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MINIMIZATION OF COMMUNICATION EXPENDITURE FOR SEASONAL PRODUCTS *

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Abstract. We consider a firm that sells seasonal goods. The firm seeks to reach a fixed level of goodwill at the end of the selling period, with the minimum total expenditure in promotional activities. We consider the linear optimal control problem faced by the firm which can only control the communication expenditure rate; communication is performed by means of advertising and sales promotion. Goodwill and sales levels are considered as state variables and word-of-mouth effect and saturation aversion are taken into account. The optimal control problem is addressed by means of the classical Pontryagin Maximum Principle and the solution can be easily found solving, in some cases numerically, a system of two non linear equations. Moreover, a parametric analysis is performed to understand how the total expenditure in communication should be divided between advertising and sales promotion.

Keywords: Optimal control, advertising, sales promotions, seasonal products.

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1. INTRODUCTION

The problem of planning media schedules and communication mix expenditure over time has received growing attention in the recent past and a number of aggregate advertising response models have been proposed in literature since the pioneering works of Vidale and Wolfe [13] and Nerlove and Arrow [9]. The key idea of Nerlove and Arrow was to take into account explicitly the goodwill level reached by a firm or product; the goodwill depends on the expenditure in advertising and is subject to a decay due to the forgetting effect. More precisely the dynamics in the model of Nerlove and Arrow is described by the linear differential equation

$$\dot{A}(t) = a(t) - \delta A(t),$$

where $A(t)$ is the goodwill level at time t , $a(t)$ is the advertising spending rate at time t and δ is the goodwill decay rate. A number of generalizations of the model, with suitably defined profit functions to be maximized, can be found in literature (see *e.g.* Feichtinger *et al.* [4]).

Here we consider a firm that sells seasonal goods and seeks for the optimal communication plan over a finite time horizon: this leads to a linear optimal control model.

Similar models were recently proposed by Favaretto and Viscolani [3], Buratto and Favaretto [2], Funari and Viscolani [5].

In particular in [3] an optimal control model for production, advertising and selling of seasonal goods is considered, where production and advertising take place in a first time period while in a second consecutive time period the firm can sell the good, continuing the advertising activity.

In our model we focus on the interaction of different promotional activities during the selling period. In fact we take into account not only advertising expenditure but also other important features of the communication mix, like sales promotion and word-of-mouth. Moreover we include in the model a market saturation effect.

We assume that the aim of the firm is to reach at the end of the selling period a level of goodwill previously defined by the management: this can be useful, for example, when reaching high levels of goodwill allows to exploit hysteresis properties of the response function [6, 7] so that it is possible to keep the level of goodwill with low economical effort. At the same time we assume that the firm requires to sell its whole inventory.

To reach its targets the firm spends in promotional activities but since, as it is well known, “many brands are overspending in advertising” ([1], p. 357), a careful minimization of costs is required and the firm tries to minimize the total expenditure in communication.

To be more precise, we assume that the communication expenditure rate is the only control allowed to the firm and that communication is performed by means of advertising and sales promotion.

In the linear optimal control model stated in Section 2, goodwill dynamics depends not only on advertising effort but also on sales level due to the effect of word-of-mouth.

High levels of goodwill improve sales, of course, but selling activity depends directly also on sales promotion and on sales level itself. In particular, the increasing level of goods already sold will reduce sales speed due to the progressive saturation of the market (see also [5]).

Sales level changes instead, depend directly on sales promotions efforts and on the reached sales level and takes into account congestion aversion.

Some technical assumptions will be discussed in Section 3 while the structure of optimal controls will be studied in Section 4.

In Section 5 we will look for admissible optimal controls. As it will be shown, the optimal control can easily be found solving a system of two non linear equations. Moreover the structure of the optimal control depends on how the combined effect of market saturation and goodwill decay is contrasted by the positive effect of goodwill and word-of-mouth. At the end of the section we also propose an algorithm to solve an instance of the problem.

Section 6 contains the analysis of the dependence of the optimal communication mix from the required initial goodwill level and from the final levels of goodwill and sales. In the same section we analyze how the total cost of the optimal promotional activities varies depending on the ratio between advertising costs and sales promotion efforts and provide a numerical example.

Finally, Section 7 contains some remarks on the economic interpretation and on the main shortcomings of the model.

Let us remark that the readers with small interest in proofs, which are rather technical, may omit them still getting a clear idea of the model and of the results obtained. The beginnings and ends of the proofs are marked by “*Proof*” and “□”, respectively.

2. FORMULATION OF THE PROBLEM

As mentioned in the previous section, we propose a linear optimal control problem to model the dynamics of selling and communication activities of a firm. Of course, linearity is a strong assumption but, since we consider a firm that sells seasonal goods, the time period in which the dynamics evolves is limited and short enough so that a linear model can be considered a sufficiently good approximation of reality.

Since the selling period is short we also assume that the marginal effects of communication activities are constant and positive both with respect to the sales and to the goodwill of the firm. The total expenditure rate in communication is bounded and divided *a priori* by the management into two parts, one for advertising and one for sales promotion.

The motion equation we consider for the goodwill generalizes the one proposed by Nerlove and Arrow [9] and is given by

$$\dot{A}(t) = \beta x(t) - \delta_A A(t) + \gamma_A \rho a(t)$$

where

$A(t)$ = goodwill level at time t ;

$x(t)$ = sales level at time t ;

$a(t)$ = communication expenditure rate at time t ;

δ_A = goodwill decay parameter, $\delta_A > 0$;

γ_A = advertising productivity in terms of goodwill, $\gamma_A > 0$;

ρ = weight of the total expenditure rate devoted to advertising (in short: advertising weight), $\rho \in [0, 1]$;

and β is the word-of-mouth productivity in terms of goodwill. Thus the word-of-mouth effect increases the goodwill rate whenever $\beta > 0$ while a negative word-of-mouth effect corresponds to $\beta < 0$ ³. In the following we will restrict our attention to the case of a favorable word-of-mouth, *i.e.* $\beta > 0$ ⁴.

The sales level dynamics will be defined by the equation

$$\dot{x}(t) = -\alpha x(t) + \delta_x A(t) + \gamma_x(1 - \rho)a(t)$$

where

δ_x = goodwill productivity in terms of sales, $\delta_x > 0$;

γ_x = promotion productivity in terms of sales, $\gamma_x > 0$;

and $\alpha > 0$ is a saturation aversion parameter: in fact this way the sales rate decreases as the sales increase, modeling the market saturation effect. Factor $(1 - \rho)$ is the part of the total expenditure rate devoted to sales promotion.

During the selling period the firm requires to reach a fixed goodwill level \tilde{A} starting from the initial level \bar{A} and to sell the total inventory \bar{m} .

In order to simplify the notation, in the following we put $\epsilon_x = \gamma_x(1 - \rho)$, $\epsilon_A = \gamma_A\rho$ and $a = a(t)$, $x = x(t)$, $A = A(t)$.

This way the following optimal control problem can be formulated

$$\begin{aligned} P : \text{ minimize} & \quad \int_{t_1}^{t_2} a(t) dt, \\ \text{subject to} & \quad \dot{x} = -\alpha x + \delta_x A + \epsilon_x a \\ & \quad \dot{A} = \beta x - \delta_A A + \epsilon_A a \\ & \quad x(t_1) = 0 \quad x(t_2) = \bar{m} \\ & \quad A(t_1) = \bar{A} \quad A(t_2) = \tilde{A} \\ & \quad a \in [0, \bar{a}] \end{aligned}$$

where $[t_1, t_2]$ is the selling period and $\bar{a} > 0$ is the upper bound for the communication expenditure rate.

³The role of parameter β is rather similar to the seller's *reputation* in Spremann's model [12].

⁴This is only done in order to restrict the number of sub-cases to consider; negative values of word-of-mouth are of course possible in practice, *e.g.* when selling a low quality product which is initially perceived by the market, due to unfair advertising, as a high quality product.

3. PRELIMINARY ASSUMPTIONS

We will assume that problem P satisfies the general position condition (GPC) [11], which guarantees the uniqueness of the solution, if any.

In order to study GPC for P , let us consider the matrices

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha & \delta_x \\ 0 & \beta & -\delta_A \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} -1 \\ \epsilon_x \\ \epsilon_A \end{pmatrix}.$$

Since the determinant of matrix

$$(\mathcal{B} \quad \mathcal{A}\mathcal{B} \quad \mathcal{A}^2\mathcal{B})$$

is

$$\mathcal{D} = (\alpha\delta_A - \beta\delta_x)(\delta_x\epsilon_A^2 - \beta\epsilon_x^2 + (\delta_A - \alpha)\epsilon_A\epsilon_x)$$

by applying GPC we obtain

Proposition 3.1. *If $\mathcal{D} \neq 0$ and an admissible control of problem P exists then there is a unique piecewise constant optimal control of the problem.*

Assumption. We assume that the hypothesis of the preceding proposition holds, *i.e.* $\mathcal{D} \neq 0$.

The ‘‘regularity’’ of the problem required by this assumption is considered quite natural in optimal control since in practical problems non-zero coefficients are known only with some approximation.

Remark that our assumption requires in particular that $\alpha\delta_A - \beta\delta_x \neq 0$ and we will see that $\alpha\delta_A - \beta\delta_x = 0$ is a threshold for the qualitative properties of the optimal control of problem P .

Another assumption that we will adopt throughout the paper is that the control is continuous from the left and continuous at the end points of the interval; this technical hypothesis is usual in optimal control (see *e.g.* [11], p. 73).

4. TYPES OF OPTIMAL CONTROL

The system of motion equations of problem P can be rewritten as

$$\dot{X} = QX + B(a), \tag{1}$$

where

$$X = X(t) = \begin{pmatrix} x(t) \\ A(t) \end{pmatrix}, \quad Q = \begin{pmatrix} -\alpha & \delta_x \\ \beta & -\delta_A \end{pmatrix}, \quad B(a) = B(a(t)) = a(t) \begin{pmatrix} \epsilon_x \\ \epsilon_A \end{pmatrix}.$$

To apply the Pontryagin's Maximum Principle [10] we need to compute the eigenvalues of $-Q^T$ which are

$$\lambda_1 = \frac{\alpha + \delta_A - \sqrt{(\alpha - \delta_A)^2 + 4\beta\delta_x}}{2}, \quad \lambda_2 = \frac{\alpha + \delta_A + \sqrt{(\alpha - \delta_A)^2 + 4\beta\delta_x}}{2}.$$

Remark that, since we assumed $\beta > 0$, we have $\lambda_1, \lambda_2 \in \mathfrak{R}$, moreover $\lambda_2 > 0$, $\lambda_2 > \lambda_1$, and the sign of λ_1 coincides with the sign of $\alpha\delta_A - \beta\delta_x$ ⁵.

Due to the GPC assumption and since the eigenvalues are real, the optimal control is “bang-bang” and the number of switches in the optimal control cannot be more than two [11].

The following proposition explains how the sign of λ_1 determines the type of optimal control:

Proposition 4.1. *If $\alpha\delta_A < \beta\delta_x$ (i.e. $\lambda_1 < 0$) then the optimal communication policy can only be of type*

$$a^* = \begin{cases} \bar{a}, & t \in [t_1, \tau_1) \\ 0, & t \in (\tau_1, \tau_2) \\ \bar{a}, & t \in (\tau_2, t_2] \end{cases} \quad (2)$$

with switch times τ_1, τ_2 such that $t_1 \leq \tau_1 \leq \tau_2 \leq t_2$; if $\alpha\delta_A > \beta\delta_x$ (i.e. $\lambda_1 > 0$) the optimal communication policy can only be of type

$$a^* = \begin{cases} 0, & t \in [t_1, \tau_1) \\ \bar{a}, & t \in (\tau_1, \tau_2) \\ 0, & t \in (\tau_2, t_2] \end{cases} \quad (3)$$

with $t_1 \leq \tau_1 \leq \tau_2 \leq t_2$.

Proof. By means of Pontryagin's Maximum Principle it is easy to show that the number of switches coincides with the number of zeroes of the so called switching function

$$F_{u_1, u_2}(t) = u_1 e^{\lambda_1 t} + u_2 e^{\lambda_2 t} - 1, \quad t \in [t_1, t_2],$$

where u_1, u_2 are arbitrary constants.

For every $\tau_1 < \tau_2$ it is immediate to show that the linear system

$$\begin{cases} u_1 e^{\lambda_1 \tau_1} + u_2 e^{\lambda_2 \tau_1} = 1 \\ u_1 e^{\lambda_1 \tau_2} + u_2 e^{\lambda_2 \tau_2} = 1 \end{cases}$$

has a unique solution (\bar{u}_1, \bar{u}_2) , therefore, if we consider $t_1 < \tau_1 < \tau_2 < t_2$, then the switching function $F_{\bar{u}_1, \bar{u}_2}(t)$ has exactly two roots in (t_1, t_2) . Moreover it is easy to show that $\bar{u}_1 > 0$.

⁵This implies that, under GPC assumption, $\lambda_1 \neq 0$; the special case in which $\lambda_1 = 0$ is considered, under some more restrictive hypotheses, in [3].

Let $\lambda_1 < 0$. Then, since $\bar{u}_1 > 0$, $\lim_{t \rightarrow -\infty} F_{\bar{u}_1, \bar{u}_2}(t) = +\infty$, therefore $F_{\bar{u}_1, \bar{u}_2}(t_1) > 0$ and the optimal control is (2). Remark that in this case all possible kinds of optimal control a^* can be written in the unified form (2) also if $t_1 \leq \tau_1 \leq \tau_2 \leq t_2$. Let $\lambda_1 > 0$; then

$$\lim_{t \rightarrow -\infty} F_{\bar{u}_1, \bar{u}_2}(t) = -1 < 0,$$

hence $F_{\bar{u}_1, \bar{u}_2}(t_1) < 0$ and the optimal control is (3). In this case all possible kinds of optimal control a^* can be written in the unified form (3) also if $t_1 \leq \tau_1 \leq \tau_2 \leq t_2$. \square

From Proposition 4.1 we have:

Corollary 4.1. *If problem P has some admissible control then one and only one of the following communication policies a^* is optimal:*

“ALTERNATE $\bar{a} - 0 - \bar{a}$ communication”: some τ_1, τ_2 exist such that

$$a^* = \begin{cases} \bar{a}, & t \in [t_1, \tau_1) \\ 0, & t \in (\tau_1, \tau_2) \\ \bar{a}, & t \in (\tau_2, t_2]; \end{cases}$$

“ALTERNATE $0 - \bar{a} - 0$ communication”: some τ_1, τ_2 exist such that

$$a^* = \begin{cases} 0, & t \in [t_1, \tau_1) \\ \bar{a}, & t \in (\tau_1, \tau_2) \\ 0, & t \in (\tau_2, t_2]; \end{cases}$$

“EARLY communication”: some τ_1 exists such that

$$a^* = \begin{cases} \bar{a}, & t \in [t_1, \tau_1) \\ 0, & t \in (\tau_1, t_2]; \end{cases}$$

“LATE communication”: some τ_1 exists such that

$$a^* = \begin{cases} 0, & t \in [t_1, \tau_1) \\ \bar{a}, & t \in (\tau_1, t_2]; \end{cases}$$

“MAXIMUM communication”: $a^* = \bar{a} \quad \forall t \in [t_1, t_2]$;

“NO communication”: $a^* = 0 \quad \forall t \in [t_1, t_2]$.

5. ADMISSIBLE OPTIMAL CONTROLS

In order to find admissible optimal controls we have to consider system (1) with t belonging to some interval $[t', t''] \subseteq [t_1, t_2]$ and the control a constant and either equal to zero or equal to \bar{a} , as stated in Proposition 4.1.

A matrix of eigenvectors of matrix Q of system (1) is

$$S = \begin{pmatrix} \delta_x & \delta_x \\ \alpha - \lambda_1 & \alpha - \lambda_2 \end{pmatrix}.$$

Remark that S is nonsingular since $\lambda_1 \neq \lambda_2$. Let us define

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = -S^{-1}QS,$$

which is a nonsingular matrix due to the GPC assumption, and

$$D(t) = e^{t\Lambda}\Lambda^{-1}S^{-1}B(\bar{a}) = \frac{\bar{a}}{\delta_x(\lambda_2 - \lambda_1)} \begin{pmatrix} [(\lambda_2 - \alpha)\epsilon_x + \delta_x\epsilon_A]e^{\lambda_1 t/\lambda_1} \\ -[(\lambda_1 - \alpha)\epsilon_x + \delta_x\epsilon_A]e^{\lambda_2 t/\lambda_2} \end{pmatrix}. \quad (4)$$

Then the solution of (1) on the interval $[t', t'']$ can be written as

$$X = \begin{cases} Se^{(t'-t)\Lambda}S^{-1}X(t'), & \text{if } a(t) = 0 \\ Se^{-t\Lambda}\{e^{t'\Lambda}S^{-1}X(t') - D(t') + D(t)\}, & \text{if } a(t) = \bar{a}. \end{cases} \quad (5)$$

At this point we can write the dynamics of the system depending on the optimal control. Moreover, exploiting continuity of the optimal trajectories, we can specify the conditions on switch times that allow to have an optimal control, as given in Proposition 5.1. To simplify notation we define the vector

$$G = e^{t_2\Lambda}S^{-1}X(t_2) - e^{t_1\Lambda}S^{-1}X(t_1). \quad (6)$$

Proposition 5.1. *The optimal trajectory for problem P is:*

a.

$$X^* = \begin{cases} Se^{-t\Lambda}\{e^{t_1\Lambda}S^{-1}X(t_1) - D(t_1) + D(t)\}, & t \in [t_1, \tau_1] \\ Se^{-t\Lambda}\{e^{t_1\Lambda}S^{-1}X(t_1) - D(t_1) + D(\tau_1)\}, & t \in [\tau_1, \tau_2] \\ Se^{-t\Lambda}\{e^{t_2\Lambda}S^{-1}X(t_2) - D(t_2) + D(t)\}, & t \in [\tau_2, t_2] \end{cases} \quad (7)$$

with τ_1, τ_2 such that $t_1 \leq \tau_1 \leq \tau_2 \leq t_1$ and

$$D(\tau_1) - D(\tau_2) = G + D(t_1) - D(t_2), \quad (8)$$

if the optimal control is “alternate $\bar{a} - 0 - \bar{a}$ communication” or “early communication” ($\tau_2 = t_2$) or “maximum communication” ($\tau_1 = \tau_2 = t_2$);

b.

$$X^* = \begin{cases} Se^{(t_1-t)\Lambda}S^{-1}X(t_1), & t \in [t_1, \tau_1] \\ Se^{-t\Lambda}\{e^{t_1\Lambda}S^{-1}X(t_1) - D(\tau_1) + D(t)\}, & t \in [\tau_1, \tau_2] \\ Se^{(t_2-t)\Lambda}S^{-1}X(t_2), & t \in [\tau_2, t_2] \end{cases} \quad (9)$$

with τ_1, τ_2 such that $t_1 \leq \tau_1 \leq \tau_2 \leq t_1$ and

$$D(\tau_2) - D(\tau_1) = G, \quad (10)$$

if the optimal control is “alternate $0 - \bar{a} - 0$ communication” or “late communication” ($\tau_2 = t_2$) or “no communication” ($\tau_1 = \tau_2 = t_2$).

Proof. Let us consider “alternate $\bar{a} - 0 - \bar{a}$ communication”. If $t \in [t_1, \tau_1)$, then (see (5)) $X = Se^{-t\Lambda}\{e^{t_1\Lambda}S^{-1}X(t_1) - D(t_1) + D(t)\}$ while if $t \in (\tau_1, \tau_2)$, then $X = Se^{(\tau_1-t)\Lambda}S^{-1}X(\tau_1)$. Therefore, by means of the continuity of X in τ_1

$$X = Se^{-t\Lambda}\{e^{t_1\Lambda}S^{-1}X(t_1) - D(t_1) + D(\tau_1)\}, \quad t \in [\tau_1, \tau_2).$$

Finally let $t \in (\tau_2, t_2]$. Then $X = Se^{-t\Lambda}\{e^{\tau_2\Lambda}S^{-1}X(\tau_2) - D(\tau_2) + D(t)\}$, *i.e.*, using continuity in τ_2 ,

$$X = Se^{-t\Lambda}\{e^{t_1\Lambda}S^{-1}X(t_1) - D(t_1) + D(\tau_1) - D(\tau_2) + D(t)\}.$$

In order to fulfill final conditions, τ_1 and τ_2 must satisfy

$$X(t_2) = Se^{-t_2\Lambda}\{e^{t_1\Lambda}S^{-1}X(t_1) - D(t_1) + D(\tau_1) - D(\tau_2) + D(t_2)\},$$

i.e. (8). Hence also (7) holds.

The other statements of the proposition can be obtained in a similar way. \square

An algorithm to solve problem P

By means of Proposition 5.1 we can outline how any instance of problem P can be solved. To this aim let us first remark that if $\tau_2 = t_2$ then equation (8) becomes

$$D(\tau_1) = G + D(t_1) \quad (11)$$

while (10) can be written as

$$D(\tau_1) = -G + D(t_2). \quad (12)$$

Moreover, if $\tau_1 = \tau_2 = t_2$ then (8) becomes

$$G = D(t_2) - D(t_1) \quad (13)$$

and (10) can be written as

$$G = (0, 0)^T. \quad (14)$$

Now it is possible to describe an algorithm to solve problem P .

First compute λ_1 . If $\lambda_1 < 0$ then check conditions (13) and (14). If one of them is satisfied we have maximum or no communication, respectively, and the

algorithm stops. Otherwise the systems of two equations (11, 12) and (8) must be considered. If one of them has solution belonging to the interval $[t_1, t_2]$ then the corresponding optimal control is determined, otherwise problem P has no solution: this way the problem is completely solved.

Remark that while systems (11) and (12) can be explicitly solved, system (8) must be solved numerically. Anyway, since at most one of them has solution, not necessarily all of them must be solved.

If $\lambda_1 > 0$ then the procedure is the same but one has to consider system (10) instead of (8).

6. PARAMETRIC ANALYSIS

In this section we will first study how optimal communication policies vary depending on the boundary conditions and then we will show how to determine the optimal value of the advertising weight ρ .

6.1. OPTIMAL CONTROLS STRUCTURE AND BOUNDARY CONDITIONS

We will look now for a graphic representation of the sets of boundary values of total inventory \bar{m} , initial and final goodwill levels \bar{A} and \tilde{A} , for which the structure of the optimal control is of the same kind.

We will use the conditions on τ_1 and τ_2 given in Proposition 5.1. More precisely: if $\lambda_1 < 0$ then, in “alternate $\bar{a} - 0 - \bar{a}$ communication” case, some τ_1 and τ_2 must exist such that $t_1 < \tau_1 < \tau_2 < t_2$ and (see (8) and (4))

$$(e^{\tau_1\Lambda} - e^{\tau_2\Lambda})\Lambda^{-1}S^{-1}B(\bar{a}) = G + (e^{t_1\Lambda} - e^{t_2\Lambda})\Lambda^{-1}S^{-1}B(\bar{a}); \quad (15)$$

while if $\lambda_1 > 0$ then, in “alternate $0 - \bar{a} - 0$ communication” case, some τ_1 and τ_2 exist such that $t_1 < \tau_1 < \tau_2 < t_2$ and (see (10) and (4))

$$(e^{\tau_2\Lambda} - e^{\tau_1\Lambda})\Lambda^{-1}S^{-1}B(\bar{a}) = G. \quad (16)$$

The space in which we will give the representation of the types of optimal control of problem P will be obtained transforming the space of parameters $\bar{m}, \bar{A}, \tilde{A}$ in a suitable two dimensional space.

In order to define this transformation let

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = S^{-1}B(\bar{a}), \quad \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = G.$$

It is easy to show that, under *GPC* assumption, $d_i \neq 0, i = 1, 2$, therefore we can also define

$$k_i = \frac{g_i}{d_i}, \quad h_i = k_i + \frac{1}{\lambda_i} (e^{\lambda_i t_1} - e^{\lambda_i t_2}), \quad i = 1, 2. \quad (17)$$

This way equation (15) can be rewritten as

$$\frac{1}{\lambda_i} (e^{\lambda_i \tau_1} - e^{\lambda_i \tau_2}) = h_i, i = 1, 2 \quad (18)$$

and (16) becomes

$$\frac{1}{\lambda_i} (e^{\lambda_i \tau_2} - e^{\lambda_i \tau_1}) = k_i, i = 1, 2. \quad (19)$$

We consider now the space of k_1, k_2 , which means that we will not work in the three-dimensional space of \bar{A}, \bar{m} and \bar{A} , but “in terms” of the boundary conditions since k_1 and k_2 depend linearly on \bar{A}, \bar{m} and \bar{A} .

To determine the set of couples (k_1, k_2) for which problem P has an admissible control, let us define

$$f_-(k_1, k_2) = \frac{1}{\lambda_1} \ln(e^{\lambda_1 t_2} - \lambda_1 k_1) - \frac{1}{\lambda_2} \ln(e^{\lambda_2 t_2} - \lambda_2 k_2), \quad (20)$$

and

$$f_+(k_1, k_2) = \frac{1}{\lambda_1} \ln(e^{\lambda_1 t_1} + \lambda_1 k_1) - \frac{1}{\lambda_2} \ln(e^{\lambda_2 t_1} + \lambda_2 k_2). \quad (21)$$

The following lemma holds:

Lemma 6.1. *If $\lambda_1 \neq 0$ and the optimal control is “alternate communication” ($\bar{a} - 0 - \bar{a}$ or $0 - \bar{a} - 0$) then k_1, k_2 must satisfy $f_-(k_1, k_2) < 0$ and $f_+(k_1, k_2) < 0$.*

Proof. Consider the case $\lambda_1 < 0$ and define the auxiliary functions

$$\begin{aligned} H_+ : [\tau_2, t_2] &\rightarrow \mathfrak{R}; & H_+(t) &= \frac{1}{\lambda_1} \ln(e^{\lambda_1 t} + \lambda_1 h_1) - \frac{1}{\lambda_2} \ln(e^{\lambda_2 t} + \lambda_2 h_2); \\ H_- : [t_1, \tau_1] &\rightarrow \mathfrak{R}; & H_-(t) &= \frac{1}{\lambda_1} \ln(e^{\lambda_1 t} - \lambda_1 h_1) - \frac{1}{\lambda_2} \ln(e^{\lambda_2 t} - \lambda_2 h_2). \end{aligned}$$

Remark that $H_-(t_1) = f_-(k_1, k_2)$ and $H_+(t_2) = f_+(k_1, k_2)$.

In “alternate $\bar{a} - 0 - \bar{a}$ communication” some τ_1 and τ_2 exist such that $t_1 < \tau_1 < \tau_2 < t_2$ and (18) holds. Hence $H_-(\tau_1) = f_-(k_1, k_2) = 0$ and $H_+(\tau_2) = f_+(k_1, k_2) = 0$.

Further, since $\tau_1 < \tau_2$ then $h_i < 0, i = 1, 2$ and $\dot{H}_-(t) > 0$ in the whole domain of H_- . Since $t_1 < \tau_1$ we have $f_-(k_1, k_2) = H_-(t_1) < 0$. In a similar way one obtains that $f_+(k_1, k_2) = H_+(t_2) < 0$.

Now consider the case $\lambda_1 > 0$ and define the functions

$$\begin{aligned} K_+ : [t_1, \tau_1] &\rightarrow \mathfrak{R}; & K_+(t) &= \frac{1}{\lambda_1} \ln(e^{\lambda_1 t} + \lambda_1 k_1) - \frac{1}{\lambda_2} \ln(e^{\lambda_2 t} + \lambda_2 k_2), \\ K_- : [\tau_2, t_2] &\rightarrow \mathfrak{R}; & K_-(t) &= \frac{1}{\lambda_1} \ln(e^{\lambda_1 t} - \lambda_1 k_1) - \frac{1}{\lambda_2} \ln(e^{\lambda_2 t} - \lambda_2 k_2). \end{aligned}$$

In this case $K_-(t_2) = f_-(k_1, k_2)$ and $K_+(t_1) = f_+(k_1, k_2)$.

In “alternate $0 - \bar{a} - 0$ communication” some τ_1 and τ_2 exist such that $t_1 < \tau_1 < \tau_2 < t_2$ and (19) are satisfied. Hence $K_+(\tau_1) = 0$ and $K_-(\tau_2) = 0$. Further, since $\tau_1 < \tau_2$, one has $k_i > 0, i = 1, 2$.

Moreover, if $\dot{K}_+(t) = 0$ then

$$\ddot{K}_+(t) = \frac{(\lambda_1 - \lambda_2)\lambda_1 k_1 e^{\lambda_1 t}}{(e^{\lambda_1 t} + \lambda_1 k_1)^2} < 0.$$

Therefore function K_+ has no stationary minimum points and has no more than one stationary maximum point. Moreover, since

$$K_+(t) = \frac{1}{\lambda_1} \ln(1 + \lambda_1 k_1 e^{-\lambda_1 t}) - \frac{1}{\lambda_2} \ln(1 + \lambda_2 k_2 e^{-\lambda_2 t})$$

one has $\lim_{t \rightarrow +\infty} K_+(t) = 0$ and, if $\dot{K}_+(t) = 0$ then

$$K_+(t) = \frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2} \ln(1 + \lambda_1 k_1 e^{-\lambda_1 t}) > 0.$$

Hence $f_+(k_1, k_2) = K_+(t_1) < 0$. By the same way one obtains that $f_-(k_1, k_2) = K_-(t_2) < 0$. \square

It is now possible to state the following proposition:

Proposition 6.1. *If $f_+(k_1, k_2) > 0$ or $f_-(k_1, k_2) > 0$ then there is no feasible control for problem P ; otherwise the optimal control of P is:*

“alternate $\bar{a} - 0 - \bar{a}$ communication” if $f_+(k_1, k_2) < 0$, $f_-(k_1, k_2) < 0$ and $\lambda_1 < 0$;

“alternate $0 - \bar{a} - 0$ communication” if $f_+(k_1, k_2) < 0$, $f_-(k_1, k_2) < 0$ and $\lambda_1 > 0$;

“early communication” if $f_+(k_1, k_2) = 0$ and $f_-(k_1, k_2) < 0$;

“late communication” if $f_+(k_1, k_2) < 0$ and $f_-(k_1, k_2) = 0$;

“maximum communication” if $f_+(k_1, k_2) = 0$, $f_-(k_1, k_2) = 0$ and $h_1 = h_2 = 0$;

“no communication” if $f_+(k_1, k_2) = 0$, $f_-(k_1, k_2) = 0$ and $k_1 = k_2 = 0$.

Proof. The two “alternate communication” cases follow from Lemma 6.1. In the “early communication” case we have, from Proposition 5.1, that the solution is the same as in “alternate $\bar{a} - 0 - \bar{a}$ communication” after putting $\tau_2 = t_2$. This implies that $f_+(k_1, k_2) = 0$ while $f_-(k_1, k_2) < 0$ as in the alternate case.

Similar considerations can be done for the “late communication”, “maximum communication” and “no communication” cases.

Finally, since these are all the admissible communication types, see Corollary 4.1, if $f_+(k_1, k_2) > 0$ or $f_-(k_1, k_2) > 0$ then there is no feasible control for P . \square

From Proposition 6.1 we have that the set

$$V_1 = \{(k_1, k_2) \mid f_+(k_1, k_2) < 0, f_-(k_1, k_2) < 0\}$$

is the region in the space of k_1 and k_2 in which the optimal control of problem P is alternate communication, “alternate $\bar{a}-0-\bar{a}$ communication” if $\lambda_1 < 0$, “alternate $0-\bar{a}-0$ communication” if $\lambda_1 > 0$.

The sets corresponding to the other kinds of communication are

$$V_2 = \{(k_1, k_2) \mid f_+(k_1, k_2) = 0, f_-(k_1, k_2) < 0\} \quad \text{early communication curve,}$$

$$V_3 = \{(k_1, k_2) \mid f_+(k_1, k_2) < 0, f_-(k_1, k_2) = 0\} \quad \text{late communication curve,}$$

$$V_4 = \left\{ \left(\frac{1}{\lambda_1}(e^{\lambda_1 t_2} - e^{\lambda_1 t_1}), \frac{1}{\lambda_2}(e^{\lambda_2 t_2} - e^{\lambda_2 t_1}) \right) \right\} \quad \text{maximum communication point,}$$

$$V_5 = \{(0, 0)\} \quad \text{no communication point.}$$

The above defined sets, $V_i, i = 1, \dots, 5$, depend only on the values of λ_1, λ_2 and t_1, t_2 . This means that given the coefficients of matrix Q , *i.e.* saturation aversion, word-of-mouth productivity, goodwill decay and goodwill productivity, the sets $V_i, i = 1, \dots, 5$, are completely determined.

The following example suggests that if a feasible control of problem P exists, then it will probably be an alternate communication:

Example 6.1. Consider the following values of the parameters of problem P :

$$\delta_A \text{ (goodwill decay parameter)} = 0.1;$$

$$\gamma_A \text{ (advertising productivity in terms of goodwill)} = 0.6;$$

$$\rho \text{ (weight of the total expenditure rate devoted to advertising)} = 0.42;$$

$$\beta \text{ (word-of-mouth productivity in terms of goodwill)} = 0.3;$$

$$\delta_x \text{ (goodwill productivity in terms of sales)} = 1.0;$$

$$\gamma_x \text{ (promotion productivity in terms of sales)} = 1.0;$$

$$\alpha \text{ (saturation aversion parameter)} = 1.0;$$

$$\bar{a} \text{ (maximal expenditure rate in communication)} = 10.0.$$

Suppose that $t_1 = 0$ and $t_2 = 1$ and consider the following boundary values:

$$\bar{A} \text{ (initial goodwill level)} = 1.0;$$

$$\tilde{A} \text{ (final goodwill level)} = 1.3;$$

$$\bar{m} \text{ (total inventory to be sold)} = 1.0.$$

In this case the matrix $-Q^T$ in system (1) is

$$-Q^T = \begin{pmatrix} \alpha & -\beta \\ -\delta_x & \delta_A \end{pmatrix} = \begin{pmatrix} 1.0 & -0.3 \\ -1.0 & 0.1 \end{pmatrix}$$

whose eigenvalues are

$$\lambda_1 \approx -0.159, \quad \lambda_2 \approx 1.259.$$

Since we have that $\alpha\delta_A < \beta\delta_x$, *i.e.* $\lambda_1 < 0$, then (see Prop. 4.1) the optimal control of P can only be of type “ $\bar{a}-0-\bar{a}$ communication”. The functions f_-

and f_+ (see (20) and (21)) that define the sets V_i are in this case

$$f_-(k_1, k_2) \approx \frac{1}{-0.159} \ln(e^{-0.159} + 0.159 k_1) - \frac{1}{1.259} \ln(e^{1.259} - 1.259 k_2)$$

and

$$f_+(k_1, k_2) \approx \frac{1}{-0.159} \ln(1 - 0.159 k_1) - \frac{1}{1.259} \ln(1 + 1.259 k_2).$$

The (feasible part of) curves $f_-(k_1, k_2) = 0$ and $f_+(k_1, k_2) = 0$ and the corresponding sets V_i are represented in Figure 1.

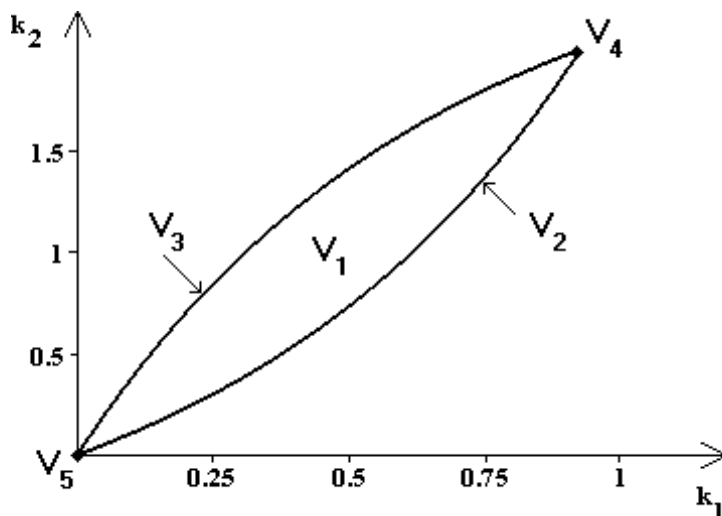


FIGURE 1. Representation of attainable boundary conditions.

We will come back to this example at the end of Section 6.2.

6.2. OPTIMAL ADVERTISING WEIGHT

Now observe that, fixing the boundary values \bar{m} , \bar{A} , \tilde{A} , we obtain, by means of (19), a couple of coordinates (k_1, k_2) . If they correspond to a point which does not belong to the set

$$V = \bigcup_{i=1}^5 V_i,$$

then problem P has no feasible solution. But consider now the case $(k_1, k_2) \in V$ and let us see what happens keeping the boundary conditions fixed and varying the parameter ρ . This way we look at the role of the weight that the firm gives to expenditure in advertising with respect to the total expenditure rate.

It is now convenient to consider the family \mathcal{F} of problems obtained fixing all parameters and boundary values in P except for the advertising weight: we denote

by P_ρ the problem of this family for which the advertising weight is ρ . We also define Ω_P as the set of values of $\rho \in [0, 1]$ such that $P_\rho \in \mathcal{F}$ is feasible.

Varying the value of ρ we change the values of k_1, k_2 since, see (17),

$$k_i = \frac{g_i}{d_i} = \frac{g_i}{b_i\rho + c_i} \quad (22)$$

where b_1, c_1, b_2 and c_2 are

$$b_1 = \frac{\bar{a}((\lambda_2 - \alpha)\gamma_x - \delta_x\gamma_A)}{\delta_x(\lambda_1 - \lambda_2)}, \quad c_1 = \frac{\bar{a}(\alpha - \lambda_2)\gamma_x}{\delta_x(\lambda_1 - \lambda_2)}, \quad (23)$$

$$b_2 = \frac{\bar{a}((\alpha - \lambda_1)\gamma_x + \delta_x\gamma_A)}{\delta_x(\lambda_1 - \lambda_2)}, \quad c_2 = \frac{\bar{a}(\lambda_1 - \alpha)\gamma_x}{\delta_x(\lambda_1 - \lambda_2)} \quad (24)$$

as can be easily found by straightforward calculations.

As ρ moves in $[0, 1]$, the couple $(k_1, k_2) \in V$ describes a part of a branch of the hyperbola (22)⁶.

If for a given problem P_ρ the corresponding point (k_1, k_2) belongs to V , then the problem is feasible and it has an optimal control a_ρ^* . For each $\rho \in \Omega_P$ it is possible to determine the optimal value $\nu(\rho)$ of the objective functional of P_ρ , and the following value function is defined:

$$\nu : \Omega_P \rightarrow \mathfrak{R}; \quad \rho \mapsto \nu(\rho) = \int_{t_1}^{t_2} a_\rho^*(t) dt.$$

It is now possible to determine the value of ρ which minimizes the total expenditure in communication with fixed boundary conditions, since this means to minimize the value function ν . To do this we first prove the following proposition:

Proposition 6.2. *Given the family of problems $\mathcal{F} = \{P_\rho : \rho \in \Omega_P\}$, and an open interval $I \subseteq \Omega_P$, if*

$$\frac{b_1\lambda_2}{b_1\rho + c_1} - \frac{b_2\lambda_1}{b_2\rho + c_2} \leq 0 \quad (\geq 0) \quad \forall \rho \in I \quad (25)$$

then the total expenditure in communication $\nu(\rho)$ is increasing (decreasing), with respect to the weight of advertising expenditure ρ , in the interval I .

Proof. Consider the case $\lambda_1 > 0$ with optimal “alternate $0 - \bar{a} - 0$ communication”; the total expenditure is $\nu(\rho) = \bar{a}(\tau_2 - \tau_1)$ where τ_1 and τ_2 are defined, see (19), by the system

$$(e^{\lambda_i\tau_2} - e^{\lambda_i\tau_1}) = k_i\lambda_i, \quad i = 1, 2.$$

⁶We recall that, due to GPC assumption, $d_i = b_i\rho + c_i \neq 0$, $i = 1, 2$.

Since k_1 and k_2 depend from ρ , also τ_1 and τ_2 are (implicit) functions of ρ . By the implicit function theorem we obtain

$$\frac{\partial \tau_1}{\partial \rho} = \frac{1}{N} \left(\frac{k_2 b_2}{d_2} e^{\lambda_1 \tau_2} - \frac{k_1 b_1}{d_1} e^{\lambda_2 \tau_2} \right), \quad \frac{\partial \tau_2}{\partial \rho} = \frac{1}{N} \left(\frac{k_2 b_2}{d_2} e^{\lambda_1 \tau_1} - \frac{k_1 b_1}{d_1} e^{\lambda_2 \tau_1} \right),$$

where $N = -e^{\lambda_1 \tau_1 + \lambda_2 \tau_2} + e^{\lambda_1 \tau_2 + \lambda_2 \tau_1} < 0$. Hence, by straightforward calculations,

$$\frac{\partial(\tau_2 - \tau_1)}{\partial \rho} = \frac{k_1 k_2}{N} \left(\frac{b_1 \lambda_2}{d_1} - \frac{b_2 \lambda_1}{d_2} \right) = \frac{k_1 k_2}{N} \left(\frac{b_1 \lambda_2}{g_1} k_1 - \frac{b_2 \lambda_1}{g_2} k_2 \right).$$

This way, since k_1 and k_2 are positive,

$$\frac{\partial(\tau_2 - \tau_1)}{\partial \rho} \geq 0 \iff \frac{b_1 \lambda_2}{g_1} k_1 - \frac{b_2 \lambda_1}{g_2} k_2 \leq 0,$$

i.e., see (22),

$$\frac{b_1 \lambda_2}{b_1 \rho + c_1} - \frac{b_2 \lambda_1}{b_2 \rho + c_2} \leq 0.$$

For $\lambda_1 < 0$ and “alternate $\bar{a} - 0 - \bar{a}$ communication” the result follows in a similar way from system (18). \square

A result similar to the one given in Proposition 6.2 can be easily proved also for the non-alternate case, *i.e.* for ρ belonging to the boundary of Ω_P .

Remark that the equation

$$\frac{b_1 \lambda_2}{g_1} k_1 - \frac{b_2 \lambda_1}{g_2} k_2 = 0 \tag{26}$$

corresponds to the case of equality in (25), taking into account (22), and defines a straight line r in the space of k_1 and k_2 . Therefore, from Proposition 6.2, it follows that if the intersection point (k_1^*, k_2^*) of the straight line r and a branch of the hyperbola defined by (22) belongs to V , then it corresponds to an extremum point ρ^* of ν . Since ρ^* satisfies (25) as an equality, we have

$$\rho^* = \frac{1}{\lambda_2 - \lambda_1} \left(\frac{c_1}{b_1} \lambda_1 - \frac{c_2}{b_2} \lambda_2 \right). \tag{27}$$

To show how the above proposition can be used, let us consider again the numerical Example 6.1.

To obtain the equation of the straight line r it is necessary to compute the values $b_1 \approx 2.406$, $b_2 \approx -12.406$ (see (23) and (24)) and g_1 , g_2 , *i.e.* the components of the vector G defined by (6). To compute g_1 and g_2 we have to calculate the inverse of the matrix S of eigenvectors of matrix Q . We obtain

$$S^{-1} = \frac{1}{\delta_x(\lambda_1 - \lambda_2)} \begin{pmatrix} \alpha - \lambda_2 & -\delta_x \\ \lambda_1 - \alpha & \delta_x \end{pmatrix} \approx \frac{1}{1.418} \begin{pmatrix} -0.259 & -1.0 \\ -1.159 & 1.0 \end{pmatrix}$$

hence

$$G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \approx \begin{pmatrix} 1.163 \\ 1.774 \end{pmatrix}.$$

The equation of the straight line r defined by (26) is therefore

$$2.604 k_1 - 1.111 k_2 = 0.$$

To write the equation of the hyperbola defined by (22) we compute also c_1 and c_2 by means of (23) and (24):

$$c_1 \approx 1.826, \quad c_2 \approx 8.174.$$

So the parametric equation of the hyperbola is

$$k_1 = \frac{1.163}{2.406 \rho + 1.826}, \quad k_2 = \frac{1.774}{-12.406 \rho + 8.174}.$$

In Figure 2 we report the set V , the hyperbola (22), *i.e.* curve h , and the straight line r .

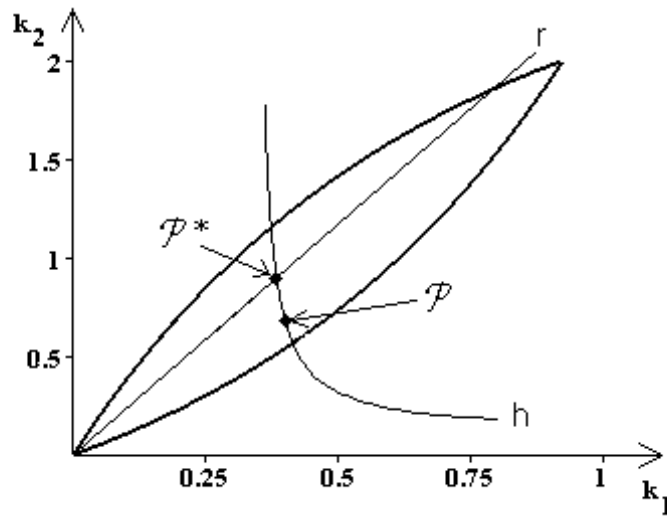


FIGURE 2. Optimal advertising weight.

The point $\mathcal{P} \in V_1$, which lies on the curve h , corresponds to the given value $\rho=0.42$, therefore the corresponding problem $P = P_\rho$ is feasible and the optimal communication is alternate. Solving system (18) we obtain the optimal switching times $\tau_1 \approx 0.406$ and $\tau_2 \approx 0.980$. The minimum expenditure in this problem is $\nu(0.42) = \bar{a}(t_2 - \tau_2) + \bar{a}(\tau_1 - t_1) \approx 4.26$.

Proposition 6.2 tells us that to minimize expenditure it is convenient to strengthen advertising with respect to sales promotion, *i.e.* to increase ρ . To do that we can move from point \mathcal{P} to point \mathcal{P}^* given by the intersection of curve h and straight line r . The corresponding advertising weight ρ^* is the minimum point of the value function ν .

By means of (27) we obtain that the optimal distribution of expenditure between advertising and sales promotion is $\rho^* = 0.5$, so the expenditure has to be equally divided into the two communication forms. Solving system (18) we obtain the optimal switching times $\tau_1^* \approx 0.196$ and $\tau_2^* \approx 0.779$. Therefore the minimum of the total expenditure for the firm, considering the best distribution among the two considered promotional activities, is $\nu(\rho^*) = \bar{a}(t_2 - \tau_2^*) + \bar{a}(\tau_1^* - t_1) \approx 4.16$.

7. CONCLUSIONS

The communication expenditure minimization model considered, is rather general and takes into consideration at once different generalizations of the classical Nerlove and Arrow model which can be found in literature. Nevertheless we did not address some typical marketing issues⁷.

Although our model responds dynamically to the increasing of communication expenditure, this happens always at the same rate, since the effect of promotion on sales (γ_x) is constant.

The linearity is another strong assumption, but we consider seasonal goods and the selling period is short therefore the problem we have faced might be considered an acceptable linear approximation of more realistic models that consider concave or S -shaped advertising return functions.

To consider only constant parameters is a shortcoming of the model too; for example we consider the goodwill effect on sales (*i.e.* the parameter γ_A) as fixed but if it would depend on time and on communication efforts (see *e.g.* Naik *et al.* [8]), this would also allow us to take into account *wearout* effects, and this way the problem would become non-linear.

Nevertheless the main meaning of the proposed model relies on its simple qualitative throughput and on putting together different promotional activities like advertising, sales promotion and the effect of word-of-mouth.

We have seen in particular that the sign of $\alpha\delta_A - \beta\delta_x$ determines the type of optimal alternate communication. This implies that if saturation aversion and goodwill decay, which can be considered as negative factors for the firm, are “stronger” than word-of-mouth and goodwill productivity, which are positive for the firm, then, in alternate communication case, it is more convenient to advertise only in the middle of the selling period. Otherwise it is convenient to advertise at the beginning of the selling period and then to refresh the goodwill of the firm at the end of the period.

⁷For a summary of the properties that an advertising response model should have, see *e.g.* [7].

Analyzing the properties of the optimal value function, we finally studied how the distribution of efforts between advertising and sales promotion can be improved in order to minimize the total expenditure in communication (see Prop. 6.2).

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