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**A new algorithm for the 2-period
Balanced Traveling Salesman Problem in euclidean graphs.**

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Abstract. In a previous paper, we proposed two heuristic algorithms for the euclidean 2-period Balanced Travelling Salesman Problem (2B-TSP). In this problem, which arises from a similar one introduced by Butler et al., a certain number of customers must be visited at minimum total cost over a period of two days: some customers must be visited daily, and the others on alternate days (even or odd days). Moreover, the number of customers visited in every tour must be ‘balanced’, i.e. it must be the same or, alternatively, the difference between the maximum and the minimum number of visited customers must be less than a given threshold: this kind of constraint does not appear explicitly in the paper by Butler. In this paper a third algorithm is presented which overcomes some inadequacy of the algorithm A2 we proposed in the previous paper. The new algorithm’s performance is then analyzed, with respect particularly with the first proposed algorithm.

Keywords: period routing problem, period travelling salesman problem, logistic, heuristic algorithms.

JEL Classification Numbers: C61.

MathSci Classification Numbers: 90B06, 90C59.

0 Introduction.

In [2], Butler et al. study the problem of finding the minimum cost path, consisting of two circuits, which visits two sets of nodes, called respectively single and double nodes: double nodes must belong to both circuits, while single nodes must be partitioned between them. The problem arises when a certain number of customers must be visited over a two days period: a subset consists of customer to be visited each day; the other one only on alternate days. This way, this second set must be partitioned between odd and even days. The objective is to minimize the total travelled distance in the period. Butler et al. solves in an exact way an instance with nearly 40 nodes in an euclidean framework, but they do not give a general algorithm. In [1], the same problem is studied inserting in it one more constraint, say, a balance constraint: the number of visited nodes must be the same over the two days or it can differ at most by a pre-definite threshold g^* . For the solution of this problem, in [1] two heuristics, A1 and A2, are proposed: they are particularly suitable for the cases in which distances satisfy the triangular property. The algorithm A1 firstly finds a (not admissible) tour, called General Tour, which visits all the nodes (both double and single ones) and takes account of the visit order of single nodes. Then it builds the two tours which form the solution, firstly (odd days) deleting from GT a certain

number of consecutive single nodes and then (even days) deleting the complementary (single node's) set.

The heuristic A2 is based on the fact that, when the two tours in a solution cross each other, it may be possible to improve the solution by an exchange procedure: this way, in A2 some nodes are definitely attributed to even or odd days in order to prevent this 'crossing'. This requires the solution of a subset sum problem which, on turn, is a well-known NP-complete problem. Moreover, in practical cases, it can have a huge number of different solutions, and a rule to overcome the testing of all of them should also be given.

In this paper a third algorithm is presented, A3. A3 is similar to A2, in that it aims to prevent the crossing of edges in the two tours, but it does not need the solution of a subset sum problem. Computational experiences show that its performance is generally better than A1.

The paper is organized as follows. In section 1, after a short review of the fundamental notations and definitions, the algorithms A1 and A2 are resumed. In section 2 we describe and state formally the algorithm A3. In the following third section we discuss some possible improvement of the solution obtained through A3. Then, in section 4, computational experiences and comparisons are given. The studied instances refer to the case in which we require that the number of nodes in the two tours is the same (the number of single nodes is supposed to be even).

It must be observed that the problem is still open on many points of view: lower bound on the value of the solution would be appreciated. It demonstrated itself a challenging test for techniques like branch and cut, even with moderate size of the instances.

1. The algorithms A1 and A2.

1.1 Notations and definitions.

Let $G = (V, E)$ a non oriented complete graph of n nodes ($n > 1$) without loops. Let c_{ij} be the weight of the edge (i, j) . In the 2-period balanced TSP the set V can be partitioned into two (disjoint) non empty subsets:

- the set of single-nodes $S = \{s_1, s_2, \dots, s_k\}$, i.e. the ones to visit once in two days;
- the set of double-nodes $D = \{d_1, d_2, \dots, d_h\}$, i.e. the ones to be visited every day.

Obviously, $h + k = n$.

In what follows, a node belonging to the set or tour X , will be called X -node. Besides this, given a set A , we shall denote with $|A|$ its cardinality.

We want to build two tours, T^1 and T^2 (one for every day), which also satisfy a balance constraint, in order to minimize the total travelled distance. Both tours visit all the D -nodes, while every single-node, i.e. every S -node, can be inserted only in one of the two tours, T^1 or T^2 .

This way, in every feasible solution, S is partitioned into two subsets, S^1 and S^2 (with $S = S^1 \cup S^2$ and $S^1 \cap S^2 = \emptyset$), the first one containing nodes visited on the first (odd) day and the second one on the second (even) day.

The core of the 2-period balanced travelling salesman problem is how to partition optimally the customers in S into the two sub-sets S^1 and S^2 .

This appears the crucial point, because, at least for instances with some hundreds of nodes, software now available allows to solve, in an exact way and very short time, the subsequent Travelling Salesman Problems, in the sub-graphs of G induced, respectively, by $D \cup S^1$ and $D \cup S^2$. At our knowledge, the best example of such software is Concorde. This enhances the use of exact procedures, which solve TSP, as a step in the achievement of an approximate solution of the 2-balanced period TSP.

Even the non balanced version of the period TSP appears quite difficult to solve in an exact way: as we pointed out above, Butler, Williams and Arrows, in [2], propose a solution for a particular problem of 42 nodes, but they do not give a fully automatic procedure. In our experience, branch and cut technique takes too long a time also for moderate size instances. So approximate algorithms are useful.

The algorithms A1 and A2, proposed in [1], both require, as a prerequisite, a hamiltonian cycle GT over all the nodes in V (General Tour): in practice, GT can be obtained by well known software, once more, for instance, Concorde.

Next, we shall describe the two algorithm, with some not relevant changes in notations. We state A1 and A2 initially for the particular case in which $g^* = 1$, giving then indications for the general case. For their justification, see [1].

1.2 A1 algorithm.

Step 1. Choose arbitrarily one of the two visit orders on the general tour GT and let us suppose that the indexes attributed to the single nodes, s_1, s_2, \dots, s_k , agree with this order. Moreover, for $i = 1, 2, \dots, k$, put $s_{i+k} \equiv s_i$. Go to step 2. Let be $k_1 = \lfloor (k+1)/2 \rfloor$ the greatest integer $\leq (k+1)/2$.

Step 2. For $i = 1$ to k : let be

$$S^1_i = \{s_i, s_{i+1}, \dots, s_{i+k_1-1}\}; \quad S^2_i = S - S^1_i.$$

Solve TSP both in the subgraphs of G induced by $S^1_i \cup D$ and $S^2_i \cup D$. Call T^1_i, T^2_i the obtained tours; $c(T^1_i)$ and $c(T^2_i)$ their costs and put

$$A_i = c(T^1_i) + c(T^2_i).$$

Go to step 3.

Step 3. Choose the best solution, i.e., the one that minimizes A_i . STOP.

Note that in case k is even, the number of different solutions in the above algorithm is $k/2$ (by symmetry!) so that in step 2 the index ‘i’ can range only from 1 to $k/2$.

In order to generalize the algorithm A1, note before all that when k is an odd number, the feasible values for the difference between the cardinality of S^1 and S^2 , i.e., $|S^1| - |S^2|$, is also an odd number, so g^* cannot be 0 . Moreover, every odd threshold \underline{g} gives the same results as $\underline{g} + 1$. On the other side, if

k is even, $|S^1| - |S^2|$ is also even and the only values that must be analysed for the threshold g^* , are even numbers.

In general, if $g^* > 1$, the algorithm must be repeated for all the (integer) values of the parameter k_l (step 1) ranging from $\lfloor (k+1)/2 \rfloor$ to $\lfloor (k + g^*)/2 \rfloor$ and the best solution between the g^* ones found in step 3 must be determined.

1.3 A2 algorithm.

The algorithm A2 is motivated by the following observation: if a path in GT linking two double nodes, say u and v , and containing some single nodes, crosses the chord (u, v) , it could be more convenient, in order to satisfy a kind of ‘triangular property’ – as shown from examples – to visit the single nodes on one side of this chord in a different day with respect to the ones on the other side. This reduces the possibility that edges belonging to different tours cross each other. As it will be shown later, in most cases, this intersection of edges implies that the actual solution is not optimal (it can be improved).

Before giving A2 in detail, the following definition is useful.

Definition. By “single-node path” $P(u, v)$, SNP for short, we mean any path in GT having at least two edges, in which the two endpoints, u and v , are double-nodes, while intermediate nodes are single ones.

Single node paths are univocally defined whenever G contains more than two double nodes. In the case in which there are only two double nodes, (including the depot) we have two single node paths: they are distinguishable introducing a visit order in GT.

Let the *cardinality* of a SNP be the number of nodes in the path and indicate it by $|P(u, v)|$.

The algorithm A2 consists of two phases. In the first one, every single node path which crosses its respective chord is analysed and its nodes are divided into two subsets; then, a subset sum problem is solved, in order to insert single nodes into two sets, Q^1 and Q^2 , containing the nodes to be visited respectively in the first and the second day, in such a way that the balancing constraint is fulfilled. In this phase some heuristics is required in order to solve this subset problem. In the second phase, the TSP is solved both in Q^1 and Q^2 in order to find the best tours T^1* and T^2* . As for A1, we give a version of A2 in which $g^* = 1$. Then we shall extend it to the general case.

Step 1. Choose a visit direction in GT. Let $Q^1 = Q^2 = D$; $q_1 = |Q^1|$ $q_2 = |Q^2|$, $q_1 = q_2 = h$. List all the SNP’s in GT as $P_1(u_1, v_1)$, $P_2(u_2, v_2)$, ..., $P_t(u_t, v_t)$. Go to step 2.

Step 2. For $i = 1, \dots, t$, consider the SNP $P_i(u_i, v_i) \subseteq GT$. Test if any edge $(s_a^i, s_b^i) \in P_i(u_i, v_i)$ crosses the chord (u_i, v_i) by a crossing-edge test: if yes, put $P_i(u_i, v_i)$ in a set W of ‘crossing SNP’s’. Otherwise, put $P_i(u_i, v_i)$ in a set Z of ‘non-crossing SNP’s’. Re-number SNP’s, both in W and in Z .

Let $w = \lfloor W \rfloor$ and $z = \lfloor Z \rfloor$. (We will suppose, without lack of generality, $W \neq \emptyset \neq Z$. In case this would not be true, the modifications to the algorithm are quite evident). Go to step 3.

Step 3. For $i = 1, \dots, w$, consider the SNP $P_i(u_i, v_i) \subseteq W$. Delete in $P_i(u_i, v_i)$ every edge (s_a^i, s_b^i) which crosses the chord (u_i, v_i) . Let $P_i^1, P_i^2, \dots, P_i^m$ be the disjoint sub-paths $\subset P_i(u_i, v_i)$ (possibly consisting of only one node) in the sequence in which they are visited in $P_i(u_i, v_i)$. Insert single nodes which belong to sub-paths P_i with odd apex into a set W^{iL} and the ones with even apex into a set W^{iR} . Let $w_{iL} = \lfloor W^{iL} \rfloor$, $w_{iR} = \lfloor W^{iR} \rfloor$. Go to step 4.

Step 4. For $i = 1, \dots, z$, consider the SNP $P_i(u_i, v_i) \subseteq Z$. Insert single nodes of this path in a set Z_i . Let $z_i = \lfloor Z^i \rfloor$. Go to step 5.

Step 5. Solve the following subset sum problem

$$\begin{aligned}
 (P) \quad & \text{Min} \quad \sum_{i=1}^w (x_{iL}w_{iL} + x_{iR}w_{iR}) + \sum_{i=1}^z y_i z_i \\
 \text{s.t.} \quad & \sum_{i=1}^w (x_{iL}w_{iL} + x_{iR}w_{iR}) + \sum_{i=1}^z y_i z_i \geq k/2 \quad (*) \\
 & x_{iL} = 1 - x_{iR} \quad (**) \\
 & x_{iL}, x_{iR}, y_i \in \{0, 1\}.
 \end{aligned}$$

Insert into Q^1 the single nodes of the sets W^{iL}, W^{iR}, Z^i for which the corresponding variable, in the optimal solution, is equal to 1. Insert into Q^2 the remaining single nodes. Update the cardinality of Q^1 and Q^2 , i.e., respectively, q_1 and q_2 . Go to step 6.

Step 6. Solve the TSP both in the sub-graphs of G induced by Q^1 and Q^2 : call, respectively, T^{1*} and T^{2*} the two optimal tour. Go to step 7.

Step 7. If $|q_1 - q_2| \leq 1$, i.e. if the balancing constraint is satisfied, go to step 10. If the balancing constraint is not fulfilled, go to step 8.

Step 8. Let us suppose, without lack of generality, $q_1 > q_2$. For every single-node s in Q^1 , consider its two neighbourhood nodes in T^{1*} : let them be called $a(s)$ and $b(s)$. Compute the transfer-cost $r(s)$ given by

$$\rho(s) = c_{a(s), b(s)} - c_{s, a(s)} - c_{s, b(s)} + \min_{(i, j)} c_{is} + c_{sj} - c_{ij},$$

where the minimum has to be computed with respect to all the edges $(i, j) \in T^{2*}$. Go to step 9.

Step 9. Find the single-node s^* with the minimum transfer-cost. Transfer s^* from Q^1 into Q^2 . Go to step 6.

Step 10. The couple (T^{1*}, T^{2*}) is the solution. STOP.

Observe that the partition (Q^1, Q^2) obtained in step 5 is the one for which the difference $|q_1 - q_2|$ attains its minimum value. However, this does not imply that the balance constraint be satisfied. This way, steps 7 to 9 are needed.

If the threshold $g^* > 1$, we must repeat the procedure from step 7 to 9 in order to analyze all the case in which $|q_1 - q_2| = g$, with g ranging from 1 to g^* . This way, if, for example, the subset sum problem gives a solution in which we have $q_1 - q_2 > g^*$, we must transfer nodes from Q^1 to Q^2 ; but if we have $q_1 > q_2$ and $q_1 - q_2 < g^*$, in order to span all the possible situations, we must both analyze cases in which we transfer nodes to Q^1 as well as cases in which we transfer nodes to Q^2 .

2. The algorithm A3.

The application of the algorithm A2 to random instances of graphs with a uniform distribution of nodes on a rectangular area and a balance constraint which states that the number of visited customer over each day be the same (even number of single nodes), highlighted some drawback of this technique.

As already observed in the introduction, A2 requires a subset sum problem to be solved: it is well known that such kind of problems is NP-hard. However, in our experiences, the difficulty lies not so much in the effort to obtain a solution in a reasonable time, but on the large number of different solutions to the same instance. This is the case, particularly, in instances with many Single Node Paths containing very few nodes. The enumeration and the analysis of all these solutions may be heavy!

These facts suggested the construction of another algorithm, which we shall call A3: A3 is similar to A2, as long as it partitions single nodes of crossing SNP's between the two visit days but in such a way to exclude the need to solve a subset sum problem and minimizing at each step the slack $|q_1 - q_2|$ between the number of yet assigned nodes. Then the remaining single nodes are attributed to the two visit days, likewise in the algorithm A1 (A3 coincides with A1 in the case in which all the SNP's are 'no-crossing' ones). This produces a certain number of different solution, at most $k-2$, when in GT there is only one crossing SNP and it has only two single nodes: among them the best one is considered. This (heuristically) optimal solution can be often improved on the basis of some geometric property.

Next we first present algorithm A3 (once more, for the case in which $g^* = 1$, reserving to a final remark the extension to the general case $g^* > 1$) and then we comment the principal steps: the improvement phase is postponed and discussed successively. Notations are the same as for A2: in particular, as before, we shall denote q_1 and q_2 the cardinality of Q^1 and Q^2 , respectively.

A3 algorithm

Step 1. Choose arbitrarily one of the two visit orders on the general tour GT. Let $Q^1 = Q^2 = D$, $q_1 = q_2 = h$. List all the SNP's in GT as $P_1(u_1, v_1)$, $P_2(u_2, v_2)$, ..., $P_t(u_t, v_t)$. Go to step 2.

Step 2. For $i = 1, \dots, t$, consider the SNP $P_i(u_i, v_i) \subseteq GT$. Test if any edge $(s_1^i, s_2^i) \in P_i(u_i, v_i)$ crosses the chord (u_i, v_i) by a crossing-edge test: if yes, put $P_i(u_i, v_i)$ in a set W of 'crossing SNP's'. Otherwise, put $P_i(u_i, v_i)$ in a set Z of 'non-crossing SNP's'. Re-number SNP's in W .

Let $w = \lfloor W \rfloor$ and $z = \lfloor Z \rfloor$. (We will suppose, without lack of generality, $W \neq \emptyset$. In case $W = \emptyset$ the algorithm A3 coincides with A1). Go to step 3.

Step 3. For $i = 1, \dots, w$, consider the SNP $P_i(u_i, v_i) \subseteq W$. Delete in $P_i(u_i, v_i)$ every edge (s_1^i, s_2^i) which crosses the chord (u_i, v_i) . Let $P_i^1, P_i^2, \dots, P_i^m$ be the disjoint sub-paths $\subset P_i(u_i, v_i)$ (possibly consisting of only one node) in the sequence in which they are visited in $P_i(u_i, v_i)$. Insert the single nodes which belong to the sub-paths P_i with odd apex into a set W^{iL} and the ones with even apex into a set W^{iR} . Let $w_{iL} = \lfloor W^{iL} \rfloor$, $w_{iR} = \lfloor W^{iR} \rfloor$. Go to step 4.

Step 4. For $i = 1, \dots, w$, calculate $\delta_i = w_{iL} - w_{iR}$;

- if $\delta_i = 0$, insert the nodes $\in W^{iL}$ in Q_1 , the nodes $\in W^{iR}$ in Q_2 .
- if $\delta_i > 0$, and $q_1 > q_2$ insert the nodes $\in W_{iL}$ in Q^2 and the nodes $\in W_{iR}$ in Q^1 ;
- if $\delta_i > 0$, and $q_1 \leq q_2$ insert the nodes $\in W_{iL}$ in Q^1 and the nodes $\in W_{iR}$ in Q^2 ;
- if $\delta_i < 0$, and $q_1 > q_2$ insert the nodes $\in W_{iL}$ in Q^1 and the nodes $\in W_{iR}$ in Q^2 ;
- if $\delta_i < 0$, and $q_1 \leq q_2$ insert the nodes $\in W_{iL}$ in Q^2 and the nodes $\in W_{iR}$ in Q^1 .

Update $q_1 = : q_1 + w_{iL}$, $q_2 = : q_2 + w_{iR}$. Go to step 5.

Step 5. Re-indexes the two sets Q^j ($j=1, 2$) in such a way that $q_1 \geq q_2$. If $q_1 > h + k/2$ (note that in this case $q_1 > q_2$), insert all the not yet assigned single nodes into Q^2 : go to **step 6**. If $q_1 = h + k/2$, insert all the not yet assigned nodes, if any, into Q^2 : go to **step 8**. If $q_1 < h + k/2$, go to **step 9**.

Step 6. Solve the TSP both in Q^1 and in Q^2 : call, respectively, T^1* and T^2* the two optimal tours. Test if the balancing constraint is satisfied. If yes, STOP (else go to the **improvement phase**). If not, go to step 7.

Step 7. For every single-node s in Q^1 , consider its two adjacent nodes in T^1* , let them be called $a(s)$ and $b(s)$. Compute the transfer-cost $r(s)$ given by

$$\rho(s) = c_{a(s),b(s)} - c_{s,a(s)} - c_{s,b(s)} + \min_{(i,j)} [c_{is} + c_{sj} - c_{ij}],$$

where the minimum has to be computed with respect to all the edges $(i, j) \in T^2*$. Find the single-node s^* with the minimum transfer-cost. Transfer s^* from Q^1 into Q^2 . Go to step 6.

Step 8. Solve the TSP both in Q^1 and in Q^2 , obtaining the couple of circuits (T^1, T^2) . Define the value of this solution Λ^* . STOP (else go to the **improvement phase**).

Step 9. Re-order the single nodes of the SNP's $\in Z$ following the order visit in GT as chosen in step 1, and call them $s_1^z, s_2^z, \dots, s_\tau^z$. Moreover, define $s_{\tau+1}^z \equiv s_1^z, s_{\tau+2}^z \equiv s_2^z, \dots, s_{2\tau-1}^z \equiv s_{\tau-1}^z$. Let be

$$r = \lfloor h + (k+1)/2 - q_1 \rfloor$$

the number of nodes to be inserted into Q^1 in order to satisfy the balance constraint (with $\lfloor Q^1 \rfloor \geq \lfloor Q^2 \rfloor$).

For $i = 1, \dots, \tau$:

- put the nodes $s_i^z, s_{i+1}^z, \dots, s_{i+r-1}^z$ in a set Q_i^1 and the remaining ones in Q_i^2 ;
- solve the TSP both in Q_i^1 and in Q_i^2 , obtaining the couple of circuits (T^1, T^2) : call the value of this solution A_i .

Go to Step 10.

Step 10. Choose the minimum solution value, i.e. the one having value $A^* = \min_i A_i$. STOP (else, go to the **improvement phase**).

In the algorithm A3, once individuated the crossing SNP's P_i , (steps 1 to 3) as well as in the algorithm A2, the nodes in W_{iL} and W_{iR} are progressively assigned to the two visit days, (the odd and the even one) in such a way that each assignment aims to balance the number of yet assigned nodes. This is accomplished attributing the maximum cardinality set, either W_{iL} or W_{iR} , to the ones of the two visit days which at the moment has the least number of yet assigned nodes (step 4; in case of ties, we assign the more numerous set to the odd day). The two so far created sets, Q^1 and Q^2 , are then completed: this is done putting in Q^2 all the remaining single nodes (step 5) if Q^1 has at least the half of the total number of the single nodes or (step 9) considering, in all the possible ways, two subsets of single nodes, not yet inserted, that are consecutively met in GT and inserting them, respectively, in Q^1 and Q^2 . If Q^1 and Q^2 have the same cardinality (step 8), it is sufficient solve the TSP in both sets. If Q^1 contains more than the half of the total number of single nodes (as checked in step 6), we proceed to check the balance constraint and, if necessary, to perform the balancing step 7. If we get more than one feasible solution (step 9), we choose the best one (step 10). Note that if the set Z is empty, step 9 is never reached.

At this point, the procedure A3 comes to an end: however, as we said above, very often the obtained solution may be improved. To this feature the next section is devoted.

The procedure can be adjusted for a generic threshold $g^* > 1$ changing steps 5 to 10 in the following sense. Consider one by one the values g , $0 \leq g \leq g^*$ (applying repeatedly the algorithm A3). Suppose, as in the above algorithm, $q_1 \geq q_2$. Let be $\sigma = q_1 - q_2$. If $\sigma > g$ apply the transfer step in order to reduce the cardinality of Q^1 ; but, on the contrary, when $\sigma < g$ transfer nodes toward Q^1 in order to reduce progressively the cardinality of Q^2 . For every value of g , calculate the optimal tours. Then choose the best among these solutions.

3. Improvement techniques.

Obviously, the well known Lin Kernighan heuristics may be applied, both to A1 and A3. But here we will put in evidence some kind of exchange techniques which can be more usefully performed handily, on the basis of the visualisation of the two circuits which form the solution, at least when the number of nodes is not very large (remember also the hypothesis of

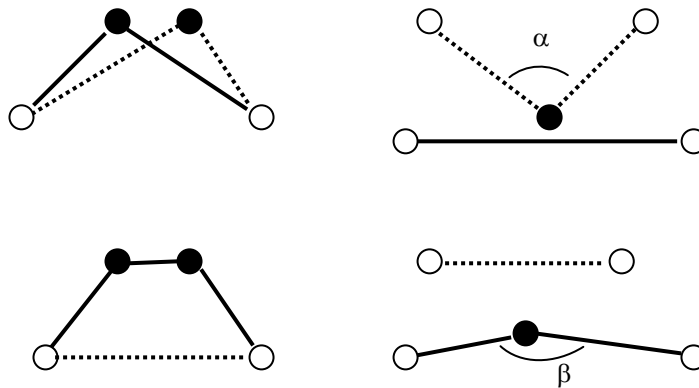
Euclidean graphs). These techniques can be seen as a useful tool to give a better answer in practical situations, but also as a proof of non-optimality of the solution so far obtained.

There are two main reasons because a solution given by A3 can be improved. In steps 1 to 4 we check if there are SNP's with the crossing property with the aim to assign their nodes to the two tours in order to forbid the presence in the latest of crossing edges.

Nonetheless, after the partition of the single nodes in the two visit days, the optimal tours do not maintain the visit order of single nodes as in GT and so some other crosses can appear.

This way, when two edges in the tours T^{1*} and T^{2*} cross each other, it is often possible to reduce the total cost through an exchange procedure. Obviously, it is worthy to eliminate the intersection, but, at the same time, when transferring a single node from, say, T^1 to T^2 , we must take account of the balance constraint and so it may be needed to contemporary transfer a single node from T^2 to T^1 .

Moreover, both to reduce the total cost as to balance the number of visits in the two days, a sort of 'angle rule', similar to the one given in [3], can be used: if the angle α between the direction from a single node, now visited in T^1 , to the two adjacent ones is less then the angle β to the two nearest nodes in the other tour T^2 , we can try to move this single node to the other tour. The following figure illustrates these situations (black nodes are single ones; the two circuits are denoted as heavy vs. dotted lines).



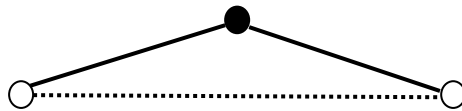
(On the left). The total cost is reduced inserting the two single nodes in the same tour.

(On the right). The angle between the single node and the other adjacent two on the dotted circuit is less then in the heavy circuit: in this case, however, we are not sure to reduce the total cost after the exchange.

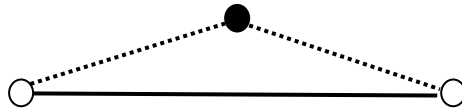
We shall call the first case an **intersection removal** and the second one a **shortening deviation**.

In both cases, we need the transfer, without additional costs, of another single node to the dotted circuit.

This can be easily accomplished if somewhere, along the couple of circuits, T^1 and T^2 , we have a structure like the one in the following figure (the meaning of symbols is the same of the preceding figure):

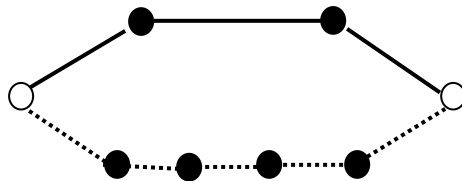


Here, the interchanging of the two paths, dotted and continuous, in such a way to include the single (black) node in the dotted circuit can compensate a transfer due to an intersection removal or of a shortening deviation.



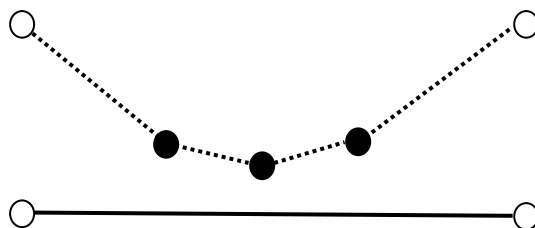
We shall call this type of structure a ‘**1-bridge**’. If there is no (adequate) 1-bridge in the solution, we can search for more general bridges, with 2 or more single nodes in the dotted path, in order to equalize the number of nodes which are transferred from one day to the other. So, we define **multiple bridges** in which we have more than one single node between the two double nodes in one tour (and none in the other) or, even more, a different number of single nodes in the two SNP’s between the double nodes: this can be useful in the case in which there are more single nodes involved in crossing edges or in deviations. We shall call such bridges **p/q-bridges**: they can compensate the transfer of exactly $|p - q|$ single nodes (in the adequate verse!).

By example, the following **4/2-bridge** can be used, exchanging dotted with continuous edges, when transferring somewhere else, two single nodes from the continuous to the dotted circuit:



This way, the number of nodes which are attributed to the two circuits does not change and the overall cost is reduced.

With reference to shortening deviations, we must note that in some case the involvement of more than one single node in a useful deviation is easily checked by visual inspection but not so much from an analytical point of view (see, once more, [3]). Consider the following example:

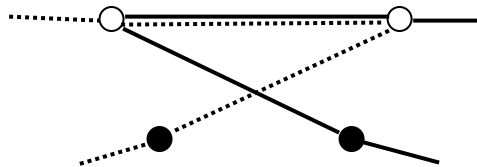


Here it may be more convenient to move the single (black) nodes in the other tour, even if the rule about the angles as quoted above does not hold.

From a practical point of view, it is easier to test whether the two circuits crosses each other then to test if there are cases of possible improvement through shortening deviations or to test if there are multiple bridges of the kind we effectively need!

In particular, the search for multiple bridges, which can compensate the transfers due to intersection removal and shortening deviations, results once more in a subset sum problem (and so it can be more useful, in practical cases, to come back to a visual inspection and an handy procedure, similarly to the technique used from Butler et al. in their particular instance). Moreover, the subset sum problem can be not feasible!

Anyway, there are also cases in which the elimination of a crossing is not possible, **even if** the solution has some bridge of the needed characteristics. This is the case of the following example:



In this case, the transfer to the other circuit of one of the single (black) nodes does not eliminate a subsequent intersection due to the necessity of visiting both the double nodes, at least as far as the visit order of double nodes in GT will be preserved, as it can easily be seen.

Formally, we could complete the algorithm A3 with the following

Step 11. (Improvement phase). Test if in the current solution the two tours T^{1*} and T^{2*} cross each other. Test then if there are cases of shorter deviation. Test then if it is possible to exchange single nodes in such a way that both crossing are eliminated and deviation are shortened. If not, search for bridges (simple ones, or multiple bridges) in order to meet the balance constraint. Perform the transfers. STOP.

4. Computational experiences.

In our computational experiences, we used the above improvement rules, above all, as a (negative) test of optimality on the solution so far obtained with A1 and A3.

The computational experiences were concerned with graphs of 48 nodes. In order to get a deeper understanding of the relation between the performance of the algorithms and the ratio between the number of single and double nodes, we studied 3 different environments, corresponding, respectively, to a number of double nodes of 8, 16 or 24. In each environment, we solved 20 randomly generated instances using both algorithms, A1 and A3 (the solution of the involved TSP was obtained using Concorde).

In every case, we tried to perform the improvement step. This was not always possible, both for the lacking of intersections or of (evident) shorter deviations, as for the presence of not eliminable intersections, as pointed out above.

On the other hand, in many case it was quite evident the possibility of improvements after other ‘intelligent’ exchange procedures, slightly different from the ones described above: as already said, we report these cases in the tables in order to state with certainty that the solution found with A1 or A3 was not the true optimum.

In all the tables, the value of the best solution is reported, as well as the presence of intersections. The symbol * denotes which, between A1 and A3, before improvements, is better; in heavy character, the value of the best found solution is indicated. In A3, the number of crossing SNP’s over the total number of SNP’s is also reported.

Table 1. Graphs with 48 nodes, 8 doubles.

Instances	A1			A3			Impr. sol.
	Value	Cross	Impr.sol.	Value	N° cr SNP	cross	
I ₂₁	263.69	Yes	259.32	252.17*	2 / 7	yes	249.37
I ₂₂	281.24	Yes	268.89	269.96*	4 / 6	-	265.49
I ₂₃	264.76*	Yes	264.22	271.68	4 / 6	yes	261.92
I ₂₄	270.10	Yes	260.91	268.02*	4 / 8	yes	260.38
I ₂₅	286.32	Yes	274.52	277.28*	3 / 8	yes	274.08
I ₂₆	251.56	Yes	242.82	243.48*	2 / 7	-	242.68
I ₂₇	273.70	Yes	268.57	273.35*	4 / 7	-	271.32
I ₂₈	252.85	Yes	242.32	246.79*	3 / 7	yes	244.93
I ₂₉	281.89	Yes	270.00	268.90*	3 / 8	yes	264.66
I ₃₀	249.78*	Yes	245.87	255.24	4 / 6	yes	242.00
I ₃₁	265.88	Yes	261.30	262.66*	4 / 8	-	261.81
I ₃₂	263.60*	Yes	260.54	266.14	1 / 8	yes	260.54
I ₃₃	268.30	Yes	267.14	264.18*	2 / 7	-	-
I ₃₄	292.41	Yes	273.76	287.33*	4 / 7	yes	270.74
I ₃₅	255.58	Yes	245.87	250.43*	2 / 8	yes	246.20
I ₃₆	271.52	Yes	271.26	271.03*	1 / 7	-	267.90
I ₃₇	257.74	Yes	252.53	256.62*	3 / 6	-	251.23
I ₃₈	266.50	Yes	257.09	264.59*	2 / 6	yes	-
I ₃₉	275.46	Yes	269.94	272.09*	5 / 8	-	-
I ₄₀	254.16*	Yes	250.54	256.35	2 / 7	yes	249.64

Table 2. Graphs with 48 nodes, 16 doubles.

Istances	A1			A3			Impr.sol.
	Value	Cross	Impr.sol.	Value	N° cr SNP	cross	
I ₁	338.04*	Yes	331.51	335.11	4 / 12	-	325.56
I ₂	309.29*	-	-	313.77	2 / 9	-	310.56
I ₃	305.96*	Yes	305.49	309.77	2 / 11	yes	303.82
I ₄	290.85	Yes	284.41	285.63*	4 / 11	-	-
I ₅	273.23*	-	-	281.18	3 / 8	yes	277.67
I ₆	330.65*	Yes	325.46	331.18	4 / 11	-	-
I ₇	334.89	Yes	324.81	328.54*	3 / 11	-	318.61
I ₈	311.13	Yes	307.51	308.50*	2 / 12	yes	305.71
I ₉	352.58	Yes	340.24	347.09*	2 / 14	-	343.90
I ₁₀	316.28	Yes	312.25	314.51*	4 / 7	-	312.01
I ₆₁	273.79	Yes	269.48	271.58*	3 / 7	yes	269.34
I ₆₂	277.74*	Yes	276.19	278.29	2 / 10	yes	276.26
I ₆₃	257.67	Yes	251.65	253.08*	3 / 11	-	249.60
I ₆₄	283.50	Yes	272.85	278.75*	5 / 8	-	-
I ₆₅	275.60	-	273.54	270.60*	1 / 9	-	(268.88)**
I ₆₆	276.92	Yes	276.34	276.77*	3 / 10	-	276.24
I ₆₇	304.81	Yes	303.43	302.68*	2 / 10	-	302.03
I ₆₈	280.33*	-	279.23	280.33*	0 / 11	-	279.23
I ₆₉	282.99	Yes	276.38	279.58*	2 / 11	yes	278.50
I ₇₀	286.57	-	284.87	284.83*	3 / 9	yes	284.27

** In this instance, a not eliminable intersection was present: an ‘intelligent’ exchange was successful (after the transfer, the visit order also was changed!).

Table 3. Graphs with 48 nodes, 24 doubles.

Istances	A1			A3			impr sol.
	Value	cross	Impr sol.	Value	N° cr SNP	cross	
I ₄₁	302.53*	yes	301.47	306.57	3 / 11	yes	304.66
I ₄₂	330.08*	-	-	333.30	2 / 12	yes	320.20
I ₄₃	315.00*	-	-	315.00*	0 / 13	-	-
I ₄₄	303.89	yes	302.06	303.13*	1 / 10	yes	301.70
I ₄₅	282.92	yes	278.57	279.86*	2 / 12	yes	278.80
I ₄₆	305.31	yes	303.55	300.73*	3 / 11	yes	300.13
I ₄₇	323.27	yes	321.70	321.06*	3 / 15	yes	-
I ₄₈	320.46	yes	316.29	310.04*	4 / 10	yes	309.42
I ₄₉	336.63*	yes	333.31	338.68	2 / 14	yes	331.85
I ₅₀	312.66*	-	-	312.66*	1 / 12	-	-
I ₅₁	340.21	yes	338.34	337.22*	2 / 13	-	336.27
I ₅₂	323.53*	yes	322.80	323.77	1 / 13	yes	322.80
I ₅₃	332.87*	-	-	334.15	1 / 11	-	332.00
I ₅₄	319.73*	yes	316.00	320.49	3 / 12	-	-
I ₅₅	339.27	-	-	338.37*	1 / 12	yes	335.07
I ₅₆	299.61	yes	298.89	296.96*	1 / 14	-	294.76
I ₅₇	316.40*	-	-	319.54	3 / 10	yes	318.65
I ₅₈	308.79	-	-	307.05*	2 / 12	-	302.46
I ₅₉	315.49*	-	-	315.49*	0 / 11	-	-
I ₆₀	328.41	yes	326.99	323.84*	4 / 12	yes	323.17

In all the three groups of instances, A3 performs better than A1: this happens more sensibly in the case with lesser double nodes. After the improvement phase, A3 is once more the best algorithm, at a lesser degree in the first and second environment. The extent of the improvement is not generally very high, but sometimes it reach 6%: it is generally higher in A1 than in A3 and in the scenario with lesser double nodes. It should be noted also the very high number of instances in which, after the first phase, in both algorithms the solution has intersections. This is more evident in the algorithm A1 and 8 or 16 double nodes.

5. Conclusions.

In this paper a new algorithm to solve the 2-period balanced travelling salesman problem is presented. Its performance, which is better in the average with respect to other heuristic techniques, is also studied in 60 random instances. The algorithm gives a solution which in most cases can be improved by exchange techniques. The underlying problem is still open, since a tight lower bound on the solution value is not yet known.

6. References.

- [1] T. Bassetto, F. Mason (2007) “The 2- period Balanced Traveling Salesman Problem”. Working Paper Series, Dept of Applied Mathematics, Venice University in Ca’ Foscari, n.154.

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- [3] J.P. Norback, R.F. Love (1977) “Geometric approaches to solving the travelling salesman problem”. *Management Science* **23** n° 11, 1208-1223.