An efficient application of the repeated Richardson extrapolation technique to option pricing

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EXTENDED ABSTRACT

Introduction

In financial engineering one has frequently to deal with approximate results that are obtained by iterative methods or computational procedures depending on some parameter (e.g. the time-step). Often the convergence of numerical schemes is slow and this may be a serious problem to their use in practice. For this reason, acceleration techniques, such as Richardson extrapolation, have been studied and applied.

In this contribution, we implement an efficient numerical method based on repeated Richardson extrapolation for the valuation of American options, paying particular attention to the choice of both the sequence of stepsizes and the order. In particular, we apply the method to the randomization approach proposed by Carr (1998), thus improving its accuracy by choosing a convenient sequence of stepsizes.

Richardson extrapolation technique and its application in finance

Richardson extrapolation has been applied to accelerate valuation schemes for American options and exotic options. Geske and Johnson (1984) first applied Richardson extrapolation in a financial context to speed up and simplify their compound option valuation model. They obtain a more accurate computational formula for the price of an American put option using the values of Bermuda options[1].

[1]Geske and Johnson approach was subsequently developed and improved by Bunch and Johnson (1992), and Ho *et al.* (1997).

Richardson extrapolation techniques were also employed to enhance efficiency of lattice methods (Breen, 1991). It is common opinion that it is not convenient to extrapolate on the number of time steps in the binomial model due to the oscillatory nature of the convergence. Broadie and Detemple (1996) successfully use Richardson extrapolation to accelerate a hybrid of the binomial and the Black-Scholes models. Tian (1999) and Heston and Zhou (2000) also apply Richardson extrapolation to binomial and multinomial approaches. Carr (1998) proposes a randomization approach for the valuation of the American put option and uses Richardson extrapolation to obtain accurate estimates of both the price and the exercise boundary of and American put option. Leisen (1999) shows that randomizing the length of the time steps in the binomial model allows the successful use of extrapolation, while Huang et al. (1996) and Ju (1998) use extrapolation methods to accelerate the integral representation of the early exercise premium.

The very natural idea of extrapolation can be summarized as follows (see Deuflhard, 1983). Consider the problem of valuing, for instance, an American put option. Instead of the exact, but unknown, solution P_0 take a discrete approximation $P(h)$ depending on the *stepsize* $h > 0$, $P(h)$ being a calculable function yielded by some numerical scheme, such that $\lim_{h\to 0} P(h) = P(0) = P_0$.

All extrapolation schemes are based on the existence of an asymptotic expansion. Under the assumption that $P(h)$ is a sufficiently smooth function, we write

$$P(h) = \alpha_0 + \alpha_1 h^{p_1} + \alpha_2 h^{p_2} + \cdots + \alpha_k h^{p_k} + O(h^{p_{k+1}})\,, \tag{1}$$

with $0 < p_1 < p_2 < \ldots$, and unknown parameters $\alpha_0, \alpha_1, \ldots$, where $h \in [0, H]$ for some $H > 0$. In particular, we have $\alpha_0 = P_0$.

One computes the function $P(h)$ over a certain *basic step* $H > 0$ a number of times with successively smaller stepsize h_i, with $h_1 > h_2 > \ldots > 0$. In such a way one obtains a sequence of approximations $P(h_1), P(h_2), \ldots$ for a given sequence of stepsizes.

We can construct extrapolation schemes of arbitrary order k by considering the following procedure[2]:

1. define $T_{i,1} = P(h_i)$, for $i = 1, 2, \ldots$;
2. for $i \geq 2$ and $j = 2, 3, \ldots, i$, compute

$$T_{i,j} = T_{i,j-1} + \frac{T_{i,j-1} - T_{i-1,j-1}}{\frac{h_{i-j+1}}{h_i} - 1}\,. \tag{2}$$

Recursion (2) is based on polynomial interpolation and an asymptotic h-expansion.

Each quantity $T_{i,j}$ is computed in terms of two successive approximations. The two point Richardson extrapolation technique is *repeated* giving rise to a numerical scheme

[2]Such a procedure is also known as Aitken-Neville algorithm and it is one of the extrapolation schemes which are commonly used.

which is extremely fast[3] and can dramatically improve accuracy. The idea behind (2) is to provide two mechanisms for enhancing the accuracy: increasing i one obtains a reduction in the stepsize parameter, while taking j large implies more accurate approximations. Both mechanisms work simultaneously, which indicates that the quantities $T_{k,k}$ are those of most interest. This provides us with the possibility of order control.

The accuracy and efficiency of the method is strictly connected with the choice of the sequence of stepsizes. Define h_i in terms of the basic step size H, such that $h_i = H/n_i$ $(i = 1, 2, \ldots)$. Any stepsize sequence is characterized by the associated sequence of integers $\{n_i\}$. In numerical experiments, we considered different sequences of the stepsize:

- harmonic sequence: $\{k\} = \{1, 2, 3, 4, 5, 6, \ldots\}$;
- double harmonic (Deuflhard) sequence: $\{2k\} = \{2, 4, 6, 8, 10, 12, \ldots\}$;
- Burlisch sequence: $\{2n_{k-2}\} = \{2, 4, 6, 8, 12, 16, \ldots\}$;
- Romberg sequence: $\{2n_{k-1}\} = \{2, 4, 8, 16, 32, 64, \ldots\}$.

All these sequences allow for convergence of the method. The first and the fourth sequence are of common use in the financial literature related to extrapolation combined with option pricing models, while we have no knowledge of an employment of the other sequences (which are well known in numerical analysis) in finance. When we consider the harmonic sequence and recurrence (2), we can directly compute the quantities $T_{k,k}$ using the formula

$$T_{k,k} = \sum_{i=1}^{k} \frac{(-1)^{k-i}\, i^k}{(k-i)!\, i\,!}\, P(h_i)\,. \tag{3}$$

Numerical results

To test the method, we applied it to the model proposed by Carr (1998) for the valuation of American put options. Carr's approach is based on a particular technique, called *randomization*, and has proved robust and quite accurate, moreover the convergence of the results is monotonic, allowing us to consider extrapolations of higher order.

Carr applies extrapolation as defined by (3), based on the harmonic sequence. We compared numerical results obtained with different sequences of the steps, and assessed the method on a large set of option valuation problems, considering different values of moneyness, maturity, volatility and risk-free interest rate[4]. In particular, when applied to Carr's model, extrapolation *based on Burlisch sequence* (see Burlisch, 1964) is preferable in terms of accuracy, the same remaining the computational effort required, obtaining a reduction of the relative error to about one third.

Finally, we investigate the possibility of applying repeated Richardson extrapolation to other models, such as that proposed by Ju (1998), to the binomial method and Monte

[3]The amount of computation required essentially corresponds to the number of function evaluations.
[4]Numerical results are not reported here in extension for seek of brevity.

Carlo simulation for American options. It is worth noting that one should be careful when employing extrapolation techniques combined with these latter approaches, because of their non-uniform convergence. To this regard, *ad hoc* smoothing procedures are needed.

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