\sim_r$ is symmetric but in general it is neither reflexive nor transitive. Moreover, it is worth notice that the effective computation of reversal bisimilarity over a finite state space chain can be implemented by exploiting the well-known algorithms that have been developed in the literature of formal models for bisimulation [28, 7].

Roughly speaking, if $i \sim_r i'$ then we say that i' is a "reversed" state of i. The following lemma shows that if two states i and j share a reversed state i' then the set of reversed states corresponding to i and j are the same.

Lemma 2 *Consider a CTMC with state space S and infinitesimal generator $\mathbf{Q}$. For all $i, j, i' \in S$ such that $i \sim_r i'$ and $j \sim_r i'$, it holds that*

$$\{i' : \ i \sim_r i'\} = \{i' : \ j \sim_r i'\}.$$

Proof Let $i'' \in S$ such that $i \sim_r i''$. We prove that also $j \sim_r i''$. From the facts that $i \sim_r i'$, $j \sim_r i'$ and $i \sim_r i''$ we have $q_i = q_{i'} = q_j = q_{i''}$ and then Condition 1 of Definition 9 is satisfied. In order to prove Condition 2 of Definition 9, consider a finite sequence of one-step transitions $j = j_1 \to j_2 \to \ldots \to j_{n-1} \to j_n$. From $j \sim_r i'$ there exist $i'_n \to i'_{n-1} \to \cdots \to i'_2 \to i'_1 = i'$ such that $(j_k, i'_k) \in \mathcal{R}$ for all $k \in \{1, \ldots, n\}$ and

$$q_{j_1 j_2} q_{j_2 j_3} \cdots q_{j_{n-1} j_n} q_{j_n j_1} = q_{i'_1 i'_n} q_{i'_n i'_{n-1}} \cdots q_{i'_3 i'_2} q_{i'_2 i'_1}.$$

From $i \sim_r i'$ there exist $i = i_1 \to i_2 \to \ldots \to i_{n-1} \to i_n$ such that $(i_k, i'_k) \in \mathcal{R}$ for all $k \in \{1, \ldots, n\}$ and

$$q_{i_1 i_2} q_{i_2 i_3} \cdots q_{i_{n-1} i_n} q_{i_n i_1} = q_{i'_1 i'_n} q_{i'_n i'_{n-1}} \cdots q_{i'_3 i'_2} q_{i'_2 i'_1}.$$

Finally, from $i \sim_r i''$ there exist $i''_n \to i''_{n-1} \to \cdots \to i''_2 \to i''_1 = i'' \in S$ such that $(i_k, i''_k) \in \mathcal{R}$ for all $k \in \{1, \ldots, n\}$ and

$$q_{i_1 i_2} q_{i_2 i_3} \cdots q_{i_{n-1} i_n} q_{i_n i_1} = q_{i''_1 i''_n} q_{i''_n i''_{n-1}} \cdots q_{i''_3 i''_2} q_{i''_2 i''_1}$$

and hence

$$q_{j_1 j_2} q_{j_2 j_3} \cdots q_{j_{n-1} j_n} q_{j_n j_1} = q_{i''_1 i''_n} q_{i''_n i''_{n-1}} \cdots q_{i''_3 i''_2} q_{i''_2 i''_1}.$$

Analogously, we can prove that for every sequence $i'' = i''_1 \to i''_2 \to \ldots \to i''_{n-1} \to i''_n$ there exist $j_n \to j_{n-1} \to \cdots \to j_2 \to j_1 = j$ such that $(i''_k, j_k) \in \mathcal{R}$ for all $k \in \{1, \ldots, n\}$ and

$$q_{i''_1 i''_2} q_{i''_2 i''_3} \cdots q_{i''_{n-1} i''_n} q_{i''_n i''_1} = q_{j_1 j_n} q_{j_n j_{n-1}} \cdots q_{j_3 j_2} q_{j_2 j_1}.$$

This concludes the proof that $j \sim_r i''$. □

Reversal bisimilarity induces an equivalence relation, named *reversal equivalence*, over the states of the CTMC equating states corresponding to the same set of reversed states.

Definition 11 (Reversal equivalence) Consider a CTMC with state space S and reversal bisimilarity $\sim_r \subseteq S \times S$. We call *reversal equivalence*, denoted by $\sim$, the relation over S defined as: for all $i, j \in S$,

$$i \sim j \quad \text{iff} \quad \{i' : \ i \sim_r i'\} = \{i' : \ j \sim_r i'\}.$$

The following proposition follows immediately by Definition 11.

Proposition 15 *Every reversal equivalence is an equivalence relation.*

Any reversal equivalence $\sim \subseteq S \times S$ induces a partition on the state space S. Let $S/\sim$ denote the set of equivalences classes generated in this way. Let $[i] \in S/\sim$ denote the equivalence class containing $i \in S$, that is $[i] = \{j \in S | \ i \sim j\}$ and n_i denote the cardinality of this set, that is $n_i = |[i]| = |\{j \in S | \ i \sim j\}|$.

We are now ready to introduce our notion of autoreversibility for a given CTMC. The following definition states that a CTMC is autoreversible if it admits a reversal bisimilarity over its states which induces a weak similarity. A further condition relating forward and reverse transitions is required.

Definition 12 (Autoreversibility) A CTMC with state space S and infinitesimal generator $\mathbf{Q}$ is said *autoreversible* if there exist

1) a reversal bisimilarity $\sim_r$ over S,
2) the reversal equivalence induced by $\sim_r$, according to Definition 11, is a weak similarity,
3) for every i, i', j, j' such that $i \sim_r i'$ and $j \sim_r j'$,

$$m_{i[j]} = m_{j'[i']}.$$

The following proposition shows that $\sim_r$ is well-defined.

Proposition 16 *Consider a CTMC with state space S and infinitesimal generator* $\mathbf{Q}$*. If there exists a reversal bisimilarity* $\sim_r \subseteq S \times S$ *then for every finite sequence of one-step transitions* $i_1 \to i_2 \to \ldots \to i_{n-1} \to i_n$ *and* $i'_n \to i'_{n-1} \to \cdots \to i'_2 \to i'_1$ *such that* $i_k \sim_r i'_k$ *for all* $k \in \{1, \ldots, n\}$ *it holds that*

$$q_{i_1 i_2} q_{i_2 i_3} \cdots q_{i_{n-1} i_n} q_{i_n i_1} = q_{i'_1 i'_n} q_{i'_n i'_{n-1}} \cdots q_{i'_3 i'_2} q_{i'_2 i'_1}.$$

Proof Let $i_1, i_2, \ldots, i_{n-1}, i_n \in S$ and $i''_1 \in S$ be a state such that $i_1 \sim_r i''_1$. By Definition 9, there exist $i''_n, i''_{n-1}, \ldots, i''_2, i''_1 \in S$ such that $(i_k, i''_k) \in \mathcal{R}$ for all $k \in \{1, \ldots, n\}$ and

$$q_{i_1 i_2} q_{i_2 i_3} \cdots q_{i_{n-1} i_n} q_{i_n i_1} = q_{i''_1 i''_n} q_{i''_n i''_{n-1}} \cdots q_{i''_3 i'_2} q_{i''_2 i''_1}.$$

By Lemma 2 and Definition 11, $i'_k \sim i''_k$ for all $k \in \{1, \ldots, n\}$. Since $\sim$ is a weak similarity, the proof follows by Condition 1 of Definition 5. □

We show that $\sim_r$ is reflexive for the class of reversible Markov processes.

Proposition 17 *For a CTMC* $X(t)$ *with state space S, if* $X(t)$ *is reversible then* $\sim_r \in S \times S$ *exists and it is reflexive.*

Proof Let $X(t)$ be reversible. Then, by Proposition 2, for any finite sequence of states $i_1, i_2, \ldots, i_{n-1}, i_n \in S$,

$$q_{i_1 i_2} q_{i_2 i_3} \cdots q_{i_{n-1} i_n} q_{i_n i_1} = q_{i_1 i_n} q_{i_n i_{n-1}} \cdots q_{i_3 i_2} q_{i_2 i_1}.$$

Consider $\mathcal{R} = \{(i, i) : i \in S\}$. It is easy to see that $\mathcal{R}$ is a reversal bisimulation and hence $\mathcal{R} \subseteq \sim_r$, i.e., $\sim_r$ is reflexive. □

In the following example Proposition 17 is illustrated by considering the well-known reversible process called Birth&Death process which is underlying to the M/M/n queues with $n \in \mathbb{N} \setminus \{0\}$ or $n = \infty$.

Example 9 (Birth&Death processes) Consider the Birth&Death process depicted in Fig. 9. The CTMC is autoreversible and its *reversal relation* is reflexive since, given an arbitrary state i, each cycle of states starting from i can be followed backwards. Moreover, the sequence of states encountered by the forward and the backward paths are trivially associated by the reversal bisimulation.

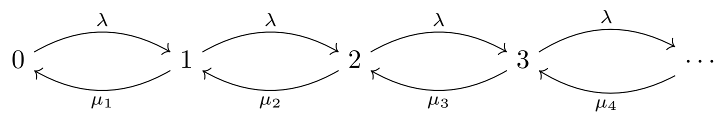


Fig. 9: Autoreversible Birth&Death process studied by Example 9.

Example 10 (Exponential queue with batch arrivals and departures) In this example we consider a Birth&Death process with constant death rate, $\mu_n = \mu$ for all $n > 0$ and constant arrival rate λ. We consider the possibility of batch arrivals of size N and batch departures of the same size.

Let us denote by λ_N and μ_N the batches' arrival and service rate. We assume that when there are less than N customers in the queue the batch departure is disabled. We can easily prove that the CTMC underlying this model is autoreversible with a reflexive *reverse relation* if $(\lambda_N/\mu_N) = (\lambda/\mu)^N$. Indeed, consider the path $n, n+1, \ldots, n+N$, then the product of the rates forming the forward cycle is $\lambda^N \mu_N$, while the product of the rates in the backward cycle is $\lambda_N \mu^N$. Since these two quantities must be identical we have the required condition dividing both hand-sides by $\mu_N \mu^N$. Notice that we can extend this analysis to batches of size N_1, N_2, $\ldots$, N_B that arrive at and leave from the queue, obtaining the condition $(\lambda/\mu)^{N_b} = \lambda_{N_b}/\mu_{N_b}$ for all $b = 1, \ldots, B$.

The following theorem proves an important property of autoreversible CTMCs since it gives an effective way to compute their steady-state distribution without solving the system of global balance equations, i.e., by inspections of the transition rates.

Theorem 5 *(Steady-state distribution) Consider a CTMC with state space S, infinitesimal generator $\mathbf{Q}$ and equilibrium distribution $\boldsymbol{\pi}$. Assume that there exists a reversal bisimilarity $\sim_r \subseteq S \times S$. Let $i_0 \in S$ be an arbitrary state. For all states $i \in S$, let $i = i_n \to i_{n-1} \to \cdots \to i_1 \to i_0$ $(n \geq 1)$ be a chain of one-step transitions and $i'_0 \to i'_1 \to \cdots \to i'_{n-1} \to i'_n = i'$ such that $i_k \sim_r i'_k$ for all $k \in \{0, \ldots, n\}$. Then*

$$\pi_i = C_{i_0} \, \frac{n_{i_0}}{n_i} \, \prod_{k=1}^{n} \frac{q_{i'_{k-1} i'_k}}{q_{i_k i_{k-1}}} \tag{16}$$

where $C_{i_0} \in \mathbb{R}^+$.

Proof First, we prove that Equation (16) gives a unique definition of π_i. Then, we will prove that it is the steady-state probability of state i as required.

Observe that for all $j, k \in S$, we can find a chain $j \to j_1 \to \cdots \to j_{n-1} \to k$ (for $n \geq 1$) of one-step transitions since the Markov process is irreducible. Now we show that π_i is well-defined. Indeed, if $i = j_m \to j_{m-1} \to \cdots \to j_1 \to j_0 = i_0$ $(m \geq 1)$ is another chain, we can always find a chain $i_0 = h_0 \to h_1 \to \cdots \to$

$h_{l-1} \to h_l = i$. Since the Markov process is autoreversible, there exists a chain $i' = h'_l \to h'_{l-1} \to \cdots h'_1 \to h'_0 = i''_0 = j'_0 \to j'_1 \to \cdots \to j'_{m-1} \to j'_m = i'$ such that $h_k \sim_r h'_k$ for all $k \in \{0, \dots, l\}$ and $j_k \sim_r j'_k$ for all $k \in \{0, \dots, m\}$, and

$$\prod_{k=1}^{m} q_{j_k j_{k-1}} \prod_{k=1}^{l} q_{h_{k-1} h_k} = \prod_{k=1}^{l} q_{h'_k h'_{k-1}} \prod_{k=1}^{m} q_{j'_{k-1} j'_k}. \tag{17}$$

Moreover, considering the one-step chain $i = i_n \to i_{n-1} \to \cdots \to i_1 \to i_0 = h_0 \to h_1 \to \cdots \to h_{l-1} \to h_l = i$, by Definition 12 there exists a chain $i'' = h''_l \to h''_{l-1} \to \cdots h''_1 \to h''_0 = i''_0 \to i''_1 \to \cdots \to i''_{n-1} \to i''_n$ such that $h_k \sim_r h''_k$ for all $k \in \{0, \dots, l\}$, $i_k \sim_r i''_k$ for all $k \in \{0, \dots, n\}$, and

$$\prod_{k=1}^{n} q_{i_k i_{k-1}} \prod_{k=1}^{l} q_{h_{k-1} h_k} = \prod_{k=1}^{l} q_{h''_k h''_{k-1}} \prod_{k=1}^{n} q_{i''_{k-1} i''_k}.$$

By Proposition 16, $h'_k \sim h''_k$ for all $k \in \{0, \dots, l\}$ and also $i'_k \sim_r i''_k$ for all $k \in \{0, \dots, n\}$. Hence by Condition 1 of Definition 5,

$$\prod_{k=1}^{l} q_{h''_k h''_{k-1}} \prod_{k=1}^{n} q_{i''_{k-1} i''_k} = \prod_{k=1}^{l} q_{h'_k h'_{k-1}} \prod_{k=1}^{n} q_{i'_{k-1} i'_k}$$

and then

$$\prod_{k=1}^{n} q_{i_k i_{k-1}} \prod_{k=1}^{l} q_{h_{k-1} h_k} = \prod_{k=1}^{l} q_{h'_k h'_{k-1}} \prod_{k=1}^{n} q_{i'_{k-1} i'_k}. \tag{18}$$

From Equations (17) and (18), we obtain

$$\prod_{k=1}^{m} \frac{q_{j'_{k-1} j'_k}}{q_{j_k j_{k-1}}} = \prod_{k=1}^{n} \frac{q_{i'_{k-1} i'_k}}{q_{i_k i_{k-1}}}.$$

Hence

$$\pi_i = C_{i_0} \, \frac{n_{i_0}}{n_i} \, \prod_{k=1}^{n} \frac{q_{i'_{k-1} i'_k}}{q_{i_k i_{k-1}}}$$

where C_{i_0} is a positive constant, is well-defined.

In order to prove that Equation (16) is the equilibrium probability of state i, we use the approach described in Section 2. Since the CTMC is stationary, we can define its reversed process whose transition matrix $\mathbf{Q}^R$ is defined according to Lemma 3. Let us assume Equation (16) and we show that the reversed process satisfies the generalised Kolmogorov's criteria of Proposition 4. By uniqueness of the steady-state distribution we will conclude the proof.

Let us consider an arbitrary transition from state i to j with rate q_{ij} in the forward chain, then the corresponding transition in the reversed process goes from j to i with rate q^R_{ji}. Observe that we have just proved that we can choose an arbitrary path from i to the reference state i_0, in particular we can choose the path going from i to j and then a path from j to i_0. By Definition 12 there will surely exist i'_0, i', j' such that $i_0 \sim_r i'_0$, $i \sim_r i'$ and $j \sim_r j'$ and a path going from i'_0 to j' and one step from j' to i' such that

$$\pi_i = \frac{\Psi'_{i_0 \to j} q_{j'i'}}{\Psi_{j \to i_0} q_{ij}} \qquad \pi_j = \frac{\Psi'_{i_0 \to j}}{\Psi_{j \to i_0}}$$

where $\Psi'_{i_0 \to j} = \prod_{k=1}^{n} q_{i'_{k-1} i'_k}$ and $\Psi_{j \to i_0} = \prod_{k=1}^{n} q_{i_k i_{k-1}}$ with $j = i_n$.
By Equation (2), we have:

$$q^R_{ji} = \frac{\pi_i}{\pi_j} q_{ij} = \frac{C_{i_0} \frac{n_{i_0}}{n_i} \frac{\Psi'_{i_0 \to j} q_{j'i'}}{\Psi_{j \to i_0} q_{ij}}}{C_{i_0} \frac{n_{i_0}}{n_j} \frac{\Psi'_{i_0 \to j}}{\Psi_{j \to i_0}}} q_{ij} = \frac{n_j}{n_i} q_{j'i'} \,.$$

The generalised Kolmogorov's criteria on the cycles is readily verified. Indeed, consider the sequence of states $i_1, \ldots, i_n$ associated with the product $q_{i_1 i_2} \cdots q_{i_{n-1} i_n} q_{i_n i_1}$, then the product of the rates in the reversed process is:

$$q^R_{i_1 i_n} q^R_{i_n i_{n-1}} \cdots q^R_{i_2 i_1} = \frac{n_{i_1}}{n_{i_n}} q_{i'_1 i'_n} \frac{n_{i_n}}{n_{i_{n-1}}} q_{i'_n i'_{n-1}} \cdots \frac{n_{i_2}}{n_{i_1}} q_{i'_2 i'_1} \,.$$

After simplifying we obtain an identity by Definition 12.

We now verify the first generalised Kolmogorov's criteria. Let us consider an arbitrary state j, then the outgoing flow from the reversed process is:

$$q^R_j = \sum_{i \in S, q^R_{ji} > 0} q^R_{ji} = \sum_{i \in S, q_{ij} > 0} \frac{n_j}{n_i} q_{j'i'} \,.$$

We prove that

$$q_{j'} = \sum_{i \in S, q_{ij} > 0} \frac{n_j}{n_i} q_{j'i'} \tag{19}$$

and this will conclude the proof since, by Definition 9, $q'_j = q_j$. Let us consider the right-hand-side of Equation (19), then we have:

$$\begin{aligned} \sum_{i \in S, q_{ij} > 0} \frac{n_j}{n_i} q_{j'i'} &= \sum_{[i] \in S/\sim} \sum_{i \in [i], q_{ij} > 0} \frac{n_j}{n_i} q_{j'i'} \\ &= \sum_{[i] \in S/\sim} m_{[i]j} \frac{n_j}{n_i} q_{j'i'} = \sum_{[i] \in S/\sim} \frac{m_{i[j]} n_i}{n_i} q_{j'i'} \,, \end{aligned}$$

where the last equality follows from Proposition 10. By Condition 3 of Definition 12 we have $m_{i[j]} = m_{j'[i']}$ and since every equivalence class has exactly one counterpart (possibly itself) by definition, we conclude the proof:

$$\sum_{[i] \in S/\sim} m_{j'[i']} q_{j'i'} = \sum_{[i'] \in S/\sim} m_{j'[i']} q_{j'i'} = \sum_{i' \in S/\sim} q_{j'i'} = q'_j \,.$$

□

The following corollaries aim at simplifying the application of Theorem 5.

Corollary 6 *Consider a CTMC with state space S, infinitesimal generator $\mathbf{Q}$ and equilibrium distribution $\boldsymbol{\pi}$. Assume that there exists a reversal bisimilarity $\sim_r \subseteq S \times S$. Then for all $i, j \in S$ such that $i = i_n \to i_{n-1} \to \cdots \to i_1 \to i_0 = j$ and $j' = i'_0 \to i'_1 \to \cdots \to i'_{n-1} \to i'_n = i'$ with $i_k \sim_r i'_k$ for $k \in \{0, \ldots, n\}$, it holds that*

$$n_i \pi_i = n_j \pi_j \prod_{k=1}^{n} \frac{q_{i'_{k-1} i'_k}}{q_{i_k i_{k-1}}}.$$

Proof By Theorem 5,

$$\pi_i = C_j \, \frac{n_j}{n_i} \, \prod_{k=1}^{n} \frac{q_{i'_{k-1} i'_k}}{q_{i_k i_{k-1}}}$$

where C_j is a positive constant. In particular, $\pi_j = C_j$ and hence

$$\pi_i = \pi_j \, \frac{n_j}{n_i} \, \prod_{k=1}^{n} \frac{q_{i'_{k-1} i'_k}}{q_{i_k i_{k-1}}},$$

i.e.,

$$n_i \pi_i = n_j \pi_j \prod_{k=1}^{n} \frac{q_{i'_{k-1} i'_k}}{q_{i_k i_{k-1}}}.$$

□

Corollary 7 *Consider a CTMC with state space S, infinitesimal generator $\mathbf{Q}$ and equilibrium probability $\boldsymbol{\pi}$. Assume that there exists a reversal bisimilarity $\sim_r \subseteq S \times S$. Then for all $i, j \in S$ with $q_{ji} > 0$ and for all $i', j' \in S$ such that $i \sim_r i'$, $j \sim_r j'$ and $q_{i'j'} > 0$ it holds*

$$n_j \pi_j q_{ji} = n_i \pi_i q_{i'j'}.$$

Proof Let $i, j \in S$ such that $i \sim_r i'$, $j \sim_r j'$ and $q_{i'j'} > 0$. Let $i_0 \in S$ be an arbitrary state, $i = i_n \to i_{n-1} \to \cdots \to i_1 \to i_0$ $(n \geq 1)$ be a chain of one-step transitions and $i'_0 \to i'_1 \to \cdots \to i'_{n-1} \to i'_n = i'$ such that $i_k \sim_r i'_k$ for all $k \in \{0, \ldots, n\}$. By Theorem 5,

$$\pi_i = C_{i_0} \, \frac{n_{i_0}}{n_i} \, \prod_{k=1}^{n} \frac{q_{i'_{k-1} i'_k}}{q_{i_k i_{k-1}}}$$

where C_{i_0} is a positive constant. Now suppose that $q_{ji} > 0$ and $q_{i'j'} > 0$. Again, by Theorem 5,

$$\pi_j = C_{i_0} \, \frac{n_{i_0}}{n_j} \, \frac{q_{i'j'}}{q_{ji}} \prod_{k=1}^{n} \frac{q_{i'_{k-1} i'_k}}{q_{i_k i_{k-1}}}$$

and hence

$$n_j \pi_j q_{ji} = n_i \pi_i q_{i'j'}.$$

□

Example 11 Let us consider again the CTMC depicted in Fig. 4 with the *reversal bisimilarity* and the *reversal equivalence* derived in Example 8. Let us choose an arbitrary reference state $i_0 = 1$ and then pick the shortest sequence of states from any other state i to 1 (whose reversed state is 5). Notice that $n_1 = 2$ since it belongs to an equivalence class of cardinality 2. For instance, take state 2 whose reversed states are either 4 or 6 and $n_2 = 1$. Then we have:

$$\pi_2 = C_1 \frac{2}{1} \frac{q_{54}}{q_{21}} = C_1 \, .$$

In a similar way we obtain:

$$\pi_3 = C_1 \, \frac{2}{2} \, \frac{q_{56}q_{62}q_{25}}{q_{36}q_{62}q_{21}} = C_1 \, \frac{\rho\mu\nu}{\lambda^2\nu} = C_1 \qquad \pi_4 = C_1 \, \frac{2}{2} \, \frac{q_{54}q_{42}}{q_{42}q_{21}} = C_1 \, \frac{\nu\mu}{\lambda\nu} = \frac{C_1}{2}$$

$$\pi_5 = C_1 \, \frac{2}{1} \, \frac{q_{54}q_{62}q_{23}}{q_{54}q_{42}q_{21}} = 2C_1 \, \frac{\lambda}{\lambda} = 2C_1 \qquad \pi_6 = C_1 \, \frac{2}{2} \, \frac{q_{54}q_{42}}{q_{62}q_{21}} = C_1 \, \frac{\mu\nu}{\nu\lambda} = \frac{C_1}{2}$$

We can now derive $\pi_1 = C_1$ and by imposing $\sum_{i \in S} \pi_1 = 1$ this gives $C_1 = 1/6$.

The next proposition states that equivalent states have the same equilibrium probability.

Proposition 18 *Consider an autoreversible CTMC $X(t)$ with state space S, infinitesimal generator $\mathbf{Q}$ and equilibrium distribution $\boldsymbol{\pi}$. Assume that there exists a reversal equivalence $\sim \subseteq S \times S$. For all states $i, j \in S$ such that $i \sim j$ it holds $\pi_i = \pi_j$.*

Proof By Definition 12, $\sim$ is a weak similarity. Hence, by Theorem 1, $\sim$ is a strict lumpability for $X(t)$. In particular, $\sim$ is an exact lumpability for $X(t)$ and the statement follows by Proposition 6. □

The following theorem establishes the relation between the equilibrium probability of a state i and that of its reversed i'. We will show that this relation highly improves the efficiency of the computation of the steady-state distribution for autoreversible processes.

Theorem 6 *Consider a CTMC with state space S, infinitesimal generator $\mathbf{Q}$ and equilibrium distribution $\boldsymbol{\pi}$. Assume that there exists a reversal bisimilarity $\sim_r \subseteq S \times S$. For all states $i, i' \in S$ such that $i \sim_r i'$ it holds $n_i \pi_i = n_{i'} \pi_{i'}$.*

Proof Consider the following chain of one step transitions: $i = i_n \to i_{n-1} \to \cdots \to i_0 = i'$. Let $i'_0 \to \cdots \to i'_{n-1} \to i'_n$ such that $i_k \sim_r i'_k$ for $k \in \{0, \ldots, n\}$. By Theorem 5,

$$n_i \pi_i = n_{i'} \pi_{i'} \prod_{k=1}^{n} \frac{q_{i'_{k-1} i'_k}}{q_{i_k i_{k-1}}} .$$

Let $i' = j_m \to j_{m-1} \to \cdots \to j_0 = i$ and $j'_0 \to \cdots \to j'_{m-1} \to j'_m$ such that $j_k \sim_r j'_k$ for $k \in \{0, \ldots, n\}$. By definition of autoreversibility,

$$\prod_{k=1}^{m} q_{j_k j_{k-1}} \prod_{k=1}^{n} q_{i'_{k-1} i'_k} = \prod_{k=1}^{n} q_{i_k i_{k-1}} \prod_{k=1}^{m} q_{j'_{k-1} j'_k} \tag{20}$$

and also

$$\prod_{k=1}^{m} q_{j_k j_{k-1}} \prod_{k=1}^{n} q_{i_k i_{k-1}} = \prod_{k=1}^{n} q_{i'_{k-1} i'_k} \prod_{k=1}^{m} q_{j'_{k-1} j'_k}. \tag{21}$$

By equations (20) and (21),

$$\prod_{k=1}^{n} \frac{q_{i'_{k-1} i'_k}}{q_{i_k i_{k-1}}} = \prod_{k=1}^{m} \frac{q_{j'_{k-1} j'_k}}{q_{j_k j_{k-1}}} = \prod_{k=1}^{m} \frac{q_{j_k j_{k-1}}}{q_{j'_{k-1} j'_k}}$$

and hence

$$\prod_{k=1}^{n} \frac{q_{i'_{k-1} i'_k}}{q_{i_k i_{k-1}}} = 1$$

which proves $n_i \pi_i = n_{i'} \pi_{i'}$. □

Example 12 Let us reconsider the steady-state probabilities derived in Example 11. Notice that from Proposition 18 and Theorem 6 we immediately know that given $\pi_1 = C_1$ we have $\pi_3 = C_1$ because they belong to the same equivalence class, and also $\pi_5 = 2C_1$ since state 5 is the reversed of state 1 (and 3) but its equivalence class has cardinality 1. Then we can compute π_5 as done in Example 11 and using again Proposition 18 we immediately derive the remaining equilibrium probabilities. In practice, Proposition 18 reduces the number of cycles one has to consider to compute the process' equilibrium distribution. Specifically, in this example we have to consider only one cycle.

We conclude this section by showing that the notion of autoreversibilty is indeed a $\lambda\rho$-reversibility.

Theorem 7 *If a CTMC with state space S is autoreversible then it is $\lambda\rho$-reversible for a strict lumpability $\sim$ and a renaming ϱ on $S/\sim$.*

Proof Let $X(t)$ be a CTMC which is autoreversible with a reversal bisimilarity $\sim_r$ over S. By Definition 12, $\sim_r$ induces a weak similarity $\sim$ over S. Hence, by Theorem 1, $X(t)$ is strictly lumpable with respect to $\sim$.

Consider now the renaming ϱ on $S/\sim$ defined by: $\varrho[i] = [i']$ whenever $i \sim_r i'$. First observe that, by Proposition 2, ϱ is well-defined. By Corollary 7, for all $i, j \in S$ with $q_{ji} > 0$ and for all $i', j' \in S$ such that $i \sim_r i'$, $j \sim_r j'$ and $q_{i'j'} > 0$ it holds

$$n_j \pi_j q_{ji} = n_i \pi_i q_{i'j'}$$

which can be rewritten as

$$\pi_{[j]} q_{ji} = \pi_{[i]} q_{i'j'}$$

by the fact that $\sim$ is an exact lumpability and then, for all equivalence classes $[i] \in S/\sim$ and for all $i_1, i_2 \in [i]$, $\pi_{i_1} = \pi_{i_2}$. Moreover,by Definition 12, for every i, i', j, j' such that $i \sim_r i'$ and $j \sim_r j'$, we have $m_{j[i]} = m_{i'[j']}$. Hence, we can write

$$\pi_{[j]} m_{j[i]} q_{ji} = \pi_{[i]} m_{i'[j']} q_{i'j'}.$$

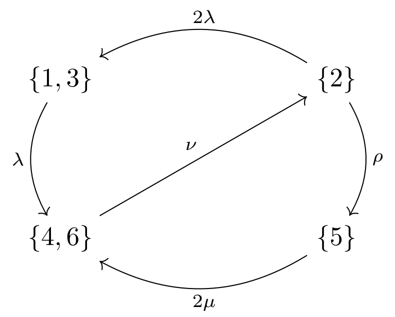


Fig. 10: The aggregated process of the CTMC in Fig 4.

By Definition 5 of weak similarity the last equality can be written as,

$$\pi_{[j]} q_{j[i]} = \pi_{[i]} q_{i'[j']}$$

and then, by Proposition 12, $X(t)$ is $\lambda\rho$-reversible with respect to $\sim$ and ϱ. $\square$

Example 13 (Autoreversibility and $\lambda\rho$-reversibility) Consider the CTMC depicted in Fig. 4. If we assume $\lambda = 2a$, $\mu = a$, $\nu = 8a$ and $\rho = 4a$, with $a \in \mathbb{R}^+$, then it can be proved that the process is autoreversible with respect to the reversal bisimilarity $\sim_r$ defined as:

$$\{(1,5),(3,5),(4,2),(6,2),(5,1),(5,3),(2,4),(2,6)\}.$$

Consider the equivalence relation induced by $\sim_r$ and partitioning the state space into the following equivalence classes:

$$\{\{1,3\},\{4,6\},\{2\},\{5\}\}.$$

The aggregated process $\widetilde{X}(t)$ is represented in Fig. 10 while $\widetilde{X}^R(t)$ is depicted in Fig. 11. Now if we consider the renaming over $S/\sim$ defined as:

$$\varrho(\{1,3\}) = \{5\} \quad \varrho(\{4,6\}) = \{2\} \quad \varrho(\{5\}) = \{1,3\} \quad \varrho(\{2\}) = \{4,6\}$$

we can prove that $\widetilde{X}(t)$ and $\varrho(\widetilde{X}^R)(t)$ are stochastically identical.

7 Applications

In this section we illustrate some examples of $\lambda\rho$-reversible processes. Clearly, all the product-form models that are reversible (see, e.g., [18,3,17]) are also $\lambda\rho$-reversible. For this reason we will focus on non-product-form models and show that the notion of $\lambda\rho$-reversibility simplifies the computation of the equilibrium distribution.

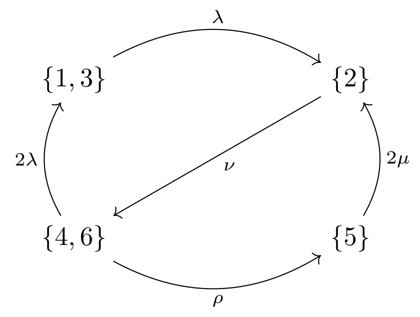


Fig. 11: Reversed of the aggregated process in Fig. 10.

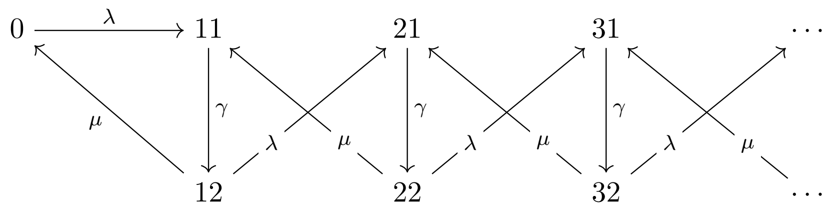


Fig. 12: Infinite state space CTMC.

7.1 Examples of $\lambda\rho$-reversible Markov chains

We first show an example of an infinite state ρ-reversible Markov chain.

Example 14 Let us consider the CTMC depicted by Fig. 12 and prove that it is ρ-reversible according to the permutation ϱ defined as:

$$\varrho(s) = \begin{cases} 0 & \text{if } s = 0 \\ n2 & \text{if } s = n1,\ n \geq 1 \\ n1 & \text{if } s = n2,\ n \geq 1 \end{cases}$$

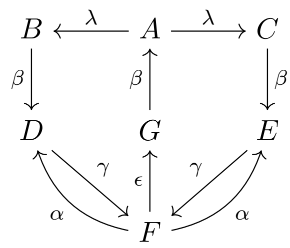
Observe that, under the assumption $\gamma = \lambda + \mu$ we have that for all the states s, it holds $q_s = q_{\varrho(s)}$ and, by exploiting the regularity of the process, we have to check the following cycles:

- $0 \xrightarrow{\lambda} 11 \xrightarrow{\gamma} 12 \xrightarrow{\mu} 0$ whose reversed is itself and hence the conditions of Proposition 13 are satisfied.
- $n1 \xrightarrow{\gamma} n2 \xrightarrow{\lambda} (n+1)1 \xrightarrow{\gamma} (n+1)2 \xrightarrow{\mu} n1$ whose reversed is $n2 \xrightarrow{\lambda} (n+1)1 \xrightarrow{\gamma} (n+1)2 \xrightarrow{\mu} n1 \xrightarrow{\gamma} n2$ whose product of the rates satisfies the conditions of Propisition 13.

Therefore, the CTMC is ρ-reversible and we have $\pi_{n1} = \pi_{n2}$ for all $n \geq 1$.

The next example illustrates the methodology for the computation of the steady-state probabilities of a $\lambda\rho$-reversible Markov chain. In particular, we show that the CTMC considered below is autoreversible.

Example 15 Consider the following CTMC:



It is easy to prove that it is autoreversible. Indeed, the state space $S = \{A, B, C, D, E, F, G\}$ is partitioned by a weak similarity relation into the following equivalence classes: $\{C, B\}$ that contains the reversed states of $\{G\}$, $\{D, E\}$ that contains the reversed states of $\{F\}$ and, finally, $\{A\}$ which is reversed of itself, provided that $2\alpha + \epsilon = \gamma$.

Let us compute the steady-state distribution starting from the reference state B, i.e., $\pi_B = \pi_C = K > 0$. We immediately derive $\pi_G = 2K$. To compute π_A we choose the path from A to C and its inverse $G \to A$, obtaining, by Theorem 5, $\pi_A = \pi_C \beta/\lambda = K\beta/\lambda$. We can derive π_F considering $F \to G$ and its reversed $C \to E$, obtaining $\pi_F = \pi_G \beta/\epsilon = 2K\beta/\epsilon$. Then we easly derive the probability of its reversed $\pi_D = \pi_E = \pi_F/2 = K\beta/\epsilon$. The value of K is obtained by normalising the probabilities.

We conclude this part about the applications by showing that the well-known model for cache analysis introduced by King in [20] is ρ-reversible.

Example 16 We consider a model for a cache with a FIFO replacing discipline. There are N objects whose requests are generated according to independent Poisson processes with rate λ_n with $1 \leq n \leq N$ (Independence Reference Model assumption - IRM). The cache size is M. When we observe a request of a class n there is a cache hit if that class is present in the cache or a cache miss otherwise. In the former case, the cache population is not changed, while in the latter the object is added in the cache and the one which has been present for the longest time is evicted. We can model the cache system in a similar way to what has been proposed in [20] but in a continuous time setting. The state of the CTMC is $c_1, \ldots, c_M$ where $1 \leq c_i \leq N$ and c_i denotes the class of the $i-$th element in the cache at a certain epoch. The Markov chain has a finite state space and is irreducible, hence it admits a steady-state distribution. Notice that the CTMC is not reversible since given the state $c_1, \ldots, c_M$ at the arrival of a class c_0 request, with $c_0 \neq c_i$, for all $i = 1, \ldots, M$ we have a transition to state $c_0, c_1, \ldots, c_{M-1}$, with class c_M being evicted from the cache. It is clear that in general there is not any class request that can restore the state $c_1, \ldots, c_M$ in one step. Nevertheless, the chain is ρ-reversible. Indeed, we can define the function $\varrho(c_1, \ldots, c_M) = (c_M, \ldots, c_1)$. Observe that since a state and its renaming have the same classes in the cache, the total outgoing rate is the same since the arrivals that cause a state transition are the same. The condition on the cycles may also be easily verified by observing that the reversed of transition:

$$(c_1, \ldots, c_M) \xrightarrow{\lambda_{c_0}} (c_0, \ldots, c_{M-1})$$

is

$$\varrho(c_0, \ldots, c_{M-1}) = (c_{M-1}, \ldots c_0) \xrightarrow{\lambda_{c_M}} (c_M, \ldots, c_1) = \varrho(c_1, \ldots, c_M)\,.$$

Finally, the equilibrium distribution of the chain can be obtained by applying Corollary 5 without the need of solving the GBE system.

7.2 An example of a ρ-reversible queue

Let us consider a queueing system defined as follows:

- Customers arrive according to a homogeneous Poisson process with rate λ;
- The service room is equipped with a single server that performs two tasks that we will call phase$_1$ and phase$_2$. The service time for each phase of service is exponentially distributed with rate 2μ and the service times are independent;
- The queueing discipline is *Last Come First Served with preemption* (LCFSP) which means that as soon as a customer arrives at the queue it starts being served and if another customer was in the service room then this is put in queue and the work done is lost. Notice that this discipline differs from the quasi-reversible one studied in [5] since LCFSP does not resume the past work once a customer returns in service.
- When a customer is resumed it enters the first phase of service with probability 1/2 and the second with probability 1/2.

The diagram of Fig. 13 shows the CTMC underlying the queueing system, where state 0 denotes the empty system, state $n1$ with $n \geq 1$ denotes that the system has n customers and the server busy in the first phase, while $n2$ denotes that the server is active in the second phase. The queue is ρ-reversible with:

$$\varrho(s) = \begin{cases} 0 & \text{if } s = 0 \\ n2 & \text{if } s = n1,\ n \geq 1 \\ n1 & \text{if } s = n2,\ n \geq 1 \end{cases},$$

in fact we have $q(n1) = q(n2) = \lambda + 2\mu$ and we have to consider the cycles:

- $0 \xrightarrow{\lambda} 11 \xrightarrow{2\mu} 12 \xrightarrow{2\mu} 0$ which is the reversed of itself;

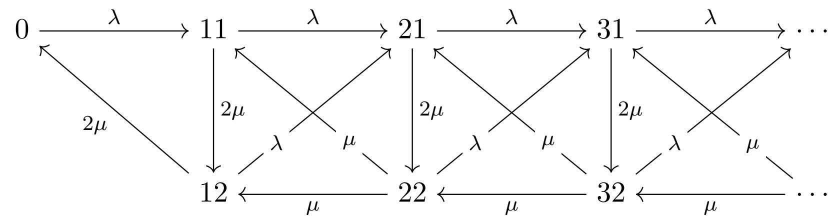


Fig. 13: A $\lambda\rho$-reversible queue

- $n2 \xrightarrow{\lambda} (n+1)1 \xrightarrow{2\mu} (n+1)2 \xrightarrow{\mu} n2$ whose reversed is $n1 \xrightarrow{\lambda} (n+1)1 \xrightarrow{2\mu} (n+1)2 \xrightarrow{\mu} n1$ and hence the product of the transition rates are the same;
- $n1 \xrightarrow{2\mu} n2 \xrightarrow{\lambda} (n+1)1 \xrightarrow{2\mu} (n+1)2 \xrightarrow{\mu} n1$ whose reversed is $n2 \xrightarrow{\lambda} (n+1)1 \xrightarrow{2\mu} (n+1)2 \xrightarrow{\mu} n1 \xrightarrow{2\mu} n2$ and hence the product of the transition rates are the same.

Therefore, we immediately derive $\pi(n1) = \pi(n2)$ and the steady-state probabilities may be obtained by the solution of the system of detailed balance equations instead of the more complex system of global balance equations.

7.3 Applications in Markovian process algebra

We briefly introduce a Markovian process algebra, i.e., the Performance Evaluation Process Algebra (PEPA) [16]. We consider a reduced syntax of PEPA as follows.

- *Prefix:* $(a, \lambda).P$ is the agent that performs an activity of type a whose duration is an exponentially distributed random variable with parameter λ and then behaves as P. We can have that instead of specifying a positive real number λ as transition rate, the symbol $\top$ is used denoting that the duration of an activity is determined by another agent.
- *Choice:* The choice operator $P + Q$ describes an agent that can choose to behave as P or Q according to the standard race policy [16] (i.e., the fastest sampled time determines the activity to carry out).
- *Constant:* A new constant agent A is defined to behave as P by writing $A \stackrel{\text{def}}{=} P$.
- *Cooperation:* The modularity of this Markovian process algebra strongly depends on the operator specifying the cooperation among two agents: $P \underset{L}{\bowtie} Q$. In this case, all the transitions in P and Q whose type belongs to the set L can be carried out only jointly. The rate of the joint transition must be decided according to the rules described in the semantics [16]. In particular, in case of cooperation on a type a between an activity with a specified rate λ (active) and one with unspecified rate $\top$ (passive), the joint activity has type a and rate λ. In the general case, the shared activity

will have the same action type as the two contributing activities and a rate reflecting the rate of the slower participant.

The following example illustrates a simple client/server scenario which is modeled as a PEPA term and whose underlying Markov chain is $\lambda\rho$-reversible.

Example 17 A client is a sequential component that repeatedly carries out a shared task *s_task* in cooperation with the server and an autonomous activity *c_task*. Similarly, a server undertakes two activities consecutively: *s_task* shared with the client and *l_task* representing a local computation.

$$\begin{aligned} Client &\stackrel{\text{def}}{=} (s\_task, \top).(c\_task, \mu).Client \\ Server &\stackrel{\text{def}}{=} (s\_task, \lambda).(l\_task, \gamma).Server \end{aligned}$$

The system composed by two clients, independent of each other but competing for the same server, is modelled as the following PEPA term:

$$S \stackrel{\text{def}}{=} (Client||Client) \underset{\{s\_task\}}{\bowtie} Server$$

The derivation graph of S has eight states as represented in Fig. 14 where s, c and l stands for *s_task*, *c_task* and *l_task*, respectively and

$$\begin{aligned}
S_1 &\stackrel{\text{def}}{=} (Client||Client) \underset{\{s_task\}}{\bowtie} (l_task, \gamma).Server \\
S_2 &\stackrel{\text{def}}{=} ((c_task, \mu).Client||Client) \underset{\{s_task\}}{\bowtie} (l_task, \gamma).Server \\
S_3 &\stackrel{\text{def}}{=} (Client||(c_task, \mu).Client) \underset{\{s_task\}}{\bowtie} (l_task, \gamma).Server \\
S_4 &\stackrel{\text{def}}{=} (Client||Client) \underset{\{s_task\}}{\bowtie} Server \\
S_5 &\stackrel{\text{def}}{=} ((c_task, \mu).Client||Client) \underset{\{s_task\}}{\bowtie} Server \\
S_6 &\stackrel{\text{def}}{=} ((c_task, \mu).Client||(c_task, \mu).Client) \underset{\{s_task\}}{\bowtie} Server \\
S_7 &\stackrel{\text{def}}{=} (Client||(c_task, \mu).Client) \underset{\{s_task\}}{\bowtie} Server \\
S_8 &\stackrel{\text{def}}{=} ((c_task, \mu).Client||(c_task, \mu).Client) \underset{\{s_task\}}{\bowtie} (l_task, \gamma).Server
\end{aligned}$$

It is worth notice that the underlying CTMC this is strictly lumpable with respect to the equivalence relation $\sim$ with equivalence classes $\{S_2, S_3\}$ and $\{S_5, S_7\}$. The corresponding aggregated process is represented in Fig. 15.

It is easy to prove that the aggregated CTMC is ρ-reversible with respect to the renaming ϱ such that $\varrho(S_1) = S_6$, $\varrho(S_6) = S_1$, $\varrho(S_4) = S_8$, $\varrho(S_8) = S_4$, $\varrho(S_{2,3}) = S_{5,7}$, $\varrho(S_{5,7}) = S_{2,3}$.

The next examples show that the CTMCs with the regular structures that are required by autoreversibility often underlie Markovian process algebra cooperations.

Example 18 This example aims at showing the simplest instance of a non-product-form cooperation between two agents that is autoreversible. Let us consider the following PEPA components:

$$\begin{aligned} P_1 &\stackrel{\text{def}}{=} (a, \alpha).P_2 \qquad Q_1 \stackrel{\text{def}}{=} (a, \top).Q_2 \\ P_2 &\stackrel{\text{def}}{=} (b, \beta).P_1 \qquad Q_2 \stackrel{\text{def}}{=} (c, \gamma).Q_1. \end{aligned}$$

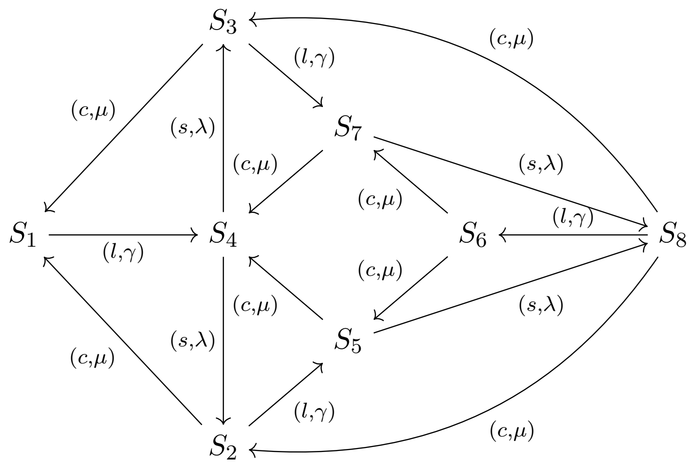


Fig. 14: Derivation graph of S.

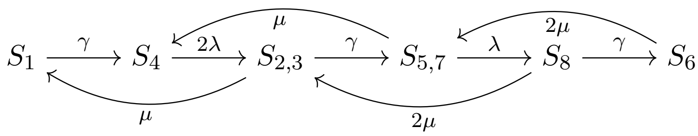
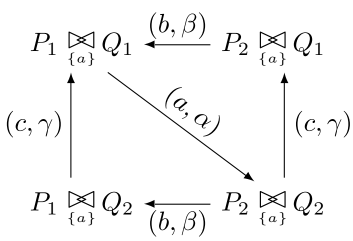
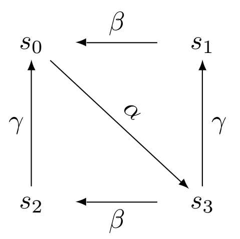


Fig. 15: Aggregated process for S.

Now consider the system Sys_1 defined by: $Sys_1 \stackrel{\text{def}}{=} P_1 \underset{\{a\}}{\bowtie} Q_1$ whose derivation graph is:



The underlying CTMC has the same state space S and is:



with $s_0 = P_1 \underset{\{a\}}{\bowtie} Q_1$, $s_1 = P_2 \underset{\{a\}}{\bowtie} Q_1$, $s_2 = P_1 \underset{\{a\}}{\bowtie} Q_2$ and $s_3 = P_2 \underset{\{a\}}{\bowtie} Q_2$. Consider the relation

$$\mathcal{R} = \{(s_0, s_3), (s_1, s_2)\}.$$

It is easy to see that if $\alpha = \beta+\gamma$ and $\beta = \gamma$ then $\mathcal{R}$ is a reversal bisimilarity over $\{s_0, s_1, s_2, s_3\}$, and the above CTMC is autoreversible (but not reversible). To derive the steady-state distribution: fix a state, e.g., s_0 with $\pi_{s_0} = C > 0$. It immediately follows that its reversed has the same equilibrium probability, i.e., $\pi_{s_3} = \pi_{s_0}$. The computation of π_{s_1} follows by Theorem 5 considering the path from the reversed of s_0, i.e., s_3 to the reversed of s_1, i.e., s_2 and dividing its rate by the transition rate of the forward path from s_1 to s_0. This gives $\pi_{s_1} = \pi_{s_0}$ and hence also $\pi_{s_2} = \pi_{s_0}$.

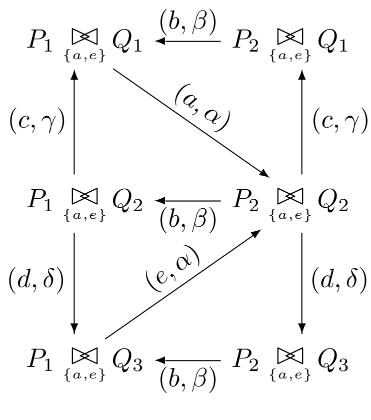
Example 19 Consider the following PEPA components:

$$P_1 \overset{\text{def}}{=} (a, \alpha).P_2 + (e, \alpha).P_2 \qquad Q_1 \overset{\text{def}}{=} (a, \top).Q_2$$
$$P_2 \overset{\text{def}}{=} (b, \beta).P_1 \qquad Q_2 \overset{\text{def}}{=} (c, \gamma).Q_1 + (d, \delta).Q_3$$
$$Q_3 \overset{\text{def}}{=} (e, \top).Q_2$$

and the system Sys_2 defined by:

$$Sys_2 \overset{\text{def}}{=} P_1 \underset{\{a,e\}}{\bowtie} Q_1.$$

The derivation graph of Sys_2 is



Notice that with the opportune rate conditions, the underlying CTMC is that of the running example depicted in Fig. 4.

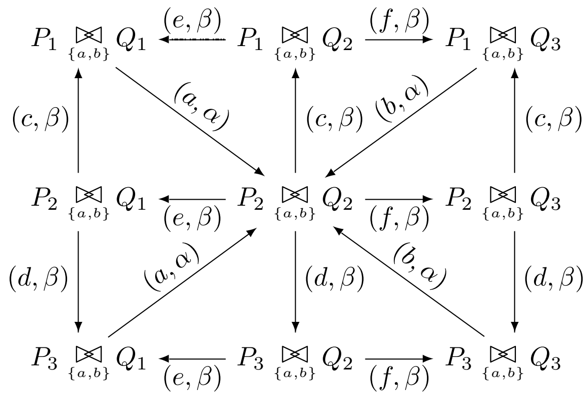
Example 20 Consider the PEPA components depicted below:

$$\begin{array}{ll} P_1 \stackrel{\text{def}}{=} (a,\alpha_1).P_2 + (b,\alpha_2).P_2 & Q_1 \stackrel{\text{def}}{=} (a,\alpha).Q_2 \\ P_2 \stackrel{\text{def}}{=} (c,\beta).P_1 + (d,\beta).P_3 & Q_2 \stackrel{\text{def}}{=} (e,\beta).Q_1 + (f,\beta).Q_3 \\ P_3 \stackrel{\text{def}}{=} (a,\top).P_2 + (b,\top).P_2 & Q_3 \stackrel{\text{def}}{=} (b,\alpha).Q_2 \end{array}$$

with $\alpha_1 \geq \alpha$ and $\alpha_2 \geq \alpha$ (Notice that these conditions allow us to determine the rate of the shared activities when the rate of the two contributing components are not unspecified. In this case the rate of the joint activities will be the rate of the slower participant). Let Sys_3 defined by:

$$Sys_3 \stackrel{\text{def}}{=} P_1 \underset{\{a,b\}}{\bowtie} Q_1.$$

The derivation graph Sys_3 is defined by:



It is easy to see that its underlying CTMC is autoreversible under the condition $\alpha = 4\beta$. Indeed, the state space is partitioned by a weak similarity relation into the following equivalence classes:

$\{P_1 \underset{\{a,b\}}{\bowtie} Q_1, P_1 \underset{\{a,b\}}{\bowtie} Q_3, P_3 \underset{\{a,b\}}{\bowtie} Q_1, P_3 \underset{\{a,b\}}{\bowtie} Q_3\}$

$\{P_2 \underset{\{a,b\}}{\bowtie} Q_1, P_2 \underset{\{a,b\}}{\bowtie} Q_3\}$

$\{P_1 \underset{\{a,b\}}{\bowtie} Q_2, P_3 \underset{\{a,b\}}{\bowtie} Q_2\}$

$\{P_2 \underset{\{a,b\}}{\bowtie} Q_2\}$

such that $\{P_1 \underset{\{a,b\}}{\bowtie} Q_1, P_1 \underset{\{a,b\}}{\bowtie} Q_3, P_3 \underset{\{a,b\}}{\bowtie} Q_1, P_3 \underset{\{a,b\}}{\bowtie} Q_3\}$ contains the reversed states of $\{P_2 \underset{\{a,b\}}{\bowtie} Q_2\}$ (and vice versa), and $\{P_2 \underset{\{a,b\}}{\bowtie} Q_1, P_2 \underset{\{a,b\}}{\bowtie} Q_3\}$ contains the reversed states of $\{P_1 \underset{\{a,b\}}{\bowtie} Q_2, P_3 \underset{\{a,b\}}{\bowtie} Q_2\}$ (and vice versa).

8 Conclusion

In this paper we have combined the notions of time-reversibility and that of lumping into a unique setting. To the best of our knowledge, the relations between the different notions of lumping and the reversed Markov chain are novel and could help the development of solution algorithms in the style of [32, 29, 31]. The class of $\lambda\rho$-reversible chains extends that of dynamically reversible chains [33, 18] by combining the idea of strict lumping and that of reversibility modulo an arbitrary permutation of the state names. We showed that this class of Markov processes unifies also the autoreversible chains studied in [23]. $\lambda\rho$-reversibility allows for an efficient computation of the equilibrium probability distribution (and consequently facilitates the derivation of the stationary model's performance indices) and can be decided by the structural analysis similar to that based on Kolmogorov's criteria for reversible chains [18].

It is worth of notice that while the notion of quasi-reversibility introduced by Kelly in [18] is used to study the product-form of stochastic networks by analysing each component in isolation, the $\lambda\rho$-reversibility provides an efficient method to derive the equilibrium distribution of large Markov chains without the need of decomposing them.

Future work includes the design of an algorithm to compute the permutation of states ϱ that makes a Markov chain ρ-reversible and the study of the compositional properties of this class of models.

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