Proximity-structured multivariate volatility models for systemic risk

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Abstract By describing the dependence structure through Granger Causality networks, we use the weights to define proximity matrices and accordingly we estimate a proximity structured BEKK model and derive a latent stability variable that could be interpreted as Systemic Risk indicator.

Key words: Systemic Risk, BEKK, Spatial Econometrics, Granger Causality

1 Introduction

The financial crisis has sparkled a renewed interest in understanding systemic risk. The approach considered in this paper is to link systemic risk with the stability of the financial system. We focus on estimating a latent variable that drives the stability of financial markets, and we describe them as a system with different interacting sectors. Due to the scarcity of publicly available data, we consider as a proxy of the dependence structure of the US financial system, a Granger causality network defined among financial stocks returns. The starting point of our effort are the results obtained in [1], since their measures are readily applicable, and they show a statistical relationship between them and anomalous market losses. To consider their networks, let us to build a model in which the dependence structure varies trough time. This is a key feature since different dependence structures can lead to stability or instability. This approach obliges us to face the curse of dimensionality problem if we want to consider a model including the whole set of institutions included in their study. For this reason we limit our analysis to Equally Weighted indexes for each of the four sectors considered, adjusting their connectedness measures accordingly. In addition, we use them as a series of weights matrices in a proximity structured volatility model [3] to reduce further the number of parameters, while retaining the possibility of spillovers. To properly discuss these kind of stable-unstable models

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it is crucial to ensure an ergodicity condition that allows statistical inference. By considering the s-BEKK model introduced in [3] and generalizing the ergodicity results obtained in [2], for the case with time varying parameters, it turns out that this condition could be imposed in a non trivial way, that allows the process to become unstable and for short time period the covariance matrix could experience exponential growth. The coincidence of those periods with anomalous market conditions allows us to interpret the latent stability variable as a systemic risk indicator and, its proximity to one, as an early warning for an incoming systemic event. The paper is structured as follows: in Section 2 we briefly review results in [1] and discuss how to define sector proximity matrices. In Section 3, we introduce our s-BEKK model with time varying parameters, discuss our main approximation and obtain ergodicity conditions. In Section 4 we report our empirical findings. Section 5 concludes with the interpretation of our stability variable as a systemic risk indicator, that can be helpful in the construction of an early warning system.

2 Granger causality networks as Sector proximity matrices

As detailed in [1], among all the measures proposed for connectedness, Granger Causality Network measures are the most concordant with losses and the ones that clearly show a dramatic increase during crisis periods. The implementation of a more statistically sounded technique considering a multivariate GARCH is difficult due to the rapidly increasing number of parameters depending on the number of series. For this reason we reduce the dimensionality of the problem by aggregating sectors in equally weighted indexes, but summarize the dependence structure in the sector proximity matrices, obtained from characteristics of all the series. Among the network measures used in [1] we focus on the following ones:

- Out Degree: number of outgoing edges/causalities from the node/institution
- Closeness: the inverse of the average shortest causality path from one node to each of the other

and introduce the sector-wide corresponding measures:

- Out Degree: number of outgoing edges/causalities from one sector to another one
- Closeness: the inverse of the average shortest causality path from all nodes of one sector to all the nodes of another one

These measures are two different ways of expressing how much the past of one sector is close to the present of the other, and thus can be interpreted as measure of similarity between sectors movements. By row-normalizing and zeroing the diagonal, we can construct proximity weights matrices (W_t) from them, that become the input of our proximity structured GARCH.

3 Proximity structured BEKK with time varying coefficients

Since we are going to use time varying weights matrices, we need to be careful to constrain them in order to be sure that the estimated model is ergodic.This is needed because we cannot obtain central limit theorems and reliable statitical inference procedures without this property.
In [3], the weights matrix W is constant and they introduce constant proximity matrices A and B:

$$\begin{aligned} A &= \operatorname{diag}(a_0) I + \operatorname{diag}(a_1) W \\ B &= \operatorname{diag}(b_0) I + \operatorname{diag}(b_1) W \end{aligned} \tag{1}$$

where I is the identity matrix. Among all the volatility specifications presented in [3] and for which they study identification and asymptotics, we choose the 1-lag s-BEKK:

$$\Sigma_t = \Omega + A R_{t-1} R_{t-1}' A' + B \Sigma_{t-1} B' \tag{2}$$

In a recent paper [2] is found that, in the case of the ordinary 1-lag BEKK, with constant A and B matrices,a unique and strictly stationary and geometrically ergodic solution exists if the classical stability condition is met:

$$\rho\left((A \otimes A) + (B \otimes B)\right) < 1 \tag{3}$$

Here $\rho(\cdot)$ is the spectral radius (the eigenvalue with the maximum absolute value) and $\otimes$ is the outer product. The cornerstone of their demonstration is the existence of the fixed point:

$$\Sigma = \mathbb{E}[\Sigma_t] = \Omega + A \mathbb{E}\left[R_{t-1} R_{t-1}'\right] A' + B \mathbb{E}[\Sigma_t] B' = \Omega + A \Sigma A' + B \Sigma B' \tag{4}$$

In our case, as already said, we have time varying weights matrices W_t and so also non constant proximity matrices A_t and B_t. Eventhough, those matrices come from the same dataset of the indexes, our working hypothesis would be that they represents different aspects of those series so that they can be considered uncorrelated to the covariances. To better express this proposition, it is convenient to introduce the vec representation of the model:

$$\begin{aligned} \Sigma_t &= \Omega + A_t R_{t-1} R_{t-1}' A_t' + B_t \Sigma_{t-1} B_t' \Leftrightarrow \\ \operatorname{vec}(\Sigma_t) &= \operatorname{vec}(\Omega) + (A_t \otimes A_t) \operatorname{vec}\left(R_{t-1} R_{t-1}'\right) + (B_t \otimes B_t) \operatorname{vec}(\Sigma_{t-1}) \end{aligned} \tag{5}$$

With this representation, our working hypothesis could be re-written as follows:

$$\begin{aligned} \mathbb{E}\left[(A_t \otimes A_t) \operatorname{vec}\left(R_{t-1} R_{t-1}'\right)\right] &\simeq \mathbb{E}[(A_t \otimes A_t)] \operatorname{vec}\left(\mathbb{E}\left[R_{t-1} R_{t-1}'\right]\right) \\ \mathbb{E}[(B_t \otimes B_t) \operatorname{vec}(\Sigma_{t-1})] &\simeq \mathbb{E}[(B_t \otimes B_t)] \operatorname{vec}(\mathbb{E}[\Sigma_{t-1}]) \end{aligned} \tag{6}$$

If we call $\tilde{A} = \mathbb{E}[(A_t \otimes A_t)]$ we can show that it is possible to construct a matrix $\bar{A}$ such that $\tilde{A} = \bar{A} \otimes \bar{A}$. This result comes from the properties of the vec operator that allows us to write: $\text{vec}^{[-1]}\left(\tilde{A}\text{vec}(I)\right) = \bar{A}\bar{A}'$ so that $\bar{A}$ can be obtained by the Cholesky decomposition of $\text{vec}^{[-1]}\left(\tilde{A}\text{vec}(I)\right)$.
Since the same is true for a matrix $\bar{B}$ coming from B_t, we can obtain the new fixed point:

$$\Sigma = \mathbb{E}[\Sigma_t] = \Omega + \mathbb{E}\left[A_{t-1}R_{t-1}R'_{t-1}A'_{t-1}\right] + \mathbb{E}\left[B_{t-1}\Sigma_t B'_{t-1}\right] = \Omega + \bar{A}\Sigma\bar{A} + \bar{B}\Sigma\bar{B}$$

and the results in [2] could be extended to time variyng uncorrelated A and B, implying the condition:

$$\rho\left(\left(\bar{A}\otimes\bar{A}\right) + \left(\bar{B}\otimes\bar{B}\right)\right) < 1. \tag{7}$$

In addition, since in our case A and B are proximity structures, our hypothesis reduces on the following three conditions on outer products of W_t

$$\begin{aligned}
\mathbb{E}\left[(I\otimes W_t)\,\text{vec}\left(R_{t-1}R'_{t-1}\right)\right] &\simeq \mathbb{E}[(I\otimes W_t)]\,\mathbb{E}\left[\text{vec}\left(R_{t-1}R'_{t-1}\right)\right]\\
\mathbb{E}\left[(W_t\otimes I)\,\text{vec}\left(R_{t-1}R'_{t-1}\right)\right] &\simeq \mathbb{E}[(W_t\otimes I)]\,\mathbb{E}\left[\text{vec}\left(R_{t-1}R'_{t-1}\right)\right]\\
\mathbb{E}\left[(W_t\otimes W_t)\,\text{vec}\left(R_{t-1}R'_{t-1}\right)\right] &\simeq \mathbb{E}[(W_t\otimes W_t)]\,\mathbb{E}\left[\text{vec}\left(R_{t-1}R'_{t-1}\right)\right]
\end{aligned} \tag{8}$$

The interesting feature of this ergodicity constraint, that we will call long run constraint, is that the stability condition

$$\rho\left((A_t\otimes A_t) + (B_t\otimes B_t)\right) < 1 \tag{9}$$

can be locally violated for short periods of time, even if the global long run constraint is satisfied, thus leading to a temporary exponential growth of the whole covariance matrix, mimicking what we can find during anomalous market conditions. This philosophy of modelling is in line with the econometric literature on the stochastic unit root models proposed by Granger [4] and also with the literature on early warnings signals for critical transitions [5]. In the following the parameter estimation and inference are conducted optimizing a Lagrangian obtained from the constrained likelihood and treating the Lagrange multiplier as a nuisance parameter.

4 Empirical Results

We used the same data as in [1], that consists in 25 monthly stock returns with the highest average market value for each period of US Banks Prime Brokers and Insurances taken from CRSP database, and returns of the 25 top AUM Hedge Funds for the same period taken from TASS database. The sample period goes from January 1994 to December 2008. The same dataset with the same frequency was used for computing the weights matrices and constructing the equally weighted indexes on which the BEKK is estimated. As an euristic justification for our working hypothesis, we compute the empirical percentage variation for each of the three outer products in (8). Consider for example $(W_t \otimes W_t)$ the empirical variation is:

$$D_{(W_t \otimes W_t)} = \frac{\frac{1}{T}\sum_{t=1}^{T}\left[(W_t \otimes W_t)\,\mathrm{vec}\left(R_{t-1}R'_{t-1}\right)\right] - \frac{1}{T}\sum_{t=1}^{T}\left[(W_t \otimes W_t)\right]\frac{1}{T}\sum_{t=1}^{T}\left[\mathrm{vec}\left(R_{t-1}R'_{t-1}\right)\right]}{\frac{1}{T}\sum_{t=1}^{T}\left[(W_t \otimes W_t)\,\mathrm{vec}\left(R_{t-1}R'_{t-1}\right)\right]}$$

The maximum absolute value for the variations are in table 1, from which we see that the approximation works better for the Out degrees measure. In table 2 we report the results of our estimations. As we can see the Out Degree weights bring a higher log-likelihood and a lower long run spectral radius.

Table 1 Maximum absolute value for $D_{(I\otimes W_t)}$, $D_{(W_t\otimes I)}$ and $D_{(W_t\otimes W_t)}$

	Out Degree			Closeness		
	$D_{(I\otimes W_t)}$	$D_{(W_t\otimes I)}$	$D_{(W_t\otimes W_t)}$	$D_{(I\otimes W_t)}$	$D_{(W_t\otimes I)}$	$D_{(W_t\otimes W_t)}$
max absolute value	0.01	0.01	0.03	0.50	0.50	0.25

Table 2 Main parameters likelihood estimation and long run spectral radius for the two sets of matrices. Boldface means significance at the 0.01 level

	Out Degree		Closeness	
	Parameter	Pvalue	Parameter	Pvalue
a_{01}	0.46	0.016	**0.58**	**0.001**
a_{02}	-0.43	0.476	**0.37**	**0.001**
a_{03}	**0.26**	**0.001**	**0.31**	**0.001**
a_{04}	0.07	0.582	**0.33**	**0.001**
a_{11}	**-0.37**	**0.001**	-0.05	0.441
a_{12}	-0.09	0.119	0.3	0.902
a_{13}	**-0.51**	**0.001**	**0.32**	**0.001**
a_{14}	-0.49	0.545	**0.24**	**0.001**
b_{01}	0.04	0.385	**0.8**	**0.001**
b_{02}	**-0.24**	**0.001**	**0.85**	**0.001**
b_{03}	**-0.79**	**0.001**	**0.91**	**0.001**
b_{04}	**1.11**	**0.001**	**0.91**	**0.000**
b_{11}	**0.2**	**0.000**	-0.02	0.834
b_{12}	**0.43**	**0.008**	**-0.33**	**0.001**
b_{13}	**0.99**	**0.001**	**-0.31**	**0.001**
b_{14}	**-1.17**	**0.001**	-0.07	0.921
log-likelihood	1676.3		1655.5	
$\rho\left(\bar{A}\otimes\bar{A}+\bar{B}\otimes\bar{B}\right)$	0.89		0.96	

5 Spectral Radius as a Systemic Risk indicator

According to previous discussion, we consider Out Degree radius more reliable and the only one for which it is worth trying an economic interpretation. In particular in figure 1 we plot the spectral radius coming from the Out Degrees, pointing out the main historical market events. Most of the times, when the spectral radius is over one, and so when the covariance experiences an exponential growth, it can be seen to correspond to important market events. Moreover, when it is not the case as the LTCM crisis the radius has a dramatic increase. So it seems that our latent spectral radius can be used as a Systemic Risk indicator and its proximity to one could be useful to develop an Early Warning signal of systemic events.

The next step in this modelling methodology would be to assume a particular matrix-valued data generating process for our weight matrices and try to forecast the radius and compute the probability that it becomes greater or equal to one. This is left for future research. Finally, we stress that this kind of methodology could be applied also to other models in which ergodicity and stability condition are different, opening new ways of modelling anomalous market conditions.

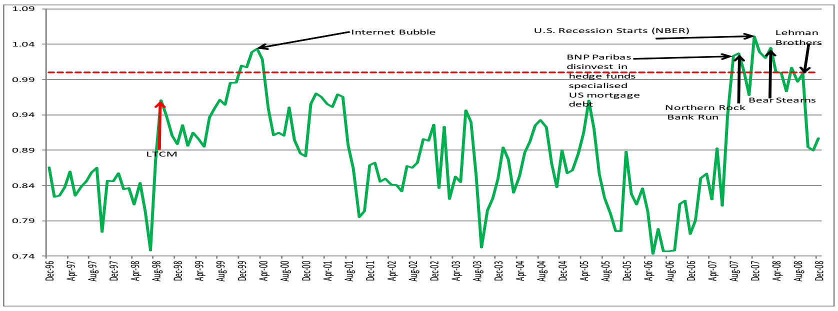

Fig. 1 Out Degree spectral radius and historical market events

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