Resource combinatory algebras

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Abstract. We initiate a purely algebraic study of Ehrhard and Regnier's resource λ-calculus, by introducing three equational classes of algebras: *resource combinatory algebras*, *resource lambda-algebras* and *resource lambda-abstraction algebras*. We establish the relations between them, laying down foundations for a model theory of resource λ-calculus. We also show that the ideal completion of a resource combinatory (resp. lambda-, lambda-abstraction) algebra induces a "classical" combinatory (resp. lambda-, lambda-abstraction) algebra, and that any model of the classical λ-calculus raising from a resource lambda-algebra determines a λ-theory which equates all terms having the same Böhm tree.

1 Introduction

There have been several attempts to reformulate the λ-calculus as a purely algebraic theory. The earliest and best known algebraic models are the *combinatory algebras* of Schönfinkel and Curry [5]. Combinatory algebras, as well as their remarkable subclass of λ-*algebras*, have a purely equational characterization but yield somewhat weak notions of models of the λ-calculus. In fact, the combinatory interpretation of λ-calculus does not satisfy the ξ-rule: under the interpretation, $M = N$ does not necessarily imply $\lambda x.M = \lambda x.N$. Thus, the class of λ-algebras is not sound for λ-theories, and one is forced to consider the non-equational class of λ-models (see [1]).

There are many advantages in using algebraic languages rather than languages with binders, particularly in connection with equational reasoning. The formers have well-understood model theory, and the models are closed under standard constructions such as cartesian products, subalgebras, quotients and free algebras. The above-mentioned problem with the ξ-rule seems to suggest that the λ-calculus is not quite equivalent to an algebraic theory.

The lattice of λ-theories is isomorphic to the congruence lattice of the term algebra of the least λ-theory $\lambda\beta$. This remark is the starting point for studying λ-calculus by universal algebraic methods, through the variety (i.e. equational class) generated by the term algebra of $\lambda\beta$. In [19] Salibra has shown that the variety generated by the term algebra of $\lambda\beta$ is axiomatized by the finite scheme of identities characterizing λ-abstraction algebras. These algebras, introduced by Pigozzi and Salibra [17], are intended as an alternative to combinatory algebras, which keeps the lambda notation and hence all the functional intuitions. In [18] the connections between the variety of λ-abstraction algebras and the other algebraic models of λ-calculus are explained; it is also shown that the free extension of a λ-algebra can be turned into a λ-abstraction algebra, thus validating all rules of the λ-calculus, including the ξ-rule. The algebraic approach to

λ-calculus has been fruitful in studying the structure of the lattice of λ-theories and in generalizing the Stone representation theorem for Boolean algebras to combinatory and λ-abstraction algebras (see [12, 13, 14]). The Stone theorem has been also applied to provide an algebraic incompleteness theorem that encompasses incompleteness results for all known semantics of λ-calculus.

In the '90s Boudol [2] introduced the *λ-calculus with multiplicities*, an extension of λ-calculus where arguments may come in limited availability and mixed together. After one decade Ehrhard and Regnier [7] introduced the *differential λ-calculus*, a conservative (see [7, Prop. 19]) extension of the λ-calculus with differential constructions, in which linear application of a term M to an argument roughly corresponds to applying the derivative of M in 0 (which is a linear function) to that argument. The presence of linear application, and linear substitution force the enrichment of the calculus with an operation of sum with a neutral element. In [8, 9] Ehrhard and Regnier introduce a simple subsystem of the differential λ-calculus, that they call *resource λ-calculus*, and establish a correspondence between *differential interaction nets*, an extension of Girard's [10] linear logic proof-nets, and resource λ-calculus. Very recently, Tranquilli [20] enriched the resource λ-calculus with a promotion operator (bearing strong similarities to Boudol's λ-calculus with multiplicities), establishing an equivalence with the differential λ-calculus. Tranquilli's resource calculus has been recently studied from the syntactical point of view by Pagani and Tranquilli [16], for confluence results, and by Pagani and Ronchi Della Rocca [15] for results about solvability. Regarding the semantics of these calculi, the first studies were conducted by Boudol et al. [3] for the λ-calculus with multiplicities. In a forthcoming paper Bucciarelli et al. [4] define categorical models for the differential λ-calculus.

In this paper we initiate a purely algebraic study of Ehrhard and Regnier's resource λ-calculus. We follow the lines of the universal-algebraic tradition in the study of λ-calculi, exploring a number of varieties which can be considered as classes of algebraic models of resource λ-calculus. We axiomatize the variety of *resource combinatory algebras* (RCAs) which are to the resource λ-calculus what combinatory algebras are to the classical λ-calculus, in the sense that they contain basic combinators which allow to define an abstraction on polynomials and to obtain a combinatory completeness result. Then establishing a parallel with the work of Curry we isolate the subvariety of *resource lambda-algebras* (RLAs) and prove that the free extension of an RLA validates the so-called *ξ-rule* for the abstraction; this is done by a construction, analogue to that of Krivine [11] for lambda-algebras, which shows that the free extension of an RLA is, up to isomorphism, an object very similar to the graded algebras which appear in module theory. Along the line of the work of Pigozzi and Salibra, we axiomatize the variety of *resource λ-abstraction algebras*. We also establish the relations between these varieties, laying down foundations for a model theory of resource λ-calculus. We then show that the ideal completion of a resource combinatory (resp. lambda-, λ-abstraction) algebra determines a "classical" combinatory (resp. lambda-, λ-abstraction) algebra, and that any model of the classical λ-calculus raising from a resource lambda-algebra induces a λ-theory which equates all terms having the same Böhm tree. Most omitted proofs can be found in Appendix.

2 Preliminaries

We identify every natural number $n \in \mathbb{N}$ with the set $n = \{0, \dots, n-1\}$. $\mathfrak{S}_n$ denotes the set of all permutations (i.e., bijections) of set $n \in \mathbb{N}$.

Sequences: The overlined letters $\bar{a}, \bar{b}, \bar{c}, \dots$ range over the set A^* of all finite sequences over A. The length of a sequence $\bar{a}$ is denoted by $|\bar{a}|$. If $\bar{a}$ is a sequence then a_i ($i \in \mathbb{N}$) denotes the i-th element of $\bar{a}$. For a sequence $\bar{a}$ of length n and a map $\sigma : k \to n$ ($k, n \in \mathbb{N}$), the composition $\sigma\bar{a}$ is the sequence $(a_{\sigma(0)}, \dots, a_{\sigma(k-1)})$. Given two sequences $\bar{a}$ and $\bar{b}$, their concatenation is denoted by $\bar{a} \cdot \bar{b}$. Sequences of length one and elements of A are identified so that $a \cdot \bar{b}$ is the concatenation of $a \in A$ and $\bar{b} \in A^*$. If $a \in A$, then a^k denotes the sequence $(a, \dots, a)$ of length k. If $\bar{i}$ is a sequence of natural numbers of length k then $\Sigma\bar{i}$ denotes $i_0 + \dots + i_k$.

Sequences of sequences will be denoted by the double over-line. Thus, $\bar{\bar{a}}$ will be a sequence of sequences, whose elements are the sequences $\bar{a}_0, \dots, \bar{a}_{|\bar{\bar{a}}|-1}$. We denote by $\prod \bar{\bar{a}}$ the sequence $\bar{a}_0 \cdot \bar{a}_1 \cdot \dots \cdot \bar{a}_{|\bar{\bar{a}}|-1}$ that is the juxtaposition of the sequences $\bar{a}_i$.

Partitions of a sequence: Let $\bar{a} \in A^n$ and $\bar{i} \in \mathbb{N}^*$ be sequences. A *$\bar{i}$-partition* of $\bar{a}$ is a sequence $\bar{\bar{b}}$ of sequences such that $|\bar{\bar{b}}| = |\bar{i}| = k$, $|\bar{b}_0| = i_0, \dots, |\bar{b}_{k-1}| = i_{k-1}$ and there exists $\sigma \in \mathfrak{S}_n$ such that $\sigma\bar{a} = \prod \bar{\bar{b}}$. We write $\mathcal{Q}_{\bar{a},\bar{i}}$ to denote the set of all $\bar{i}$-partitions of $\bar{a}$. $\mathcal{Q}_{\bar{a},\bar{i}} \neq \emptyset$ if, and only if, $\Sigma\bar{i} = |\bar{a}|$. $\mathcal{Q}_{\bar{a},k}$ to denote the set $\cup_{\bar{i} \in \mathbb{N}^*}\{\mathcal{Q}_{\bar{a},\bar{i}} : |\bar{i}| = k\}$.

Let $\bar{x}, \bar{y}$ be sequences of the same length and let $\bar{\bar{a}} \in \mathcal{Q}_{\bar{x},\bar{n}}$. We say that $\bar{\bar{b}} \in \mathcal{Q}_{\bar{y},\bar{n}}$ is the *partition of $\bar{y}$ induced by $\bar{\bar{a}}$* iff $\prod \bar{\bar{a}} = \sigma\bar{x}$ and $\prod \bar{\bar{b}} = \sigma\bar{y}$.

Free extensions: Let $\mathcal{V}$ be a variety and $\mathbf{A} \in \mathcal{V}$. The *free extension* $\mathbf{A}[X]$ of $\mathbf{A}$ in $\mathcal{V}$ by X is characterized by: (i) $X \subseteq A[X]$; (ii) for any homomorphism $f : \mathbf{A} \to \mathbf{B} \in \mathcal{V}$ and any function $g : X \to B$ there exists a unique homomorphism $f^* : \mathbf{A}[X] \to \mathbf{B}$ extending both f and g.

Direct sums of join-semilattices: Let $(\mathbf{A}_i)_{i\in I}$ be a family of join-semilattices. We say that $\mathbf{B}$ is the *direct sum* of the family $(\mathbf{A}_i)_{i\in I}$, notation $\mathbf{B} = \oplus_{i\in I}\mathbf{A}_i$, if $\mathbf{B} \leq \prod_{i\in I}\mathbf{A}_i$ is the subalgebra of the sequences $(a_i \in A_i : i \in I)$ such that $\{i : a_i \neq 0\}$ is finite.

3 Bunch-applicative algebras

Let $\mathbf{R}$ be a semiring with unit. We introduce an *algebraic signature Γ* constituted by a binary operator "+", a nullary operator "0", a family of unary operators r ($r \in \mathbf{R}$), and a family of operators $\cdot_k$ ($k \in \mathbb{N}$) of ariety $k+1$, called collectively *applications.*

The prefix notation for application is indeed cumbersome for common use so that each operation $\cdot_n(a, b_0, \dots, b_{n-1})$ will be replaced by the lighter $a[b_0, \dots, b_{n-1}]$, so that, for example, $\cdot_0(a) = a[\,]$ and $\cdot_2(a, b, c) = a[b, c]$. Another reason for this choice is that, when we write $a[b_0, \dots, b_{n-1}]$, we think to the element a applied to the "bag" $[b_0, \dots, b_{n-1}]$. We will also adopt the usual convention that application associates to the left. For a sequence $\bar{b}$ of length n, we adopt the further notational simplification to write $a\bar{b}$ instead of $a[b_0, \dots, b_{n-1}]$. By a^k we indicate the sequence $(a, \dots, a)$ of length k, thus $ba^k = b[a, \dots, a]$ (a repeated k times), with the convention that $ba^0 = b[\ ]$. Note that the above conventions lead to write just ab for $a[b]$: clearly in case b is itself an application cd, we are obliged to write $a[cd]$ in order to avoid any ambiguity.

Definition 1. *A Γ-algebra is called a* bunch-applicative algebra *if it satisfies the following axioms, which are universally quantified.*

Commutative Monoid Axioms :
$(x + y) + z = x + (y + z);\qquad x + y = y + x;\qquad 0 + x = x$
Module Axioms $(r, s \in R)$:
$r(x + y) = rx + ry;\quad (r + s)x = rx + sx;\quad (rs)x = r(sx);\quad 1x = x;\quad 0x = 0$
Multiset Axiom : $x[y_0, \dots, y_{k-1}] = x[y_{\sigma(0)}, \dots, y_{\sigma(k-1)}]\quad (\sigma \in \mathfrak{S}_k)$
Multilinearity Axioms :
$x[y_0 \dots, 0, \dots, y_{k-1}] = 0;\qquad 0[y_0, \dots, y_{k-1}] = 0$
$(ax + by)[y_0, \dots, y_{k-1}] = a(x[y_0, \dots, y_{k-1}]) + b(y[y_0, \dots, y_{k-1}])$
$x[\dots, ay + bz, \dots] = a(x[\dots, y, \dots]) + b(x[\dots, z, \dots])$

If a signature Δ extends Γ, we say that a Δ-algebra $\mathbf{A}$ is a bunch-applicative Δ-algebra if it is so the Γ-reduct of $\mathbf{A}$.

4 The linear resource lambda calculus

We will now briefly introduce the linear fragment of resource λ-calculus ($r\lambda$-calculus, for short).

Let V be an infinite countable set of *names* which represent the variables of λ-calculus. The set Λ^r of *$r\lambda$-terms* is described by the following grammar ($x \in V$, $r \in R$): $M, N, L ::= x \mid \lambda x.M \mid M[N_1, \dots, N_k] \mid rM \mid M + L \mid 0$. As usual, $r\lambda$-terms are to be considered modulo α-conversion.

The definition of the linear substitution of a "bag" $[N_1, \dots, N_k]$ for x in M, notation $M\langle[N_1, \dots, N_k]/x\rangle$, can be found in [16]. Notice that the linear substitution is a "meta-operation" as the usual substitution in classical λ-calculus.

Finally we present the system of equations governing the $r\lambda$-calculus:

Axioms of a bunch-applicative algebra (instantied by $r\lambda$-terms)
βr-conversion: $(\lambda x.M)[N_1, \dots, N_k] = M\langle[N_1, \dots, N_k]/x\rangle$
Multilinearity for λ $(r, s \in R)$: $\lambda x.0 = 0;\ \ \lambda x.(rM + sN) = r\lambda x.M + s\lambda x.N$

Fig 1: Equational axioms for linear $r\lambda$-calculus over the semiring $\mathbf{R}$

Warning: In the remaining part of this paper we take $\mathbf{R}$ to be the semiring $\mathbf{2} = (\{0, 1\}, +, \cdot, 0, 1)$ of truth values with $0 < 1$, where $+$ and $\cdot$ are the usual lattice operation. This means that the module axioms can be substituted by the single identity $x + x = x$. In this way, the commutative monoid becomes a join semilattice. Working with coefficients from an arbitrary semiring would increase significantly the complexity of notations in statements and proofs, covering the essence of the results presented in this paper.

Theorem 1. *[7, 21] The $r\lambda$-calculus over the semiring* $\mathbf{2}$ *is consistent, because it enjoys confluence and strong normalization.*

5 The $r\lambda$-calculus from the algebraic point of view

The variable-binding properties of λ-abstraction prevent names in $r\lambda$-calculus from operating as real algebraic variables. The same problem occurs in classic λ-calculus and was solved by Pigozzi and Salibra [17] by introducing the variety of λ-abstraction algebras. We adopt here their solution and transform the names (i.e., elements of V) into constants.

Definition 2. *The signature Γ_λ is an extension of the signature Γ of bunch-applicative algebra by a family of nullary operators $x \in V$, one for each element of V, and a family of unary operators λx ($x \in V$), called collectively λ-abstractions.*

The $r\lambda$-terms are just the Γ_λ-terms without occurrences of algebraic variables.

The absolutely free Γ_λ-algebra is the algebra $\mathbf{\Lambda^r} = (\Lambda^r, +, 0, \cdot_k, \lambda x, x)_{x \in V, k \in \mathbb{N}}$, where Λ^r is the set of $r\lambda$-terms and the operations are just the syntactical operations of construction of the $r\lambda$-terms.

Definition 3. *A $r\lambda$-theory is any congruence on $\mathbf{\Lambda^r}$ (with respect to all the involved operations) including all the identities of Figure 1.*

The least $r\lambda$-theory, denoted by $\lambda\beta r$, is consistent by Theorem 1. If T is a $r\lambda$-theory, we denote by $\mathbf{\Lambda^r_T} \equiv \mathbf{\Lambda^r}/T$ the quotient of the absolutaly free Γ_λ-algebra $\mathbf{\Lambda^r}$ modulo the $r\lambda$-theory T. $\mathbf{\Lambda^r_T}$ is called the *term algebra* of T.

We now abstract the notion of term algebra by introducing the variety of resource λ-abstraction algebras as a pure algebraic theory of $r\lambda$-calculus. The term algebras of $r\lambda$-theories are the first example of resource λ-abstraction algebra. Another example will be given in Section 9.

Definition 4. *A* resource λ-abstraction algebra *(*RLAA*, for short) is a bunch-applicative Γ_λ-algebra satisfying the following identities (for all $a \in A$, $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in A^*$, and $x \neq y \in V$):*

$$(r\alpha)\quad (\lambda x.a)x^k = a,\ (\lambda y.a)[\,] = a \ \Rightarrow\ \lambda x.a = \lambda y.(\lambda x.a)y^k$$

$$(r\beta_1)\quad (\lambda x.x)\bar{a} = \begin{cases} a_0 & \text{if } |\bar{a}| = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$(r\beta_2)\quad (\lambda x.y)\bar{a} = \begin{cases} y & \text{if } |\bar{a}| = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(r\beta_3)\quad (\lambda x \lambda x.a)\bar{b} = \begin{cases} \lambda x.a & \text{if } |\bar{b}| = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(r\beta_4)\quad (\lambda y.\bar{b})[\,] = \bar{b} \ \Rightarrow\ (\lambda x.\lambda y.a)\bar{b} = \lambda y.(\lambda x.a)\bar{b}$$

$$(r\beta_5)\quad (\lambda x.a\bar{b})\bar{c} = \Sigma_{\bar{d} \in \mathcal{Q}_{\bar{c},k}} (\lambda x.a)\bar{d}_0[(\lambda x.b_0)\bar{d}_1, \ldots, (\lambda x.b_{k-1})\bar{d}_k]\ (|\bar{b}| = k)$$

$$(r\gamma)\quad (\lambda x.a)x^n \leq a$$

$$(r\lambda)\quad \lambda x.0 = 0; \quad \lambda x.(a+b) = \lambda x.a + \lambda x.b.$$

Zero-dimensional elements are a generalization of the $r\lambda$-terms which have no free occurrence of names. We say that a *name $x \in V$ does not occur free in $a \in A$* if $(\lambda x.a)[\,] = a$. An element $a \in A$ is *zero-dimensional* if $(\lambda x.a)[\,] = a$ for all $x \in V$. The set of zero-dimensional elements will be denoted by $\mathrm{Zd}\,\mathbf{A}$.

Finite-dimensional elements are a generalization of the $r\lambda$-terms. An element x is *finite-dimensional* if there exists $X \subseteq_{\mathrm{f}} V$ such that $(\lambda a.x)[\,] = x$ for all $a \in V \backslash X$ and, for all $a \in X$, there is exactly one $n \neq 0$ such that $(\lambda a.x)a^n = x$ (in such a case $(\lambda a.x)a^k = 0$ for all $k \neq n$). Of course, every zero-dimensional element is also finite-dimensional. However, in general, a RLAA may have elements where all the names occur free. These elements are a generalization of infinite λ-terms.

A RLAA $\mathbf{A}$ is called *locally finite* if it is generated by its finite-dimensional elements (through the join operator). Every RLAA $\mathbf{A}$ contains a canonical locally finite RLAA, which is the subalgebra of $\mathbf{A}$ generated by all its finite-dimensional elements. This algebra will be denoted by $\mathsf{Lf}\mathbf{A}$.

Proposition 1. *(i) The term algebra $\mathbf{\Lambda^r_T}$ is a locally finite* RLAA.
(ii) The minimal subalgebra of a RLAA $\mathbf{A}$ *is isomorphic to $\mathbf{\Lambda^r_T}$ for some T.*

6 Resource combinatory algebras

In this section we introduce a class of algebras which are to the $r\lambda$-calculus what combinatory algebras are to the classical λ-calculus.

The signature $\mathcal{G}_c$ of is an extension of the signature Γ of bunch-applicative algebras by a nullary operator K and a family of nullary operators $S_{\bar{n}}$ ($\bar{n} \in \mathbb{N}^*$).

Recall the definition of the set $\mathcal{Q}_{\bar{z},\bar{n}}$ from the preliminaries.

Definition 5. *A* resource combinatory algebra *(*RCA*, for short) is a bunch-applicative Γ_c-algebra satisfying the following identities:*

(K) $K\bar{x}\bar{y} = x_0$ *if* $|\bar{x}| = 1$ *and* $|\bar{y}| = 0$*; it is equal to* 0*, otherwise.*

$$(S_{\bar{n}})\quad S_{\bar{n}}\bar{x}\bar{y}\bar{z} = \begin{cases} \sum_{\bar{\bar{c}} \in \mathcal{Q}_{\bar{z},\bar{n}}} x_0\bar{c}_0[y_0\bar{c}_1, \dots, y_{k-1}\bar{c}_k] & \textit{if } |\bar{x}| = 1,\ k = |\bar{y}| = |\bar{n}| - 1, \\ & \textit{and } |\bar{z}| = \Sigma\bar{n} \\ 0 & \textit{otherwise.} \end{cases}$$

The variety of resource combinatory algebras is denoted by RCA.

We secretly think of K and $S_{\bar{n}}$ as the following $r\lambda$-terms:

$$K_\lambda \equiv \lambda xy.y; \quad S_{\bar{n},\lambda} \equiv \lambda xyz.xz^{n_0}[yz^{n_1}, \dots, yz^{n_{k-1}}] \quad (|\bar{n}| = k) \tag{1}$$

We define *(resource) monomials* with names in V and constant in A by the following grammar: $t ::= x \mid c_a \mid K \mid S_{\bar{n}} \mid t_0[t_1, \dots t_n]$ ($\bar{n} \in \mathbb{N}, a \in A$). A *(resource) polynomial* is a finite sum of monomials: $t_1 + \dots + t_n$. We denote by $P(A)$ the set of all polynomials with names in V and constant in A.

For a monomial t we define the *degree* $deg_x(t)$ of $x \in V$ in t as the number of occurrences of the name x in t.

We define an abstraction operation on polynomials, with which the abstraction of $r\lambda$-calculus can be simulated. First of all we need to define the combinator $I \equiv S_{(1)}K[\,]$. It is immediate to see that $I\bar{x} = x_0$ if $|\bar{x}| = 1$ and $I\bar{x} = 0$ otherwise.

Definition 6. *Let $t, t_1, \dots, t_n$ be monomials. We define a new monomial $\lambda^* x.t$ as follows:*

(i) $\lambda^* x.t \equiv Kt$ *if* $deg_x(t) = 0$
(ii) $\lambda^* x.x \equiv I$
(iii) $\lambda^* x.t_0[t_1, \dots, t_k] \equiv S_{\bar{n}}[\lambda^* x.t_0][\lambda^* x.t_1, \dots, \lambda^* x.t_k]$ $(\bar{n} = (deg_x(t_0), \dots, deg_x(t_k)))$ *if* $\exists i\ deg_x(t_i) \neq 0$.

We extend the definition of abstraction to polynomials: $\lambda^* x.\Sigma_{i=1}^n t_i = \Sigma_{i=1}^n \lambda^* x.t_i$.

Let t be a monomial with $deg_x(t) = n$, $\bar{p}$ be a sequence of n polynomials and $\sigma \in \mathfrak{S}_n$ be a permutation. Then the expression $t\{\bar{x} := \sigma\bar{p}\}$ denotes the *simultaneous* substitution of the i-th occurrence x^i of x in t by the polynomial $p_{\sigma(i)}$ $(i = 1, \dots, n)$.

Lemma 1. *Let* $\mathbf{A}$ *be a* RCA. *For any monomial* t, *any sequence* $\bar{p}$ *of polynomials, and any name* x *we have:* $\mathbf{A} \vDash (\lambda^* x.t)\bar{p} = \Sigma_{\sigma \in \mathfrak{S}_n} t\{\bar{x} := \sigma\bar{p}\}$ *if* $n = deg_x(t) = |\bar{p}|$; *it is equal to* 0, *otherwise.*

Let $\mathbf{A}$ be a RLAA. The *combinatory reduct* of $\mathbf{A}$ is defined as the algebra $\mathrm{Cr}\,\mathbf{A} = (A, \cdot_k, K_\lambda^{\mathbf{A}}, S_{\bar{n},\lambda}^{\mathbf{A}})$, where the $r\lambda$-terms K_λ and $S_{\bar{n},\lambda}$ are defined in (1) above. The subalgebra of $\mathrm{Cr}\,\mathbf{A}$ constituted by the zero-dimensional elements of $\mathbf{A}$ will be denoted by $\mathbf{Zd}\,\mathbf{A}$.

Proposition 2. *Let* $\mathbf{A}$ *be a locally finite* RLAA. *Then,* $\mathrm{Cr}\,\mathbf{A}$ *is a* RCA.

The proof of the above proposition is trivial because of the hypothesis of locally finiteness. If we drop this hypothesis, then we cannot always apply α-conversion because elements may exist where all variables occur free.

The $r\lambda$-term t_λ associated with a polynomial t can be easily defined by induction: K, S are respectively translated into K_λ and $S_{\bar{n},\lambda}$ (see (1) above); $(\lambda^* x.t)_\lambda = \lambda x.t_\lambda$; $(t[s_1, \dots, s_n])_\lambda = t_\lambda[s_{1,\lambda}, \dots, s_{n,\lambda}]$; $(\Sigma t_i)_\lambda = \Sigma t_{i,\lambda}$.

The following lemma can be shown by induction over the complexity of the polynomial p. If $\mathbf{A}$ is a RLAA, then $p^{\mathrm{Cr}\,\mathbf{A}}$ denotes the interpretation of p into $\mathrm{Cr}\,\mathbf{A}$.

Lemma 2. *Let* $\mathbf{A}$ *be a* RLAA *and* p *be a polynomial. Then,* $p^{\mathrm{Cr}\,\mathbf{A}} = p_\lambda^{\mathbf{A}}$ *and* $(\lambda^* x.p)^{\mathrm{Cr}\,\mathbf{A}} = \lambda x^{\mathbf{A}}.p_\lambda^{\mathbf{A}}$.

7 Resource $\boldsymbol{\lambda}$-algebras

In this section we axiomatize the variety of resource λ-algebras ($r\lambda$-algebras for short), and prove that the free extension of an $r\lambda$-algebra in the variety of $r\lambda$-algebras can be turned into a RLAA, so that it validates all the rules of $r\lambda$-calculus.

For the subsequent developments, it turns out very important to isolate a particular family of combinators: for $n \in \mathbb{N}$, the *n-homogenizer* is the combinator $H_n \equiv S_{(0,n)}[KI]$. Using the equation schemata of RCAs we obtain that $H_n\bar{x}\bar{y} = x_0\bar{y}$ if $|\bar{x}| = 1$ and $|\bar{y}| = n$; it is equal to 0, otherwise.

The elements of the form $H_n a$ are the semantical counterpart of monomials of the form $\lambda^* x.t$, with $deg_x(t) = n$. Via H_n is in fact possible to give a semantical notion of degree: $a \in A$ is called *homogeneous of degree* n iff $H_n a = a$.

We now define $r\lambda$-algebras. We advice the reader that some identities defining $r\lambda$-algebras are difficult to read, but we need them to show that the free extension of a $r\lambda$-algebra satisfies the ξ-rule (see Theorem 2 below).

An identity $p = q$, where $p = \Sigma_{i\in I} t_i$ and $q = \Sigma_{j\in J} u_j$, is *homogeneous* if, for every $x \in V$, $deg_x(t_i) = deg_x(u_j)$ for all $i \in I$, $j \in J$. The λ^*-closure of $p = q$ is the identity $\lambda^*\bar{x}.p = \lambda^*\bar{x}.q$, where $\bar{x}$ is the sequence of all names occurring in p, q. If $p = q$ is homogeneous then $\mathbf{A} \vDash \lambda^*\bar{x}.p = \lambda^*\bar{x}.q$ implies that $\mathbf{A} \vDash p = q$.

Definition 7. *A* RCA $\mathbf{A}$ *is a* $r\lambda$-algebra *if it satisfies the* λ^**-closure of the following identities:*

(R0) $H_n[H_m x] = H_m x$ *if* $n = m$*; it is equal to* 0*, otherwise.*
(R1) $K = H_1 K$; $\quad Kx = H_0[Kx]$
(R2) $S_{\bar{n}} = H_1 S_{\bar{n}}$; $\quad S_{\bar{n}} x = H_{|\bar{n}|-1}[S_{\bar{n}} x]$; $\quad S_{\bar{n}} x\bar{y} = H_{\Sigma\bar{n}}[S_{\bar{n}} x\bar{y}]$
(R3) $S_{\bar{m}}[S_{\bar{n}}[KK]\bar{x}]\bar{y} = \begin{cases} H_{n_1} x_0 \text{ if } |\bar{x}| = 1,\ |\bar{y}| = 0,\ \bar{n} = (0, n_1),\ \bar{m} = (n_1) \\ 0 \quad \text{otherwise} \end{cases}$
(R4) $S_{\bar{n}}[S_{\bar{m}}[S_{\bar{p}}[KS_{\bar{l}}]\bar{x}]\bar{y}]\bar{z} =$
$\Sigma_{\bar{\bar{s}}\in\mathcal{Q}_{\bar{z},\bar{l}}} S_{(\Sigma(m_0\cdot\bar{o}_0),\dots,\Sigma(m_k\cdot\bar{o}_k))}[S_{m_0\cdot\bar{o}_0} x_0 \bar{s}_0][S_{m_1\cdot\bar{o}_1} y_0 \bar{s}_1, \dots, S_{m_k\cdot\bar{o}_k} y_{k-1} \bar{s}_k]$,
if $|\bar{x}| = 1$, $\bar{p} = (0, p_1)$, $|\bar{y}| + 1 = |\bar{m}| = k$, $\bar{n} = (\Sigma\bar{m}) \cdot \bar{n}'$, $|\bar{z}| = |\bar{n}'| = \Sigma\bar{l}$, $\bar{m} = p_1 \cdot \bar{l}$*, and, for each* $\bar{\bar{s}} \in \mathcal{Q}_{\bar{z},\bar{l}}$, $\bar{\bar{o}} \in \mathcal{Q}_{\bar{n}',\bar{l}}$ *is the partition of* $\bar{n}'$ *induced by* $\bar{\bar{s}}$*; it is equal to* 0*, otherwise.*
(R5) $K[x\bar{y}] = S_{0^{k+1}}[Kx][Ky_0, \dots, Ky_{k-1}]$ $(|\bar{y}| = k)$
(R6) $H_k x = S_{0\cdot 1^k}[Kx]I^k$.

The variety of $r\lambda$-algebras will be denoted by RLA.

The next lemma shows the aforementioned connection between homogenizers and the induced λ-abstraction on polynomials.

Lemma 3. *Let* $\mathbf{A}$ *be a* RLA *and* t *be a monomial. Then* $\mathbf{A} \vDash H_n[\lambda^* x.t] = \lambda^* x.t$ *if* $n = deg_x(t)$*; it is equal to* 0*, otherwise.*

Theorem 2. *The free extension* $\mathbf{A}[V]$ *of a* $r\lambda$*-algebra* $\mathbf{A}$ *by the set* V *of names in the variety* RCA *satisfies the following* ξ*-rule, for all polynomials* $p, q \in P(A)$*:*

$$(\xi) \qquad \mathbf{A}[V] \vDash p = q \ \Rightarrow\ \mathbf{A}[V] \vDash \lambda^* x(p) = \lambda^* x(q).$$

We apply the above theorem to define λ-abstraction operators on $A[V]$. For any $e \in A[V]$, we define $\lambda x.e = \lambda^* x(p)$, for some polynomial $p \in e$. Rule ξ validates the above definition of λx. Define the algebra $\mathbf{A}[V]_\lambda = (A[V], +, 0, \cdot_k, \lambda x, x)_{x\in V}$, where $(A[V], +, 0, \cdot_k,)$ is the Γ-reduct of the free extension $\mathbf{A}[V]$, λx is defined as above and the name $x \in V$ is viewed as a nullary operator.

Corollary 1. $\mathbf{A}[V]_\lambda$ *is a locally finite* RLAA *such that* $K_\lambda^{\mathbf{A}[V]_\lambda} = K^{\mathbf{A}}$ *and* $S_{\bar{n},\lambda}^{\mathbf{A}[V]_\lambda} = S_{\bar{n}}^{\mathbf{A}}$.

$\mathbf{A}[V]_\lambda$ is called the RLAA *freely generated by the* $r\lambda$*-algebra* $\mathbf{A}$.

Corollary 2. *Let* $\mathbf{A}$ *be a* RCA*. Then,* $\mathbf{A}$ *is a* $r\lambda$*-algebra if, and only if,* $\mathbf{A}$ *is isomorphic to the zero-dimensional reduct* $\mathbf{Zd}\,\mathbf{B}$ *of some* RLAA $\mathbf{B}$.

This corollary is very useful to prove when a RCA is a RLA (see Section 9).

The construction of the free extension will turn out to be the construction of a graded algebra as a direct sum of specific join semilattices. We now provide the proof of Theorem 2. The proof generalizes a construction by Krivine [11].

Lemma 4. *Let $B_n = \{a \in A : H_n a = a\}$. Then $(B_n, +, 0)$ is a join subsemilattice of $(A, +, 0)$ such that $B_n \cap B_m = \{0\}$.*

We now define an algebra $\mathbf{B} = (B, +, 0, \bullet_k, KK, KS_{\bar{n}})_{k\in\mathbb{N},\bar{n}\in\mathbb{N}^*}$, called the $\mathbb{N}$-*graded algebra generated by* $\mathbf{A}$ in the similarity type of RCA by setting:

1. $(B, +, 0) = \oplus_{n\in\mathbb{N}}(B_n, +, 0)$ is the direct sum of the join semilattices $(B_n, +, 0)$.
2. each application "$\bullet_k$" is the extension by linearity of the following operation: $a_0 \bullet_k [a_1, \dots, a_k] = S_{\bar{p}} a_0[a_1, \dots, a_k]$, with $a_i \in B_{p_i}$.

Lemma 5. *The $\mathbb{N}$-graded algebra $\mathbf{B}$ is a RCA which satisfies the following conditions: (i) KK and $KS_{\bar{n}}$ are elements of B_0; (ii) $B_{d_0} \bullet_k [B_{d_1}, \dots, B_{d_{k-1}}] \subseteq B_{\Sigma\bar{d}}$, for all $\bar{d} \in \mathbb{N}^{k+1}$.*

Lemma 6. *The map ι, defined by $\iota(a) = Ka$, is an embedding of $\mathbf{A}$ into $\mathbf{B}$.*

Proof. By (R5) $\iota(a_0[a_1, \dots, a_n]) = K[a_0[a_1, \dots, a_n]] = S_{0^{n+1}}[Ka_0][Ka_1, \dots, Ka_n] = \iota(a_0) \bullet [\iota(a_1), \dots, \iota(a_n)]$. The other properties are trivial. □

By Lemma 6 $\mathbf{B}$ is a RLA. We are now going to show the connection between the $\mathbb{N}$-graded algebra $\mathbf{B}$ and the free extension $\mathbf{A}[x]$ by one name x.

Lemma 7. *The $\mathbb{N}$-graded algebra $\mathbf{B}$ is the free extension of $\mathbf{A}$ by one name in the variety RLA. Consequently, $\mathbf{B} \cong \mathbf{A}[x]$.*

Proof. We prove that $\mathbf{B} \cong \mathbf{A}[x]$. Let $\mathbf{C}$ be a RCA, $c \in C$ and $f : \mathbf{A} \to \mathbf{C}$ be a homomorphism. Define a family of functions $f_k : B_k \to C$ ($k \in \mathbb{N}$) as follows: $f_k(a) = f(a) \cdot^{\mathbf{C}} c^k$ for all $a \in B_k$. Let $f^* : B \to C$ be the unique extension by linearity of the family of functions f_k, that is, $f^*(0) = 0$ and $f^*(\Sigma_{i=1}^m a_i) = \Sigma_{i=1}^m f_{d_i}(a_i)$ for $a_i \in B_{d_i}$.

We now prove that f^* is a homomorphism. It is immediate to check that f^* is a monoid homomorphism, using multi-linearity of application. Since f^* extends f by linearity, it suffices to prove the following:

$$\begin{aligned} f_{\Sigma\bar{e}}(a \bullet \bar{b}) &= f(S_{\bar{e}}a\bar{b})c^{\Sigma\bar{e}} = S^{\mathbf{C}}_{\bar{e}} f(a)[f(b_0), \dots, f(b_{n-1})]c^{\Sigma\bar{e}} \text{ , since } f \text{ is hom,} \\ &= f(a)c^{e_0}[f(b_0)c^{e_1}, \dots, f(b_{n-1})c^{e_n}], \text{ by } (S_{\bar{n}}) \text{ and idempotence of sum,} \\ &= f_{e_0}(a)[f_{e_1}(b_0), \dots, f_{e_n}(b_{n-1})]. \end{aligned}$$

We have: $f_0(K^{\mathbf{A}}K^{\mathbf{A}}) = f(K^{\mathbf{A}}K^{\mathbf{A}})[\,] = K^{\mathbf{C}}K^{\mathbf{C}}[\,] = K^{\mathbf{C}}$. A similar argument shows that $f^*(K^{\mathbf{A}}S^{\mathbf{A}}_{\bar{n}}) = S^{\mathbf{C}}_{\bar{n}}$. This shows that f^* is a homomorphism.

We have: $f^*(\iota(a)) = f_0(Ka) = f(Ka)[\,] = K^{\mathbf{C}}f(a)[\,] = f(a)$; and $f^*(I) = f_1(I) = f(I)c = I^{\mathbf{C}}c = c$. This proves $f^* \circ \iota = f$ and $f^*(I) = c$.

Finally suppose $h : \mathbf{B} \to \mathbf{C}$ is another homomorphism satisfying $h \circ \iota = f$ and $h(I) = c$. The uniqueness of f^* is shown as follows:

$$\begin{aligned} h(a) &= h(H_k a) \text{ , for some } k \in \mathbb{N}, \\ &= h(S_{0\cdot 1^k}[Ka]I^k), \text{by axiom (R6)} \\ &= h((Ka) \bullet I^k) = h(Ka)(h(I))^k = h(\iota(a))c^k = f_k(a)c^k = f^*(a). \quad \square \end{aligned}$$

We denote by ι^* the unique isomorphism from $\mathbf{A}[x]$ onto $\mathbf{B}$ extending the embedding $\iota : \mathbf{A} \to \mathbf{B}$ defined in Lemma 6, and such that $\iota^*(x) = I$.

Lemma 8. *For all $a, b \in A$ we have $\mathbf{A}[x] \models ax^n = bx^k$ iff $\mathbf{A} \models H_n a = H_k b$.*

Proof. $\iota^*(ax^n) = \iota(a) \bullet I^n = S_{0 \cdot 1^n}[Ka]I^n = H_n a$, by (R6). We conclude since ι^* is an isomorphism. Of course, if $n \neq k$, then $H_n a = H_k b = 0$. □

Lemma 9. *For all polynomials p, q with at most the name x we have that $\mathbf{A}[x] \models p = q$ implies $\mathbf{A}[x] \models \lambda^* x.p = \lambda^* x.q$.*

Proof. First we prove the result for monomials t, u. Let $n = deg_x(t)$ and $k = deg_x(u)$. By Lemma 1 $\mathbf{A}[x] \models (\lambda^* x.t)x^n = t = u = (\lambda^* x.u)x^k$. Now by Lemma 8 and by Lemma 3 it follows that $\mathbf{A} \models \lambda^* x.t = H_n[\lambda^* x.t] = H_k[\lambda^* x.s] = \lambda^* x.s$; therefore trivially $\mathbf{A}[x] \models \lambda^* x.t = \lambda^* x.s$.

We now use the algebra $\mathbf{B}$, which is an isomorphic copy of $\mathbf{A}[x]$ through the isomorphism ι^* defined above. First we consider an element $a \in B_n$. We define $\lambda x.a = \iota^*(\lambda^* x.ax^n)$. The definition is well done because $\lambda^* x.ax^n$ is a monomial and we have shown the statement of the theorem for monomials. If b is an arbitrary element of B, then b can be written in a unique way as a finite sum $\Sigma_{i \in I} b_i$ of elements $b_i \in B_{n_i}$. Then we define $\lambda x.b = \Sigma_{i \in I} \lambda x.b_i$. □

The extension of the above lemma to polynomials with an arbitrary number of names is standard, because $\mathbf{A}[x, y] \cong \mathbf{A}[x][y]$ and $\mathbf{A}[x]$ is a RLA.

8 From resource to classical lambda calculus

After having introduced a number of structures which algebrize the resource λ-calculus, we show how, by some standard constructions, we can recover the algebraic models of classical λ-calculus. This is done, as often happens in mathematics, by the method of *ideal completion*.

Let $\mathbf{A}$ be a bunch-applicative Γ-algebra. An *ideal* is a downward closed subset X of A closed under join. For a subset $X \subseteq A$, $\downarrow X = \{y : \exists x_1, \dots, x_n \in X.\ y \leq \sum_{i=1}^n x_i\}$ is the *ideal generated* by X. We denote by $Ide(\mathbf{A})$ the collection of all ideals of $\mathbf{A}$.

Let $\mathbf{A}$ be a RCA. Define an algebra $Ide(\mathbf{A}) = (Ide(A), *, \underline{K}, \underline{S})$ by setting $\underline{K} = \downarrow \{K\}$; $\underline{S} = \downarrow \{S_{\bar{n}} : \bar{n} \in \mathbb{N}^*\}$; $X * Y = \downarrow \{a\bar{b} : a \in X,\ \bar{b} \in Y^*\}$.

If $\mathbf{B}$ is a RLAA we define the structure $Ide(\mathbf{A}) = (Ide(A), *, \underline{\lambda} x, \underline{a})_{x \in V}$ by setting $\underline{a} = \downarrow \{a\}$; $\underline{\lambda} x.X = \downarrow \{\lambda x.a : a \in X\}$ and the application $*$ as above.

Theorem 3. *(i) If $\mathbf{A}$ is a RCA, then $Ide(\mathbf{A})$ is a combinatory algebra.*

(ii) Let $\mathbf{A}$ be a RLAA and $\mathsf{Lf}\mathbf{A}$ be the subalgebra of $\mathbf{A}$ generated by its locally finite elements. Then $Ide(\mathsf{Lf}\mathbf{A})$ is a λ-abstraction algebra.

(iii) If $\mathbf{A}$ is a RLA, then $Ide(\mathbf{A})$ is a λ-algebra.

According to [6], we now define a translation of λ-terms to sets of $r\lambda$-terms. Following [1], we identify a Böhm tree with an ideal (downwards closed and directed set) of normal terms in the λ-calculus extended with a constant $\perp$ subject to the equations $\perp M = \perp$ and $\lambda x.\perp = \perp$. This way we can also translate Böhm trees into subsets of Λ^r.

As a matter of terminology, a $r\lambda$-term t is: *simple* if none of its subterms (including t) contains either "+" or "0"; *normal* if none of its subterms (including t) is of the form $(\lambda x.t')\bar{s}$; in *canonical form* if it is a sum of simple terms. By an easy argument involving the multilinearity axioms of the $r\lambda$-calculus and Theorem 1, we can argue that for every term $t \in \Lambda^r$, there exists a unique normal term s in canonical form which is equal to t.

We let $\mathrm{NF}(t) \subseteq \Lambda^r$ to be the set of all simple subterms of the normal canonical form of t.

Definition 8. *[6] Let $M \in \Lambda$ be a λ-term, possibly containing $\perp$. The set $\mathcal{T}(M) \subseteq \Lambda^r$ is defined inductively as follows:*

$$\mathcal{T}(x) = \{x\} \qquad \mathcal{T}(\perp) = \emptyset;$$
$$\mathcal{T}(\lambda x.N) = \{\lambda x.t : t \in \mathcal{T}(N)\} \qquad \mathcal{T}(PQ) = \{t\bar{s} : t \in \mathcal{T}(P),\ \bar{s} \in \mathcal{T}(Q)^*\}$$

Now $\mathcal{T}(BT(M)) \subseteq \Lambda^r$ is defined as $\cup\{\mathcal{T}(B) : B \in BT(M)\}$.

Lemma 10. *Let $\mathbf{A}$ be a locally finite* RLAA *and let $\mathbf{B} = Ide(\mathbf{A})$ be the* LAA *built over $\mathbf{A}$. Then for all $M \in \Lambda$, $|M|^{\mathbf{B}} = \downarrow \{|t|^{\mathbf{A}} : t \in \mathcal{T}(M)\}$.*

Theorem 4. *[6] Let M be a λ-term and let u be a normal simple $r\lambda$-term. Then $u \in \mathcal{T}(BT(M))$ iff there exists $s \in \mathcal{T}(M)$ such that $u \in NF(s)$.*

Lemma 11. *Let $\mathbf{A}$ be a locally finite* RLAA*. Then for all terms $M, N \in \Lambda$ we have $BT \vdash M = N$ implies $|M|^{Ide(\mathbf{A})} = |N|^{Ide(\mathbf{A})}$. In particular all λ-theories induced by the ideal completions of* RLAA*s are sensible.*

Proof. Suppose $BT(M) = BT(N)$. Then obviously $\mathcal{T}(BT(M)) = \mathcal{T}(BT(N))$. We conclude by applying Theorem 4 and Lemma 10 as follows: $|M|^{\mathbf{B}} = \downarrow \{|u|^{\mathbf{A}} : u \in \mathrm{NF}(t),\ t \in \mathcal{T}(M)\} = \downarrow \{|u|^{\mathbf{A}} : u \in \mathrm{NF}(t),\ t \in \mathcal{T}(N)\} = |N|^{\mathbf{B}}$. □

9 An example

$\mathcal{M}_{\mathrm{f}}(D)$ is the set of all finite multisets with elements in D, where $m \in \mathcal{M}_{\mathrm{f}}(D)$ is a function from D into $\mathbb{N}$ such that $m(a) = 0$ for all a belonging to a cofinite subset of D. The natural number $\sharp m = \Sigma_{a\in D} m(a)$ is the cardinality of m. The union $m \uplus p$ of two finite multisets is defined by $(m \uplus p)(a) = m(a) + p(a)$ for all $a \in D$.

Let D be a set together with an injection $\to: \mathcal{M}_{\mathrm{f}}(D) \times D \to D$. We adopt the convention that the operator "$\to$" associates to the right, i.e., $p \to (q \to \gamma)$ is abbreviated by $p \to q \to \gamma$.

We define an algebra $\mathbf{D} = (\mathcal{P}(D), \cup, \emptyset, \cdot_n, K, S_{\bar{k}})_{n\in\mathbb{N},\ \bar{k}\in\mathbb{N}^*}$ in the similarity type of RCA, where $K = \{[\alpha] \to [\,] \to \alpha : \alpha \in D\}$,

$$S_{\bar{k}} = \{[p_0 \to [\beta_1, \dots, \beta_n] \to \beta_0] \to [p_1 \to \beta_1, \dots, p_n \to \beta_n] \to (\uplus_{i=0}^n p_i) \to \beta_0 : \\ \beta_i \in D,\ p_i \in \mathcal{M}_{\mathrm{f}}(D),\ \sharp p_i = k_i, |\bar{k}| = n+1\}$$

and application is the extension by linearity of the following map on singleton sets (we write γ for $\{\gamma\}$, etc.): $\gamma[\beta_1, \dots, \beta_n] = \alpha$ if $\gamma = [\beta_1, \dots, \beta_n] \to \alpha$; it is equal to $\emptyset$, otherwise.

It is an easy calculation to show that $\mathbf{D}$ is a RCA. To prove that $\mathbf{D}$ is indeed a RLA, by Corollary 2 it is sufficient to embed $\mathbf{D}$ into the combinatory reduct of a suitable RLAA $\mathbf{E}$. We now define the algebra $\mathbf{E}$.

(i) $\mathcal{M}_{\mathrm{f}}(D)^{(V)} = \{\rho : V \to \mathcal{M}_{\mathrm{f}}(D) : \rho(x) = [\,] \text{ for almost all } x \in V\}$ is the set of *environments*;

(ii) $\perp$, defined by $x \mapsto [\,]$, is the *empty environment*, while, for an environment ρ and a finite multiset m, we define a new environment $\rho\{x := m\}$ as follows: $\rho\{x := m\}(x) = m$ and $\rho\{x := m\}(y) = \rho(y)$ if $y \neq x$.

We now construct the algebra $\mathbf{E} = (\mathcal{P}(\mathcal{M}_{\mathrm{f}}(D)^{(V)} \times D), \cup, \emptyset, \cdot_k, \lambda x^{\mathbf{E}}, x^{\mathbf{E}})_{x \in V, k \in \mathbb{N}}$ by defining application and abstraction as the extension by linearity of the following family of functions defined over the singletons (we write (ρ, α) for $\{(\rho, \alpha)\}$): $\lambda x^{\mathbf{E}}(\rho, \alpha) = (\rho\{x := [\,]\}, \rho(x) \to \alpha)$; $x^{\mathbf{E}} = \{(\perp\{x := [\alpha]\}, \alpha) : \alpha \in D\}$; $(\rho_0, \alpha_0)[(\rho_1, \alpha_1)\ \dots, (\rho_n, \alpha_n)] = (\uplus_{i=0}^{n} \rho_i, \alpha)$ if $\alpha_0 = [\alpha_1, \dots, \alpha_n] \to \alpha$; it is equal to $\emptyset$, otherwise. Notice that $(\lambda x(\rho, \alpha))x^n = (\rho, \alpha)$ if, and only if, $\sharp\rho(x) = n$.

Theorem 5. *Algebra* $\mathbf{E}$ *is a* RLAA *and map* $h : \mathcal{P}(D) \to \mathrm{Zd}\, E$*, defined by* $h(X) = \{(\perp, \alpha) : \alpha \in X\}$ *is an embedding from* $\mathbf{D}$ *into* $\mathrm{Cr}\, \mathbf{E}$*, making* $\mathbf{D}$ *an* RLA.

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10 Appendix: omitted proofs

Proof of Lemma 1. The proof is an easy induction. The only interesting case is that in which $t \equiv t_0[t_1, \ldots, t_k]$ and there exists i such that $deg_x(t_i) \neq 0$.

$$\begin{aligned}
&(\lambda^* x.t_0[t_1, \ldots, t_k])\bar{p} \\
&= S_{\bar{n}}[\lambda^* x.t_0][\lambda^* x.t_1, \ldots, \lambda^* x.t_k]\bar{p}, \quad \text{by Def. 6(iii)} \\
&= \Sigma_{\bar{p} \in \mathcal{Q}_{\bar{p},\bar{n}}}(\lambda^* x.t_0)\bar{p}_0\big[(\lambda^* x.t_1)\bar{p}_1, \ldots, (\lambda^* x.t_k)\bar{p}_k\big], \quad \text{by Def. 5} \\
&= \Sigma_{\bar{p} \in \mathcal{Q}_{\bar{p},\bar{n}}}(\Sigma_{\sigma_0 \in \mathfrak{S}_{n_0}} t_0\{\bar{x} := \sigma_0 \bar{p}_0\})[\Sigma_{\sigma_1 \in \mathfrak{S}_{n_1}} t_1\{\bar{x} := \sigma_1 \bar{p}_1\}, \ldots], \text{ by ind. hyp.} \\
&= \Sigma_{\bar{p} \in \mathcal{Q}_{\bar{p},\bar{n}}} \Sigma_{(\sigma_0, \ldots, \sigma_k) \in \prod_{j=0}^k \mathfrak{S}_{n_j}} t_0\{\bar{x} := \sigma_0 \bar{p}_0\}[t_1\{\bar{x} := \sigma_1 \bar{p}_1\}, \ldots] \\
&= \Sigma_{\bar{p} \in \mathcal{Q}_{\bar{p},\bar{n}}} t_0\{\bar{x} := \bar{p}_0\}[t_1\{\bar{x} := \bar{p}_1\}, \ldots], \text{ by } (\sigma_0 \bar{p}_0, \sigma_1 \bar{p}_1, \ldots) \in \mathcal{Q}_{\bar{p},\bar{n}} \\
&= \Sigma_{\sigma \in \mathfrak{S}_m}(t_0[t_1, \ldots, t_k])\{\bar{x} := \sigma \bar{p}\} \ (m = \Sigma \bar{n}).
\end{aligned}$$

Of course if $|\bar{p}| \neq \Sigma \bar{n}$, then the above calculation yields 0 as a result. ☐

Proof of Lemma 3. If $deg_x(t) = 0$, then

$$\lambda^* x.t = Kt = H_0[Kt] \text{ (by (R1)) } = H_0[\lambda^* x.t].$$

If $t \equiv x$, then

$$\lambda^* x.x = I = S_{(1)}K[\,] = H_1[S_{(1)}K[\,]] \text{ (by (R2) } = H_1[\lambda^* x.x].$$

Let $t \equiv t_0[t_1, \ldots, t_k]$ with $deg_x(t_i) \neq 0$ for some i, and $\bar{n} = (deg_x(t_0), \ldots, deg_x(t_k))$. Then we have:

$$\begin{aligned}
\lambda^* x.t_0[t_1, \ldots, t_k] &= S_{\bar{n}}[\lambda^* x.t_0][\lambda^* x.t_1, \ldots, \lambda^* x.t_k] \\
&= H_{\Sigma \bar{n}}[S_{\bar{n}}[\lambda^* x.t_0][\lambda^* x.t_1, \ldots, \lambda^* x.t_k]] \quad \text{by (R2)} \\
&= H_{deg_x(t)}[\lambda^* x.t_0[t_1, \ldots, t_k]]. \qquad \square
\end{aligned}$$

Proof of Corollary 1. It is straightforward to check the axioms of RLAA. We now prove the last part of the corollary.

$$\begin{aligned}
\lambda^* xy.x &= S_{(0,1)}[KK]I \text{ by definition} \\
&= H_1 K \qquad\quad \text{by (R6)} \\
&= K \qquad\qquad \text{by (R1).}
\end{aligned}$$

Let $\bar{l} = (l_0, \ldots, l_r)$ and $t \equiv \lambda^* xyz.xz^{l_0}[yz^{l_1}, \ldots, yz^{l_r}]$. Then we have:

$t = \lambda^* xy.S_{\bar{l}}[S_{0\cdot 1^{l_0}}[Kx]I^{l_0}][S_{0\cdot 1^{l_1}}[Ky]I^{l_1},\dots,S_{0\cdot 1^{l_r}}[Ky]I^{l_r}]$ by definition
$= \lambda^* xy.S_{0\cdot 1^{\Sigma\bar{l}}}[S_{0^{|\bar{l}|}}[S_{(0,0)}[KS_{\bar{l}}][Kx]](Ky)^{|\bar{l}|-1}]I^{\Sigma\bar{l}}$, where we have applied (R4) as follows: $\bar{n} \equiv 0\cdot 1^{\Sigma\bar{l}}$, $\bar{m} \equiv 0^{|\bar{l}|}$, $\bar{p} = (0,0)$, $\bar{x} \equiv Kx$, $\bar{y} \equiv (Ky)^{|\bar{l}|-1}$, $\bar{z} \equiv I^{\Sigma\bar{l}}$
Note that we have just one $\bar{l}$-partition of $\bar{z} \equiv I^{\Sigma\bar{l}}$
$= \lambda^* xy.S_{0\cdot 1^{\Sigma\bar{l}}}\left[S_{0^{|\bar{l}|}}[K[S_{\bar{l}}x]](Ky)^{|\bar{l}|-1}\right]I^{\Sigma\bar{l}}$ by (R5)
$= \lambda^* xy.S_{0\cdot 1^{\Sigma\bar{l}}}\left[K[S_{\bar{l}}xy^{|\bar{l}|-1}]\right]I^{\Sigma\bar{l}}$ by (R5)
$= \lambda^* xy.H_{\Sigma\bar{l}}[S_{\bar{l}}xy^{|\bar{l}|-1}]$ by (R6)
$= \lambda^* xy.S_{\bar{l}}xy^{|\bar{l}|-1}$ by (R2)
$= \lambda^* x.S_{(0,1,\dots,1)}[K[S_{\bar{l}}x]]I^{|\bar{l}|-1}$ by definition
$= \lambda^* x.H_{|\bar{l}|-1}[S_{\bar{l}}x]$ by (R6)
$= \lambda^* x.S_{\bar{l}}x$ by (R2)
$= S_{(0,1)}[KS_{\bar{l}}]I$ by definition
$= H_1 S_{\bar{l}}$ by (R6)
$= S_{\bar{l}}$ by (R2). □

Proof of Corollary 2. (⇒) Consider the RLAA $\mathbf{A}[V]_\lambda$ freely generated by $\mathbf{A}$. We show that $\mathbf{A} \cong \mathbf{Zd}\,\mathbf{A}[V]_\lambda$.

($A \subseteq \mathrm{Zd}\,\mathbf{A}[V]_\lambda$): For all $a \in A$ and all $x \in V$ we have: $(\lambda x.a)[\,] = (\lambda^* x.c_a)[\,] = Kc_a[\,] = c_a$, so that every $a \in A$ is zero-dimensional.

($\mathrm{Zd}\,\mathbf{A}[V]_\lambda \subseteq A$): Let $e \in \mathrm{Zd}\,\mathbf{A}[V]_\lambda$, let $p = \sum_{i\in I} t_i \in e$ be a polynomial and let $x_1,\dots,x_n$ be all names occurring in p. Let y be one of the above names. Since e is zero-dimensional, then $(\lambda^* y.p)[\,] = \sum_{i\in I}(\lambda^* y.t_i)[\,] = p$. By Lemma 1 we have that either $(\lambda^* y.t_i)[\,] = t_i$ or $(\lambda^* y.t_i)[\,] = 0$. Let $J = \{i \in I : (\lambda^* y.t_i)[\,] = t_i\} = \{i \in I : deg_y(t_i) = 0\}$. Then we have: $p = \sum_{j\in J}(\lambda^* y.t_j)[\,] = \sum_{j\in J} t_j$. Since $deg_y(t_j) = 0$ ($j \in J$), then p is equivalent to a polynomial without occurrences of name y. By iterating the reasoning with all other names, at the end of the process we get that $p = \sum_{r\in K} t_r$ is equivalent to a polynomial without names, whose interpretation is of course in A.

(⇐) By Lemma 2 is sufficient to verify that the RLAA $\mathbf{A}[V]$ freely generated by $\mathbf{A}$ satisfies all identities $t_\lambda = u_\lambda$, where $t = u$ is one of the axioms (R0)-(R6). This is a tedious but straightforward verification. □

Proof of Lemma 5. B is closed under applications by axiom (R2). By axiom (R1) KK and $KS_{\bar{n}}$ are elements of B_0. We now show that $\mathbf{B}$ is a RCA. Let $H_{n_{i+1}}a_i = a_i$ and $H_{m_{j+1}}b_j = b_j$.

$$\begin{aligned}
(KK)\bullet\bar{a}\bullet\bar{b} &= (S_{\bar{n}}[KK]\bar{a})\bullet\bar{b}, && \text{where } n_0 = 0,\ H_{n_{i+1}}a_i = a_i\\
&= S_{\bar{m}}[S_{\bar{n}}[KK]\bar{a}]\bar{b}, && \text{where } m_0 = \Sigma\bar{n},\ H_{m_{j+1}}b_j = b_j\\
&= H_{n_1}a_0, && \text{by (R3), if } |\bar{a}| = 1 \text{ and } |\bar{b}| = 0\\
&= a_0, && \text{by assumption.}
\end{aligned}$$

The axiom $(S_{\bar{l}})$ of RCA follows directly from an application of (R4). We assume directly that $\bar{a} = (a_0)$. Let $t \equiv (KS_{\bar{l}})\bullet a_0\bullet\bar{b}\bullet\bar{c}$, where $H_{m_0}a_0 = a_0$, $H_{m_{i+1}}b_i = b_i$

$(i = 0, \dots, k-1)$ and $H_{l_j} c_j = c_j$ $(j = 0, \dots, \Sigma\bar{l} - 1)$, $k = |\bar{b}| = |\bar{m}| - 1$, $|\bar{c}| = \Sigma\bar{l}$ and $\bar{n} = (\Sigma\bar{m}) \cdot \bar{n}'$. Then we have:

$$
\begin{aligned}
t &= S_{\bar{n}}[S_{\bar{m}}[S_{\bar{p}}[KS_{\bar{l}}]\bar{a}]\bar{b}]\bar{c} \\
&= \Sigma_{\bar{\bar{s}} \in \mathcal{Q}_{\bar{c},\bar{l}}} S_{(\Sigma(m_0 \cdot \bar{o}_0), \dots, \Sigma(m_k \cdot \bar{o}_k))}[S_{m_0 \cdot \bar{o}_0} a_0 \bar{s}_0][S_{m_1 \cdot \bar{o}_1} b_0 \bar{s}_1, \dots, S_{m_k \cdot \bar{o}_k} b_{k-1} \bar{s}_k], \text{ by (R4)} \\
&\qquad \text{where for each } \bar{\bar{s}} \in \mathcal{Q}_{\bar{c},\bar{l}},\ \bar{\bar{o}} \in \mathcal{Q}_{\bar{n}',\bar{l}} \text{ is the partition of } \bar{n}' \text{ induced by } \bar{\bar{s}} \\
&= \Sigma_{\bar{\bar{s}} \in \mathcal{Q}_{\bar{c},\bar{l}}} (a_0 \bullet \bar{s}_0) \bullet [b_0 \bullet \bar{s}_1, \dots, b_{k-1} \bullet \bar{s}_k].
\end{aligned}
$$

Proof of Theorem 3.
(i) We prove the axioms of a combinatory algebra.

$$\underline{K} * X * Y = \downarrow \{K\bar{b}\bar{c} : \bar{b} \in X^*,\ \bar{c} \in Y^*\} = \downarrow \{b : b \in X\} = X, \text{ by axiom } (K) \text{ of } \mathsf{RCA}.$$

$$
\begin{aligned}
\underline{S} * X * Y * Z &= \downarrow \{S_{\bar{n}} a\bar{b}\bar{c} : a \in X, \bar{b} \in Y^*, \bar{c} \in Z^*, \bar{n} \in \mathbb{N}^*\} \\
&= \downarrow \{\textstyle\sum_{\bar{\bar{d}} \in \mathcal{Q}_{\bar{c},\bar{n}}} a\bar{d}_0[b_0\bar{d}_1, \dots, b_{k-1}\bar{d}_k] : a \in X,\ \bar{b} \in Y^k,\ k = |\bar{n}| - 1,\ \bar{c} \in Z^{\Sigma\bar{n}}\} \\
&\quad \text{by axiom } (S_{\bar{n}}), \\
&= \downarrow \{a\bar{d}_0[b_0\bar{d}_1, \dots, b_{k-1}\bar{d}_k] : a \in X,\ \bar{b} \in Y^k,\ \bar{d}_0, \dots, \bar{d}_k \in Z^*,\ k \in \mathbb{N}\} \\
&\quad \text{since ideals are closed under joins and downward closed,} \\
&= X * Z * (Y * Z)
\end{aligned}
$$

(ii) Before proving the axioms of a λ-abstraction algebra, we recall from [17] that a λ-*abstraction algebra* (LAA, for short) is a structure $\mathbf{A} = (A, \cdot, \lambda x, x)_{x \in V}$ such that λx is a unary operation (for each $x \in V$), $\cdot$ is a binary operation, $x \in V$ is a nullary operation and the following identities hold (for all $a, b, c \in A$, $x \neq y \in V$):

(β_1) $(\lambda x.x)a = a$
(β_2) $(\lambda x.y)a = y$
(β_3) $(\lambda x.a)x = a$
(β_4) $(\lambda x.\lambda x.a)b = \lambda x.a$
(β_5) $(\lambda x.ab)c = (\lambda x.a)c((\lambda x.b)c)$
(β_6) $(\lambda y.b)x = b \ \Rightarrow\ (\lambda x.\lambda y.a)b = \lambda y.(\lambda x.a)b$
(α) $(\lambda y.a)x = a \ \Rightarrow\ \lambda x.a = \lambda y.(\lambda x.a)y.$

We now prove of the axioms.

$$
\begin{aligned}
(\beta_1)\ (\underline{\lambda} x.\underline{x})X &= \downarrow \{(\lambda x.x)\bar{a} : \bar{a} \in X^*\} && \text{by linearity} \\
&= \downarrow \{(\lambda x.x)a : a \in X\} && \text{by } (r\beta_1) \\
&= X && \text{by } (r\beta_1).
\end{aligned}
$$

$$
\begin{aligned}
(\beta_2)\ (\underline{\lambda} x.\underline{y})X &= \downarrow \{(\lambda x.y)\bar{a} : \bar{a} \in X^*\} \\
&= \downarrow \{y\} && \text{by } (r\beta_2) \\
&= \underline{y}
\end{aligned}
$$

$$
\begin{aligned}
(\beta_3)\ (\underline{\lambda} x.X)\underline{x} &= \downarrow \{(\lambda x.a)x^n : a \in X, n \in \mathbb{N}\} \\
&= X,
\end{aligned}
$$

because by $(r\gamma)$ we have that $\downarrow \{(\lambda x.a)x^n : a \in X, n \in \mathbb{N}\} \subseteq X$, while the opposite direction is obtained by the hypothesis of locally finiteness. Indeed, the ideal X is generated by its locally finite elements and each locally finite element a satisfies the condition $(\lambda x.a)x^n = a$ for some n.

$$
\begin{aligned}
(\beta_4)\ (\underline{\lambda} x.\underline{\lambda} x.X)Y &= \downarrow \{(\lambda x.\lambda x.a)\bar{b} : a \in X,\ \bar{b} \in Y^*\} \\
&= \downarrow \{\lambda x.a : a \in X\} && \text{by } (r\beta_3) \\
&= \underline{\lambda} x.X.
\end{aligned}
$$

$$
\begin{aligned}
(\beta_5)\ (\underline{\lambda} x.XY)Z &= \downarrow \{(\lambda x.a\bar{b})\bar{c} : a \in X, \bar{b} \in Y^*, \bar{c} \in Z^*\} \\
&= \downarrow \{\textstyle\sum_{\bar{d} \in \mathcal{Q}_{\bar{c},|\bar{b}|}} (\lambda x.a)\bar{d}_0[(\lambda x.b_0)\bar{d}_1, \dots, (\lambda x.b_{|\bar{b}|-1})\bar{d}_{|\bar{b}|}] \\
&\qquad : a \in X, \bar{b} \in Y^*, \bar{c} \in Z^*\},\ \text{by } (r\beta_5) \\
&= \downarrow \{(\lambda x.a)\bar{d}_0[(\lambda x.b_0)\bar{d}_1, \dots, (\lambda x.b_{|\bar{b}|-1})\bar{d}_{|\bar{b}|}] \\
&\qquad : a \in X,\ \bar{b} \in Y^*,\ \bar{d}_0, \dots, \bar{d}_{|\bar{b}|} \in Z^*\} \\
&= (\underline{\lambda} x.X)Z((\underline{\lambda} x.Y)Z).
\end{aligned}
$$

(β_6) Suppose now $(\underline{\lambda} y.Y)\underline{x} = Y$, that is, $Y = \downarrow \{(\lambda y.b)x^n : b \in Y, n \in \mathbb{N}\}$. Then it is easy to show that $(\lambda y.(\lambda y.b)x^n)[\,] = (\lambda y.b)x^n$.

$$
\begin{aligned}
(\underline{\lambda} x.\underline{\lambda} y.X)Y &= \downarrow \{(\lambda x.\lambda y.a)\bar{b} : a \in X,\ \bar{b} \in Y^*\} \\
&= \downarrow \{(\lambda x.\lambda y.a)[(\lambda y.b_1)x^{n_1}, \dots, (\lambda y.b_k)x^{n_k}] : a \in X,\ b_i \in Y\} && \text{by hyp.} \\
&= \downarrow \{\lambda y.(\lambda x.a)[(\lambda y.b_1)x^{n_1}, \dots, (\lambda y.b_k)x^{n_k}] : a \in X,\ b_i \in Y\} && \text{by } (r\beta_4) \\
&= \downarrow \{\lambda y.(\lambda x.a)[b_1, \dots, b_k] : a \in X,\ b_i \in Y\} && \text{by hyp.} \\
&= \underline{\lambda} y.(\underline{\lambda} x.X)Y.
\end{aligned}
$$

(α) Suppose now $(\underline{\lambda} y.X)\underline{z} = X$, that is, $X = \downarrow \{(\lambda y.a)z^n : a \in X, n \in \mathbb{N}\}$. It is easy to check that the hypotheses of rule $(r\alpha)$ are satisfied for the element $(\lambda y.a)z^n$, that is, $(\lambda y.(\lambda y.a)z^n)[\,] = (\lambda y.a)z^n$ and there exists k such that $(\lambda x.(\lambda y.a)z^n)x^k = (\lambda y.a)z^n$. Then by $(r\alpha)$ we have: $\lambda x.(\lambda y.a)z^n = \lambda y.(\lambda x.(\lambda y.a)z^n)y^k$.

$$
\begin{aligned}
\underline{\lambda} y.(\underline{\lambda} x.X)\underline{y} &= \downarrow \{\lambda y.(\lambda x.a)y^k : a \in X,\ k \in \mathbb{N}\} \\
&= \downarrow \{\lambda y.(\lambda x.(\lambda y.a)z^n)y^k : a \in X,\ k, n \in \mathbb{N}\} \\
&\qquad \text{by } X = \downarrow \{(\lambda y.a)z^n : a \in X, n \in \mathbb{N}\} \\
&= \downarrow \{\lambda x.(\lambda y.a)z^n : a \in X, n \in \mathbb{N}\} \\
&\qquad \text{by } \lambda x.(\lambda y.a)z^n = \lambda y.(\lambda x.(\lambda y.a)z^n)y^k \\
&= \underline{\lambda} x.X
\end{aligned}
$$

(iii) Assume that $\mathbf{A}$ is a RLA. Then $\mathbf{A}$ can be embedded into the combinatory reduct of the RLAA $\mathbf{A}[V]_\lambda$ freely generated by $\mathbf{A}$. It is an easy matter to show that $Ide(\mathbf{A})$ can be embedded into the combinatory reduct of $Ide(\mathbf{A}[V]_\lambda)$, that we know to be an LAA from item (ii) of this theorem. $\square$

Proof of Lemma 10. The proof is by induction on $M \in \Lambda$ to show that $|M|^{\mathbf{B}} = \downarrow \{|t|^{\mathbf{A}} : t \in \mathcal{T}(M)\}$. Then the statement of the lemma follows from the simple observation that $\downarrow \{|t|^{\mathbf{A}} : t \in \mathcal{T}(M)\} = \downarrow \{|u|^{\mathbf{A}} : u \in \mathrm{NF}(t),\ t \in \mathcal{T}(M)\}$. The case in which M is a variable is trivial.

$$\begin{aligned}|\lambda x.N|^{\mathbf{B}} &= \downarrow \{\lambda x^{\mathbf{A}}.a : a \in |N|^{\mathbf{B}}\}\\ &= \downarrow \{\lambda x^{\mathbf{A}}.a : \exists \bar{t} \in \mathcal{T}(N)^*.\ a \leq \textstyle\sum_i |t_i|^{\mathbf{A}}\}, \text{ using the ind. hyp.,}\\ &= \downarrow \{\lambda x^{\mathbf{A}}.|t|^{\mathbf{A}} : t \in \mathcal{T}(N)\}\\ &= \downarrow \{|s|^{\mathbf{A}} : s \in \mathcal{T}(\lambda x.N)\}\end{aligned}$$

$$\begin{aligned}|PQ|^{\mathbf{B}} &= \downarrow \{a\bar{b} : a \in |P|^{\mathbf{B}},\ \bar{b} \in (|Q|^{\mathbf{B}})^*\}\\ &= \downarrow \{a\bar{b} : \exists \bar{t} \in \mathcal{T}(P)^*.\exists \bar{s}^1, \ldots, \bar{s}^n \in \mathcal{T}(Q)^*.\ a \leq \textstyle\sum_i |t_i|^{\mathbf{A}},\ b_i \leq \sum_h |s^j_h|^{\mathbf{A}}\}\\ &\quad \text{using the ind. hyp.,}\\ &= \downarrow \{|t'\bar{s}|^{\mathbf{A}} : t' \in \mathcal{T}(P),\ \bar{s} \in (\mathcal{T}(Q))^*\}\ \text{, using the linearity of operations in } \mathbf{A},\\ &= \downarrow \{|s|^{\mathbf{A}} : s \in \mathcal{T}(PQ)\}. \qquad \square\end{aligned}$$

Proof of Theorem 5. The proof that $\mathbf{E}$ is a RLAA is a verification of axioms $(r\beta_1)$-$(r\beta_5)$, $(r\alpha)$, $(r\gamma)$. As an example, we show the calculation for $(r\beta_4)$. By linearity it is sufficient to verify for singleton sets that we write without braced parenthesis. The assumption $(\lambda y.(\rho_i, \beta_i))[\ ] = (\rho_i, \beta_i)$ means that $\rho(y) = [\ ]$. Then we have

$$\begin{aligned}(\lambda xy(\sigma, \alpha))[(\rho_1, \beta_1), \ldots] &= (\sigma\{x, y := [\ ]\}, \sigma(x) \to \sigma(y) \to \alpha)[(\rho_1, \beta_1), \ldots]\\ &= (\uplus_{i=1}^n \rho_i \uplus \sigma\{x, y := [\ ]\}, \sigma(y) \to \alpha)\\ &\qquad \text{assuming } \sigma(x) = [\beta_1, \ldots, \beta_n]\\ &= \lambda y.(\uplus_{i=1}^n \rho_i \uplus \sigma\{x := [\ ]\}, \alpha) \text{ because } \rho_i(y) = [\ ]\\ &= \lambda y.(\lambda x(\sigma, \alpha))[(\rho_1, \beta_1), \ldots]\end{aligned}$$

If $\sigma(x) \neq [\beta_1, \ldots, \beta_n]$ then both the expression give $\emptyset$ as result.

Now by direct calculations we observe that in the algebra $\mathbf{E}$:

$$\begin{aligned}&(\lambda xy.x)^{\mathbf{E}} = \{(\perp, \alpha) : \alpha \in K^{\mathbf{D}}\}\\ &(\lambda xyz.xz^{n_0}[yz^{n_1}, \ldots, yz^{n_k}])^{\mathbf{E}} = \{(\perp, \alpha) : \alpha \in S^{\mathbf{D}}_{\bar{n}}\}\end{aligned}$$

The second part of the theorem is trivial. $\square$