



Trade Discount Policies in the Differential Games Framework

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Abstract: We consider a vertical control distribution channel in which a manufacturer sells a single kind of good to a retailer. We assume that a wholesale price discount increases the retailer's sale motivation thus improving sales. The optimal control of manufacturer's profit via trade discounts is embedded in a differential game framework; in the special case of constant controls we compare the Stackelberg equilibria obtained considering manufacturer and retailer respectively as leaders of the game with Nash equilibrium points.

Keywords: Optimal control, differential games, motivation, trade discounts

1 Introduction

To earn a reasonable profit the members of a distribution channel often adopt rather simple pricing techniques. For example, manufacturers may use cost-plus pricing, simply defining the price adding a desired profit margin to (variable) production costs; in a similar fashion, retailers very often use to determine shelf prices adding a fixed percentage markup to the wholesale price.

The main advantage of simple policies is that they are...easy to be applied. But this blind approach to pricing does not provide tools to manufacturers in order to encourage retailers to sell and retailers, in turn, cannot adequately stimulate consumer to buy.

Differential games are used to represent problems of conflict and cooperation when decisions are made in real time. Their use in marketing, e.g. in advertising, pricing, promotion policies optimization, has a long tradition dating over 30 years ago (see e.g. [4],[7] and also [8],[5],[6]).

In this paper we will focus on the effects of trade promotions, a widely used dynamic pricing strategy that manufacturers can exploit to raise sales. With trade promotions an incentive mechanism is used to drive other channel members' behaviors.

In particular we investigate the relationships between the members of a distribution channel by means

of optimal control models in a stylized vertical distribution channel: a manufacturer serves a single segment market through a single retailer and a contract fixes a trade discount policy which will be followed by the contractors.

Trade discounts have usually a double positive effect on sales since part of the wholesale price reduction may be transferred to the shelf price (pass-through) and part of the discount will be kept by the retailer who will be more motivated (see [9],[10]) and higher motivation means higher effort in selling the product.

In Section 2 we recall an optimal control problem in which the retailer's performance is explicitly modeled as a function of retailer's skill and motivation.

There is an obvious trade-off between manufacturer's and retailer's goals, each of the two firms aims at maximizing its own profit and this led us to the differential game models of the distribution channel pricing policy decisions which are reported at the end of Section 2. In Section 3 we consider the special case in which trade discount and pass-through (the two controls of the model) are assumed to be constant during the selling period. In this case we compare Stackelberg equilibria (considering either the manufacturer or the retailer as the leader) with Nash equilibrium points.

2 Two optimal control models

Our starting point is given by a couple of models presented in [1] and [2] where we considered a stylized vertical channel in which a manufacturer sells a single product during the limited time period $[t_1, t_2]$. In those models the manufacturer sells to a single re-

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tailer, her aim is to maximize the total profit in the given time period. By means of trade discounts the manufacturer can raise her sales both because the retailer transfers a part of the discount to the shelf price (pass-through) and because if the retailer will keep part of the incentive for himself he will be more motivated in selling the specific product, thus giving another upward push to sales.

This way two optimal control models can be considered in which the controls are, respectively, trade discount (the manufacturer's control) and pass-through (the retailer's control). The state variables in the models are the cumulative sales and the retailer's motivation.

Let us define the details of the models. Define

$x(t)$ = cumulated sales during the time period $[t_1, t]$,

$p_w(t)$ = wholesale price at time t ,

c_0 = unit production cost,

$\alpha(t)$ = trade discount at time t , $\alpha(t) \in [\alpha_1, \alpha_2] \subseteq [0, 1]$,

$\beta(t)$ = pass-through at time t , $\beta(t) \in [\beta_1, \beta_2] \subseteq [0, 1]$.

Constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ represent the boundary values of trade discount and pass-through that manufacturer and retailer require not to be exceeded in order to participate in the selling activity of the channel. In particular manufacturer establishes the values of α_2 and β_1 while the retailer fixes the values of α_1 and β_2 .

Considering the trade discount explicitly, the wholesale price can be rewritten as $p_w(t) = p(1 - \alpha(t))$ where p is the wholesale price when no trade discount is applied.

Remark that $\dot{x}(t)$ represents the sales rate at time t ; we suppose that it coincides with the consumer's demand at time t and that the firm will produce exactly the quantity to be sold.

The total profit of the manufacturer can be written as

$$\int_{t_1}^{t_2} (p_w(t) - c_0)\dot{x}(t)dt,$$

or, since $x(t_1) = 0$,

$$J_M = qx(t_2) - p \int_{t_1}^{t_2} \dot{x}(t)\alpha(t)dt,$$

where $q = p - c_0$. In order to obtain a non negative profit the manufacturer will ask $\alpha_2 \leq q/p$ as it is shown in [1].

The total profit of the retailer is then

$$J_R = p \int_{t_1}^{t_2} \alpha(t)(1 - \beta(t))\dot{x}(t)dt.$$

If the retailer's sales motivation at time t , summarized by the state variable $M(t)$, is increasing with respect to consumer's demand and to trade discount then its dynamics can be described by

$$\dot{M}(t) = \gamma\dot{x}(t) + \epsilon(\alpha(t) - \bar{\alpha}),$$

where γ and ϵ are strictly positive constants. Constant $\bar{\alpha} \in (\alpha_1, \alpha_2)$ takes into account the fact that the retailer has some expectations on trade prices, his motivation decreases if his expectations are disappointed, i.e. if $\alpha(t) < \bar{\alpha}$, and increase if $\alpha(t) \geq \bar{\alpha}$.

The dynamics of the total amount of sales at time t is given by

$$\dot{x}(t) = -\theta x(t) + \delta M(t) + \tilde{\eta}\beta(t)\alpha(t),$$

where $\delta, \tilde{\eta}$, and θ are strictly positive. Constant δ represents the retailer's selling skill while θ is needed to model the market saturation effect (e.g. large markets will display low values of θ). Parameter $\tilde{\eta}$ represents the productivity of the retail price discount on sales.

The manufacturer's profit maximization problem requires then to deal with the following optimal control problem (see [1])

Problem M : maximize J_M

subject to

$$\dot{x}(t) = -\theta x(t) + \delta M(t) + \tilde{\eta}\beta(t)\alpha(t),$$

$$\dot{M}(t) = \gamma\dot{x}(t) + \epsilon(\alpha(t) - \bar{\alpha}),$$

$$x(t_1) = 0, \quad M(t_1) = \bar{M},$$

$$\alpha(t) \in [\alpha_1, \alpha_2] \subseteq [0, 1],$$

$$\beta(t) \in [\beta_1, \beta_2] \subseteq [0, 1],$$

where \bar{M} is the initial motivation of the retailer (we assume $\bar{M} > 0$).

For the case of constant $\beta(t)$ problem M has been investigated in [1], [2].

In a similar way it is possible to formulate (see [2]) a corresponding retailer's optimal control problem, keeping the same motion equations and constraints and with objective functional J_R .

In this paper we will address some preliminary considerations on the differential game, which will be denoted by MR , defined by the objective functionals

$$J_M, \quad J_R,$$

by the motion equations

$$\dot{x}(t) = -\theta x(t) + \delta M(t) + \tilde{\eta}\beta(t)\alpha(t),$$

$$\dot{M}(t) = \gamma\dot{x}(t) + \epsilon(\alpha(t) - \bar{\alpha}),$$

$$x(t_1) = 0, \quad M(t_1) = \bar{M},$$

and by the constraints

$$\begin{aligned}\alpha(t) &\in [\alpha_1, \alpha_2] \subseteq [0, 1], \\ \beta(t) &\in [\beta_1, \beta_2] \subseteq [0, 1].\end{aligned}$$

3 The case in which both controls are constant

In this paper we study the game MR in a simplified framework in which both controls must take a constant value in the whole time period $[t_1, t_2]$ and these values are decided at time t_1 . In this case the solution of problems M and R becomes straightforward and allows to obtain some properties of the differential game MR .

With constant controls $\alpha(t) = \alpha$ and $\beta(t) = \beta$ the manufacturer's profit is

$$J_M = J_M(\alpha, \beta) = (q - p\alpha)x(t_2),$$

while the profit of retailer is

$$J_R = J_R(\alpha, \beta) = p\alpha(1 - \beta)x(t_2).$$

The total volume of sales during $[t_1, t_2]$, i.e. $x(t_2)$, depends explicitly from α and β . More precisely

$$x(t_2) = (H\beta + L)\alpha + K,$$

where we have defined

$$\begin{aligned}a &= \theta - \gamma\delta \\ T &= a(t_1 - t_2) \\ H &= \frac{\tilde{\eta}}{a}(1 - e^T) \\ L &= -\frac{\delta\epsilon}{a^2}(1 - e^T + T) \\ K &= \frac{\delta}{\tilde{\eta}}\overline{M}H - \overline{\alpha}L.\end{aligned}$$

In this paper we assume $a > 0$ (cf. [1]), this means, for example, that the retailer is less sensitive to consumer demand rather than to the fulfillment of his expectations (i.e. γ has a low value). This way we also have $T < 0$, $H > 0$ and $L > 0$.

The manufacturer's profit can now be rewritten as

$$J_M(\alpha, \beta) = (q - p\alpha)[(H\beta + L)\alpha + K],$$

while the profit of the retailer is

$$J_R(\alpha, \beta) = p\alpha(1 - \beta)[(H\beta + L)\alpha + K].$$

3.1 Nash equilibria

Let us look now for the Nash equilibria of the differential game MR . Functional J_M is concave with respect to α , while functional J_R is concave with respect to β . Nash equilibria are therefore the solutions of the system

$$\begin{cases} \frac{\partial J_M}{\partial \alpha} = 0 \\ \frac{\partial J_R}{\partial \beta} = 0 \end{cases} \quad (1)$$

where

$$\begin{aligned}\frac{\partial J_M}{\partial \alpha} &= (H\beta + L)(q - 2p\alpha) - pK, \\ \frac{\partial J_R}{\partial \beta} &= p\alpha[(-2H\beta + H - L)\alpha - K].\end{aligned}$$

In order to simplify notation let us define

$$\Gamma = \sqrt{1 - \frac{8pK}{q(H + L)}}.$$

One solution of the system is (α^0, β^0) where

$$\alpha^0 = 0, \quad H\beta^0 + L = \frac{pK}{q}.$$

But this equilibrium has poor economical meaning in practice. In fact, a constant trade discount $\alpha = \alpha^0 = 0$ means that the profit of the retailer is zero, in that case the retailer will not participate in the selling activity if the manufacturer will not allow him some other incentive.

If $q(H + L) - 8pK \geq 0$ then system (1) provides other two Nash equilibria, (α^+, β^+) and (α^-, β^-) , where

$$\alpha^+ = \frac{q(1 + \Gamma)}{4p}, \quad H\beta^+ + L = \frac{(H + L)(1 + \Gamma)}{4},$$

$$\alpha^- = \frac{q(1 - \Gamma)}{4p}, \quad H\beta^- + L = \frac{(H + L)(1 - \Gamma)}{4}.$$

It is easy to compute the values of the profits in the equilibria:

$$J_M^0 \equiv J_M(\alpha^0, \beta^0) = qK,$$

$$J_R^0 \equiv J_R(\alpha^0, \beta^0) = 0,$$

$$J_M^+ \equiv J_M(\alpha^+, \beta^+) = \frac{q^2(H + L)(1 + \Gamma)(3 - \Gamma)^2}{64p},$$

$$J_R^+ \equiv J_R(\alpha^+, \beta^+) = \frac{q^2(H+L)^2(1+\Gamma)^2(3-\Gamma)^2}{256pH},$$

where, for each fixed α , β is (the) optimal solution of maximize

$$J_M^- \equiv J_M(\alpha^-, \beta^-) = \frac{q^2(H+L)(1-\Gamma)(3+\Gamma)^2}{64p},$$

$$J_R(\alpha, \beta) = p\alpha(1-\beta)[(H\beta+L)\alpha+K], \beta \in [\beta_1, \beta_2].$$

The Stackelberg equilibrium, when the manufacturer is the channel leader, is (α^M, β^M) , where

$$J_R^- \equiv J_R(\alpha^-, \beta^-) = \frac{q^2(H+L)^2(1-\Gamma)^2(3+\Gamma)^2}{256pH}.$$

$$\alpha^M = \frac{q(H+L) - pK}{2p(H+L)} = \frac{q(7+\Gamma^2)}{16p},$$

Therefore

$$J_M^0 \leq J_M^- < J_M^+, \quad J_R^0 \leq J_R^- < J_R^+.$$

$$H\beta^M + L = \frac{(H+L)\alpha^M - K}{2\alpha^M} = \frac{(H+L)(5+3\Gamma^2)}{2(7+\Gamma^2)}.$$

From now on we will not consider anymore the equilibria (α^0, β^0) due to its poor economical meaning.

It is easy to compute the values of the profits in the equilibria:

Remark that it is rather easy to find the necessary and sufficient conditions under which the Nash equilibria (α^+, β^+) and (α^-, β^-) are feasible, i.e. belong to $[\alpha_1, \alpha_2] \times [\beta_1, \beta_2]$: details are reported in [3].

$$\begin{aligned} J_M^{ML} \equiv J_M(\alpha^M, \beta^M) &= \frac{[q(H+L) + pK]^2}{8p(H+L)} = \\ &= \frac{q^2(H+L)(3-\Gamma)^2(3+\Gamma)^2}{512p}, \end{aligned}$$

$$\begin{aligned} J_R^{ML} \equiv J_R(\alpha^M, \beta^M) &= \frac{[q(H+L) + pK]^2}{16pH} = \\ &= \frac{q^2(H+L)^2(3-\Gamma)^2(3+\Gamma)^2}{1024p}. \end{aligned}$$

3.2 Stackelberg equilibrium when the manufacturer is leader

A different point of view on the channel marketing activity can be obtained considering the manufacturer and the retailer as the two players of a Stackelberg game (see [4] and [7]).

Remark again that we can rather easily find the necessary and sufficient conditions under which equilibria (α^M, β^M) is feasible (see [3]).

We first consider the manufacturer as the channel leader: in this case we assume that she can only choose a constant trade discount during the whole sales period. This way we formulate the following Stackelberg game:

Game ML : maximize

$$qx(t_2) - p\alpha \int_{t_1}^{t_2} \dot{x}(t) dt, \quad \alpha \in [\alpha_1, \alpha_2],$$

where, for each fixed α , functions $x(t)$, $M(t)$, $\beta(t)$ are optimal solution of

$$\begin{aligned} \text{maximize} \quad & p\alpha \int_{t_1}^{t_2} \dot{x}(t)(1-\beta(t)) dt, \\ \text{subject to} \quad & \dot{x}(t) = -\theta x(t) + \delta M(t) + \tilde{\eta}\alpha\beta(t), \\ & \dot{M}(t) = \gamma\dot{x}(t) + \epsilon(\alpha - \bar{\alpha}), \\ & x(t_1) = 0, \quad M(t_1) = \bar{M}, \\ & \beta \in [\beta_1, \beta_2]. \end{aligned}$$

Since in our case both controls are constant, we can rewrite the Stackelberg game this way:

Game ML : maximize

$$J_M(\alpha, \beta) = (q - p\alpha)[(H\beta + L)\alpha + K], \quad \alpha \in [\alpha_1, \alpha_2],$$

3.3 Stackelberg equilibrium when the retailer is leader

Consider the retailer as the channel leader: we assume in this case that he can only choose constant pass-through during the whole sales period. This way a new Stackelberg game can be formulated as follows:

Game RL : maximize

$$(1-\beta)p \int_{t_1}^{t_2} \dot{x}(t)\alpha(t) dt, \quad \beta \in [\beta_1, \beta_2],$$

where, for each fixed β , functions $x(t)$, $M(t)$ and $\alpha(t)$ are optimal solution of

$$\begin{aligned} \text{maximize} \quad & qx(t_2) - p \int_{t_1}^{t_2} \dot{x}(t)\alpha(t) dt, \\ \text{subject to} \quad & \dot{x}(t) = -\theta x(t) + \delta M(t) + \tilde{\eta}\beta\alpha(t), \\ & \dot{M}(t) = \gamma\dot{x}(t) + \epsilon(\alpha(t) - \bar{\alpha}), \\ & x(t_1) = 0, \quad M(t_1) = \bar{M}, \\ & \alpha(t) \in [\alpha_1, \alpha_2]. \end{aligned}$$

Since in our case both controls are constant, the Stackelberg game can be formulated as follows:

Game RL: maximize

$$J_R(\alpha, \beta) = p\alpha(1 - \beta)[(H\beta + L)\alpha + K], \quad \beta \in [\beta_1, \beta_2],$$

where, for each fixed β , α is the optimal solution of maximize

$$J_M(\alpha, \beta) = (q - p\alpha)[(H\beta + L)\alpha + K], \quad \alpha(t) \in [\alpha_1, \alpha_2].$$

Deriving J_M with respect to α we obtain the manufacturer best response to retailers policy β :

$$\alpha^R = \frac{q}{2p} - \frac{K}{2(H + L)v},$$

where we used the variable transformation

$$v = \frac{H\beta + L}{H + L}.$$

Setting

$$C = \frac{pK}{q(H + L)}. \quad (2)$$

and deriving $J_R(\alpha^R, \beta)$ with respect to β , and using the above variable transformation, we obtain that J_R is maximized when v^R is the positive root of the cubic equation

$$2v^3 - v^2 = C^2.$$

This way we can compute the Stackelberg equilibrium (α^R, β^R) , when the retailer is the channel leader.

By straightforward calculations, one has

$$J_M^{RL} \equiv J_M(\alpha^R, \beta^R) = \frac{q^2(H + L)(v^R + C)^2}{4pv^R},$$

$$\begin{aligned} J_R^{RL} \equiv J_R(\alpha^R, \beta^R) &= \\ &= \frac{q^2(H + L)^2[(v^R)^2 - C^2](1 - v^R)}{4pHv^R}. \end{aligned}$$

Again, it is possible (but not very easy in this case) to find the necessary and sufficient conditions, when Stackelberg equilibrium (α^R, β^R) is inside the feasible region (see [3]).

3.4 Comparison of the profit in Nash and Stackelberg equilibria

One has

$$J_M^+ - J_M^{ML} = - \frac{q^2(H + L)(3 - \Gamma)^2(1 - \Gamma)^2}{512p} < 0,$$

$$J_M^- - J_M^{ML} = - \frac{q^2(H + L)(3 + \Gamma)^2(1 + \Gamma)^2}{512p} < 0.$$

Further,

$$J_R^+ - J_R^{ML} = - \frac{q^2(H + L)^2(3 - \Gamma)^2(5 + 3\Gamma)(1 - \Gamma)}{1024pH},$$

$$J_R^- - J_R^{ML} = - \frac{q^2(H + L)^2(3 + \Gamma)^2(5 - 3\Gamma)(1 + \Gamma)}{1024pH}.$$

Remark that $J_R^+ - J_R^{ML} < 0 \Leftrightarrow \Gamma < 1 \Leftrightarrow C > 0$ while $J_R^- - J_R^{ML} < 0$ since $\Gamma < 1$ for every $\alpha^- > 0$.

Now let us compare manufacturer's and retailer's profits in the Stackelberg equilibrium in two cases: when the manufacturer is leader and when the retailer is leader. One has

$$J_M^{ML} - J_M^{RL} =$$

$$= \frac{q^2(H + L)}{8p} \cdot \left[(1 - C)^2 - 2 \left(v^R + \frac{C^2}{v^R} \right) \right],$$

$$J_R^{RL} - J_R^{ML} =$$

$$= \frac{q^2(H + L)^2}{16pH} \cdot \left[4(1 - v^R) \left(v^R - \frac{C^2}{v^R} \right) - (1 + C)^2 \right].$$

It can be shown that if

$$(\alpha^M, \beta^M) \in [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]$$

and

$$(\alpha^R, \beta^R) \in [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]$$

then

$$C \in [-1, 1].$$

Depending on the sign of C , manufacturer and retailer both prefer either to act as the leader or as the follower. More precisely:

1) if $C < 0$ then $J_M^{ML} > J_M^{RL}$ and $J_R^{RL} > J_R^{ML}$, i.e. both manufacturer and retailer want to be leader (**the struggle to be leader**);

2) if $C > 0$ then $J_M^{ML} < J_M^{RL}$ and $J_R^{RL} < J_R^{ML}$, i.e. both manufacturer and retailer want to be follower (**the struggle to be follower**).

Remark that if $C = 0$ then $\alpha^M = \alpha^R$ and $\beta^M = \beta^R$.

Observe that the sign of C coincides with the sign of K : this sign is positive if the initial motivation of the retailer is already high (\bar{M} is high) or his expectations are low ($\bar{\alpha}$ is low). In this case, i.e. the retailer is rather unassuming and optimistic, neither member of the channel needs to lead it, they would prefer to be followers. In case of a less well-disposed retailer the situation upsets and both would like to lead the channel.

4 Conclusions

In this paper we explore a bilevel programming approach to study the discount policies in a supply channel. We found that to be leader or follower in the channel is preferred by both the manufacturer and the retailer depending on the value of parameter C (see (2)) which economical meaning should be investigated in a future research.

A further theme of our future research will be to consider the case when feasible controls (trade discount $\alpha(t)$ and pass-through $\beta(t)$) are not constant, but piece-wise constant.

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