FACTOR VARIETIES

ANTONINO SALIBRA, ANTONIO LEDDA, FRANCESCO PAOLI

ABSTRACT. The universal algebraic literature is rife with generalisations of discriminator varieties, whereby several investigators have tried to preserve in more general settings as much as possible of their structure theory. Here, we modify the definition of discriminator algebra by having the switching function project onto its third coordinate in case the ordered pair of its first two coordinates belongs to a designated relation (not necessarily the diagonal relation). We call these algebras factor algebras and the varieties they generate factor varieties. Among other things, we provide an equational description of these varieties and match equational conditions involving the factor term with properties of the associated factor relation. Factor varieties include, apart from discriminator varieties, several varieties of algebras from quantum and fuzzy logics.

1. Introduction

A discriminator variety [32] is a variety $\mathcal{V}$ with a quaternary term $s(x, y, z, w)$ that realises the switching function

$$s(a, b, c, d) = \begin{cases} c \text{ if } a = b \\ d, \text{ otherwise} \end{cases}$$

on any subdirectly irreducible member of $\mathcal{V}$. Over the years, the idea of discriminator variety has proved immensely successful: on the one hand, in fact, it is a unifying notion for many important classes of algebras arising in algebra and logic, including the varieties of Boolean algebras, monadic algebras, n-dimensional cylindric algebras, Post algebras, and n-valued MV algebras. On the other hand, discriminator varieties are very convenient to work with, and are characterised by extremely useful and strong algebraic properties, such as a Boolean product representation.

The universal algebraic literature is rife with generalisations of this concept in different directions, whereby several investigators have tried to preserve in more general settings as much as possible of the structure theory of discriminator varieties. By way of example, we mention here dual discriminator varieties [16], binary discriminator varieties [10], varieties with a ternary deductive term [5], fixedpoint discriminator varieties [5], quasi-discriminator varieties [25].

The very definitions of switching term and switching function, however, suggest a natural move. The switching function projects onto its third coordinate whenever its first two coordinates are identical, else onto its fourth coordinate. One could meaningfully wonder what happens if the role of the identity relation in this definition is played by a generic binary relation R. In other words, we could define a *factor function* on a set A as a function $u : A^4 \to A$ such that

$$u\,(a,b,c,d) = \begin{cases} c \text{ if } \ (a,b) \in R \\ d, \text{otherwise} \end{cases}$$

and define a *factor variety* as a variety $\mathcal{V}$ with a quaternary term $u(x,y,z,w)$ (called *factor term*) that realises the factor function on any subdirectly irreducible member of $\mathcal{V}$.

This notion is not motivated only by a drive for ever more general abstractions of the concept of discriminator variety. There are, in fact, at least two additional motives of interest behind this concept:

- On the one hand, factor varieties are closely tied to decomposition operations. Given a type ν and a quaternary term u of type ν, a variety $\mathcal{V}$ of type ν is a factor variety with common factor term u iff the binary function $f_{x,y}\,(a,b) = u^{\mathbf{A}}\,(x,y,a,b)$ is a decomposition operation for all $\mathbf{A} \in \mathcal{V}$ and all $x, y \in A$. This paper, in a sense, can thus be seen as a first step towards an abstract theory of decomposition operations.
- On the other hand, we will see that the class of factor varieties includes notable examples of varieties arising both in fuzzy logic (Gödel algebras, product algebras, the variety of MV algebras generated by Chang's algebra) and in quantum logics (Jauch-Piron orthomodular lattices with states, the varieties of modular ortholattices generated by $\mathbf{MO_k}$ for $k \in N$).

The paper is organised as follows: in §2 we go over several preliminary notions, partly concerning the terminology and the notation to be used in the sequel, partly regarding the key concepts that features prominently in the rest of the paper, namely, Church algebras and skew Boolean algebras. In §3 and §4 we introduce factor algebras and varieties, providing some examples and proving some elementary properties of these notions. We also show under what conditions the quaternary factor function can be equivalently replaced by a ternary one. In §5 we develop a sort of "correspondence theory", matching equational conditions involving the factor term and properties of the associated factor relation. In particular, we give an equational description of factor varieties along the lines of the one suggested by Vaggione for discriminator varieties [31]. In §6 we investigate one-pointed and double-pointed factor varieties.

We show that (i) a double-pointed variety is a $(0,1)$-factor variety if, and only if, it is a semi-Boolean-like variety with a binary term d satisfying some natural conditions (see Theorem 38); (ii) a one-pointed variety is a 0-factor variety if, and only if, it is a variety of skew Boolean algebras with additional regular operations and with a binary term d satisfying the identities $d(x,0)=x$ and $d(0,0)=0$ (see Theorem 41). One-pointed and double-pointed discriminator varieties are also characterized in terms of factor varieties. We conclude the paper by investigating pure factor algebras, that have the factor term and nothing else in their type.

2. Preliminaries

2.1. **Terminology and Notation.** In this subsection we clarify some terminology and notation that will be frequently used throughout the paper. All the unexplained notions and symbols are in keeping with the standard universal algebraic jargon (see e.g. [7]). Superscripts marking the difference between an operation and an operation symbol will be dropped whenever this does not prejudice clarity.

$\mathrm{Con}(\mathbf{A})$ and $\mathrm{Con}^*(\mathbf{A})$ will respectively denote the set (occasionally, the lattice) of congruences of an algebra $\mathbf{A}$, and the set $\mathrm{Con}(\mathbf{A}) \setminus \{\Delta, \nabla\}$. The *monolith* of the algebra $\mathbf{A}$ is the congruence $\mu^{\mathbf{A}} = \bigcap \mathrm{Con}^*(\mathbf{A})$; $\mathbf{A}$ is subdirectly irreducible if, and only if, $\mu^{\mathbf{A}} \neq \Delta$. The notation $\theta(a,b)$ refers to the least congruence including (a,b).

If $\mathcal{V}$ is a variety, then $\mathcal{V}_{\mathrm{di}}$ (resp. $\mathcal{V}_{\mathrm{si}}$, $\mathcal{V}_{\mathrm{s}}$) denotes the class of all directly indecomposable (resp. subdirectly irreducible, simple) members of $\mathcal{V}$.

We recall that, due to a result by Vaggione, discriminator varieties can be equationally described in terms of conditions on the switching term. In fact:

Theorem 1. [31] *A variety $\mathcal{V}$ of type ν is a discriminator variety iff there is a quaternary term u satisfying the following identities:*

- *(D_1) $u(x,x,y,z) \approx y$;*
- *(D_2) $u(x,y,z,z) \approx z$;*
- *(D_3) $u(x,y,u(x,y,v,w),z) \approx u(x,y,v,z) \approx u(x,y,v,u(x,y,w,z))$;*
- *(D_4) $u(x,y,\sigma(\overline{v}),\sigma(\overline{w})) \approx \sigma(u(x,y,v_1,w_1),\dots,u(x,y,v_n,w_n))$ (for all $\sigma \in \nu$)*
- *(D_5) $u(x,y,x,y) \approx y$.*

If $\mathbf{A}$ is an algebra of type ν and u is a quaternary term, then, for all $a,b \in A$, the binary function $u^{\mathbf{A}}(a,b,-,-)$ is a decomposition operation (see [24, Thm. 4.33]) iff $\mathbf{A} \models D_2, D_3, D_4$.

2.2. **Church Algebras.** The key observation motivating the introduction of *Church algebras* [23] is that many algebras arising in completely different fields of mathematics — including Heyting algebras, rings with unit, or combinatory algebras — have a term operation q satisfying the fundamental properties of the if-then-else connective:

$$q\left(1,x,y\right)\approx x \text{ and } q\left(0,x,y\right)\approx y.$$

As simple as they may appear, these properties are enough to yield rather strong results. This motivates the next definition.

Definition 2. *An algebra* $\mathbf{A}$ *of type* ν *is a* Church algebra *if there are term definable elements* $0^{\mathbf{A}},1^{\mathbf{A}}\in A$ *and a term operation* $q^{\mathbf{A}}$ *s.t., for all* $a,b\in A$

$$q^{\mathbf{A}}\left(1^{\mathbf{A}},a,b\right)=a \textit{ and } q^{\mathbf{A}}\left(0^{\mathbf{A}},a,b\right)=b.$$

A variety $\mathcal{V}$ *of type* ν *is a* Church variety *if every member of* $\mathcal{V}$ *is a Church algebra with respect to the same term* $q\left(x,y,z\right)$ *and the same constants* $0,1$*.*

Examples of Church algebras include FL_{ew}-algebras (commutative, integral and double-pointed residuated lattices, for which see [17]) and, in particular, Heyting algebras and thus also Boolean algebras; ortholattices; rings with unit; combinatory algebras. Expanding on an idea due to Vaggione [30], we also define:

Definition 3. *An element* e *of a Church algebra* $\mathbf{A}$ *is called* central *if the pair* $\left(\theta\left(e,0\right),\theta\left(e,1\right)\right)$ *is a pair of complementary factor congruences on* $\mathbf{A}$*. A central element* e *is* nontrivial *if* $e\notin\{0,1\}$*. By* $\mathrm{Ce}\left(A\right)$ *we denote the* centre *of* $\mathbf{A}$*, i.e. the set of central elements of the algebra* $\mathbf{A}$*.*

It is proved in [28] that Church algebras have Boolean factor congruences and that, by defining

$$x\wedge y=q(x,y,0),\ x\vee y=q(x,1,y) \text{ and } x'=q(x,0,1), \tag{1}$$

we get:

Theorem 4. *Let* $\mathbf{A}$ *be a Church algebra. Then* $c\left[\mathbf{A}\right]=\left(\mathrm{Ce}\left(A\right),\vee,\wedge,',0,1\right)$ *is a Boolean algebra which is isomorphic to the Boolean algebra of factor congruences of* $\mathbf{A}$*.*

It clearly follows that a Church algebra is directly indecomposable iff $\mathrm{Ce}\left(A\right)=\{0,1\}$. This result, together with theorems by Comer [13] and Vaggione [30], implies:

Theorem 5. *Let* $\mathbf{A}$ *be a Church algebra,* S *be the Boolean space of maximal ideals of* $c[\mathbf{A}]$ *and* $f : A \to \Pi_{I\in S} A/\theta_I$ *be the map defined by*

$$f(a) = (a/\theta_I : I \in S),$$

where $\theta_I = \bigvee_{e\in I} \theta(0, e)$*. Then we have:*

(1) f *gives a weak Boolean representation of* $\mathbf{A}$*.*

(2) f *provides a Boolean representation of* $\mathbf{A}$ *iff, for all* $a \neq b \in A$*, there exists a least central element* e *such that* $q(e, a, b) = a$ *(i.e.,* $(a, b) \in \theta(0, e)$*).*

In general, not much can be said about the factors in this representation. However, these factors are guaranteed to be directly indecomposable provided that the d.i. members of $\mathcal{V}$ form a universal class. In fact, following [30], it is shown in [28] that:

Theorem 6. *Let* $\mathcal{V}$ *be a Church variety of type* ν*. Then, the following conditions are equivalent:*

(i) *For all* $\mathbf{A} \in \mathcal{V}$*, the stalks* $\mathbf{A}/\theta_I$ *(*$I \in S$ *a maximal ideal) are directly indecomposable.*

(ii) *The class* $\mathcal{V}_{\mathrm{di}}$ *of directly indecomposable members of* $\mathcal{V}$ *is a universal class.*

In a generic Church algebra, of course, there is no need for the set of central elements to comprise all elements of the algebra — not any more than an arbitrary ortholattice needs to be a Boolean algebra, or a ring with unit a Boolean ring. In [28], Church algebras where the set of central elements comprises all elements of the algebra were introduced and investigated under the label of *Boolean-like algebras*, while the name of *semi-Boolean-like algebras* was reserved for the concept defined below:

Definition 7. *We say that a Church algebra* $\mathbf{A}$ *of type* ν *is a* semi-Boolean-like algebra *(or a* SBlA*, for short) if it satisfies the following conditions, for all* $e, a, a_1, a_2 \in A$, for every n-ary $g \in \nu$, and for every $\overline{b}, \overline{c} \in A^n$*:*

Ax$_0$. $q(1, a, b) = a = q(0, b, a)$

Ax$_1$. $q(e, a, a) = a$

Ax$_2$. $q(e, q(e, a_1, a_2), a) = q(e, a_1, a) = q(e, a_1, q(e, a_2, a))$

Ax$_3$. $q(e, g(\overline{b}), g(\overline{c})) = g(q(e, b_1, c_1), \ldots, q(e, b_n, c_n))$.

If $\mathbf{A}$ *satisfies* Ax_0*-*Ax_3 *plus*

Ax_4. $q(a,1,0)=a$

then we say that $\mathbf{A}$ *is a* Boolean-like algebra *(or a* BlA*, for short).*

Definition 8. *A variety* $\mathcal{V}$ *of type* ν *is a* (semi-)Boolean-like variety *if every member of* $\mathcal{V}$ *is a (semi-)Boolean-like algebra with respect to the same term* $q(x,y,z)$ *and the same constants* $0,1$.

Semi-Boolean-like algebras and varieties were further investigated in [20]. It turns out that, if we define $c(x)=q(x,1,0)$, an element a in a SBlA is central just in case $c(a)=a$. By Ax_4, therefore, BlAs are precisely those SBlAs where every element is central. Moreover:

Proposition 9. *For a Church variety* $\mathcal{V}$ *(w.r.t. the term* q*), the following are equivalent:*

(1) $\mathcal{V}$ *is semi-Boolean like;*

(2) $\mathcal{V}$ *satisfies the conditions:*

(i): *for all* $a,b,c\in\mathbf{A}\in\mathcal{V}$, $q(a,b,c)=q(c(a),b,c)$;

(ii): *for all* $a\in\mathbf{A}\in\mathcal{V}$, $c(a)$ *is central;*

(3) $\mathcal{V}$ *satisfies the condition 2(i) and the universal formula* $c(0)\approx 0 \;\bar{\wedge}\; c(1)\approx 1 \;\bar{\wedge}\; \forall x(c(x)\approx 0 \,\underline{\vee}\, c(x)\approx 1)$ *holds in every subdirectly irreducible member of* $\mathcal{V}$.

The "pure semi-Boolean-like" variety $\mathcal{SBlA}_0$, consisting of all the term reducts of the form $(A,q,0,1)$ of SBlAs, and axiomatised by Ax_0-Ax_3 above, is of independent interest. We say that a term t is $\mathcal{V}$*-idempotent* if $\mathcal{V}\vDash t(t(x))\approx t(x)$, and $\mathcal{V}$*-compatible* in case $t^{\mathbf{A}}$ is an endomorphism in every $\mathbf{A}\in\mathcal{V}$. It can be shown that the term c is $\mathcal{SBlA}_0$-compatible and $\mathcal{SBlA}_0$-idempotent and thus, if $\mathbf{A}$ is a member of $\mathcal{SBlA}_0$, $c[\mathbf{A}]$ is a *retract* of $\mathbf{A}$. $\mathcal{SBlA}_0$ is generated by the algebras in the next two examples:

Example 10. *Let* $\mathbf{3}=(\{0,1,2\},q,0,1)$ *be the Church algebra completely specified by the stipulation that* $q(0,a,b)=q(2,a,b)$ *for all* $a,b\in\{0,1,2\}$. *It can be checked that* $\mathbf{3}$ *is semi-Boolean-like. However,* $c(2)=q(2,1,0)=0\neq 2$. *Moreover,* $\mathbf{3}$ *is a nonsimple subdirectly irreducible algebra, with the middle congruence corresponding to the partition* $\{\{1\},\{0,2\}\}$. *Therefore* $\mathcal{V}(\mathbf{3})$ *is not a discriminator variety, although it is a binary 1-discriminator variety in the sense of* [10] *with binary 1-discriminator term* $y'\vee x$.

Example 11. *Let* $\mathbf{3}'=(\{0,1,2\},q,0,1)$ *be the Church algebra completely specified by the stipulation that* $q(1,a,b)=q(2,a,b)$ *for all* $a,b\in\{0,1,2\}$. *It can be checked that*

$\mathbf{3}'$ is semi-Boolean-like. However, $c(2) = q(2,1,0) = 1 \neq 2$. Moreover, $\mathbf{3}'$ is a non-simple subdirectly irreducible algebra, with the middle congruence corresponding to the partition $\{\{0\},\{1,2\}\}$. Therefore $\mathcal{V}(\mathbf{3}')$ is not a discriminator variety, although it is a binary 0-discriminator variety with binary 0-discriminator term $y' \wedge x$.

A semi-Boolean-like variety is said to be *meet-idempotent* if it satisfies the identity

$$Ax_5 : x \wedge x \approx x.$$

Double-pointed discriminator varieties coincide with 0-regular meet-idempotent semi-Boolean-like varieties (see [28, Theorem 5.6]).

2.3. **Skew Boolean Algebras.** Weakenings of lattices where the meet and join operations may fail to be commutative attracted from time to time the attention of various communities of scholars, including ordered algebra theorists (for their connection with preordered sets) and semigroup theorists (who viewed them as structurally enriched bands possessing a dual multiplication). Here we will review some basic definitions and results on one such generalisation, probably the most interesting and successful: the concept of *skew lattice* [21], in fact, along with the related notion of *skew Boolean algebra*, has important connections with discriminator varieties; the interested reader is referred to [22, 29, 14] for far more comprehensive accounts.

Definition 12. A *skew lattice* is an algebra $\mathbf{A} = (A, +, \cdot)$ of type $(2,2)$ satisfying:

- Associativity: $x + (y + z) \approx (x + y) + z; \qquad x(yz) \approx (xy)z$
- Idempotence: $xx \approx x \approx x + x$
- Absorption: $x + xy \approx x \approx x(x + y); \qquad yx + x \approx x \approx (y + x)x$

It is not difficult to see that the absorption condition is equivalent to the following pair of biconditionals:

$$x + y \approx y \text{ iff } xy \approx x\text{; and } x + y \approx x \text{ iff } xy \approx y.$$

In any skew lattice $\mathbf{A}$ we define a partial ordering as follows:

$$a \leq b \text{ iff } ab = a = ba.$$

Observe that $aba \leq a \leq a + b + a$ for every a, b.
Now, define

$$\ulcorner x \urcorner = \{y : y \leq x\} \text{ and } \llcorner x \lrcorner = \{y : y \geq x\}.$$

It can be seen that $\ulcorner x \urcorner = \{xyx : y \in A\}$ and $\llcorner x \lrcorner = \{x + y + x : y \in A\}$ are subalgebras of $\mathbf{A}$, by means of which we can identify two interesting subclasses of skew lattices:

Definition 13. A skew lattice $\mathbf{A}$ is *normal* if $\ulcorner x \urcorner$ is a lattice for every x; it is *Boolean* if $\ulcorner x \urcorner$ is a Boolean lattice for every $x \in A$.

Normal skew lattices are a variety, axiomatised relative to skew lattices by the equation

$$xyzx \approx xzyx.$$

In any skew lattice $\mathbf{A}$ we define a preorder:

$$a \leq_d b \text{ iff } aba = a \text{ iff } b + a + b = b.$$

Observe that $ab \leq_d a \leq_d a + b$ and $ba \leq_d a \leq_d b + a$, for every a, b. The equivalence induced by $\leq_d$, denoted as $\mathcal{D}$, is in fact a congruence, and $\mathbf{A}/\mathcal{D}$ is the maximal lattice image of $\mathbf{A}$. The $\mathcal{D}$-equivalence class $\mathcal{D}_a$ of an element a is $\{a\}$ iff, for all $b \in A$, $ab = ba$. Remark that $\mathcal{D}_a \subseteq \mathcal{D}_b$ iff $aba = a$.

Two further preorders can be defined on a skew lattice:

(1) $x \leq_l y$ iff $xy = x$;

(2) $x \leq_r y$ iff $yx = x$.

The equivalences $\mathcal{L}$ and $\mathcal{R}$, respectively induced by $\leq_l$ and $\leq_r$, are again congruences; moreover, $\mathcal{L}$ is the minimal congruence making $\mathbf{A}/\mathcal{L}$ a right-zero skew lattice.

Definition 14. A skew lattice is *left-handed* (*right-handed*) if $\mathcal{L} = \mathcal{D}$ ($\mathcal{R} = \mathcal{D}$).

Lemma 15. *The following conditions are equivalent for a skew lattice* $\mathbf{A}$*:*

(1) $\mathbf{A}$ *is left-handed;*

(2) for all $a, b \in A$*,* $a\mathcal{D}b$ *implies* $ab = a$*;*

(3) for all $a, b \in A$*,* $aba = ab$*.*

If we expand skew lattices by a subtraction operation and a constant 0, we get the following noncommutative variant of Boolean algebras.

Definition 16. A *skew Boolean algebra* (*SBA*, for short) is an algebra $\mathbf{A} = (A, +, \cdot, -, 0)$ of type $(2, 2, 2, 0)$ such that:

- its reduct $(A, +, \cdot)$ is a normal skew lattice satisfying the identities $x(y+z) \approx xy + xz$ and $(y + z)x \approx yx + zx$;

- 0 is left and right absorbing w.r.t. multiplication;
- the operation $-$ satisfies the identities

$$xyx + (x - y) \approx x \approx (x - y) + xyx;$$
$$xyx(x - y) \approx 0 \approx (x - y)xyx.$$

Definition 17. An algebra of type ν is a *SBA with additional operations* if it is a *SBA* satisfying the following identity for every $g \in \nu$:

$$(x \cdot g(\bar{y})) + (g(\bar{z}) - x)) \approx g((x \cdot y_1) + (z_1 - x), \dots, (x \cdot y_n) + (z_n - x)).$$

One-pointed discriminator varieties coincide with varieties of 0-regular right-handed SBAs with additional operations (see [14]).

We conclude the section by introducing the notion of a semicentral right Church algebra. In [14] it is shown that the variety of right-handed SBAs with additional operations is term equivalent to the variety of semicentral right Church algebras.

Definition 18. A pointed algebra $\mathbf{A}$ of type ν is called a *semicentral right Church algebra* (*SRCA*, for short) if it admits a ternary term operation q satisfying the following identities:

$$q(0, x, y)) \approx y. \qquad q(x, x, 0)) \approx x. \qquad q(x, y, y)) \approx y.$$
$$q(w, q(w, x, y), z)) \approx q(w, x, z)) \approx q(w, x, q(w, y, z)).$$
$$\forall w, q(w, -, -) : A \times A \to A \text{ is a homomorphism w.r.t. the } \nu\text{-operations.}$$

Lemma 19. *Let* $\mathbf{A}$ *be an* SRCA*. Then* $\mathbf{A}$ *is a directly indecomposable if, and only if, for all* $x, y, z \in A$*:*

$$q(x, y, z) = \begin{cases} y & \text{if } x \neq 0; \\ z & \text{if } x = 0. \end{cases}$$

3. Factor Algebras

The aim of this section is introducing the concept of a factor algebra and proving some basic results thereabout.

Definition 20. An algebra $\mathbf{A}$ of a given type ν is a *factor algebra* if there is a quaternary term u of type ν (called a *factor term*) s.t. the following condition is satisfied for all $x, y \in A$:

$$\forall ab(u(x, y, a, b) = a) \ \veebar \ \forall ab(u(x, y, a, b) = b).$$

Observe that in any factor algebra $\mathbf{A}$ the binary relation

$$R_u = \big\{(a,b) : u^{\mathbf{A}}(a,b,c,d) = c \text{ for all } c,d \in A\big\}$$

is well-defined. The function $u^{\mathbf{A}}$ and the relation R_u are interdefinable. (Note that we could define another binary relation

$$S_u = \big\{(a,b) : u^{\mathbf{A}}(a,b,c,d) = d \text{ for all } c,d \in A\big\}$$

but then $R_u \cup S_u = A^2$ and $R_u \cap S_u = \emptyset$, so $S_u = A^2 \setminus R_u$ and then the relation S_u is rendundant). The relation R_u is called the *factor relation* of u, while the term operation $u^{\mathbf{A}}$ is called the *factor function* of $\mathbf{A}$. We denote by $\overline{R}_u$ the relation $A^2 \setminus R_u$.

Discriminator algebras are, of course, factor algebras where the switching function acts as a factor function; in this particular case $R_u = \Delta$.

A factor algebra $\mathbf{A}$ is:

- *proper* if $\emptyset \subset R_u \subset A^2$;
- *degenerate* if either $R_u = \emptyset$ or $R_u = A^2$;
- *reflexive* (resp. *co-reflexive*) if $\Delta \subseteq R_u$ (resp. $\Delta \subseteq \overline{R}_u$);
- *diagonal* if $R_u \subseteq \Delta$*;*
- *antisymmetric* (resp. *symmetric*, *transitive*, *compatible, equivalential, congruential*, *ordered*) if the factor relation is antisymmetric (resp. symmetric, transitive, compatible, an equivalence, a congruence, a partial ordering).

If $\bar{a} = a_0, \dots, a_{n-1}$ is a sequence and $0 \le i \le n-1$, then we write $\bar{a}[b/i]$ for the sequence $a_0, \dots, a_{i-1}, b, a_{i+1}, \dots, a_{n-1}$.

Definition 21. Let $\mathbf{A}$ be a factor algebra. We say that *a pair* $(b,c) \in A^2$ *splits* $\mathbf{A}$ if there exists $\bar{a} = (a_0, a_1) \in A^2$ and $0 \le i \le 1$ such that $u(\bar{a}[b/i], x, y) = u(\bar{a}[c/i], y, x)$ for all x, y.

Note that a pair $(b_0, b_1) \in A^2$ splits $\mathbf{A}$ if there exists $a \in A$ and $0 \le i \le 1$ such that either $((a, b_i) \in R_u \Leftrightarrow (a, b_{1-i}) \in \overline{R}_u)$ or $((b_i, a) \in R_u \Leftrightarrow (b_{1-i}, a) \in \overline{R}_u)$.

Lemma 22. *Let* $(a_1, a_2), (b_1, b_2) \in A^2$*. If* $\mathbf{A} \models u(a_1, a_2, x, y) = u(b_1, b_2, y, x)$*, then there exists* $1 \le i \le 2$ *such that* (a_i, b_i) *splits* $\mathbf{A}$*.*

Proof. If $a_1 \ne b_1$ and $u(a_1, a_2, x, y) = u(b_1, a_2, y, x)$ then the pair (a_1, b_1) splits $\mathbf{A}$. Otherwise, we have $u(a_1, a_2, x, y) = u(b_1, a_2, x, y)$; thus $u(b_1, a_2, x, y) = u(b_1, b_2, y, x)$ and (a_2, b_2) splits $\mathbf{A}$. $\square$

A pair is *unsplitting* if it does not split $\mathbf{A}$. We denote by $\varepsilon_{\mathbf{A}}$ the set of all unsplitting pairs of $\mathbf{A}$.

Proposition 23. *Let $\varepsilon_{\mathbf{A}}$ be the set of all unsplitting pairs.*

(1) The set $\varepsilon_{\mathbf{A}}$ is a congruence of the algebra $(A, u^{\mathbf{A}})$.

(2) $\varepsilon_{\mathbf{A}} = A \times A$ if, and only if, $\mathbf{A}$ is degenerate.

(3) Every proper congruence of $\mathbf{A}$ is contained within $\varepsilon_{\mathbf{A}}$ (In symbols, $\bigcup \mathrm{Con}(\mathbf{A})^* \subseteq \varepsilon_{\mathbf{A}}$).

(4) If R_u is reflexive, then $\varepsilon_{\mathbf{A}} \subseteq R_u$.

(5) If R_u is an equivalence relation, then $\varepsilon_{\mathbf{A}} = R_u$.

(6) If $R_u \neq \nabla$ is a congruence, then $\varepsilon_{\mathbf{A}} = R_u$ is the unique coatom of $\mathrm{Con}(\mathbf{A})$.

(7) If R_u is antisymmetric, then $(\varepsilon_{\mathbf{A}} \setminus \Delta) \subseteq \overline{R}_u$.

Proof. (1) The relation $\varepsilon_{\mathbf{A}}$ is trivially an equivalence relation. We now show that it is compatible with respect to u. If $a_i \varepsilon_{\mathbf{A}} b_i$ and $x_i \varepsilon_{\mathbf{A}} y_i$ $(i = 0, 1)$, then by Lemma 22 $u(a_0, a_1, x_0, x_1) = x_i$ iff $u(b_0, b_1, y_0, y_1) = y_i$.

(2) By Lemma 22.

(3) Let ϕ be a proper congruence. If $b\phi c$ and (b, c) splits $\mathbf{A}$, then there exists $\bar{a} \in A^2$ and i such that $u(\bar{a}[b/i], x_0, x_1) = u(\bar{a}[c/i], x_1, x_0)$. Then we get

$$x_j = u(\bar{a}[b/i], x_0, x_1) \phi u(a[c/i], x_0, x_1) = x_{1-j};$$

thus $\phi = \nabla$.

(4) If $(b, c) \in \overline{R}_u$ then by $(b, b) \in R_u$ we get $x = u(b, b, x, y) = u(b, c, y, x)$. It follows that (b, c) splits $\mathbf{A}$. Then $\varepsilon_{\mathbf{A}} \subseteq R_u$.

(5) If $(b, c) \in R_u$ splits $\mathbf{A}$ then there exist $\bar{a} = (a_0, a_1) \in A^2$ and i such that $u(\bar{a}[b/i], x_0, x_1) = u(\bar{a}[c/i], x_1, x_0)$. Without loss of generality, we assume $i = 1$. Then $(a_0, b) \in R_u$ iff $(a_0, c) \in \overline{R}_u$. By simmetry and transitivity of R_u and by $(b, c) \in R_u$ we get the contradiction $(a_0, b) \in R_u$ iff $(a_0, c) \in R_u$.

(6) By (3).

(7) If $(a, b) \in R_u$ with $a \neq b$ then $(b, a) \in \overline{R}_u$ and $u(a, b, x, y) = u(b, a, y, x)$. From Lemma 22 it follows that (a, b) splits $\mathbf{A}$. □

Proposition 24. *If $\mathbf{A}$ is a directly decomposable factor algebra then $\mathbf{A}$ is degenerate.*

Proof. Let $(\phi, \bar{\phi})$ be a pair of nontrivial complementary factor congruences. By Proposition 23(3) we have $\phi \cup \bar{\phi} \subseteq \varepsilon_{\mathbf{A}}$. Assume, by the way of contradiction, that $\mathbf{A}$ is not degenerate. Then by Proposition 23(2) there exist $c, d \in A$ such that (c, d) splits $\mathbf{A}$. This means that, for example, $u(a, c, x, y) = u(a, d, y, x)$ for some $a \in A$ and all x, y. Since $\phi \circ \bar{\phi} = \nabla$, then there exists z such that $c\phi z\bar{\phi}d$. Since $(c, z), (z, d) \in \varepsilon_{\mathbf{A}}$ and $\varepsilon_{\mathbf{A}}$ is an equivalence, then we get the contradiction $(c, d) \in \varepsilon_{\mathbf{A}}$. ☐

4. Factor Varieties

In full analogy with the case of discriminator algebras, it is natural to lift the concept of factor algebra to the level of varieties, along the following lines:

Definition 25. If $\mathcal{K}$ is a class of factor algebras with a common factor term u, the variety $\mathcal{V}(\mathcal{K})$ generated by $\mathcal{K}$ is called a *factor variety.*

If $\mathcal{V}$ is a factor variety, then it is consistent to mimick our previous notational conventions and denote by $\mathcal{V}_{\mathrm{fa}}$ the class of factor algebras in $\mathcal{V}$. Henceforth, we will also say that a factor variety $\mathcal{V}$ has the property P (e.g. reflexivity) if the factor relation of every subdirectly irreducible member of $\mathcal{V}$ has the property P.

Next, we list some examples of factor varieties.

Example 26. [25] Let $\mathcal{V}$ be a variety whose type ν includes a unary term $\Box$. Moreover, suppose that $\Box$ is $\mathcal{V}$-idempotent and $\mathcal{V}$-compatible. $\mathcal{V}$ is a quasi-discriminator variety w.r.t. $\Box$ if there exists a quaternary term s such that, for every subdirectly irreducible member $\mathbf{A}$ of $\mathcal{V}$ and for all $a, b, c, d \in A$,

$$s(a, b, c, d) = \begin{cases} c & \text{if } \Box a = \Box b; \\ d & \text{otherwise.} \end{cases}$$

Quasi-discriminator varieties include discriminator varieties as well as many non-discriminator examples, like (1) Glivenko MTL algebras with the Boolean retraction property [12], hence in particular Gödel algebras, Product algebras [19], and the variety of MV algebras generated by Chang's algebra [11]; (2) regular Nelson residuated lattices [9]; (3) Jauch-Piron orthomodular lattices with states [15].

Clearly, every quasi-discriminator variety is a congruential factor variety, with s as a factor term and $\ker \Box = R_s$.

Example 27. Varieties $\mathcal{V}$ of SBlAs (§ 2.2), where c is $\mathcal{V}$-compatible, are congruential factor varieties with $u(x, y, a, b) = q(c(x) \oplus c(y), b, a)$ and $R_u = \ker c$.

Example 28. For $k \geq 2$, let $\mathbf{MO}_k$ be the modular ortholattice of height 3 with $2k$ atoms [8, 27, 2]. The subdirectly irreducible members of $V(\mathbf{MO}_k)$ are the 2-element Boolean algebra and $\mathbf{MO}_l$, for all l s.t. $2 \leq l \leq k$. Recall that in any ortholattice the *commutation relation* C is defined as follows:

$$C = \{(a, b) : a = (a \wedge b) \vee (a \wedge b')\},$$

while the *commutator* of the elements a and b is

$$\gamma(a, b) = (a \wedge b) \vee (a \wedge b') \vee (a' \wedge b') \vee (a' \wedge b).$$

Now, it is easy to see that in any $\mathbf{MO}_k$ we have that

$$C = \{(a, b) : a \in \{0, 1\} \text{ or } b \in \{0, 1\}\},$$

and that

$$\gamma(a, b) = \begin{cases} 0 \text{ if } aCb \\ 1 \text{ otherwise.} \end{cases}$$

Therefore, for all $k \geq 2$, $V(\mathbf{MO}_k)$ is a symmetric factor variety with factor term

$$u(x, y, w, z) = (\gamma(x, y) \wedge z) \vee \left(\gamma(x, y)' \wedge w\right).$$

Observe that, if $\mathcal{V}$ is a factor variety, then $\mathcal{V}_{\mathrm{fa}}$ is axiomatised, relative to $\mathcal{V}$, by the universal formula

$$\forall xy[\forall zv(u(x, y, z, v) = z) \veebar \forall zv(u(x, y, z, v) = v)],$$

whence

$$\mathcal{V}_{\mathrm{s}} \subseteq \mathcal{V}_{\mathrm{si}} \subseteq \mathcal{V}_{\mathrm{di}} \subseteq \mathcal{V}_{\mathrm{fa}}.$$

In an arbitrary factor variety, however — as witnessed by the examples mentioned in Example 26 — the classes of factor algebras, simple members, subdirectly irreducible members, and directly indecomposable members need not coincide with one another, a fact that marks a contrast with the discriminator case. In some special cases, nonetheless, we are in a position to regain, at least in part, this property.

Lemma 29. $\mathcal{V}_{\mathrm{fa}}$ *is closed under subalgebras, ultraproducts and homomorphic images.*

Proof. Since $\mathcal{V}_{\mathrm{fa}}$ is a universal class, then it is closed under subalgebras and ultraproducts. Let $\mathbf{A}$ be a nontrivial factor algebra and ϕ be a proper congruence. Assume, by the way of contradiction, that $\mathbf{A}/\phi$ is not a factor algebra. Then there exist $x, y, a, b \in A$ such that $a/\phi \neq b/\phi$, $c/\phi \neq d/\phi$, $u(x/\phi, y/\phi, a/\phi, b/\phi) = a/\phi$, and $u(x/\phi, y/\phi, c/\phi, d/\phi) = d/\phi$. Then $u(x, y, a, b) = a$ and $u(x, y, c, d) = d$. Contradiction. □

Proposition 30. *(1) If a factor variety $\mathcal{V}$ is antisymmetric and reflexive, then $\mathcal{V}$ is semisimple and $\mathcal{V}_{\mathrm{fa}} = \mathcal{V}_{\mathrm{di}} = \mathcal{V}_{\mathrm{si}} = \mathcal{V}_{\mathrm{s}}$ is a universal class.*

(2) If a factor variety $\mathcal{V}$ is proper and congruential, then $\mathcal{V}_{\mathrm{di}} = \mathcal{V}_{\mathrm{fa}}$ is a universal class and every subdirectly irreducible algebra in the variety is either simple or has a unique coatom in its congruence lattice.

Proof. (1) Let $\mathbf{A} \in \mathcal{V}$ be a factor algebra. Then, by Proposition 23(4-6) we have that $\varepsilon_{\mathbf{A}} = \Delta$; thus every pair (a, b) $(a \neq b)$ splits $\mathbf{A}$. It follows that $\mathbf{A}$ is simple. Then $\mathcal{V}$ is semisimple and $\mathcal{V}_{\mathrm{s}} \subseteq \mathcal{V}_{\mathrm{si}} \subseteq \mathcal{V}_{\mathrm{di}} \subseteq \mathcal{V}_{\mathrm{fa}} \subseteq \mathcal{V}_{\mathrm{s}}$; our observation above implies that $\mathcal{V}_{\mathrm{s}}$ is a universal class.

(2) By hypothesis and by Proposition 23(5) $\varepsilon_{\mathbf{A}} = R_u$ is a congruence in every factor algebra $\mathbf{A} \in \mathcal{V}$. Since $\mathbf{A}$ is proper, R_u is the unique coatom of $\mathrm{Con}(\mathbf{A})$. Then $\mathcal{V}_{\mathrm{di}} = \mathcal{V}_{\mathrm{fa}}$ and, again, $\mathcal{V}_{\mathrm{di}}$ is a universal class. Any nonsimple subdirectly irreducible member of $\mathcal{V}$ is a factor algebra and thus has a unique coatom in its congruence lattice. ☐

Example 31. The concept of an ordered algebra is a direct generalisation of the classical notion of an algebra (see [6]): an ordered algebra of type ν is a pair $(\mathbf{A}, \leq_{\mathbf{A}})$, where $\mathbf{A}$ is an algebra of type ν, $(A, \leq_{\mathbf{A}})$ is a poset and $f^{\mathbf{A}} : A^n \to A$ is a monotone operation for every $f \in \nu$ of arity n. There exists a bijective correspondence between the class of ordered algebras $(\mathbf{A}, \leq_{\mathbf{A}})$ of type ν and the class of ordered factor algebras $(\mathbf{A}, u^{\mathbf{A}})$ of type $\nu' = \nu \cup \{u\}$, where $u^{\mathbf{A}}$ is the factor function defined as follows:

$$u^{\mathbf{A}}(x, y, z, t) = \begin{cases} z & \text{if } x \leq_{\mathbf{A}} y; \\ t & \text{otherwise.} \end{cases}$$

By Proposition 30(1) the factor variety generated by all ordered factor algebras of type ν' is semisimple.

4.1. **The ternary factor term.** Discriminator varieties admit another equivalent definition (actually, the customary one) in terms of a *ternary discriminator*, to be used in place of the switching term. An analogous possibility is available for the framework of equivalential factor varieties.

Definition 32. Let $\mathbf{A}$ be an algebra of type ν. A function $t^{\mathbf{A}} : A^3 \to A$ is called a *ternary factor function* if there is a relation $R_{t^{\mathbf{A}}}$ such that

$$t^{\mathbf{A}}(x, y, z) = \begin{cases} z & \text{if } (x, y) \in R_{t^{\mathbf{A}}}; \\ x & \text{otherwise.} \end{cases}$$

The relation $R_{t^{\mathbf{A}}}$ is called the *ternary factor relation associated with* $t^{\mathbf{A}}$. We call a ternary factor function *equivalential* iff its associated factor relation is such. A ternary term t of type ν that realises the ternary factor function on $\mathbf{A}$ is called the *ternary factor term for* $\mathbf{A}$.

Lemma 33. *An algebra* $\mathbf{A}$ *is an equivalential factor algebra iff it has an equivalential ternary factor function.*

Proof. ($\Leftarrow$) Given a ternary factor term t, define quaternary terms p and q by

$p(x, y, z, w) = t(w, t(x, y, w), t(x, y, z))$ and $q(x, y, z, w) = t(w, t(y, x, w), t(y, x, z))$.

Then it can be checked that

$$u(x, y, z, w) = t(p(x, y, z, w), x, q(x, y, z, w))$$

is the quaternary factor term.
($\Rightarrow$) Just set, as in the case of discriminator varieties, $t(x, y, z) = u(x, y, z, x)$. $\square$

5. Correspondence Theory

The term "correspondence theory" is widely employed, in the Kripke-style semantics for modal logic, to describe associations between modal axioms and formulas of a first order language that includes individual variables for possible worlds and a binary predicate for the accessibility relation of Kripke frames (see e.g. [1], [18]). Below, we try to develop something vaguely related for factor varieties — namely, we match properties of the factor relation R_u and equational conditions involving the factor term u.

By Theorem 1 discriminator varieties can be equationally described in terms of conditions on the switching term. It is therefore to be expected that an equational characterisation is available for factor varieties as well. Indeed, this is precisely the case. We have that:

Proposition 34. *The following conditions are equivalent for a variety* $\mathcal{V}$*:*

(1) $\mathcal{V} = \mathcal{V}(\mathcal{K})$ *is a factor variety (resp. reflexive factor variety, diagonal factor variety, discriminator variety) w.r.t. a class* $\mathcal{K}$ *of factor algebras with a common factor term* u*;*

(2) $\mathcal{V}$ *satisfies* (D_2-D_4) *(resp.* D_1-D_4*,* D_2-D_5*,* D_1-D_5*).*

Proof. ($1 \Rightarrow 2$) Let $\mathbf{A} \in \mathcal{K}$ be a factor algebra. We show that the identities (D_2-D_4) are satisfied by $\mathbf{A}$.

(D_2) $u(b, c, a, a) = a$ because the factor function projects either to its third or to its fourth component.

(D_3) If $(x, y) \notin R_u$ then $u(x, y, a, u(x, y, b, c)) = u(x, y, a, c) = u(x, y, u(x, y, a, b), c) = c$. If $(x, y) \in R_u$ then $u(x, y, u(x, y, a, b), c) = u(x, y, a, c) = u(x, y, a, u(x, y, b, c)) = a$.

(D_4) If $(x, y) \notin R_u$ then $u(x, y, \sigma(\overline{a}), \sigma(\overline{b})) = \sigma(u(x, y, a_1, b_1), \dots, u(x, y, a_n, b_n)) = \sigma(\overline{b})$. If $(x, y) \in R_u$ then $u(x, y, \sigma(\overline{a}), \sigma(\overline{b})) = \sigma(u(x, y, a_1, b_1), \dots, u(x, y, a_n, b_n)) = \sigma(\overline{a})$.

Moreover, if R_u is reflexive, then by definition of factor function $u(c, c, a, b) = a$ for all $a, b, c \in A$, and therefore D_1 is satisfied. If $R_u \subseteq \Delta$, then $u(a, b, a, b) = b$ for all $a, b \in A$, taking care of D_5.

$(2 \Rightarrow 1)$ Let $\mathbf{A} \in \mathcal{V}_{\mathrm{di}}$ and let $x, y \in A$. Since the operation $f(a, b) = u(x, y, a, b)$ is a decomposition operation and $\mathbf{A}$ is d.i., then either $\{(a, b) : u(x, y, a, b) = b\} = A^2$ or $\{(a, b) : u(x, y, a, b) = a\} = A^2$. Moreover, if $u(c, c, a, b) = a$ for all $c \in A$, then $(c, c) \in R_u$ for all c, so that R_u is reflexive. If $u(a, b, a, b) = b$ for all $a, b \in A$, then $(a, b) \in \overline{R}_u$ for all $a \neq b$, so that $R_u \subseteq \Delta$. □

Proposition 34 gives an equational characterisation of four classes of factor varieties. Notice however that the identities (D_1-D_4) are satisfied by setting u to be the projection onto the third argument, and (D_2-D_5) are satisfied by setting u to be the projection onto the fourth argument. This shows that every variety $\mathcal{V}$ is a diagonal factor variety as well as a reflexive factor variety, so it suggests that a further narrowing of the notion is necessary. The notion of a factor variety is a terminological convenience, to be supplemented by further demands in the main results of the paper.

Corollary 35. *Let $\mathbf{A}$ be an algebra in a factor variety and $x, y \in A$. Then we have for all $a_0, a_1 \in A$:*

$$u(x, y, a_0, a_1) = a_i \Leftrightarrow u(x, y, a_1, a_0) = a_{1-i}$$

Proof. It follows from Proposition 34 because $f(a_0, a_1) = u(x, y, a_0, a_1)$ is a decomposition operation and $\{(a_0, a_1) : u(x, y, a_0, a_1) = a_i\}$ $(i = 0, 1)$ is a pair of complementary factor congruences. □

Proposition 34 provides us with a start. The next proposition yields additional matches.

Proposition 36. *Let $\mathcal{V}$ be a factor variety.*

(1) $\mathcal{V}$ is symmetric iff $\mathcal{V} \models u(x,y,z,w) \approx u(y,x,z,w)$;

(2) $\mathcal{V}$ is transitive iff

$$\mathcal{V} \models u(u(x,y,x,y), u(y,z,z,y), w_1, u(x,y,w_2,w_1)) \approx w_1;$$

(3) $\mathcal{V}$ is antisymmetric iff

$$\mathcal{V} \models u(x,y,x,y) \approx u(y,x,y,u(x,y,x,y));$$

(4) if $\mathcal{V}$ is equivalential, then $\mathcal{V}$ is congruential iff, for any n-ary term f, any $i \leq n$, and sequences $\bar{s} = s_1 \ldots s_{i-1}$, $\bar{t} = t_{i+1} \ldots t_n$ of distinct variables, we have

$$\mathcal{V} \models f(\bar{s}, u(x,y,z,w), \bar{t}) \approx u(f(\bar{s},x,\bar{t}), f(\bar{s},y,\bar{t}), u(x,y,f(\bar{s},z,\bar{t}), f(\bar{s},w,\bar{t})), f(\bar{s},w,\bar{t}))$$

(5) if $\mathcal{V}$ is ordered, then $\mathcal{V}$ is compatible at coordinate $i \leq k$ of the function symbol g of arity k iff

$$\mathcal{V} \models u(g(z_1,\ldots,z_{i-1},x,z_{i+1},\ldots,z_k), g(z_1,\ldots,z_{i-1},y,z_{i+1},\ldots,z_k), w, u(x,y,t,w)) = w.$$

Proof. (1) Suppose that $\mathcal{V}$ is symmetric and $\mathbf{A} \in \mathcal{V}_{\mathrm{fa}}$. Notice that, if R_u is symmetric, then $\overline{R}_u$ is also symmetric. Then, if $(a,b) \in R_u$, $c = u(a,b,c,d) = u(b,a,c,d)$. A similar reasoning works if $(a,b) \in \overline{R}_u$. Conversely, if $(a,b) \in \overline{R}_u$, then $c = u(a,b,c,d) = u(b,a,c,d)$, i.e. $(b,a) \in \overline{R}_u$.

(2) Let $\mathcal{V}$ be transitive and $\mathbf{A} \in \mathcal{V}_{\mathrm{fa}}$. Then, for $a,b,c,d,e \in A$, the following cases are possible:

- if $(a,b),(b,c) \in R_u$, then by transitivity $u(u(a,b,a,b), u(b,c,c,b), d, u(a,b,e,d)) = u(a,c,d,e) = d$;
- if $(a,b) \in R_u$ and $(b,c) \notin R_u$, then $u(u(a,b,a,b), u(b,c,c,b), d, u(a,b,e,d)) = u(a,b,d,e) = d$;
- if $(a,b) \notin R_u$ and $(b,c) \in R_u$, then $u(u(a,b,a,b), u(b,c,c,b), d, u(a,b,e,d)) = u(b,c,d,d) = d$;
- if $(a,b),(b,c) \notin R_u$, then $u(u(a,b,a,b), u(b,c,c,b), d, u(a,b,e,d)) = u(b,b,d,d) = d$.

Conversely, suppose $\mathbf{A} \in \mathcal{V}_{\mathrm{fa}}$ is s.t. R_u is not transitive. Then, for $a,b,c \in A$, $(a,b),(b,c) \in R_u$, but $(a,c) \notin R_u$. Now, if $\mathbf{A}$ is nontrivial, we can consider distinct $d,e \in A$ and argue as follows: $u(u(a,b,a,b), u(b,c,c,b), d, u(a,b,e,d)) = u(a,c,d,e) = e \neq d$.

(3) Let $\mathcal{V}$ be antisymmetric and $\mathbf{A} \in \mathcal{V}_{\mathrm{fa}}$. Then, for $a, b \in A$, the following cases are possible:

- if $(a,b),(b,a) \in R_u$, then $u(a,b,a,b) = a$ and $u(b,a,b,u(a,b,a,b)) = b$, and by antisymmetry $a = b$;
- if $(a,b) \in R_u$ and $(b,a) \notin R_u$, then $u(a,b,a,b) = a$, and $u(b,a,b,u(a,b,a,b)) = u(a,b,a,b) = a$;
- if $(a,b) \notin R_u$ and $(b,a) \in R_u$, then $u(a,b,a,b) = b$, and $u(b,a,b,u(a,b,a,b)) = b$;
- if $(a,b),(b,a) \notin R_u$, then $u(a,b,a,b) = b$, and $u(b,a,b,u(a,b,a,b)) = b$.

On the other hand, suppose $\mathbf{A} \in \mathcal{V}_{\mathrm{fa}}$ is s.t. R_u is not antisymmetric. Then, for some distinct $a, b \in A$, $u(a,b,a,b) = a$, and $u(b,a,b,u(a,b,a,b)) = b$.

(4) Let $\mathcal{V}$ be congruential and $\mathbf{A} \in \mathcal{V}_{\mathrm{fa}}$. Then, for $a, b \in A$ and f a unary polynomial over $\mathbf{A}$, the following cases are possible:

- if $(a,b),(f(a),f(b)) \in R_u$, then $f(u(a,b,c,d)) = f(c)$ and
$$u(f(a),f(b),u(a,b,f(c),f(d)),f(d)) = u(a,b,f(c),f(d)) = f(c);$$
- if$(a,b) \notin R_u$, $(f(a),f(b)) \in R_u$, then $f(u(a,b,c,d)) = f(d)$ and also
$$u(f(a),f(b),u(a,b,f(c),f(d)),f(d)) = f(d).$$
- if $(a,b),(f(a),f(b)) \notin R_u$, then $f(u(a,b,c,d)) = f(d)$ and
$$u(f(a),f(b),u(a,b,f(c),f(d)),f(d)) = f(d).$$

On the other hand, let $\mathbf{A} \in \mathcal{V}_{\mathrm{fa}}$ be s.t. R_u is equivalential but not congruential. So, there are distinct $a, b \in A$ and a unary polynomial f over $\mathbf{A}$ such that $(a,b) \in R_u$ and $(f(a),f(b)) \notin R_u$. Since f yields a counterexample to congruentiality, then there are $c, d \in A$ for which $f(c) \neq f(d)$. Therefore,
$$f(u(a,b,c,d)) = f(c) \neq f(d) = u(f(a),f(b),u(a,b,f(c),f(d)),f(d)).$$

(5) Let $\mathcal{V}$ be ordered and compatible. We show that $\mathcal{V}$ satisfies the identity
$$u(g(z_1,\ldots,z_{i-1},x,z_{i+1},\ldots,z_k),g(z_1,\ldots,z_{i-1},y,z_{i+1},\ldots,z_k),w,u(x,y,t,w)) = w.$$
If $(x,y) \notin R_u$ then the conclusion is trivial because $u(x,y,t,w) = w$. If $(x,y) \in R_u$ then by hypothesis $(g(z_1,\ldots,z_{i-1},x,z_{i+1},\ldots,z_k),g(z_1,\ldots,z_{i-1},y,z_{i+1},\ldots,z_k)) \in R_u$, so that we get the conclusion.

Let $\mathcal{V}$ be ordered satisfying the above identity. We show that g is compatible at coordinate i. Let $(x, y) \in R_u$. Then $u(x, y, t, w) = t$, so that we have:
$u(g(z_1, \dots, z_{i-1}, x, z_{i+1}, \dots, z_k), g(z_1, \dots, z_{i-1}, y, z_{i+1}, \dots, z_k), w, t) = w$. Then $(g(z_1, \dots, z_{i-1}, x, z_{i+1}, \dots, z_k), g(z_1, \dots, z_{i-1}, y, z_{i+1}, \dots, z_k)) \in R_u$. □

6. Pointed factor varieties

Recall that a double-pointed variety is a variety with two residually distinct constants $0, 1$ in its type. In the double-pointed case, some interesting connections between factor and semi-Boolean-like varieties can be pointed out.

Definition 37. A double-pointed variety is called a $(1, 0)$-factor variety if it is a factor variety satisfying one of the following equivalent conditions:

(1) $\mathcal{V} \models u(1, 0, y, x) = x = u(0, 0, x, y)$.

(2) $(0, 0) \in R_u$ and $(1, 0) \in \overline{R}_u$ in every factor member of $\mathcal{V}$.

A semi-Boolean-like variety is called a *d-semi-Boolean-like variety* if there is a binary term d satisfying the following conditions, for every $\mathbf{A} \in \mathcal{V}$, and every $x, y \in A$:

$$c^{\mathbf{A}}(d^{\mathbf{A}}(x, y)) = d^{\mathbf{A}}(x, y); \quad d^{\mathbf{A}}(0, 0) = 0 \text{ and } d^{\mathbf{A}}(1, 0) = 1.$$

By Proposition 9 the identity $c^{\mathbf{A}}(d^{\mathbf{A}}(x, y)) = d^{\mathbf{A}}(x, y)$ implies that $d^{\mathbf{A}}(x, y)$ is a central element.

Theorem 38. *A double-pointed variety $\mathcal{V}$ is a $(1, 0)$-factor variety iff it is a d-semi-Boolean-like variety. Moreover we have that $\mathcal{V}$ is:*

(1) a reflexive $(1, 0)$-factor variety iff it is a d-semi-Boolean-like variety satisfying $d(x, x) \approx 0$.

(2) a symmetric $(1, 0)$-factor variety iff it is a d-semi-Boolean-like variety satisfying $d(x, y) \approx d(y, x)$.

(3) a transitive $(1, 0)$-factor variety iff it is a d-semi-Boolean-like variety satisfying $d(x, y) \approx 0, d(y, z) \approx 0 \Rightarrow d(x, z) \approx 0$.

(4) a diagonal $(1, 0)$-factor variety iff it is a d-semi-Boolean-like variety satisfying $d(x, y) \approx 0 \Rightarrow x \approx y$.

(5) a double-pointed discriminator variety iff it is a d-semi-Boolean-like variety satisfying $d(x, y) \approx 0 \Rightarrow x \approx y$ and $d(x, x) \approx 0$ (i.e., 0-regular).

Proof. ($\Rightarrow$) Define $q(e,x,y) = u(e,0,y,x)$. Then by hypothesis we have $q(0,x,y) = u(0,0,y,x) = y$ and $q(1,x,y) = u(1,0,y,x) = x$. Thus $\mathcal{V}$ is a Church variety. By (D_2) we have $q(e,x,x) = u(e,0,x,x) = x$. By (D_3) we have $q(e,q(e,x,y),z) = u(e,0,z,u(e,0,y,x)) = u(e,0,z,x) = q(e,x,z)$. Similarly for the other identities characterizing SBlAs. Finally, define $d(x,y) = u(x,y,0,1)$. It is easy to show that

$$\begin{array}{rcll} c(d(x,y)) & = & u(d(x,y),0,0,1) & \\ & = & u(u(x,y,0,1),0,0,1) & \\ & = & u(x,y,u(0,0,0,1),u(1,0,0,1)) & \text{by (D4)} \\ & = & u(x,y,0,1) & \text{by hypothesis} \\ & = & d(x,y). & \end{array}$$

Moreover, we have: $d(0,0) = u(0,0,0,1) = 0$ and $d(1,0) = u(1,0,0,1) = 1$.

($\Leftarrow$) Define

$$u(x,y,z,w) = (d(x,y) \wedge z) \vee (d(x,y)' \wedge w),$$

where $x \vee y = q(x,1,y)$, $x \wedge y = q(x,y,0)$ and $x' = q(x,0,1)$, as in Conditions (1) on page 4. Since in every directly indecomposable algebra $\mathbf{A}$ of $\mathcal{V}$ we have that the sole central elements of $\mathbf{A}$ are $0,1$, then $d^{\mathbf{A}}(x,y) \in \{0,1\}$ and $\mathcal{V}$ is a factor variety.

The "moreover" part is easy. $\square$

Algebras in $(1,0)$-factor varieties are amenable to weak Boolean product representations.

Theorem 39. *Every algebra in a $(1,0)$-factor variety $\mathcal{V}$ is representable as a weak Boolean product of directly indecomposable algebras.*

Proof. By Theorem 38 an element $x \in \mathbf{A} \in \mathcal{V}$ is central if and only if $q(x,1,0) = u(x,0,0,1) = x$; thus the class of all subdirectly irreducible members of $\mathcal{V}$ is axiomatised by the universal formula $\forall x(q(x,1,0) = x \to (x = 1 \vee x = 0))$. Then the conclusion follows from Theorem 6. $\square$

In the one-pointed case, some interesting connections between factor varieties and the variety of skew Boolean algebras (see [4, 14]) can be pointed out.

Definition 40. A one-pointed variety is called a 0-*factor variety* if it is a factor variety satisfying one of the following equivalent conditions:

(1) $\mathcal{V} \models u(0,0,y,x) = y = u(y,0,0,y)$.

(2) $(0,0) \in R_u$ and $\forall y(y \neq 0 \Rightarrow (y,0) \in \overline{R}_u)$ in every factor member of $\mathcal{V}$.

A variety of skew Boolean algebras with additional operations is called a d-SBA-variety if there is a binary term d satisfying the following conditions, for every $\mathbf{A} \in \mathcal{V}$, and every $x \in A$: $d^{\mathbf{A}}(0,0) = 0$ and $d^{\mathbf{A}}(x,0) = x$.

Theorem 41. *A one-pointed variety $\mathcal{V}$ is a 0-factor variety iff it is a d-SBA-variety. Moreover we have that $\mathcal{V}$ is:*

(1) a reflexive 0-factor variety iff it is a d-SBA-variety satisfying $d(x,x) \approx 0$.

(2) a symmetric 0-factor variety iff it is a d-SBA-variety satisfying $d(x,y) \approx d(y,x)$.

(3) a transitive 0-factor variety iff it is a d-SBA-variety satisfying $d(x,y) \approx 0, d(y,z) \approx 0 \Rightarrow d(x,z) \approx 0$.

(4) a diagonal 0-factor variety iff it is a d-SBA-variety satisfying $d(x,y) \approx 0 \Rightarrow x \approx y$.

(5) a one-pointed discriminator variety iff it is a d-SBA-variety satisfying $d(x,y) \approx 0 \Rightarrow x \approx y$ and $d(x,x) \approx 0$.

Proof. In [14] the variety of skew Boolean algebras with additional operations has been shown equivalent to the variety of SRCAs (see Definition 18). Without loss of generality, in this proof we use SRCAs.

($\Rightarrow$) Define $q(e,x,y) = u(e,0,y,x)$. Then by hypothesis we have $q(0,x,y) = u(0,0,y,x) = y$ and $q(x,x,0) = u(x,0,0,x) = x$. From (D2)-(D4) it follows the other conditions defining semicentral right Church algebras. Finally, define $d(x,y) = u(x,y,0,x)$. By definition of 0-factor variety we immediately have $d(0,0) = 0$ and $d(x,0) = u(x,0,0,x) = x$.

($\Leftarrow$) Define
$$u(x,y,z,w) = q(d(x,y),w,z).$$
We have: $u(0,0,x,y) = q(d(0,0),y,x) = q(0,y,x) = x$ and $u(x,0,0,x) = q(d(x,0),x,0) = q(x,x,0) = x$. By Lemma 19 in every directly indecomposable algebra $\mathbf{A}$ of $\mathcal{V}$ we have that
$$q(x,y,z) = \begin{cases} y & \text{if } x \neq 0; \\ z & \text{if } x = 0. \end{cases}$$
Then in every directly indecomposable algebra $\mathbf{A}$ of $\mathcal{V}$ we have:
$$u(x,y,z,w) = \begin{cases} w & \text{if } d(x,y) \neq 0; \\ z & \text{if } d(x,y) = 0 \end{cases}$$
and $\mathcal{V}$ is a factor variety.

The "moreover" part is easy. □

7. Pure Factor Algebras

A *pure factor algebra* is a pair $(A, u^{\mathbf{A}})$, where A is a set and $u^{\mathbf{A}} : A^4 \to A$ is a factor function. In other words, there exists a binary relation $R_u \subseteq A^2$ such that:

$$u^{\mathbf{A}}(x, y, z, w) = \begin{cases} z & \text{if } (x, y) \in R_u; \\ w & \text{if } (x, y) \notin R_u. \end{cases}$$

Pure factor algebras are on a par with pure semi-Boolean-like algebras (§ 2.2) or with pure discriminator algebras (see e.g. [3] or [4]) insofar as they enjoy extra properties above and beyond those of ordinary factor algebras. In this short section, we make a note of a few facts about these algebras.

For a start, just observe that if $\mathbf{A}$ is a pure factor algebra with factor function u, then every subset of A is a subalgebra of $\mathbf{A}$, and

$$\theta(a, b) = \begin{cases} \nabla, & \text{if } (a, b) \text{ splits } \mathbf{A}; \\ \{(a, b), (b, a)\} \cup \Delta, & \text{otherwise.} \end{cases}$$

Therefore $\mathrm{Con}(\mathbf{A})$ is an atomic lattice, whose atoms are the principal congruences. Upon recalling that every algebra of cardinality 2 is simple, we further obtain from the previous observation that:

Proposition 42. *Let $\mathbf{A}$ be a pure factor algebra of cardinality > 2. Then we have:*

(1) $\mathbf{A}$ is simple iff every pair (a, b) $(a \neq b)$ splits $\mathbf{A}$ iff $\varepsilon_{\mathbf{A}} = \Delta$;

(2) If $\mathbf{A}$ is reflexive, then $\mathbf{A}$ is simple iff it is antisymmetric;

(3) If $\mathbf{A}$ is congruential, then $\mathbf{A}$ is simple iff $\mathbf{A}$ is a discriminator algebra.

A characterisation of subdirectly irreducible pure factor algebras is readily forthcoming.

Proposition 43. *Let $\mathbf{A}$ be a pure factor algebra. Then the following conditions are equivalent:*

(1) $\mathbf{A}$ is subdirectly irreducible;

(2) Either $\varepsilon_{\mathbf{A}} = \Delta$ or $\varepsilon_{\mathbf{A}} = \Delta \cup \{(a, b), (b, a)\}$ for some $a \neq b$;

(3) $\mathrm{Con}(\mathbf{A})$ is isomorphic to either the three-element chain or the two-element chain;

(4) *Either* $\mathbf{A}$ *is simple or there exists* a, b *such that* $A \setminus \{a, b\}$ *is a simple subalgebra.*

Proof. We just prove the equivalence between (1) and (2), because the remaining items immediately follow given our previous description of principal congruences. Let $\mathbf{A}$ be subdirectly irreducible but not simple, and let μ be its monolith. If a, b are distinct elements such that $(a, b) \in \mu$, then $\mu = \theta(a, b) = \Delta \cup \{(a, b), (b, a)\}$. Let $(c, d) \neq (a, b), (b, a)$. Since $(a, b), (b, a) \in \theta(c, d)$, then by the above characterization of $\theta(c, d)$ this is possible only if $\theta(c, d) = \nabla$. The converse is clear. □

We round off this subsection with two results concerning the *pure factor variety* $\mathcal{PF}$, generated by all pure factor algebras and its subvarieties.

Theorem 44. $\mathcal{PF}$ *is locally finite.*

Proof. By [26, Theorem 1], it suffices to show that there exists a fixed integer-valued function f such that, for each nonnegative integer n, it is the case that every n-generated subdirectly irreducible algebra has at most $f(n)$ elements. This is certainly the case for $\mathcal{PF}$, since any n-generated subdirectly irreducible algebra has cardinality n. □

Lemma 45. *Let* $\mathcal{V}$ *be a subvariety of* $\mathcal{PF}$. $\mathcal{V}$ *admits a non-trivial discriminator subvariety iff there exists a subdirectly irreducible member* $\mathbf{A}$ *of* $\mathcal{V}$ *s.t. for at least two distinct element* $a, b \in A$: $(a, b), (b, a) \notin R_u$.

Proof. If $\mathcal{V}$ admits a non-trivial discriminator subvariety, then there is at least a non-trivial subdirectly irreducible member $\mathbf{A}$ of $\mathcal{V}$ where u is the switching term. Hence, there are distinct elements $a, b \in A$ s.t. $u(a, b, x, y) = u(b, a, x, y) = y$, i.e. $(a, b), (b, a) \notin R_u$. Conversely, suppose that there exists a subdirectly irreducible member $\mathbf{A}$ of $\mathcal{V}$ s.t. for at least two distinct elements $a, b \in A$, $(a, b), (b, a) \notin R_u$. Then one can readily verify that the set $\{a, b\}$ is closed w.r.t. the factor function. Therefore, $\langle \{a, b\}, u \rangle$ is a discriminator algebra. □

References

[1] van Benthem J.,“Correspondence theory”, in D. Gabbay and F. Guenthner (Eds.), *Handbook of Philosophical Logic*, 1st ed., vol. 2. Reidel, Dordrecht, 1984.

[2] Berman J., ”Upper bounds on the sizes of finitely generated algebras”, *Demonstratio Mathematica*, 44, 3, 2011.

[3] Bignall R., “A non-commutative multiple-valued logic”, in *Multiple-Valued Logic, 1991, Proceedings of the Twenty-First International Symposium*, Victoria, BC, 1991, pp. 49-54.

[4] Bignall R.J., Leech J., "Skew Boolean algebras and discriminator varieties", *Algebra Universalis*, 33, 1995, pp. 387-398.

[5] Blok W.J., Pigozzi D., "On the structure of varieties with equationally definable principal congruences III", *Algebra Universalis*, 32, 4, 1994, pp. 545-608.

[6] Bloom S. L., "Varieties of ordered algebras", J. Comput. System Sci. 13(2), 1976, pp. 200-212.

[7] Burris S., Sankappanavar H. P., *A Course in Universal Algebra*, Graduate Text in Mathematics, Vol. 78. Springer Verlag, Berlin, 1981.

[8] Bruns G., Harding J., "Algebraic aspects of orthomodular lattices", in B. Coecke et al. (Eds.), *Current Research in Operational Quantum Logic: Algebras, Categories, Languages*, Kluwer, Dordrecht, 2000, pp. 37-65.

[9] Busaniche M., Cignoli R.,"Constructive logic with strong negation as a substructural logic", *Journal of Logic and Computation*, 20, 4, 2010, pp. 761-793.

[10] Chajda I., Halaš R., Rosenberg I.G., "Ideals and the binary discriminator in universal algebra", *Algebra Universalis,* 42, 1999, pp. 239–251.

[11] Cignoli R., D'Ottaviano I.M.L., Mundici D., *Algebraic Foundations of Many-Valued Reasoning*, Kluwer, Dordrecht, 1999.

[12] Cignoli R., Torrens A., "Free algebras in varieties of Glivenko MTL-algebras satisfying the equation $2(x^2) = (2x)^2$, *Studia Logica,* 83, 1-3, 2006, pp. 157-181.

[13] Comer S., "Representations by algebras of sections over Boolean spaces", *Pacific Journal of Mathematics*, 38, 1971, pp. 29-38.

[14] Cvetko-Vah K., Salibra A., "The connection of skew Boolean algebras and discriminator varieties to Church algebras", *Algebra Universalis* (to appear).

[15] Domenech G., Freytes H., de Ronde C., "Equational characterization for two-valued states in orthomodular quantum systems", *Reports on Mathematical Physics*, 68, 1, 2011, pp. 65-83.

[16] Fried E., Pixley A.F., "The dual discriminator function in universal algebra", *Acta Sci. Math.*, 41, 1979, pp. 83–100.

[17] Galatos N., Jipsen P., Kowalski T., Ono H., *Residuated Lattices: An Algebraic Glimpse on Substructural Logics*, Elsevier, Amsterdam, 2007.

[18] Goldblatt R., "First-order definability in modal logic", *Journal of Symbolic Logic*, 40, 1975, pp. 35–40.

[19] Hájek P., *Metamathematics of Fuzzy Logic*, Kluwer, Dordrect, 1998.

[20] Ledda A., Paoli F., Salibra A., "On Semi-Boolean like algebras", *Acta Univ. Palack. Olomuc.*, 52, 1, 2013, pp. 101-120.

[21] Leech J., "Skew lattices in rings", *Algebra Universalis*, 26, 1989, pp. 48–72.

[22] Leech J., "Recent developments in the theory of skew lattices", *Semigroup Forum*, 52, 1996, pp. 7-24.

[23] Manzonetto G., Salibra A., "From λ-calculus to universal algebra and back", in *MFCS'08*, volume 5162 of LNCS, 2008, pp. 479-490.

[24] McKenzie R.N., McNulty G.F., Taylor W.F., *Algebras, Lattices, Varieties, Volume I.* Wadsworth Brooks, Monterey, California, 1987.

[25] Paoli F., Ledda A., Kowalski T., Spinks M., "Quasi-discriminator varieties", submitted for publication.

[26] Raftery J., "Representable idempotent commutative residuated lattices", *Transactions of the AMS*, 359, 9, 2007, pp. 4405-4427.

[27] Roddy M., "Varieties of modular ortholattices", *Order*, 3, 4, 1987, pp. 405-426.

[28] Salibra A., Ledda A., Paoli F., Kowalski T., "Boolean-like algebras", *Algebra Universalis*, 69, 2, 2013, pp. 113-138.
[29] Spinks M., *On the Theory of Pre-BCK Algebras*, PhD Thesis, Monash University, 2003.
[30] Vaggione D., "Varieties in which the Pierce stalks are directly indecomposable", *Journal of Algebra*, 184, 1996, pp. 424-434.
[31] Vaggione D., "Equational characterization of the quaternary discriminator", *Algebra Universalis*, 43, 2000, pp. 99-100.
[32] Werner H., *Discriminator Algebras*, Studien zur Algebra und ihre Anwendungen, Band 6, Akademie-Verlag, Berlin, 1978.