Siberian Mathematical Journal, Vol. 45, *No.* 6, *pp.* 1053–1062, 2004
Original Russian Text Copyright © 2004 *Dolfi S., Jabara E., and Lucido M. S.*

$C55$-GROUPS

S. Dolfi, E. Jabara, and M. S. Lucido UDC 512.54

Abstract: We classify the $C55$-groups, i.e., finite groups in which the centralizer of every 5-element is a 5-group.

Keywords: group, finite group, centralizer, Frobenius group

1. Introduction

It is well known that the centralizers of involutions play a fundamental role in the study of finite groups. The case of the groups has been of great interest in which the centralizer of every involution is a 2-group. These groups are called $C22$*-groups* or CIT*-groups*. In 1900, Burnside characterized the finite groups of even order in which the order of every element is either 2 or odd (see [1, pp. 208–209; 2, p. 316]). It is not difficult to characterize the soluble $C22$-groups whereas the classification of the simple $C22$-groups is a deep result due to Suzuki. In [3] he classified the simple CN-groups and then in [4] he proved that a simple $C22$-group is a CN-group. A CN-group is a group in which the centralizer of every nontrivial element is nilpotent.

A natural generalization of the concept of $C22$-group is the concept of Cpp-group, meaning a group whose order is divisible by p and in which the centralizer of a p-element is a p-group.

The first result in this direction was obtained by Feit and Thompson: in [5] they classified the simple groups with a self-centralizing subgroup of order 3 (see also Theorem 9.2 of [6]). Then Stewart proved a more general result (see Theorem A of [7]), which, together with the classification of the simple groups without elements of order 6 in [8], gives a complete description of the nonsoluble $C33$-groups.

In this paper we classify the finite $C55$-groups.

Let G be one of the groups in the following lists (it is easy to verify that G is a $C55$-group):

List A.

(A1) G is a 5-group;

(A2) G is a soluble Frobenius group such that either the Frobenius kernel or a Frobenius complement is a 5-group;

(A3) G is a 2-Frobenius group such that $\mathrm{Fit}(G)$ is a $5'$-group and $G/\mathrm{Fit}(G)$ is a Frobenius group, whose kernel is a cyclic 5-group and whose complement has order 2 or 4;

(A4) G is a 2-Frobenius group such that $\mathrm{Fit}(G)$ is a 5-group and $G/\mathrm{Fit}(G)$ is a Frobenius group, whose kernel is a cyclic $5'$-group and whose complement is a cyclic 5-group.

All groups in List A are soluble.

List B.

(B1) $G \simeq PSL(2, 5^f)$, with f a nonnegative integer;

(B2) $G \simeq PSL(2, p)$, with p prime, $p = 2 \cdot 5^f \pm 1$, and f a nonnegative integer;

(B3) $G \simeq PSL(2, 9) \simeq A_6$ or $PSL(2, 49)$;

(B4) $G \simeq PSL(3, 4)$;

(B5) $G \simeq Sz(8)$ or $Sz(32)$;

(B6) $G \simeq PSU(4, 2) \simeq PSp(4, 3)$ or $PSU(4, 3)$ or $PSp(4, 7)$;

(B7) $G \simeq A_7$ or M_{11} or M_{22}.

All groups in List B are simple.

The authors were supported by the MURST Project "Group Theory and Applications."

Florence; Venice; Udine (Italy). Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 45, No. 6, pp. 1285–1298, November–December, 2004. Original article submitted June 26, 2003.

List C.

(C1) $G \simeq PGL(2, 5^f)$ or $G \simeq M(5^{2f})$, with f a nonnegative integer;

(C2) $G \simeq M(9)$ or $PSL(2, 9)\langle\alpha\rangle \simeq S_6$, with α a field automorphism of order 2;

(C3) $G \simeq M(49)$ or $PSL(2, 49)\langle\alpha\rangle$, with α a field automorphism of order 2;

(C4) $G \simeq PSL(3, 4)\langle\alpha\rangle$, with α a field or graph-field automorphism of order 2.

All groups in List C are almost simple.

We conclude with a list of nonsoluble groups in which the Fitting subgroup $\mathrm{Fit}(G)$ is not trivial.

List D.

$\mathrm{Fit}(G) \neq 1$, every element of order 5 of G acts by conjugation fixed point freely on $\mathrm{Fit}(G)$ and $G/\mathrm{Fit}(G)$ is isomorphic to:

(D1) $PSL(2, 5) \simeq A_5$ or S_5 and $\mathrm{Fit}(G)$ is a direct product of a 2-group of class at most 3 and an abelian $2'$-group;

(D2) $PSL(2, 9) \simeq A_6$ or S_6 or $M(9)$ and $\mathrm{Fit}(G)$ is a direct product of an elementary abelian 2-group and an abelian 3-group;

(D3) $PSL(2, 49)$, $M(49)$ or $PSL(2, 49)\langle\alpha\rangle$, with α a field automorphism of order 2, and $\mathrm{Fit}(G)$ is an abelian 7-group;

(D4) $Sz(8)$ or $Sz(32)$ and $\mathrm{Fit}(G)$ is an elementary abelian 2-group;

(D5) $PSU(4, 2) \simeq PSp(4, 3)$ and $\mathrm{Fit}(G)$ is an elementary abelian 2-group;

(D6) A_7 and $\mathrm{Fit}(G)$ is an elementary abelian 2-group.

We can state our main result:

Theorem 1. *G is a finite $C55$-group if and only if G is isomorphic to one of the groups in Lists* A–D.

2. Notations and Preliminary Results

All groups in this article are finite. We use the following notations:

- $q = p^f$, with p a prime and f a nonnegative integer;
- $\mathrm{IBr}_r(G)$ is the set of irreducible Brauer characters of G in characteristic r, where r is a prime;
- $M(q)$ is the nonsplit extension of $PSL(2, q)$, with $|M(q) : PSL(2, q)| = 2$, if p is an odd prime and $q = p^{2f}$.

A group G is *almost simple* if there exists a finite nonabelian simple group S such that $S \leq G \leq \mathrm{Aut}(S)$. A group G is called 2-*Frobenius* if it has two normal subgroups N and K with $N < K$, such that K is a Frobenius group with kernel N and G/N is a Frobenius group with kernel K/N.

If G is a group then we define its *prime graph* $\Gamma(G) = \Gamma$ as follows: the set of vertices of Γ is $\pi(G)$, the set of primes dividing $|G|$. Two vertices p and q are connected if and only if in G there exists an element of order pq. Let $\pi_1, \pi_2, \dots, \pi_t$ be connected components of Γ and let $t(G) = t$ be the number of these components. Moreover if $2 \in \pi(G)$ then we suppose $2 \in \pi_1$.

A group G is a *Cpp*-group if and only if $\{p\}$ is a connected component of $\Gamma(G)$, the prime graph of G. We observe that if a group G has the *Cpp*-property then every subgroup of G of order divisible by p also has the *Cpp*-property. The same is true if we consider a quotient of G of order divisible by p.

The groups have been studied in which the prime graph is not connected. In particular Gruenberg and Kegel proved in an unpublished paper (see [9]) that these groups have the following structure:

Proposition 2 [9]. *If G is a group whose prime graph has more than one connected component then*

(a) *G is a Frobenius or 2-Frobenius group;*

(b) *G is simple;*

(c) *G is simple by π_1;*

(d) *G is π_1 by simple by π_1.*

It is clear that items (a)–(d) correspond respectively to Lists A–D, except for the 5-groups.

3. Some Number Theoretic Lemmas

To classify the simple $C55$-groups, we need to know the prime powers $q = p^f$ such that $q = 2 \cdot 5^n \pm 1$. If $f = 1$ then it is unknown whether there are finitely many primes of that form. We are interested in the case $f > 1$. We begin with

Lemma 1. *The diophantine equation*

$$X^2 + 1 = 2Y^3 \tag{$*$}$$

admits the only solutions $(1, 1)$ *and* $(-1, 1)$.

Proof. We work in the ring $\mathbb{Z}[i]$ which is a factorial domain. Let (x, y) be a solution of $(*)$. Then x is odd and therefore $1 + ix$ is divisible by $1 + i$ but not by 2. So the greatest common divisor of $1 + ix$ and $1 - ix$ is $1 + i$. From the fact that $(1 + ix)(1 - ix) = 2y^3$, and that the units of $\mathbb{Z}[i]$ are ± 1 and $\pm i$, which are all cubes, we obtain the factorization

$$1 + ix = \epsilon(1 + i)(a' + ib')^3 = (1 + i)(a + ib)^3,$$

with ϵ a unit of $\mathbb{Z}[i]$.

Adding the conjugates and dividing by 2, we find

$$1 = (a + b)(a^2 - 4ab + b^2)$$

and therefore $a = \pm 1$ and $b = 0$ or $a = 0$ and $b = \pm 1$ from which follows $x = \pm 1$ and the lemma is proved.

We can now prove

Lemma 2. *Let p be a prime number and $n, t \in \mathbb{N}$, $t > 0$. Then*

(i) *if $p^n + 1 = 2 \cdot 5^t$ then either $n = 1$ or $n = 2$; if $n = 2$ then either $t = 1$, $p = 3$ or $t = 2$, $p = 7$;*

(ii) *if $p^n - 1 = 2 \cdot 5^t$ then $n = 1$;*

(iii) *if $2^n \pm 1 = 5^t$ then $t = 1$, $n = 2$.*

Proof. (i) We suppose that $n > 1$. Let $n = 2^k \cdot d$ with d odd. If $d > 1$ then we put $q = p^{2^k}$ so that $p^n + 1 = q^d + 1 = (q + 1) \cdot (q^{d-1} - q^{d-2} + \cdots + 1)$ and therefore $(p^n + 1)/2$ is divisible by two distinct primes. So $d = 1$. Since $p^4 \equiv 1 (\mathrm{mod}\, 5)$, we hence have $k = 1$ and $n = 2$.

We now distinguish the two cases:

(A) $t = 2k + 1$ is odd. Then $p^2 = 10 \cdot 25^k - 1 \equiv 0 (\mathrm{mod}\, 3)$, p^2 is divisible by 3 and $p = 3$, since p is a prime.

(B) $t = 2k$ is even. Then

- if $k \equiv 1 (\mathrm{mod}\, 3)$ then $2 \cdot 25^k - 1 \equiv 0 (\mathrm{mod}\, 7)$; therefore, 7 divides p^2 and so $p = 7$.
- if $k \equiv 2 (\mathrm{mod}\, 3)$ then $2 \cdot 25^k - 1 \equiv 3 (\mathrm{mod}\, 7)$, which is impossible since 3 is not a square $(\mathrm{mod}\, 7)$.
- if $k \equiv 0 (\mathrm{mod}\, 3)$ then $k = 3h$ and $p^2 = 2 \cdot (25^h)^3 - 1$, which is impossible by the preceding lemma.

(ii) If $n > 1$ then there exists a Zsigmondy prime divisor q of $p^n - 1$ that does not divide $p - 1$ (see [10]). Then $q = 5$ does not divide $p - 1$, $p - 1 = 2$ and again n is an odd prime number. Therefore if $n \equiv 1 (\mathrm{mod}\, 4)$ then $3^n - 1 \equiv 2 (\mathrm{mod}\, 5)$, while if $n \equiv 3 (\mathrm{mod}\, 4)$ then $3^n - 1 \equiv 1 (\mathrm{mod}\, 5)$. This proves $n = 1$.

(iii) If $t \geq 2$ then $5^t - 1$ is divisible by an odd Zsigmondy prime (see [10]). If $5^t = 2^m - 1$ then m is a prime; otherwise $2^m - 1$ is divisible by two distinct primes. We can suppose $m \geq 3$ and then $(2^m - 1, 2^4 - 1) = 2^{(m,4)} - 1 = 1$. Therefore $2^m - 1$ is never a power of 5.

We now state some very easy results that will be helpful in the next section.

Lemma 3. *Let s be a natural number. Then*

(i) *5 divides $s(s^4 - 1)$;*

(ii) *if 5 does not divide $s(s^2 - 1)$ then 5 does not divide $s^6 - 1$;*

(iii) *if f is a prime number and r is a prime dividing $s - 1$ then r^2 does not divide $(s^f - 1)/(s - 1)$ and r divides $(s^f - 1)/(s - 1)$ if and only if $r = f$.*

Proof. (i) It is a consequence of Fermat's little theorem.

(ii) If 5 does not divide $s(s^2-1)$ then by (i) 5 divides s^2+1, which implies that 5 does not divide $s^2 \pm s+1$. This concludes the proof, since $s^6-1=(s+1)(s^2-s+1)(s-1)(s^2+s+1)$.

(iii) If r divides $s-1$ then $s=1+rm$ for some $m \in \mathbb{N}$. Then

$$\frac{(s^f-1)}{(s-1)} = s^{f-1}+s^{f-2}+\cdots+s+1 = (1+rm)^{f-1}+\cdots+(1+rm)+1$$
$$= f+rm\sum_{i=1}^{f-1} i + r^2 l = f+rmf\frac{f-1}{2}+r^2 l = f\left(1+rm\frac{f-1}{2}\right)+r^2 l$$

for some $l \in \mathbb{N}$. This implies that r^2 does not divide $(s^f-1)/(s-1)$ and r divides $(s^f-1)/(s-1)$ if and only if $r=f$.

4. Simple and Almost Simple $C55$-Groups

We now begin to study the simple groups that are $C55$. We observe that Theorem 4 of [9] is a particular case of the next proposition which is a straightforward corollary of Williams and Kondrat′ev results (see [11]).

Proposition 3. *Let G be a simple $C55$-group. Then G is one of the following:*

$$PSL(2,q), \quad \text{with } q=5^f, 9, 49 \quad \text{or} \quad q=p=2\cdot 5^t \pm 1, \quad p \text{ prime},$$
$$Sz(8),\ Sz(32),\ PSL(3,4),\ PSp(4,3),\ PSp(4,7),\ PSU(4,3),\ A_7,\ M_{11},\ M_{22}.$$

Proof. For the sporadic and alternating groups it is enough to check the connected components of the prime graph $\Gamma(G)$ in [9]. We observe that $A_5 \simeq PSL(2,5)$ and $A_6 \simeq PSL(2,9)$.

Now let G be a simple group of Lie type, $G={}^dL_n(q)$ of rank n. It is easily seen, checking the tables in [9, 11, 12], that if $n \geq 3$ then $\pi(q(q^4-1)) \subseteq \pi_1(G)$, except for ${}^3D_4(q)$, $PSU(4,2)$, and $PSU(4,3)$. Moreover $\pi(q(q^4-1)) \subseteq \pi_1(G)$ also if $G=Ree(q)={}^2G_2(q)$.

Then by Lemma 3 (i) the prime 5 is in π_1, except for $PSL(2,q)$, $PSL(3,q)$, $PSp(4,q)$, $PSU(3,q)$, $Sz(q)$, $G_2(q)$, ${}^3D_4(q)$, $PSU(4,2)$, and $PSU(4,3)$.

If $G=PSL(2,q)$ and $q \neq 5^f$ is odd then either $(q+1)/2=5^f$ or $(q-1)/2=5^f$. By Lemma 2 (i) or (ii) we can conclude that either $q=p$ for some prime p or $q=9$ or 49. If q is even then $2^n+1=5^t$ or $2^n-1=5^t$. Then by Lemma 2 (iii) we can conclude $G=PSL(2,4)\simeq PSL(2,5)\simeq A_5$.

Let G be $PSL(3,q)$, $PSU(3,q)$ or $G_2(q)$. We can suppose that $G \neq PSL(3,4)$. If $5 \notin \pi_1$ then 5 does not divide $q(q^2-1)$. By Lemma 3 (ii), 5 does not divide q^6-1, which implies that 5 does not divide $|G|$.

Let G be $PSp(4,q)$. Then $\pi_2(G)=\pi((q^2+1)/(2,q-1))$. If q is odd then by Lemma 2 (i) we have $q=3$ or 7. If q is even then by Lemma 2 (iii) we have $q=2$. But $PSp(4,2)$ is not a simple group. We observe that $PSU(4,2)\simeq PSp(4,3)$.

For the groups ${}^3D_4(q)$ we see that since 5 does not divide $q(q^2-1)$; therefore, $q^2 \equiv -1 (\bmod 5)$ and $q^4-q^2+1 \equiv 3 (\bmod 5)$ so that q^4-q^2+1 cannot be a power of 5.

If $G \simeq Sz(q)$ then $q=2^f$ with $f=2m+1$ an odd number ($m \in \mathbb{N}$). Then $\pi_3(G)=\pi(q-\sqrt{2q}+1)$, $\pi_4(G)=\pi(q+\sqrt{2q}+1)$, and $(q-\sqrt{2q}+1)(q+\sqrt{2q}+1)=(q^2+1)$. We observe that $5=2^2+1$ divides $2^{2f}+1=q^2+1$ and therefore either $\pi_3(G)=\{5\}$ or $\pi_4(G)=\{5\}$.

We first suppose that f is a prime number. From Lemma 3 (iii) with $r=5$, $s=16$, we obtain then that the highest power of 5 dividing $2^{2f}+1$ is 25 and this happens if and only if $f=5$. Therefore, if f is a prime, we conclude that $f=3$ or $f=5$. In fact for $f=3$, $\pi_3(G)=\pi(5)=\{5\}$.

Let now $f=rn$, with $1<r<f$ and r a prime number. If we put $q_0=2^r$ then $q=q_0^n$. We recall that

- if $n \equiv 1, 7 (\bmod 8)$ then $(q_0-\sqrt{2q_0}+1)$ divides $(q-\sqrt{2q}+1)$ and $(q_0+\sqrt{2q_0}+1)$ divides $(q+\sqrt{2q}+1)$;

or

- if $n \equiv 3, 5 (\bmod 8)$ then $(q_0-\sqrt{2q_0}+1)$ divides $(q+\sqrt{2q}+1)$ and $(q_0+\sqrt{2q_0}+1)$ divides $(q-\sqrt{2q}+1)$. (This is in the proof of Theorem 5 for Type ${}^2B_2$ of [13, 14].)

We now observe that if $r \neq 3, 5$ then $\pi_i(Sz(q_0)) \neq \{5\}$ for $i = 3, 4$. Therefore by the preceding remark, we conclude $\pi_i(Sz(q)) \neq \{5\}$ for $i = 3, 4$.

If $f = 9$, 15, 25 then by direct computation $\pi_i(Sz(q)) \neq \{5\}$, for $i = 3, 4$. Using again the preceding remark, we conclude that $Sz(q)$ is a $C55$-group if and only if $q = 8$, 32.

From this we easily obtain

Proposition 4. *Let G be an almost simple $C55$-group, which is not simple. Then G is one of the following:*

(i) *$PGL(2, 5^f)$ or $M(5^{2f})$, with f a nonnegative integer;*
(ii) *$M(9)$ or $PSL(2, 9)\langle\alpha\rangle \simeq S_6$, with α a field automorphism of order 2;*
(iii) *$M(49)$ or $PSL(2, 49)\langle\alpha\rangle$, with α a field automorphism of order 2;*
(iv) *$PSL(3, 4)\langle\alpha\rangle$, with α a field or graph-field automorphism of order 2.*

Proof. We have to consider the groups G such that $S < G \leq \mathrm{Aut}(S)$, with S as in Proposition 3. These can be found in [15], except for $S \simeq PSL(2, q)$ and $PSp(4, 7)$. The connected components of $\Gamma(G)$ for these groups are described in [14]. It is easily seen that if $G = \mathrm{Aut}(PSp(4, 7))$ then $\Gamma(G)$ is connected.

For the groups $PSL(2, q)$ we see that if $G = PGL(2, q)$, $q = p^f$ then the only prime not belonging to $\pi_1(G)$ is p for p an odd prime. Therefore G is a $C55$-group if and only if $p = 5$.

The connected components of $G = M(p^{2f})$, with f a nonnegative integer are exactly the same of $S = PSL(2, p^{2f})$ and therefore $M(9)$, $M(49)$ and $M(5^{2f})$ are $C55$-groups. Finally, if $G = PSL(2, q)\langle\alpha\rangle$, with α a field automorphism of order $n > 1$, then in the cases of Proposition 3 we have $q = 5^f$, 9, 49. If $q \neq 9$ then $\pi(q(q-1)) \subseteq \pi_1(G)$ and so the only possible remaining cases are $PSL(2, 9)\langle\alpha\rangle$ and $PSL(2, 49)\langle\alpha\rangle$ with α a field automorphism of order 2, which are in fact $C55$-groups.

5. Fixed Point Free Actions

If the Fitting subgroup of G is a $5'$-group then an element of order 5 of $G \setminus \mathrm{Fit}(G)$ acts fixed point freely on $\mathrm{Fit}(G)$. We therefore need some results on fixed point free actions.

In this section we use the character tables of some simple groups described in [15, 16], without further reference.

Lemma 4. *Let N be a nontrivial normal subgroup of a group G, such that $G/N \simeq S$, with S a simple group. If there is an element $g \in G$ of prime order that acts fixed point freely on N then, for every prime r dividing $|N|$, there exists some $\chi \in \mathrm{IBr}_r(S)$ such that $[\chi_T, 1_T] = 0$, where $T = \langle gN\rangle$.*

Proof. N is nilpotent, as g induces on N a fixed point free automorphism of prime order (see [17, V.8.14].) As $\langle g\rangle$ acts fixed point freely on each primary component of N, we can assume that N is an r-group for some prime $r \neq |g|$.

Since $\langle g\rangle$ acts fixed point freely on each G-composition factor in N, we can reduce to the case that N is a minimal normal subgroup of G.

We can further assume that N is an absolutely irreducible and faithful S-module. Namely, as S is simple and acts nontrivially on N, N is a faithful S-module. Let now K be a finite extension of $F = GF(r)$, such that K is a splitting field for S and let $M = K \otimes_F N$. Then for every $x \in S$ we have $C_M(x) = 0$ if and only if $C_N(x) = 0$, since x has a fixed point if and only if 1 is a root of the characteristic polynomial of x. So we can assume that N is a $K[S]$-module, i.e., N is absolutely irreducible. Since $T = \langle gN\rangle$ is a nontrivial group that acts fixed point freely on N, the restriction N_T does not contain the trivial module 1_T as a constituent. If $\chi \in \mathrm{IBr}_r(S)$ is the Brauer character associated to N, that amounts to $[\chi_T, 1_T] = 0$, as $(r, |T|) = 1$ and χ_T is an ordinary (complex) character of T.

Proposition 5. *Let N be a normal subgroup of a group G, such that $G/N \simeq S$, with S one of the following almost simple groups. Suppose further that every 5-element of G acts fixed point freely on N. Then*

(i) *if $S \simeq PSL(2, p)$, where p is an odd prime such that $(p+1)/2$ or $(p-1)/2$ is a power of 5, then $N = 1$;*

(ii) *if* $S \simeq PSL(2, 5^f)$, *with* $f \geq 2$, *then* $N = 1$;

(iii) *if* $S \simeq PSL(2,5) \simeq A_5$ *or* S_5 *then* N *is the direct product of a* 2*-group of class at most* 3 *and an abelian* $2'$*-group;*

(iv) *if* $S \simeq PSL(2,9) \simeq A_6$ *or* S_6 *or* $M(9)$ *then* N *is a direct product of an elementary abelian* 2*-group and an abelian* 3*-group;*

(v) *if* $S \simeq PSL(2,49)$ *or* $M(49)$ *or* $PSL(2,49)\langle\alpha\rangle$, *with* α *a field automorphism of order* 2, *then* N *is an abelian* 7*-group;*

(vi) *if* $S \simeq Sz(8)$, $Sz(32)$, $PSp(4,3)$, A_7 *then* N *is an elementary abelian* 2*-group;*

(vii) *if* $S \simeq PSL(3,4)$, $PSU(4,3)$, $PSp(4,7)$, M_{11} *or* M_{22} *then* $N = 1$;

Proof. As N is nilpotent, we can assume that N is an r-group, $r \neq 5$.

(i) Let $g \in G$ be an element of order 5 that acts fixed point freely on N. Let $S = G/N$ and $T = \langle gN \rangle \leq S$. By Lemma 4 to prove that N is trivial it is enough to show that

$$[\phi_T, 1_T] > 0$$

for every $\phi \in \mathrm{IBr_r}(S)$ and for each prime r, $r \neq 5$.

We denote by A a cyclic subgroup of S of order $(p-1)/2$ and by B, a cyclic subgroup of S of order $(p+1)/2$.

I. We first suppose that $r = p$. It is well known that the degrees of the p-Brauer characters of $PSL(2,p)$ are of the form $m+1$ where $0 \leq m \leq p-1$ and m is even. Further, if $\phi \in \mathrm{IBr}_p(PSL(2,p))$ has degree $2k+1$ then the restrictions of ϕ to A and B decompose in the following way:

$$\phi_A = \eta^k + \eta^{k-1} + \eta^{k-2} + \cdots + \eta^{-(k-1)} + \eta^{-k},$$
$$\phi_B = \delta^k + \delta^{k-1} + \delta^{k-2} + \cdots + \delta^{-(k-1)} + \delta^{-k}$$

where η and δ are generators of the dual groups $\widehat{A}$ and $\widehat{B}$.

As $(|T|, r) = 1$, up to conjugation we have $T \leq A$ or $T \leq B$ and hence it follows that ϕ_T has 1_T as a constituent.

So we can assume $r \neq p$.

We can also assume that $(p+1)/2$ is a power of 5 and that, up to conjugation, $T \leq B$. If namely $(p-1)/2$ is a power of 5 then T (as a conjugate to a subgroup of the “diagonal” subgroup of S) normalizes a Sylow p-subgroup P of S and T acts fixed point freely on PN. Hence PN is nilpotent and then, as $r \neq p$, P centralizes N, which implies $N = \{1\}$.

II. Let us consider first the case in which $r = \mathrm{char}(N)$ does not divide $|S|$. Then $\mathrm{IBr}_r(S) = \mathrm{Irr}(S)$.

Also, $p \equiv 1 (\mathrm{mod}\, 4)$ and the part of the character table of S which is significant for us is

	1	$\ldots$	$b \in B \setminus \{1\}$
1_G	1	$\ldots$	1
α	p	$\ldots$	-1
χ_i	$p+1$	$\ldots$	0
θ_j	$p-1$	$\ldots$	$-(\delta_j(b) + \overline{\delta_j}(b))$
γ_1	$\frac{1}{2}(p+1)$	$\ldots$	0
γ_2	$\frac{1}{2}(p+1)$	$\ldots$	0

for $1 \leq i \leq (p-5)/4$, $1 \leq j \leq (p-1)/4$, and $1_B \neq \delta_j \in \mathrm{Irr}(B)$.

We have:

(a) $[\alpha_T, 1_T] = \frac{1}{|T|}(p - |T| + 1) = \frac{p+1}{|T|} - 1 \geq 2 - 1 > 0$ as $|T|$ divides $|B| = (p+1)/2$.

(b) If $\chi = \gamma_1, \gamma_2$ or χ_i, for some $1 \leq i \leq (p-5)/4$, then $[\chi_T, 1_T] = \frac{\chi(1)}{|T|} > 0$.

(c) Let, for some $1 \leq j \leq (p-1)/4$, $\theta = \theta_j$ and $1_B \neq \delta = \delta_j \in \mathrm{Irr}(B)$. Thus,

$$[\theta_T, 1_T] = \frac{1}{|T|}(p - 1 + 2 - |T|([\delta_T, 1_T] + [\overline{\delta}_T, 1_T])) \geq \frac{p+1}{|T|} - 2.$$

Observe that $|T| = 5$ is a proper divisor of $|B| = (p+1)/2$, as $5 = (p+1)/2$ implies $p = 9$, against the assumption that p is prime. Hence, it follows $[\theta_T, 1_T] > 0$.

Let us now assume that r divides $|S| = \frac{1}{2}(p-1)p(p+1)$. Since $r \neq p$, we can assume r divides $p-1$.

III. Suppose first that $r \neq 2$. By [18, Case III], every $\phi \in \mathrm{IBr}_r(S)$ has a lift in $\mathrm{Irr}(S)$ and hence from part II it follows that $[\phi_T, 1_T] > 0$.

If $r = 2$, by [18, Case VIII (a)], every r-Brauer character ϕ that belongs to a nonprincipal block of S has a lift in $\mathrm{Irr}(S)$ and hence, again by part II, $[\phi_T, 1_T] > 0$. On the other hand, the principal block contains three Brauer characters 1, β_1, β_2 and the decomposition matrix in [18, p. 90] gives $\beta_i = \gamma_i^o - 1$ where γ_i^o is the restriction to the r-regular elements of S of the above-mentioned complex character γ_i $(i = 1, 2)$.

Since $T \leq B$, we hence obtain for $\beta = \beta_1, \beta_2$

$$[\beta_T, 1_T] = \frac{1}{|T|}\left(\frac{p-1}{2} - (|T| - 1)\right) = \frac{p+1}{2|T|} - 1 > 0$$

because $|T| = 5 \neq (p+1)/2$.

(ii) Let H be a Sylow 5-subgroup of G. If $N \neq 1$ then NH is a Frobenius group and therefore H is a Frobenius complement, and so it is cyclic. But the Sylow 5-subgroups of $PSL(2, 5^f)$ are cyclic if and only if $f = 1$.

(iii) If $r = 2$ then Theorem 2 of [19] and Theorem 1 of [20] give the conclusion.

We consider the following presentation of A_5:

$$\langle \alpha, \beta, \gamma \mid \alpha^2 = \beta^3 = \gamma^5,\ \gamma = \alpha\beta \rangle.$$

A_5 has a natural representation of dimension 4 on $\mathbb{Z}$, in which α, β, and γ are mapped respectively to the matrices A, B, and C:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad C = A \cdot B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

If $r \neq 2$ the only irreducible modular representation of A_5 in which the elements of order 5 act fixed point freely is the one just described, that can be realized over $GF(r)$, as can be checked in the character tables. We will denote by Σ the module obtained by this representation. Every composition factor of N is isomorphic to Σ, as a $GF(r)A_5$-module, and has therefore order r^4.

A simple computation shows that the exterior product $\Sigma \wedge \Sigma$ is of dimension 6 over $GF(r)$ and decomposes, in a quadratic extension of $GF(r)$, in the sum of two absolutely irreducible $GF(r^2)A_5$-modules of dimension 3. In each of these, an element of order 5 of A_5 has nontrivial fixed points. In particular there exists no nontrivial homomorphism of $GF(r)A_5$-modules $\Sigma \wedge \Sigma \to \Sigma$.

We now prove by contradiction that N is abelian. Let N be a minimal counterexample. Then N' is elementary abelian of order r^4 and isomorphic to Σ as $GF(r)A_5$-modulo. We now distinguish two cases:

(a) $N/Z(N)$ has order r^4 and it is therefore isomorphic to Σ. Then the map $\Sigma \times \Sigma \to N'$ defined by $(Z(N)x, Z(N)y) \mapsto [x, y]$ is well defined and it induces a surjective homomorphism $\psi : \Sigma \wedge \Sigma \to N' \simeq \Sigma$. This is a contradiction by the preceding remark.

(b) $|N/Z(N)| > r^4$. Since N has class 2 and N' has exponent r, for all $x, y \in N$ we have $[x, y^r] = [x, y]^r = 1$ and therefore $\Phi(N) = \langle N', N^r \rangle \leq Z(N)$. Then $N/Z(N)$ decomposes in a direct sum of a certain number of modules $\overline{N_i}$ isomorphic to Σ, with $i \in I$, a set of indices. Let N_i be the subgroup of N such that $N_i/Z(N) = \overline{N_i}$. Since $N_i < N$; therefore, N_i is abelian for all $i \in I$. Since N is not abelian by hypothesis, there exist N_1 and N_2 such that $[N_1, N_2] \neq 1$. By the minimality of $|N|$ we then have $N = N_1 N_2$, $[N_1, N_2] = N'$ and moreover $N_1 \cap N_2 = Z(N)$.

Fix a basis $\overline{x_i} = x_i Z(N)$, $i = 1, \ldots, 4$, of $\overline{N_1}$, such that $\alpha, \beta, \gamma \in A_5$ are represented by the matrices A, B, and C. Moreover, we can choose the elements x_1, x_2, x_3, x_4 of N_1, such that $x_i^\gamma = x_{i+1}$ for $i = 1, 2, 3$ and $x_4^\gamma = x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1}$.

Similarly we choose elements y_1, y_2, y_3, y_4 of N_2.

It is easy to verify that $N_3 = \langle x_1y_1, x_2y_2, x_3y_3, x_4y_4, Z(N)\rangle$ is a G-invariant subgroup of N and since $N_3 < N$, N_3 is again abelian. In particular, since N has class 2 and both N_1 and N_2 are abelian, we obtain $1 = [x_iy_i, x_jy_j] = [x_i, y_j][y_i, x_j]$ and therefore

$$[x_i, y_j] = [x_j, y_i] \quad \text{for all} \quad i, j \in \{1, 2, 3, 4\}.$$

We put

$$\epsilon_{i,j} = \begin{cases} 1 & \text{if} \quad i < j, \\ 0 & \text{if} \quad i = j, \\ -1 & \text{if} \quad i > j. \end{cases}$$

Let s_1, s_2, s_3, s_4 be a basis of Σ, chosen such that $\alpha, \beta, \gamma \in A_5$ are represented by the matrices A, B, and C, as before. We consider the map $\psi : \Sigma \times \Sigma \to N'$ of $GF(r)A_5$-modules, defined by $\psi(s_i, s_j) = [x_i, y_j]^{\epsilon_{i,j}}$.

It is easy to verify that ψ is alternating, but there does not exist nontrivial maps $\Sigma \wedge \Sigma \to N' \simeq \Sigma$ and therefore

$$[x_i, y_j] = 1, \quad i, j \in \{1, 2, 3, 4\}, \ i \neq j.$$

The only nontrivial commutators of this generating set of N' are therefore $[x_i, y_i]$ with $i = 1, 2, 3, 4$. We recall that if an automorphism γ of order 5 of a finite group T acts fixed point freely, then for all $t \in T$, we have $tt^{\gamma}t^{\gamma^2}t^{\gamma^3}t^{\gamma^4} = 1$. Then

$$[x_4, y_4]^{\gamma} = \left[x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}, y_1^{-1}y_2^{-1}y_3^{-1}y_4^{-1}\right] = [x_1, y_1][x_2, y_2][x_3, y_3][x_4, y_4]$$

because N has class 2. Therefore,

$$[x_1, y_1][x_1, y_1]^{\gamma}[x_1, y_1]^{\gamma^2}[x_1, y_1]^{\gamma^3}[x_1, y_1]^{\gamma^4} = [x_1, y_1]^2[x_2, y_2]^2[x_3, y_3]^2[x_4, y_4]^2 \neq 1$$

since $r \neq 2$. This contradiction completes the proof.

If $S \simeq S_5$ then similar methods can be used to prove the statement.

(iv) If $r = 2$ then the claim follows by Theorem 2 of [20].

If $r > 5$ then $\mathrm{IBr}_r(A_6) = \mathrm{Irr}(A_6)$ and, just checking the character table of A_6, by Lemma 4 it follows that $N = 1$.

If $r = 3$ then there exists a representation of dimension 4 over $GF(3)$, such that the 5-elements act fixed point freely and, since $A_5 < A_6$, by (iii), N is abelian.

If $S \simeq S_6$ or $M(9)$ then similar methods can be used to prove the statement.

(v) Using the character tables of $PSL(2, 49)$ and Lemma 4 we can easily conclude that the only possible case is $r = 7$. It is well known that $PSL(2, 49)$ can be represented with matrices 4×4 with coefficients in $GF(7)$ and in such a representation each element of order 5 acts fixed point freely. Since $PSL(2, 49)$ contains a subgroup isomorphic to A_5, by (iii), it follows that the 7-group N is abelian.

If $S \simeq M(49)$ or $PSL(2, 49)\langle\alpha\rangle$ with α a field automorphism of order 2, similar methods can be used to prove the statement.

(vi) Let $S \simeq Sz(8)$ or $Sz(32)$. If $r \neq 2$ then $N = 1$, as proved in [21].

If $r = 2$ then N is an elementary abelian 2-group, and the action is the natural action as proved in [22].

In $PSp(4, 3)$ there is a maximal subgroup H, which is the semidirect product of an elementary abelian 2-group K with a group isomorphic to A_5. Moreover, H is a $C55$-group. Then NK is nilpotent and therefore N is a 2-group. Since $PSp(4, 3)$ has also a subgroup isomorphic to A_6, by (iv) we conclude that N is elementary abelian.

Since $A_6 \leq A_7$ by (iv), N is an abelian $\{2, 3\}$-group. Using the 3-modular character table of A_7, by Lemma 4 the 3-component of N is trivial.

(vii) Using the character tables of $PSL(3, 4)$ and Lemma 4, we can easily conclude that $N = 1$.

$PSU(4,3)$ contains a Frobenius subgroup, with an elementary abelian kernel of order 2^4 and a complement of order 5 and a Frobenius subgroup, with an elementary abelian kernel of order 3^4 and a complement of order 5. This implies that N should be a 2-group and 3-group. Then $N = 1$.

$PSp(4,7)$ contains a subgroup isomorphic to $PSL(2,49)$ therefore, by (v), N should be a 7-group. But $PSp(4,7)$ contains also a subgroup isomorphic to A_7 therefore, by (vi), N should be a 2-group. Then $N = 1$.

Both M_{11} and M_{22} contain a subgroup isomorphic to A_6 and a subgroup isomorphic to the Frobenius group of order 55. Then N should be both a $\{2,3\}$-group and 11-group. Therefore, $N = 1$.

6. Proof of the Theorem and Concluding Remarks

We can now easily complete the proof of our theorem.

Proof of Theorem 1. We suppose that G is not a 5-group. Therefore, $\Gamma(G)$ is not connected and so by Proposition 2 G is one of the following groups:

(a) G is a Frobenius or 2-Frobenius group. In the first case either the Frobenius kernel or the Frobenius complement are 5-groups, since the Frobenius kernel as well as the Frobenius complement has nontrivial center. In the second case, if $F = \mathrm{Fit}(G)$ is a 5-group then $G/\mathrm{Fit}(G)$ is a Frobenius group whose kernel $\overline{K}$ is a cyclic $5'$-group. In fact if K is the subgroup of G containing F such that $\overline{K} = K/F$ is the Fitting subgroup of G/F, then $K = FH$ is a Frobenius group, with H a nilpotent Frobenius complement. Therefore H is either a cyclic subgroup or the product of a cyclic group with a generalized quaternion group. Moreover, $\pi_1(G) = \pi(K/F)$ and $\pi_2(G) = \pi(F) \cup \pi(G/K) = \{5\}$. Since $\overline{K} = FH/F \simeq H$ and G/K is a 5-group acting fixed point freely on $\overline{K}$, we conclude that H is a cyclic group, because the outer automorphism group of the generalized quaternion group Q_{2^n} is a 2-group, if $n > 3$ and $\mathrm{Out}(Q_8) \simeq S_3$.

If F is a $5'$-group then $G/\mathrm{Fit}(G)$ is a Frobenius group whose kernel $\overline{K}$ is a cyclic 5-group and therefore the Frobenius complement can only be a cyclic group of order 2 or 4.

We remark that a Frobenius $C55$-group is necessarily soluble. Otherwise the Frobenius complement contains a subgroup isomorphic to $SL(2,5)$, which is not a $C55$-group.

(b) G is a simple group, and then the claim follows from Proposition 3.

(c) G is a simple by π_1 group. This implies that G is an almost simple group, and again we conclude by Proposition 4.

(d) G is a π_1 by simple by π_1 group.

It can be easily deduced from the results in [9] that $F = \mathrm{Fit}(G) = O_{\pi_1}(G)$ and G/F is isomorphic to an almost simple group. Moreover if S is the only simple nonabelian section of G, we have $\pi_i(G) = \pi_i(S)$ for $i \geq 2$. Therefore this is the case in which $F \neq 1$ and G/F is an almost simple $C55$-group, and the conclusion comes from Proposition 4.

If G is a soluble nonnilpotent $C55$-group we can give a more detailed description of the structure of G. In particular, if we put $\pi_*(G) = \pi(G) \setminus \{5\}$ and $p_* = \min(\pi_*(G))$, we have the following

Proposition 6. *If G is a soluble nonnilpotent $C55$-group then*

(i) *the derived length of G is bounded by a function of p_*, in particular if $p_* = 2$ then $G^{(5)} = 1$;*

(ii) *if $p_* > 2$ then G'' is nilpotent.*

Proof. It is well known that a finite group with a fixed point free automorphism of prime order p is nilpotent and its nilpotency class is bounded by a function $f(p)$ of p. We can suppose $p > 2$, otherwise the group is abelian. We have $f(p) \leq 1 + (p-1) + \cdots + (p-1)^{2^p-2}$ (see Theorem VIII.10.12 of [23]); moreover G. Higman conjectured that if p is odd, $f(p) = \frac{p^2-1}{4}$ and proved its conjecture for $p = 5$: in particular $f(5) = 6$ (see Remark VIII.10.13.b of [23]).

We study the different cases following List A.

(1) G is a Frobenius group (case A2). Let N be the Frobenius kernel and let K be a Frobenius complement of G. We can distinguish two subcases:

(1a) N is a 5-group. If $2 \in \pi(K)$ then N is abelian and K has derived length at most 4. In fact a soluble Frobenius complement has derived length at most 4, as it can be easily deduced from Chapter 18 of [24]. Therefore G has derived length at most 5. If $2 \notin \pi(K)$ then K is metacyclic and therefore $G'' \leq N$. Moreover, as we have observed, the nilpotency class of N is bounded by $f(p_*)$. Therefore the derived length of G is bounded by a function of p_*.

(1b) N is a $5'$-group. Then K is a cyclic 5-group and N is nilpotent of class at most $f(5) = 6$. In particular the derived length of N is at most 3 and since $G' \leq N$ we have $G^{(4)} = 1$.

(2) G is a 2-Frobenius group. Let $N = \mathrm{Fit}(G)$. We can distinguish two subcases:

(2a) N is a 5-group (case A4). Then $G'' \leq N$ and we conclude as in (1a). We observe that in this case the order of G is necessarily odd.

(2b) N is a $5'$-group (case A3). Then $G'' \leq N$ and N is nilpotent of class at most $f(5) = 6$. In particular $G^{(5)} = 1$.

References

1. *Curtis C. W.*, "Pioneers of representation theory," Hist. Math., **15**, 75–96 (1999).
2. *Solomon R.*, "A brief history of classification of the finite simple groups," Bull. Amer. Math. Soc., **38**, 315–352 (2001).
3. *Suzuki M.*, "Finite groups with nilpotent centralizers," Trans. Amer. Math. Soc., **99**, 425–470 (1961).
4. *Suzuki M.*, "On a class of doubly transitive groups," Ann. Math. (2), **75**, 105–145 (1962).
5. *Feit W. and Thompson J. G.*, "Finite groups which contain a self-centralizing subgroup of order 3," Nagoya Math. J., **21**, 185–197 (1962).
6. *Higman G.,* Odd Characterizations of Finite Simple Groups, Univ. of Michigan, Michigan (1968).
7. *Stewart W. B.*, "Groups having strongly self-centralizing 3-centralizers," Proc. London Math. Soc., **26**, 653–680 (1973).
8. *Fletcher L. R., Stellmacher B., and Stewart W. B.*, "Endliche Gruppen, die kein Element der Ordnung 6 enthalten," Quart. J. Math. Oxford, **28**, 143–154 (1977).
9. *Williams J. S.*, "Prime graph components of finite groups," J. Algebra, **69**, 487–513 (1981).
10. *Zsigmondy K.*, "Zur Theorie der Potenzreste," Monatsh. Math. Phys., **3**, 265–284 (1892).
11. *Kondrat′ev A. S.*, "Prime graph components of finite simple groups," Math. of the USSR, **67**, 235–247 (1990).
12. *Iiyori N. and Yamaki H.*, "Prime graph components of the simple groups of Lie type over the field of even characteristic," J. Algebra, **155**, 335–343 (1993).
13. *Lucido M. S.*, "Addendum to 'Prime graph components of finite almost simple groups'," Rend. Sem. Mat. Univ. Padova, **107**, 1–2 (2002).
14. *Lucido M. S.*, "Prime graph components of finite almost simple groups," Rend. Sem. Mat. Univ. Padova, **102**, 1–22 (1999).
15. *Conway J., Curtis R., Norton S., Parker R., and Wilson R.*, Atlas of Finite Groups, Clarendon Press, Oxford (1985).
16. *Jansen C., Lux K., Parker R., and Wilson R.,* An Atlas of Brauer Characters, Clarendon Press; Oxford Univ. Press, New York (1995). (L. M. S. Monographs. New Series, 11. Oxford Sci. Publ.)
17. *Huppert B.*, Endliche Gruppen. I, Springer-Verlag, Berlin (1967).
18. *Burkhardt R.*, "Die Zerlegungsmatrizen der Gruppen $PSL(2, p^f)$," J. Algebra, **40**, 75–96 (1976).
19. *Holt D. F. and Plesken W.*, "A_5-invariant 2-groups with no trivial sections," Quart. J. Math. Oxford, **37**, 39–47 (1986).
20. *Prince A. R.*, "On 2-groups admitting A_5 or A_6 with an element of order 5 acting fixed point freely," J. Algebra, **49**, 374–386 (1977).
21. *Martineau P.*, "On representations of the Suzuki groups over fields of odd characteristic," J. London Math. Soc., **6**, 153–160 (1972).
22. *Martineau P.*, "On 2-modular representations of the Suzuki groups," Amer. J. Math., **94**, 55–72 (1972).
23. *Blackburn N. and Huppert B.*, Finite Groups. II, Springer-Verlag, Berlin; Heidelberg; New York (1982).
24. *Passman D.*, Permutation Groups, W. A. Benjamin Inc., New York; Amsterdam (1968).