Searching for Fractal Structure in Agricultural Futures Markets

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The four parameters of the Pareto stable probability distribution for six agricultural futures are estimated. The behavior of these estimates for different time-scaled distributions is consistent with the conjecture that the stochastic processes generating these agricultural futures returns are characterized by a fractal structure. In particular, it is empirically verified that the six futures returns satisfy the property of statistical self-similarity. Moreover, the same time series is analyzed by using the so-called rescaled range analysis. This analysis is able to detect both the fractal structure and the presence of long-term dependence within the observations. The Hurst exponent with the use of two methods, the classical and modified rescaled analysis, is estimated and tested. Finally, with the use of Mandelbrot's result on the existence of a link between the characteristic ex-

We are grateful to two anonymous referees for extensive comments and to Professor Giovanni Zambruno, Department of Mathematics and Statistics, University of Milan (Italy), for his useful suggestions. We are also thankful to the editor, Dr. Mark Powers, for his support and encouragement.

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The Journal of Futures Markets, Vol. 17, No. 4, 433–473 (1997)

CCC 0270-7314/97/040433-41

ponent of a stable distribution and the Hurst exponent, further empirical confirmation is found that the processes generating agricultural futures returns are fractal.

INTRODUCTION

Despite the early contributions of Working (1934) about the random behavior of futures prices, financial economists have studied systematically the behavior of asset pricing only during the past three decades. Asset pricing is assumed to mean the dynamic, period-by-period, change in the price level of an asset, such as the closing price of a given stock or the settlement price of a certain futures contract. The exhaustive literature on the random walk behavior of asset prices is known as market efficiency. Despite the existence of several puzzling and conflicting results, in general, and in futures markets in particular, the theory of efficient markets remains a central pillar of modern financial economics.

Samuelson (1965) developed the efficient market hypothesis to rationalize the random walk behavior, whereby the current price, $P(t)$, fully reflects all relevant information. Because the flow of such information between now and the next period cannot be anticipated, efficient market price changes are serially uncorrelated. In other words, the randomness in price changes originates in the random flow of unanticipated information.

During the past 20 years, the theory of market efficiency has been refined analytically, mathematically and statistically; the concept of information has been made more precise; and the notion of random walk has been generalized to martingales and Itô processes. Numerous sophisticated statistical tests were employed to test the theory. Moreover, a very large literature has been developed concerning the statistical distribution of the changes in spot or futures prices: Are they log-normal, or are they leptokurtic and if leptokurtic, how fat are the tails?

The actual distribution of spot or futures price changes or returns is an issue of great importance to financial economists. In an efficient market it follows that such returns are random. Furthermore, these random returns are postulated to be normally distributed. The theoretical foundations underlying these studies are not always clear. Grossman and Stiglitz (1980) addressed several important analytical issues of the theory of efficient markets. They argued that the notion of market efficiency is inconsistent with the reality of costly arbitrage. They developed a simple model with a constant absolute risk-aversion utility function and showed that costless information is both necessary and sufficient for prices to fully reflect all available information. Efficient markets theorists realize that costless information is a sufficient condition for market efficiency. However, they are not always clear that it is also a necessary condition. The cost of acquiring and acting on new information means that prices in a competitive market need not follow a random walk. Tomek (1994) explained well how persistence in price behavior can occur in rational markets.

It is not surprising to find that along with numerous studies confirming market efficiency, there are numerous studies rejecting it, and that there is no agreement concerning the statistical distribution functions of price changes. Nevertheless, the most convenient and widely acceptable paradigm postulates that returns are normally distributed, which means that asset prices follow log-normal distributions. Both modern portfolio theory and the Black–Scholes methodology of pricing derivative assets are founded on such a paradigm.

Although market efficiency remains the central theory of financial economics, numerous studies question its twin foundations: random walk and log-normal distribution of asset prices. Notice, however, that it is not enough to reject randomness or log-normality. To make scientific progress, alternatives to randomness and log-normality must be specified.

The purpose of this paper is to investigate these twin issues of randomness and log-normality by empirically examining the behavior of six agricultural futures prices.

It is shown that returns are neither log-normally distributed nor statistically independent.

The classical approach of time-series analysis concerning financial markets [initiated by Bachelier (1900)] investigates the distribution of security price increments. Most models are based on the hypothesis of a normal distribution for the variation of such prices; that is,

$$P(t + dt) - P(t) \sim N(\mu\, dt, \sigma^2\, dt) \tag{1}$$

As a result of the empirical work of Osborne (1959), such normal distributions were replaced by the notion that asset prices are independent and log-normally distributed as

$$\log[P(t + dt)] - \log[P(t)] = \log[P(t + dt)/P(t)] \sim N(\mu\, dt, \sigma^2\, dt) \tag{2}$$

This idea had an important impact on financial theory. The Black–Scholes option pricing model is one of its most celebrated results. However, as the following review describes, several studies showed that (2) does not hold empirically, primarily because of reasons such as fatter tails, the instability in the variance level (accounting for the relatively many outliers), and issues of asymmetry. The Pareto–Lévy stable distribution family is proposed as a way of correcting for these realities.

HYPOTHESIS

This study's hypothesis rejects the null hypothesis that the distribution of agricultural futures prices is log-normal, as was postulated by Helms and Martell (1985). Then the four parameters of the characteristic function of the stable random variable are estimated with the use of U.S. agricultural futures data. The behavior of the values of the estimates for different time-scaled distributions leads to the formulation of the hypothesis: The stochastic-process underlying futures returns are characterized by a fractal structure, as proposed by Peters (1989, 1991a, 1991b and 1994) and Walter (1990).

This hypothesis is analyzed also with the use of the so-called rescaled range analysis proposed by Hurst (1951) to detect both a possible fractal structure and long-run dependence in the observations. In particular, the memory effect emphasizes the existence of a nonzero temporal linear or nonlinear correlation among the observations, contradicting the usual hypothesis of their independence by the efficient market hypothesis. The fractality and the dependence within the observations is measured with the use of the Hurst exponent.

The Hurst exponent is estimated by two methods: the classical R/S analysis and the modified $R/\bar{S}$ one, proposed by Lo (1991), which adjusts the classical rescaled range statistic mainly for short-term dependence. Then, the goodness fit of the estimate of the Hurst exponent is checked with the use of an asymptotic test proposed also by Lo (1991). Finally, with the use of the Mandelbrot and Taqqu (1979) result on the existence of a link between the characteristic exponent of the Pareto–Lévy stable distribution and the Hurst exponent, it is confirmed that the processes generating the six agricultural futures returns are fractal.

REVIEW OF THE LITERATURE

A brief review of the literature on randomness and log-normality follows. Only a few key references are discussed, because these ideas are generally well known.

The two fundamental reviews are Fama (1970) and (1991), and the book by Guimaraes, Kingsman, and Taylor (1989). These apply to asset prices in general, rather than to futures prices, in particular. Studies that deal with the appropriateness of the random walk or the martingale model in futures markets include: the investigation of the treasury bill and treasury bond futures markets by Chance (1985), Klemkosky and Lasser (1985), Cole, Impson, and Reichenstein (1991), and MacDonald and Hein (1993); the investigation of the agricultural commodities by Bigman, Goldfarb, and Schechtman (1983), Canarella and Pollard (1985), Maberly (1985), Bird (1985), Elam and Dixon (1988); the investigation of the metal futures market by Gross (1988); and the investigation of the foreign currency markets by Glassman (1987), Saunders and Mahajan (1988), Harpaz, Krull, and Yagil (1990).

Many of these studies hold positive opinions on market efficiency. Chance (1985) believes that the treasury bond futures market correctly anticipates the information contained in the announcement of the rate of change of the Consumer Price Index. MacDonald and Hein (1993) comment that the T-bill futures market may not be as inefficient as once presumed in terms of weak form efficiency, though it does not provide optimal forecasts. Maberly (1985) demonstrates that, in the grains, the inference that the market is inefficient for more distant futures contracts is due to the bias that results from using ordinary least squares to estimate parameters. Elam and Dixon (1988) attack the inefficiency grain market argument by conducting several Monte Carlo experiments to find out that very often the *F* test tends to wrongly reject the true model. The research of Canarella and Pollard (1985) suggests that the efficient market hypothesis cannot be rejected for corn, wheat, soybeans, and soybean oil. Gross (1988) claims that the hypothesis of efficient copper and aluminum markets cannot be rejected on the evidence of his semistrong efficiency tests. Saunders and Mahajan (1988) show that the index futures pricing is efficient.

However, numerous authors offer negative evidence on market efficiency. Bird (1985) discovers that for coffee and sugar the efficient market hypothesis is invalid, and for cocoa there is some evidence of inefficiency, but of limited economic significance. Harpaz et al. (1990) perform tests for efficiency of the USDX futures contracts during the period, 1985–1988, which result in their rejection of the null hypothesis that the USDX futures market is efficient during that period.

Finally, quite a few authors, instead of totally supporting or rejecting the efficient market hypothesis, offer different answers under different situations. Bigman et al. (1983) believe that the market can be generally characterized as efficient for the futures contracts on wheat, soybeans, and corn six weeks before delivery or less. For longer-term futures contracts, their tests reject the efficiency hypothesis. The results of the T-bond market efficiency tests of Klemkosky and Lasser (1985) do not agree totally with the conclusions drawn from earlier studies. Glassman (1987) reports evidence of multimarket and joint multimarket inefficiency in foreign currency futures markets during some of the 38 contract periods studied. Much of the inefficiency appeared to be short term in duration (one week or less). Cole et al. (1991) conclude that the T-bill futures rates provide rational one- and two-quarters-ahead forecasts of futures spot rates, which are the forecast horizons that seem to be of most interest to the public. However, they believe the rationality of four-quarters-ahead futures forecasts should be rejected.

The various empirical studies that have rejected the theory of market efficiency have encouraged financial economists to seek alternative explanations for the time-series behavior of asset returns. This literature is known as the *chaotic dynamics* approach to asset returns, and several studies, such as Decoster, Labys, and Mitchell (1992) offer evidence that futures prices appear to follow low dimensional chaotic dynamics.

Observe that the majority of research concentrates on stock returns. After the seminal articles by Osborne (1959), Fama (1965), Mandelbrot (1963), Fama and Roll (1968 and 1971), and Mandelbrot and Taylor (1967), numerous other articles have followed. These are carefully reviewed in Akgiray and Booth (1988). Although most articles reject the normal distribution in favor of the stable Lévy–Paretian, studies exist that reject the stable Lévy–Paretian distribution, but not in favor of normality.

Earlier, Stevenson and Bear (1970) and Dusak (1973) offered evidence in support of the stable Lévy–Paretian distribution. More recently, Helms and Martell (1985), using data for all commodities traded on the Chicago Board of Trade, conclude that returns on futures prices, although they are not normally distributed, are closer to normal than to any other member of the family of Pareto distributions. Contrary to their results, Cornew, Town, and Crowson (1984) claim that the stable Lévy–Paretian distribution offers a better fit for futures returns of several contracts than the normal distribution. So (1987) confirms that currency futures and spot returns are stable Lévy–Paretian, whereas Hall, Brorsen, and Irwin (1989) and Hudson, Leuthold, and Sarassoro (1987) claim that futures returns are not stable Lévy–Paretian. Finally, Gribbin, Harris, and Lau (1992) use a newly developed statistical methodology to conclude that futures prices are not stable Lévy–Paretian distributed. Their methodology, however, is not powerful enough to distinguish a stable distribution from other distributions. Simulation results show that the method will almost always reject any stable distribution.

DATA

The data used in this study correspond to returns of daily settlement prices for the time period, January 2, 1981 to October 24, 1991, for the following six agricultural futures contracts: corn, oats, soybeans, soybean meal, soybean oil, and wheat. These contracts are traded at the Chicago Board of Trade.

FRACTAL PARETO–LÉVY STABLE DISTRIBUTIONS: THEORETICAL ASPECTS

Lévy (1925) introduced the stable distributions as a generalization of the Brownian motion. Recall that a Brownian motion is simply a continuous-time random walk. Falconer (1990) gives the following definition regarding the stable process.

Definition 5.1. A random process $X(t)$, with $t \in [0, +\infty)$, is stable if the increments $X(t + \Delta t) - X(t)$ are stationary; that is, they depend only on Δt, and independent; that is, for all $0 < t_1 < t_2 < \cdots < t_{2m}$, the increments $X(t_2) - X(t_1), \ldots, X(t_{2m}) - X(t_{2m-1})$ are independent.

This class of distributions allows a generalization of the central limit theorem under weaker hypotheses. In particular, the stable distribution represents a generalization of the normal one when the moment of order 2 or the moments of order 1 and 2 do not exist.

Generally speaking, it is not possible to give a closed form for the density functions of the stable class. This family can be characterized by its characteristic function:

$$\phi(t) = \exp\{i\,\delta t - |\gamma t|^{\alpha}[1 + j\beta\, sgn(t)\omega(t,\alpha)]\} \tag{3}$$

where

$$j = \begin{cases} -i, & \text{if } \alpha \neq 1 \\ +i, & \text{if } \alpha = 1 \end{cases}$$

$$\omega(t,\alpha) = \begin{cases} \tan(\alpha\pi/2), & \text{if } \alpha \neq 1 \\ (2/\pi)\log(\mathrm{t}), & \text{if } \alpha = 1 \end{cases}$$

Note that (3) is characterized by four parameters α, β, γ, and δ. In particular:

1. $\alpha \in (0,2]$ is the characteristic exponent that accounts for the relative importance of the tails. If $\alpha = 2$, then (3) corresponds to the normal distribution with finite mean and variance. When $\alpha \in (1,2]$, the random variable has only finite mean.

2. $\beta \in [-1,1]$ is a skewness parameter. In particular, when $\beta = 0$, the distribution is symmetric.
3. $\gamma \in (0,+\infty)$ is a scale parameter. In particular, some authors use $c = \gamma^{\alpha}$ (when $\alpha = 2$, the distribution has variance $2c$),
4. $\delta \in (-\infty,+\infty)$ is a location parameter. When $\alpha \in (1,2]$, δ is the mean of the distribution and when $\beta = 0$, δ is the median of the distribution.

The density probability function of the stable distribution can be written in the following integral form:

$$f(x) = \frac{1}{\pi}\int_{0}^{+\infty} \cos[-xt + \delta t - (\gamma t)^{\alpha}\beta\omega(t,\alpha)] \exp[-(\gamma t)^{\alpha}]\, dt. \qquad (4)$$

Using *Definition 5.1* of the stable distribution, Falconer (1990) and Peters (1991b) cite an important theorem that gives a property of this distribution, namely, that the stable distribution is fractal. Falconer (1990) offers a detailed mathematical definition of fractal and then summarizes it intuitively as follows. A set, F, is fractal if it satisfies four conditions:

1. F has a fine structure. This means the set is very detailed on arbitrary small scales.
2. F is too irregular both locally and globally.
3. F has some form of self-similarity; that is, parts of F resemble the whole F in some way.
4. The fractal dimension of F, defined in some way, is greater than its topological dimension.
5. F is often described recursively.

The interest in fractal objects is motivated by the central question of financial economics; that is, what is the behavior of asset prices? To show that asset prices follow fractal processes is to show more than random walk. Fractal processes generalize random walks because, in addition to their irregularity they are also self-similar, and the dimension of the set can be computed. In other words, fractal processes have a fine structure that often cannot be detected by various low-power tests of random walk. Furthermore, because fractal processes are quite complex and cannot be detected easily, they are consistent with the paradigm of market efficiency.

To characterize the fractal nature of a Pareto–Lévy stable distribution, Falconer (1990) and Peters (1991b) give the following theorem.

Theorem 5.1. Let $X(t)$, with $t \in [0, +\infty)$, be a Pareto–Lévy stable stochastic process with characteristic exponent, α, and let $S_{X(t)} = \{X(t_i), t_i \in [0, +\infty), i \in \{1, \ldots, T\}\}$ be its time series. Then, with probability 1, the $\dim(S_{X(t)})$, is equal to α.

Moreover, to point out another property of the stochastic fractal objects; that is, the self-similarity, Mandelbrot and Taqqu (1979), Feder (1988), and Falconer (1990) give the following definition.

Definition 5.2. Let $X(t)$, with $t \in [0, +\infty)$, be a continuous stochastic process. This process is called self-similar if, for fixed t, it has the same distribution as $\lambda^{-K}X(\lambda t)$, with $K \in (-\infty, +\infty)$ and $\lambda \in (0, +\infty)$.

Notice that the characteristic functions of these two random variables must depend on the same parameters. Therefore, the random variable, $X(t)$, is affected neither by an expansion of the time scale (λt) nor by the contemporaneous homothety of the space scale, (λ^{-K}). In particular, if the random variable is Pareto–Lévy stable, Mandelbrot and Taqqu (1979) prove that $K = 1/\alpha$.

FRACTAL PARETO–LÉVY STABLE DISTRIBUTIONS: EMPIRICAL RESULTS

To obtain the values of the four parameters and to verify the statistical self-similarity property, the time series of the daily settlement prices of the six agricultural futures contracts are used to compute scaled returns

$$X_{t,n} = \lambda^{-1/\alpha} \log[P(t + n)/P(t)] \tag{5}$$

where $n = \lambda t$ is a time scale, with $t = 1$ and $\lambda > 0$. $P(t)$ is the settlement price at time t. Notice that, if $n = 1$, then $X_{t,n}$ is the usual logarithmic return of daily settlement prices.

Possible dependence in the data must be eliminated. Of course, the true autocorrelative relationship characterizing each time series is not known, so it is approximated by an autoregressive model of order q (AR(q)). From this the corresponding uncorrelated residuals time series is obtained by fitting an ordinary least-squares (OLS) regression. In this OLS regression a crucial role is played by q, and the Andrews (1991) data-dependent rule, $q = \text{Int}\{(3T/2)^{1/3}[\mu/(1 - \mu^2)]^{2/3}\}$, where T is the time series size and μ is its sample first-order autocorrelation coefficient, is used to detect it. Notice that this data-dependent rule is mainly able to detect the short-term autocorrelative length and so, by using it, a second source of approximation is introduced. The results obtained by using the Andrews' rule are the following: $q = 3$ for corn, $q = 3$ for oats, $q =$

TABLE I

Tests for Log-Normality Using a $\chi^2(g)$ test; g Denotes Degrees of Freedom

Futures	$\chi_0^2(13)$	$\chi_q^2(13)$	$\chi_0^2(27)$	$\chi_q^2(27)$
Corn	450.337	425.461	477.513	455.028
Oats	249.722	231.980	286.482	262.884
Soybeans	248.087	248.087	276.800	276.800
Soybean meal	310.154	305.477	336.552	325.947
Soybean oil	158.493	143.318	173.902	165.264
Wheat	352.703	352.218	368.403	369.448

TABLE II

Estimates of the Four Parameters of the Stable Distribution Assuming Independence

Futures	$\alpha_{1,0}$	R_α^2	$\bar{R}_\alpha^2$	$\beta_{1,0}$	R_β^2	$\bar{R}_\beta^2$	$c_{1,0}$	$\delta_{1,0}$
Corn	1.64	1.00	1.00	0.05	0.91	0.90	0.01	0.00
Oats	1.77	1.00	1.00	−0.02	0.99	0.99	0.01	0.00
Soybeans	1.74	1.00	1.00	−0.17	1.00	1.00	0.01	0.00
Soybean meal	1.73	1.00	1.00	0.15	0.96	0.96	0.01	0.00
Soybean oil	1.81	1.00	1.00	0.17	0.97	0.96	0.01	0.00
Wheat	1.77	1.00	1.00	0.18	0.92	0.92	0.01	0.00

TABLE III

Estimates of the Four Parameters Assuming Dependence

Futures	$\alpha_{1,q}$	R_α^2	$\bar{R}_\alpha^2$	$\beta_{1,q}$	R_β^2	$\bar{R}_\beta^2$	$c_{1,q}$	$\delta_{1,q}$
Corn	1.64	1.00	1.00	0.05	0.95	0.94	0.01	0.00
Oats	1.77	1.00	1.00	0.01	0.98	0.98	0.01	0.00
Soybean	1.74	1.00	1.00	−0.17	1.00	1.00	0.01	0.00
Soybean meal	1.73	1.00	1.00	0.15	0.97	0.97	0.01	0.00
Soybean oil	1.82	1.00	1.00	0.16	0.98	0.98	0.01	0.00
Wheat	1.77	1.00	1.00	0.18	0.93	0.92	0.01	0.00

0 for soybeans, $q = 1$ for soybean meal, $q = 4$ for soybean oil, and $q = 1$ for wheat.

Next the null hypothesis that the distribution of daily price changes of the agricultural futures contracts is log-normal is rejected. Specifically, this study tests for log-normality using an χ_q^2 (g) distributed fit test, where g are the degrees of freedom, assuming both independence ($q = 0$) and dependence as previously described. The results of this test are presented in Table I.

In Tables II and III the estimates, $\alpha_{n,q}$, $\beta_{n,q}$, $c_{n,q}$ and $\delta_{n,q}$, are reported with the time scale, $n = 1$, assuming both independence and dependence as previously described. To estimate these parameters, the method developed by Koutrouvelis (1980, 1981) is used, where the starting values of the parameters are determined by the MacCulloch method (1986) instead of by the simple Fama and Roll one (1971). The algorithm for the estimation of these parameters using the Koutrouvelis' methodology is found in Canestrelli, Cipriani, and Corazza (1993). In particular, the MacCulloch method eliminates some restrictions arising from the (strong) *a priori* hypotheses assumed by the Fama and Roll method, that is, the symmetry of the probability distribution ($\beta = 0$) and the existence of a mean ($\alpha \in (1,2]$).

To detect the statistical self-similarity, following the *Definition* 5.2, it is empirically verified if the four estimated parameters are affected by an expansion of the time scale on the contemporaneous homothety of the space scale. In particular, the time scale from 1 day ($n = 1$) to about 1 month ($n = 25$) is considered. Notice that, for every fixed time scale, n, it is possible to extract n different sequences from the original time series. To get better estimates, the value of each parameter is determined by calculating the mean of the different sequences. The results of this analysis are reported in Figures 1–6.

From Tables I–III and Figures 1–6 one can deduce the following.

First, the analyzed time series are significantly nonnormal because of the (wide) rejection of the null hypothesis that the sample distribution of daily (i.e., $n = 1$) price changes is normal (see Table I). In particular, the results of the test are qualitatively the same assuming both independence ($q = 0$) and dependence; therefore, one can conjecture that the influence of the simple autocorrelative structure assumed earlier is negligible. Moreover, the estimated characteristic exponent for different time-scaled distributions (i.e., from $n = 1$ to $n = 25$) are, in general, less than 2 and, so, nonnormal.

Second, the values of the statistics, R^2 and $\bar{R}^2$, associated with both the characteristic exponents and the skewness parameters of every analyzed time series, are elevated ($\bar{R}^2_\alpha = 1.00$, $\bar{R}^2_\alpha = 1.00$, $R^2_\beta \in [0.91,1.00]$ and $\bar{R}^2_\beta \in [0.90,1.00]$). Notice that, because the characteristic exponent is greater than 1, the location parameter gives the sample mean of the distribution and, in particular, it is close to 0 for all the analyzed time series. Notice also that, because the characteristic exponent is less than 2, the probability distribution does not have finite variance, so, one can-

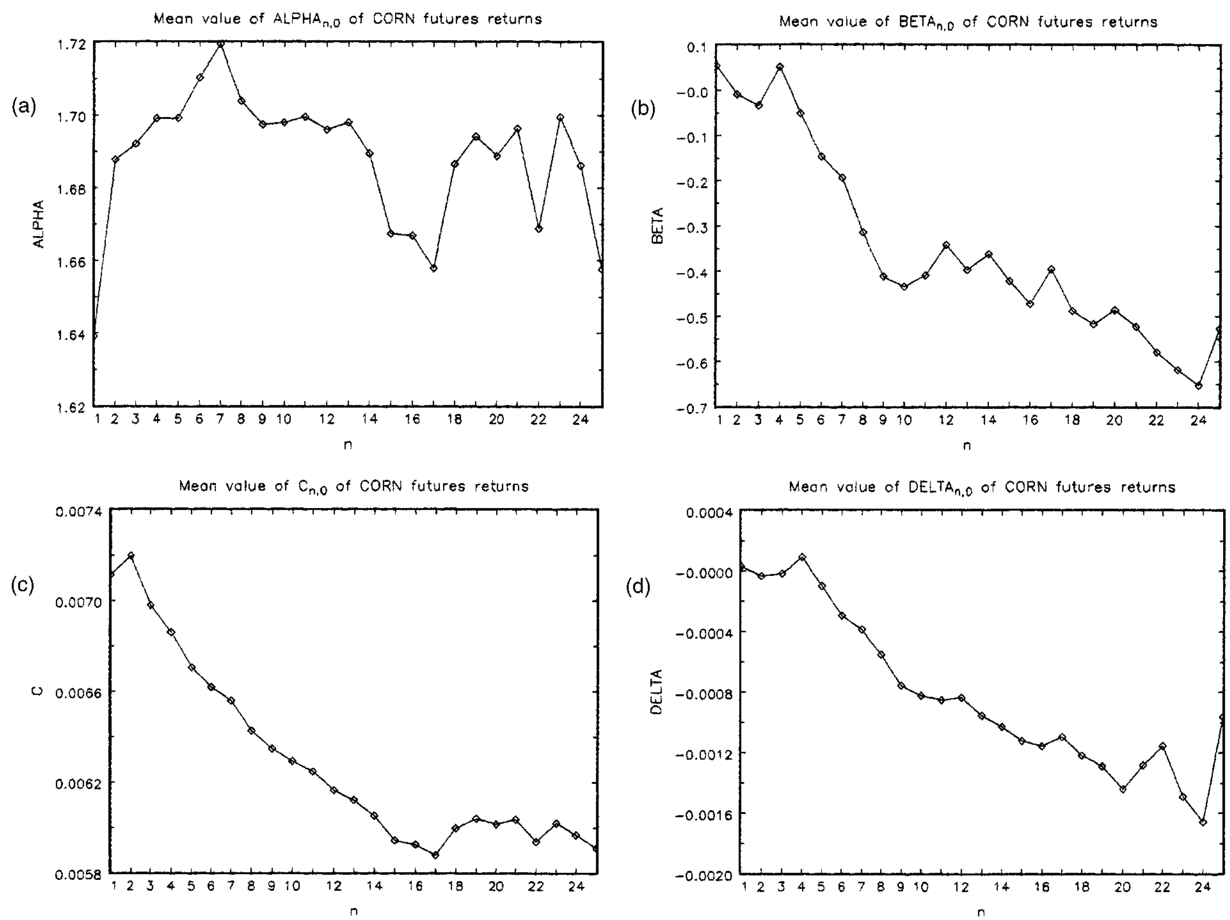


FIGURE 1

(a) Mean value of ALPHA$_{n,0}$ of corn futures returns. (b) Mean value of BETA$_{n,0}$ of corn futures returns. (c) Mean value of $C_{n,0}$ of corn futures returns. (d) Mean value of DELTA$_{n,0}$ of corn futures returns.

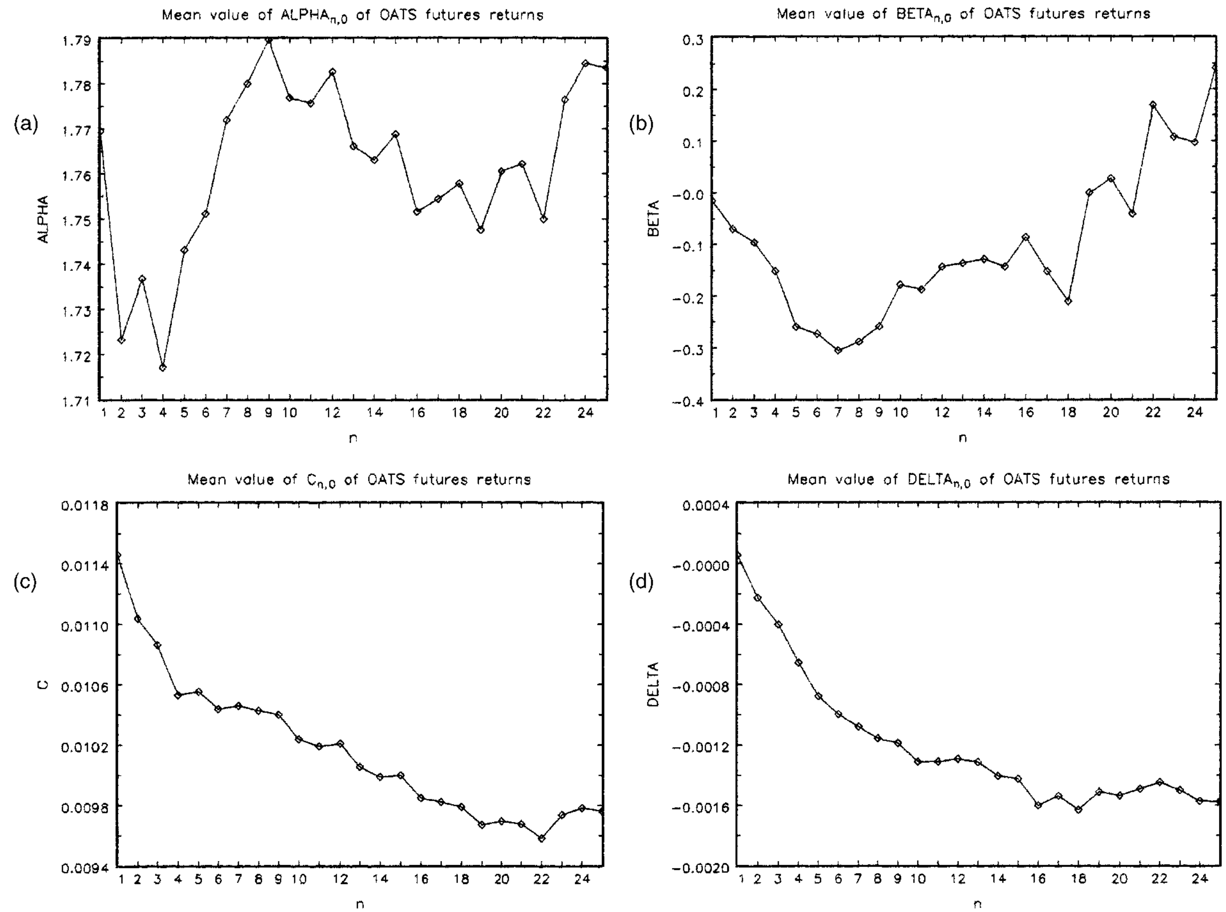


FIGURE 2

(a) Mean value of ALPHA$_{n,0}$ of oats futures returns. (b) Mean value of BETA$_{n,0}$ of oats futures returns. (c) Mean value of C$_{n,0}$ of oats futures returns. (d) Mean value of DELTA$_{n,0}$ of oats futures returns.

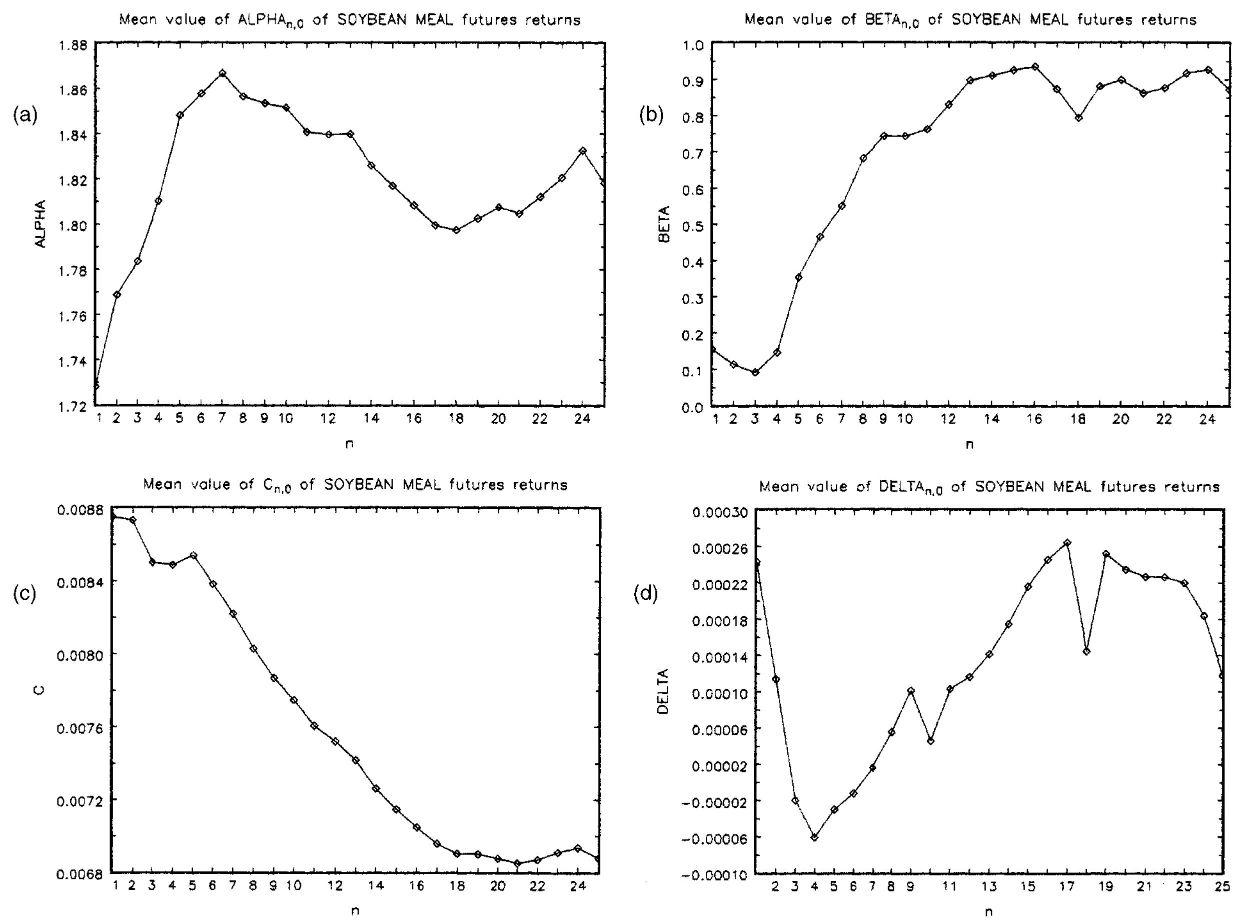


FIGURE 3

(a) Mean value of ALPHA$_{n,0}$ of soybeans futures returns. (b) Mean value of BETA$_{n,0}$ of soybeans futures returns. (c) Mean value of $C_{n,0}$ of soybeans futures returns. (d) Mean value of DELTA$_{n,0}$ of soybeans futures returns.

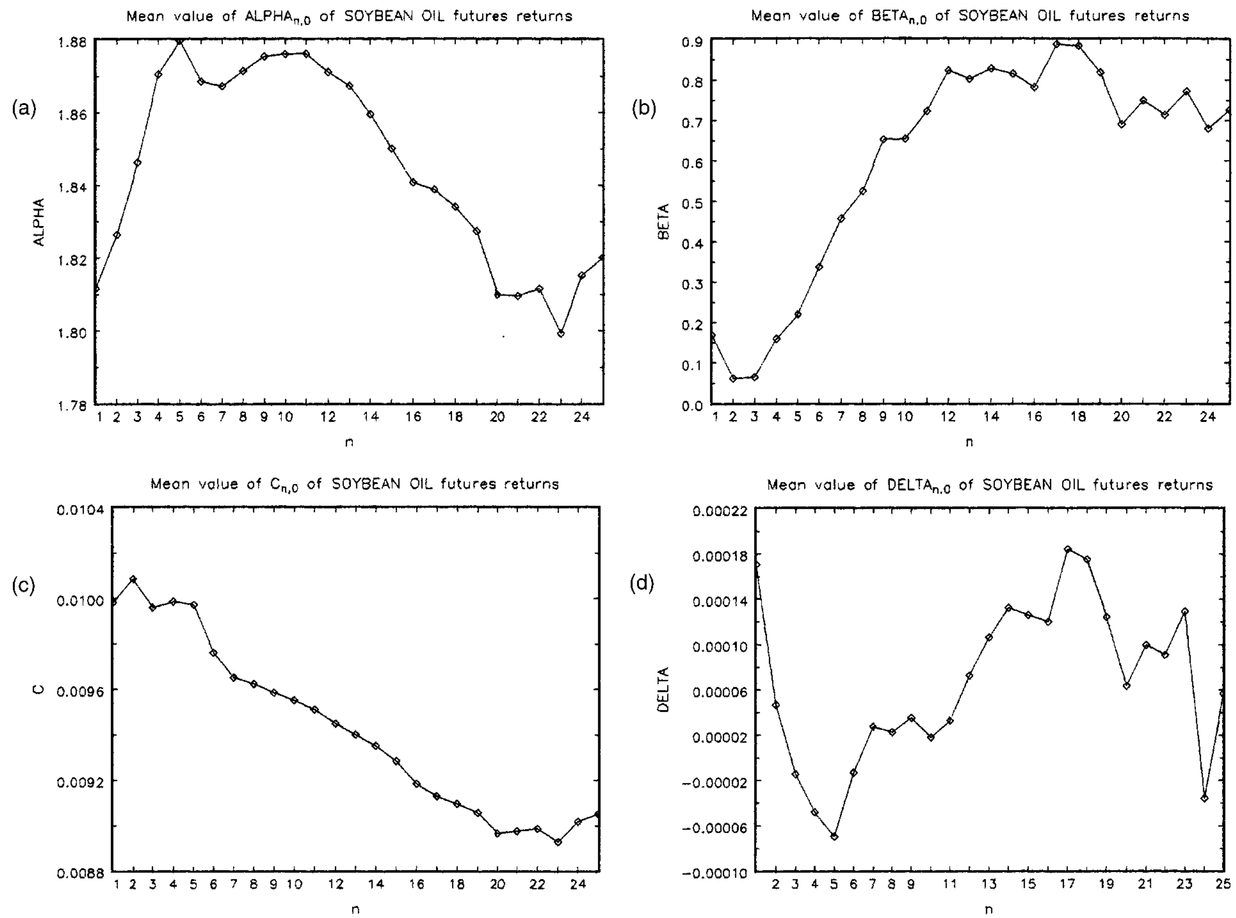


FIGURE 4

(a) Mean value of ALPHA$_{n,0}$ of soybean meal futures returns. (b) Mean value of BETA$_{n,0}$ of soybean meal futures returns. (c) Mean value of $C_{n,0}$ of soybean meal futures returns. (d) Mean value of DELTA$_{n,0}$ of soybean meal futures returns.

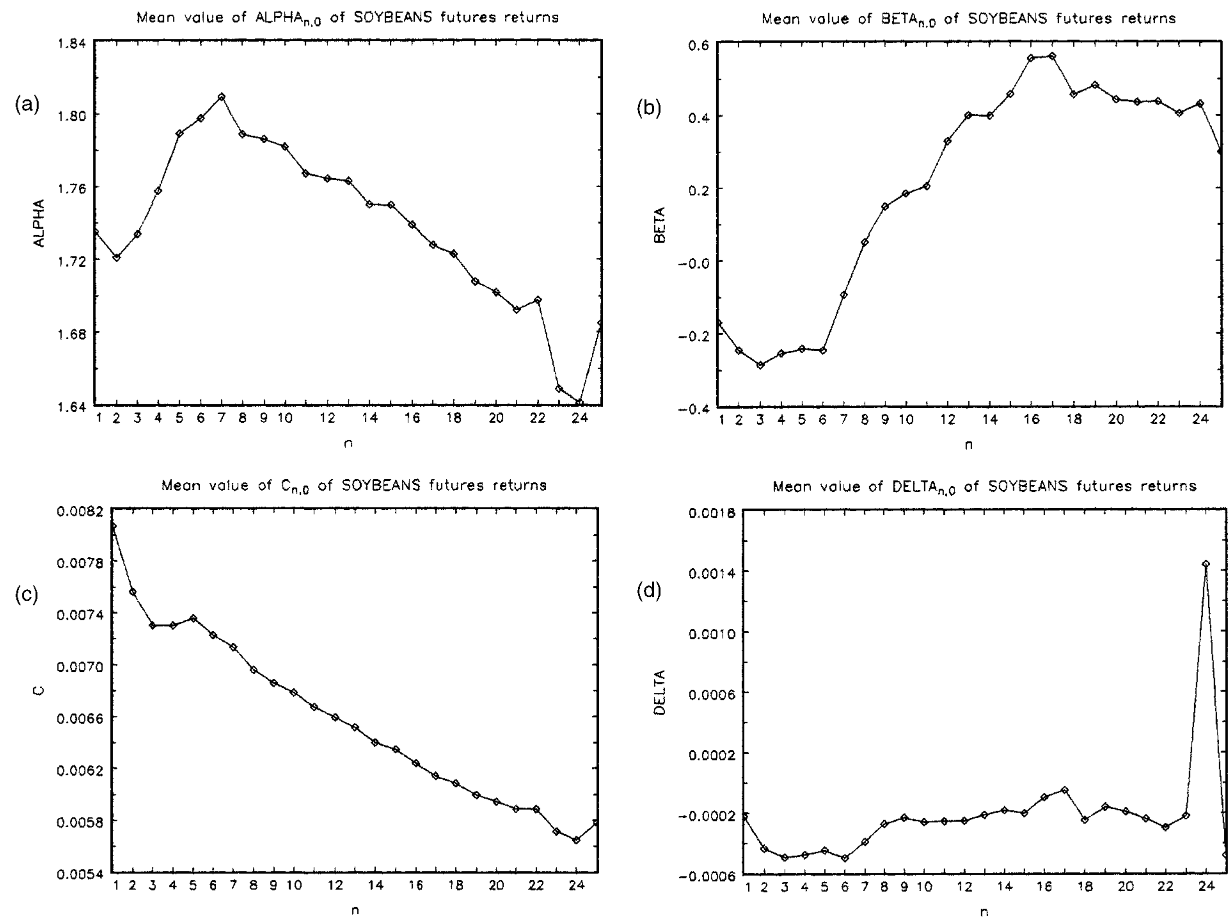


FIGURE 5

(a) Mean value of $\text{ALPHA}_{n,0}$ of soybean oil futures returns. (b) Mean value of $\text{BETA}_{n,0}$ of soybean oil futures returns. (c) Mean value of $C_{n,0}$ of soybean oil futures returns. (d) Mean value of $\text{DELTA}_{n,0}$ of soybean oil futures returns.

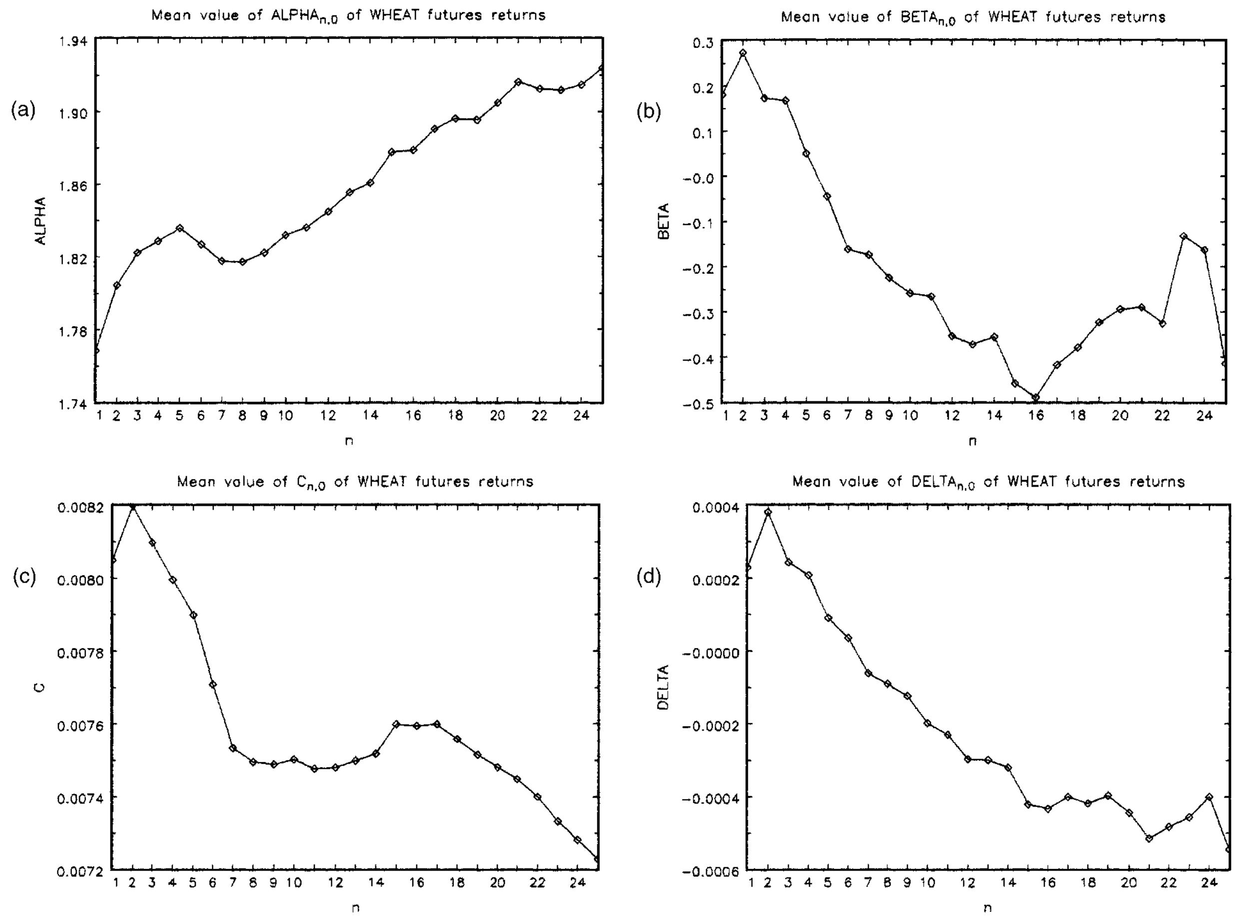


FIGURE 6

(a) Mean value of ALPHA$_{n,0}$ of wheat futures returns. (b) Mean value of BETA$_{n,0}$ of wheat futures returns. (c) Mean value of C$_{n,0}$ of wheat futures returns. (d) Mean value of DELTA$_{n,0}$ of wheat futures returns.

not use the volatility as a proper index of risk. Nevertheless, because the probability distribution has finite mean, it is always possible to define an index of variability based on the first-order moment, like, for instance, the mean absolute error (i.e., $[\Sigma_{i=1}^{T}|x_i - \delta|]/T$).

Third, the estimates of the four parameters are quite similar assuming both independence ($q = 0$) and dependence (see Tables II and III), so one can conjecture also the negligibility of the sample autocorrelative structure assumed in section 6.

Because of $\alpha_{1,q} \in [1.64,1.82]$ (see Tables II and III), following *Theorem 5.1,* it is confirmed empirically that the dimensions of all time series analyzed are not integer. It is a first confirmation of the fractal nature of the process generating the returns.

Following *Definition* 5.2 and the expression (6), one can deduce empirically the property of the invariability by time-scale change, or statistical self-similarity, for corn, oats, soybeans, soybean meal, and soybean oil (see Figures 1–5) because their estimated characteristic exponents are between 1.64 and 1.81, as n goes from 1 to 25. This variability range is similar to the one ($\alpha_{n,0} \in [1.6,1.8]$) found by Walter (1990), by which he accepted the statistical self-similarity property for the French stock market. On the other hand, only the variability range concerning wheat (see Figure 6) is wider than the others. Notice that the characteristic exponents of the different time series analyzed do not show the same behavior when n increases from 1 to 25. In particular, for soybeans it shows a decreasing behavior, for corn, oats, soybean meal, and soybean oil they show an oscillating one, and for wheat they show an increasing one, taking values progressively closer to 2 as n increases. Furthermore, the presence of invariability by time scale change for all time series analyzed is confirmed empirically by the steadiness of their scale and location parameters behavior. Only in the skewness parameter behavior are significant differences found among the analyzed time series, probably due to the estimation method [for more details see Walter (1990)]. Indeed, such behaviors are characterized by a variability range equaling 40% of its domain. In particular, for corn and wheat the skewness parameter shows a decreasing behavior and for oats, soybeans, soybean meal, and soybean oil it shows an increasing one.

From an economics point of view, the empirical results verify the invariability by time-scale change property. This means that the analyzed agricultural futures markets are characterized by (self-) similar liquidity, risk, and trading levels. It implies the contemporaneous presence of investors with different time horizons (probably due to different evaluations of the same new information arriving to the market). Moreover, it makes the matching between supply and demand easier, and, consequently, makes the market able to avoid panics and/or stampedes when the supply and demand become imbalanced. Peters (1994) calls this property, the fractal market hypothesis.

HURST EXPONENT *H:* THEORETICAL ASPECTS

Hurst (1951) discovered that a large class of natural phenomena show a behavior over time that can be described through a particular biased stochastic process. Such a process was called fractional Brownian motion (FBM) by Mandelbrot and van Ness (1968). This process implies the presence of some long-term dependence in its realizations. Falconer (1990) defines the FBM as follows.

Definition 7.1. A real stochastic process $X(t)$, with $t \in [0, +\infty)$, is a fractional Brownian motion with index $H \in (0,1)$, called the Hurst exponent, if

(a) $X(0) = 0$ with probability one,

(b) $X(t)$ is continuous almost everywhere for all $t \in [0, +\infty)$,

(c) The increments $X(t + \Delta t) - X(t)$ are normally distributed with mean zero and variance Δt^{2H} for all $t \in [0, +\infty)$ and $\Delta t \in [0, +\infty)$.

This means that, if $H \neq 0.5$, the increments of the FBM are stationary but dependent random variables. In particular, it is not the short-term (Markovian-like) memory, but the long-term memory that is influenced the most by the latest increments.

Notice that for FBM the Hurst exponent, H, provides double information on the underlying stochastic process. Indeed, remembering *Definition 5.2,* it is possible to prove that H is also equal to the statistical self-similarity parameter, K. This result is found in Mandelbrot and Taqqu (1979). Notice, also, that the Hurst exponent, H, qualifies the nature of the long-term memory. In particular, for $H \in (0,0.5)$ there is a negative dependence between the increments; that is, if the graph of $X(t)$ increases/decreases for $t_0 \geq 0$, then it probably decreases/increases for some $t > t_0$. In this case the process has an antipersistent behavior and the time series of the realizations is qualified as ergodic or mean reverting. For $H \in (0.5,1)$ there is a positive dependence between the increments; that is, if the graph of $X(t)$ increases/decreases for $t_0 \geq 0$, then it probably continues to increase/decrease for some $t > t_0$. In this case the process has a persistent or trend-reinforcing behavior.

The case, $H = 0.5$, is the standard Brownian motion (sBm) with independent increments. Moreover, the Hurst exponent, H, also gives a kind of measure of the long-term memory intensity; that is, the period and the strength of the antipersistent/persistent behavior increases as H approaches to 0/1. In particular, for a FBM, it is possible to quantify the link between the Hurst exponent, H, and the long-term dependence by the following autocorrelation function:

$$C(H) = 2^{2H-1} - 1$$

$$\begin{cases} <0 & \text{if } H \in (0,0.5), \quad C(H) \to -0.5 \text{ as } H \to 0 \\ =0 & \text{if } H = 0.5 \\ >0 & \text{if } H \in (0.5,1), \quad C(H) \to 1 \text{ as } H \to 1 \end{cases} \tag{7}$$

To determine the value of the Hurst exponent, H, Hurst used the R/S analysis, based on the range of partial sums of deviations of a time series from its mean, rescaled by its standard deviation. From a qualitative point of view it gives a standardized measure of the path length covered over a given time interval by the stochastic process. It is a statistical method used to study a wide range of phenomena. Mandelbrot and Wallis (1969) show that the R/S analysis is robust to highly nonnormal distribution of the process generating the considered time series. Moreover, it is possible to prove its almost-sure convergence for stochastic process with infinite variance [for more details see Mandelbrot and Taqqu (1979)]. In particular, it is possible to prove the following link between the R/S statistic and the Hurst exponent, H. [For more details see Cutland, Kopp, and Willinger (1993) and Peters (1994).]

$$\lim_{T \to +\infty} E[R_T/S_T]/(aT^H) = 1 \tag{8}$$

where

R_T is the range of the partial sums of deviations of the time series from its sample mean

S_T is the sample standard deviation of the original time series

$a \in (0, +\infty)$ is a constant.

From this link, it is possible to obtain the approximate relationship

$$\ln\{E[R_T/S_T]\} \simeq \ln(a) + H \ln(T). \tag{9}$$

In particular, some techniques proposed by Greene and Fielitz (1977), Mandelbrot and Taqqu (1979), Feder (1988), and Peters (1991b) are improved upon and coordinated. The algorithm used can be summarized as follows:

Step 1. Consider the original time series, $Y = \{y_i, i = 1, \ldots, T\}$.

Step 2. Fix subtime series of length, $N = N_0 \leq T$.

Step 3. Determine all possible nonoverlapping subtime series, $Y_{1,N} = \{y_1, \ldots, y_N\}$, $Y_{2,N} = \{y_{N+1}, \ldots, y_{2N}\}, \ldots, Y_{i,N} = \{y_{(i-1)N+1}, \ldots, y_{iN}\}, \ldots, Y_{i,Z} = \{y_{(i-1)Z+1}, \ldots, y_{iZ}\}$, where Z is the integer part of the ratio, T/N.

Step 4. For every subtime series, $Y_{i,N}$, $i = 1, \ldots, Z$, calculate the sample mean, $m_{i,N}$, determine subtime series, $X_{i,N}$, of the cumulative sums of deviations

$$X_{i,N} = \{x_{(i-1)N+t} = \sum_{j=1}^{t} (y_{(i-1)N+j} - m_{i,N}), t = 1, \ldots, N\} \quad (10)$$

compute the sample variation range, $R_{i,N}$,

$$R_{i,N} = \max_{1 \leq t \leq N} X_{i,N} - \min_{1 \leq t \leq N} X_{i,N} \quad (11)$$

and calculate its standard deviation, $S_{i,N}$, and ratio, $R_{i,N}/S_{i,N}$.

Step 5. Calculate the mean value, R_N/S_N, from $R_{i,N}/S_{i,N}$.

Step 6. Set new subtime series length, N by $N_{\text{Ser}} = N + S$ $(S > 0)$ and $N = N_{\text{Ser}}$.

Step 7. If $N \leq T$, then go to *Step* 3; otherwise, go to *Step* 8.

Step 8. Fit OLS regression between $\{\log(R_1/S_1), \ldots, \log(R_j/S_j)\}$ and $\{\log(1), \ldots, \log(j)\}$ for every $j = 2, \ldots, Z$.

Step 9. Determine the unique value of the Hurst exponent, H, among the Z-2 estimates making joined use of the graphic approach proposed in Peters (1989 and 1991b) and of the statistical one proposed in Lo (1991).

Notice that, if the time series possesses a natural cycle of length M [called *mean orbital period* (MOP)], R/S analysis identifies it. In particular, it is possible to show that the Hurst exponent, H, tends to 0.5 as T becomes greater than M and tends to $+\infty$; it indicates that, for such a large time lag, the stochastic process is losing its long-term memory.

Finally, to point out another property of the stochastic fractal objects, Greene and Fielitz (1977) report that the (fractal) dimension of a process probability distribution is $1/H$, with $H \in [0.5,1)$. Notice that, if $H = 0.5$ (i.e., sBm) the corresponding dimension is 2. Moreover, it is possible to prove the existence of the following relationship between the statistical self-similarity of a FBM ($K = H$) and one of a Pareto–Lévy stable process; that is, $K = 1/\alpha$ [for more details see Mandelbrot and Taqqu (1979)]:

$$H = 1/\alpha, \qquad H \in (0{,}0.5) \tag{12}$$

It means that, although the two considered stochastic processes are quite different, their behaviors, from a fractal point of view, are the same.

Lo (1991) proposes a modification of the classical R/S analysis mainly because of its sensitivity to short-term dependence. Indeed, it is possible to prove that, because of such a sensitivity, the long-term memory results from the classical R/S method can merely be due to short-term memory. In particular, the modified statistic is robust to both short-term dependence and highly nonnormal innovations and its behavior is invariant over a general class of short-term memory processes but deviates for long-term memory processes. Moreover, unlike the classical R/S statistics, it has well-defined distribution properties.

Lo modified the classical statistic R/S by using $R/\bar{S}$, with

$$\bar{S}^2 = S^2 + 2\sum_{i=1}^{q} w_i(q)\gamma_i, \qquad q < T \tag{13}$$

where

$w_i(q) = 1 - i/(q + 1)$, $i = 1, \ldots, q$ are weights depending on the short-term memory length q

$$\gamma_i = \sum_{j=i+1}^{T} (Y_j - m_N)(Y_{j-1} - m_N)$$

$i = 1, \ldots, q$ are autocorrelation estimators.

Observe that the $R/\bar{S}$ statistic differs from the R/S one only in its modified standard deviation, which is the square root of a correct and consistent estimator of the sample variance. In fact, if the analyzed time series is characterized by short-term dependence, the modified variance also includes the autocovariances weighted up to lag q. In particular, to detect q the Andrews data-dependent rule is used and to determine the weights, the Newey and West proposal (1987) is used, always yielding a nonnegative $\bar{S}^2$. Moreover, Lo (1991) determines the distribution properties of the modified statistic and identifies some link between it and the classical one. In particular, he proves the following asymptotic relationship for the modified statistics

$$Q_T(q) = \frac{1}{\sqrt{T}} R/\bar{S} \sim V \tag{14}$$

where

the tilde denotes weak convergence
V is a random variable with the following probability distribution

$$F_v(v) = 1 + 2 \sum_{K=1}^{+\infty} (1 - 4k^2v^2)\exp(-2k^2v^2). \tag{15}$$

Lo (1991) also proves the asymptotic relationship for the classical statistics

$$Q_T(0) = \frac{1}{\sqrt{T}} R/S \sim \xi V, \tag{16}$$

where ξ is a function depending on the short-time memory structure

Finally, by using the fractiles of the distribution of $Q_T(q)$ [also calculated by Lo (1991)], it is possible to determine the values for different levels of significance to test the null hypothesis of no long-term dependence.[1]

HURST EXPONENT *H* IN THE FUTURES RETURNS

To determine the values of the Hurst exponent, H, the time series of the futures returns are used. Tables IV and V report the values of $H_{n,q}$ obtained with the use of the algorithm summarized in the previous section, with the time scale, $n = 1$, 5, and 25, assuming both independence; that is, $q = 0$ (classical Hurst exponent) and dependence (modified Hurst exponent). Notice that, for time scale $n = 5$ and 25, $q = 0$ (independence) is used because it is found that $0 \leq q \leq 4$ for all analyzed time series. Notice also that, because the stability of the results depends on the nonoverlapping subtime series, N (see *Steps* 2, 6, and 7), and because detection of the MOP requires that the Hurst exponent tends to 0.5, in general, a time series of sufficient size is necessary.

Second, the same tables report the results obtained from testing the null hypothesis of no long-term dependence by the statistic, $Q_T(q)$, $q = 0$ and $q \neq 0$ [see relationship (14)].

[1]Notice that a rejection of such a null hypothesis does not necessarily imply that long-range memory is present, but merely that the underlying stochastic process does not satisfy simultaneously all the conditions stated by Lo (1991).

TABLE IV

Estimates of $H_{n,q}$ for $n = 1$, $q = 0$, and $q \neq 0$ with Estimates of Corresponding Mean Orbital Periods

Futures	$H_{1,0}$		*MOP*	$H_{1,q}$		*MOP*
Corn	0.60	Lo	1000	0.57	Lo	950
Corn	0.61	–	1250	0.59	–	1300
Oats	0.55	Me	1450	0.53	Me	1450
Oats	0.57	–	1350	0.55	–	1350
Soybeans	??	??	??	??	??	??
Soybeans	0.62	–	1350	0.62	–	1350
S. Meal	??	??	??	??	??	??
S. Meal	0.59	–	1200	0.59	–	1200
S. Oil	0.65	Me	1100	0.62	Me	1100
S. Oil	0.65	–	1150	0.62	–	1200
Wheat	0.43	Hi	1800	0.43	Hi	1800
Wheat	0.42	–	1950	0.42	–	1950

Note: The first row for each commodity contract reports estimates of the classical or modified Hurst exponent from eq. (19) for which the null hypothesis of no long-term dependence is rejected. Hi denotes rejection at the 95% or 99% confidence level, Me denotes rejection at the 90%, and Lo rejection at the 80% confidence level. If the null hypothesis of no long-term dependence is not rejected, the row is filled with question marks. The second row reports the descriptive results from the rescaled range analysis obtained from the graphical approach for which no hypothesis testing can be performed.

TABLE V

Estimates of $H_{n,q}$ for $n = 5, 25$ and $q = 0$, with Estimates of Corresponding Mean Orbital Periods

Futures	$H_{5,0}$		*MOP*	$H_{25,0}$		*MOP*
Corn	??	??	??	??	??	??
Corn	0.66	–	1300	0.76	–	1200
Oats	0.60	Me	1450	0.66	Me	1500
Oats	0.62	–	1300	0.70	–	1250
Soybeans	??	??	??	0.60	Lo	2700
Soybeans	0.68	–	1250	0.74	–	1250
Soybean meal	??	??	??	??	??	??
Soybean meal	0.65	–	1050	0.76	–	900
Soybean oil	0.71	Lo	1150	??	??	??
Soybean oil	0.71	–	1150	0.80	–	1150
Wheat	0.50	Hi	1750	0.60	Hi	1500
Wheat	0.50	–	1750	0.65	–	1300

Note: The first row for each commodity contract reports estimates of the classical Hurst exponent from eq. (19) for which the null hypothesis of no long-term dependence is rejected. Hi denotes rejection at the 95% or 99% confidence level, Me denotes rejection at the 90%, and Lo rejection at the 80% confidence level. If the null hypothesis of no long-term dependence is not rejected, the row is filled with question marks. The second row reports the descriptive results from the rescaled range analysis obtained from the graphical approach for which no hypothesis testing can be performed.

TABLE VI

Estimates of $\alpha_{n,q}H_{n,q}$ for $n = 1, 5, 25$, $q = 0$, and $q \neq 0$

Futures	$\alpha_{1,0}H_{1,0}$	$\alpha_{1,q}H_{1,q}$	$\alpha_{5,0}H_{5,0}$	$\alpha_{25,0}H_{25,0}$
Corn	0.98	0.93	1.12	1.26
Oats	0.97	0.94	1.04	1.24
Soybeans	1.08 G	1.03 G	1.22	1.01
Soybean meal	1.02 G	1.02 G	1.20 G	1.38 G
Soybean oil	1.18	1.13	1.33	1.46 G
Wheat	??	??	0.92	1.15

Note: The row for each commodity contract reports the empirical verification of the relationship (12) by using the $H_{n,q}$ values obtained from the statistical analysis when the null hypothesis of no long-term memory is rejected and by using the $H_{n,q}$ values obtained from the graphical analysis when the same null hypothesis is accepted. In the last case the result is marked by a *G*. If $H_{n,q} < 0.5$ the row is filled with question marks.

Third, Table VI reports the results of the empirical proof of the relationship (12) obtained by determining the values of $\alpha_{n,q}H_{n,q}$ (equal to one, from a theoretical point of view).

Finally, Figures 7–16 report the (typical) behavior of some Hurst exponent $H_{n,q}$'s estimates *versus* N. In particular, from this analysis, note that:

- There is a starting interval ($N_0 \leq N \leq N^*$) of estimate arrangement, in which $H_{n,q}$ decreases as N increases (probably due to the low power of the rescale range analysis for small samples).
- There is a second interval ($N^* < N \leq T$) in which both the classical Hurst exponent estimate and the modified one obtain a relative minimum/maximum or are constant (in general, corresponding to the true value of the Hurst exponent).
- The underlying memory structure is confirmed by reanalyzing the same time series after a random alteration of their time order, because the reestimated $H_{n,q}$ values tend to 0.5 (i.e., no more long-term correlation) and, so reveal the destruction of a long-term dependence that exists in the unscrambled original time series.

The previous three tables allow one to deduce several things. The graphical analysis indicates evidence of long-term memory for all the daily (i.e., $n = 1$) returns time series (see Table IV). In particular, the results of this analysis are qualitatively the same, assuming both independence ($q = 0$) and dependence, so one can conjecture that the influence of the short-term memory is negligible. Moreover, such results are analogous to the ones found by Peters (1989) for the U.S. stock market. Notice that the graphical approach detects positive dependence [i.e., $H \in (0.5,1)$]

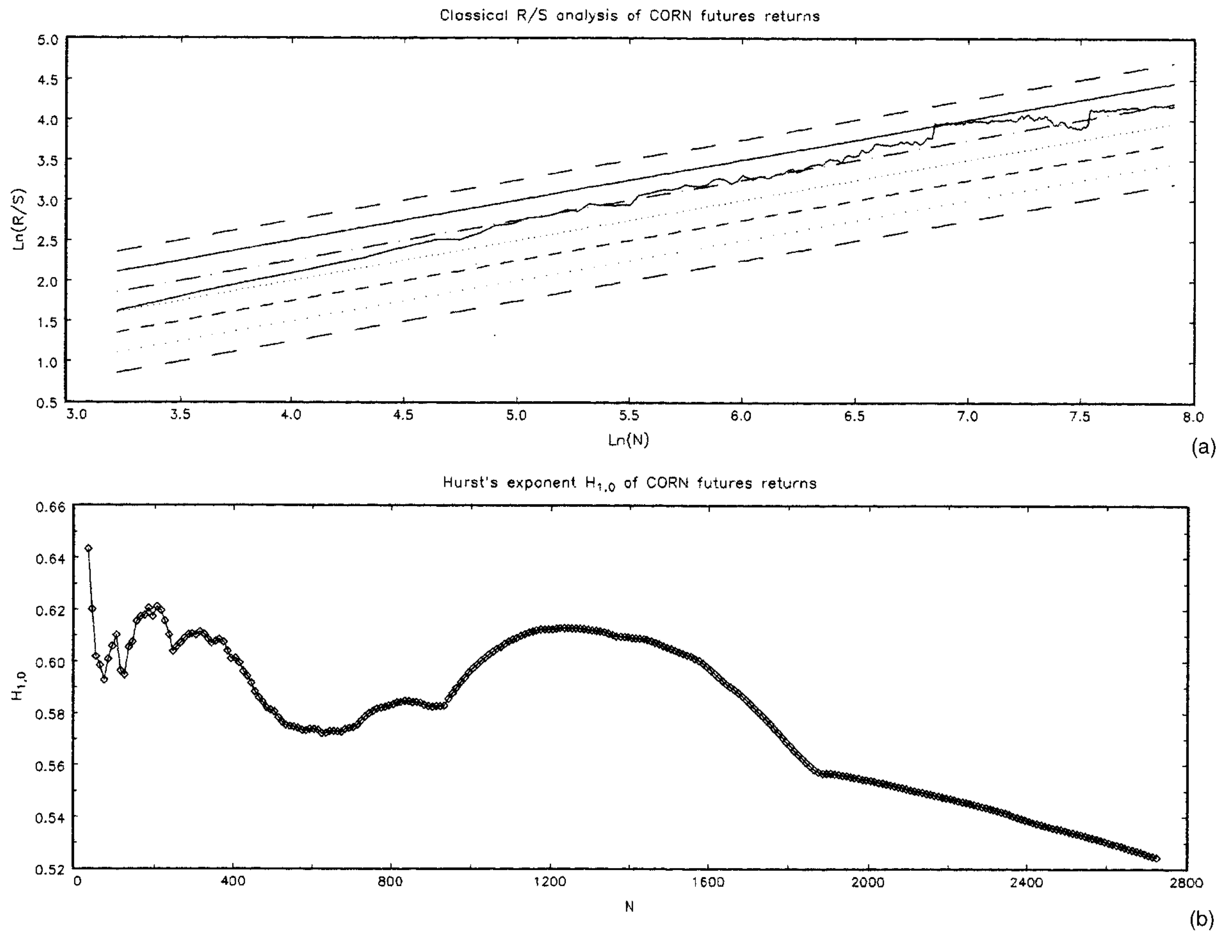


FIGURE 7

(a) Classical *R/S* analysis of corn futures returns. (b) Hurst's exponent $H_{1,0}$ of corn futures returns.

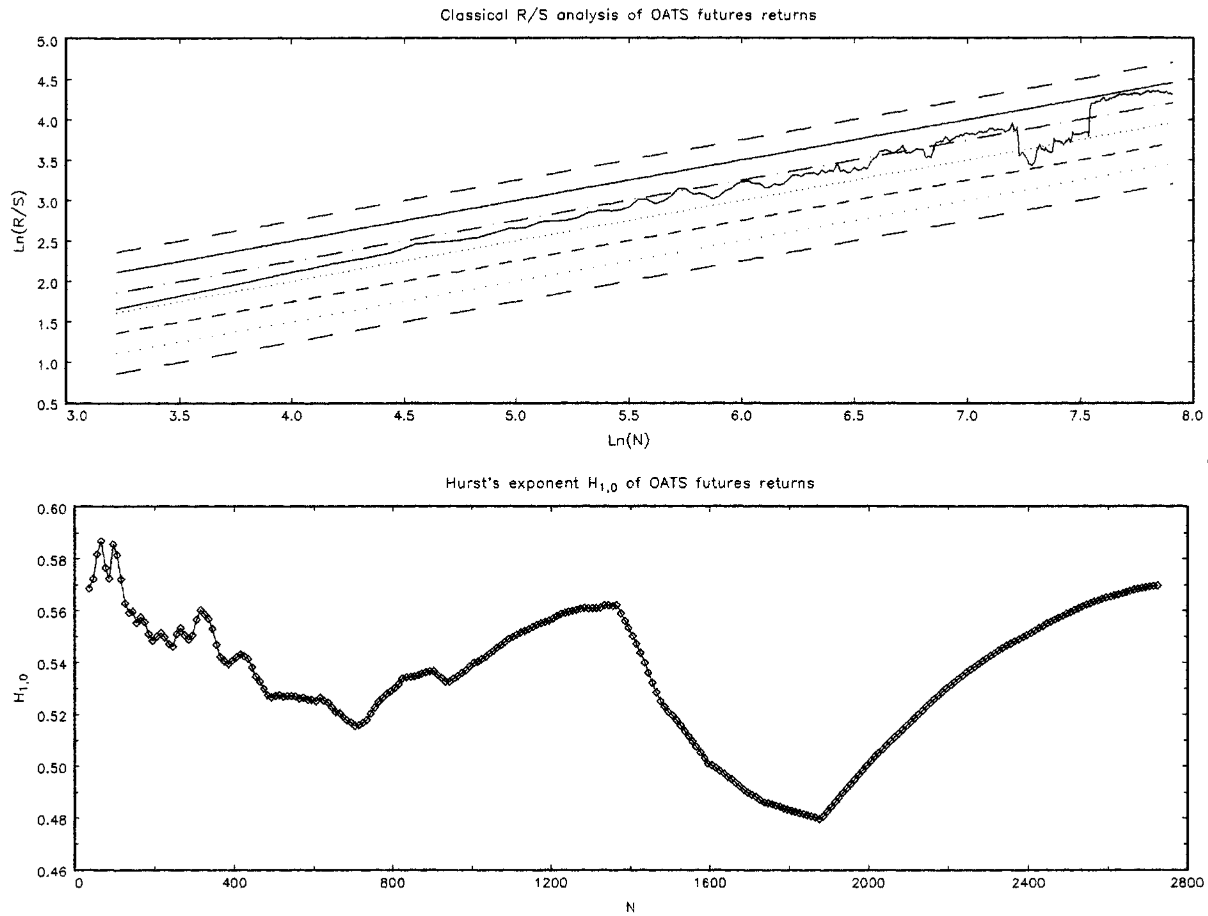


FIGURE 8
(a) Classical R/S analysis of oats futures returns. (b) Hurst's exponent $H_{1,0}$ of oats futures returns.

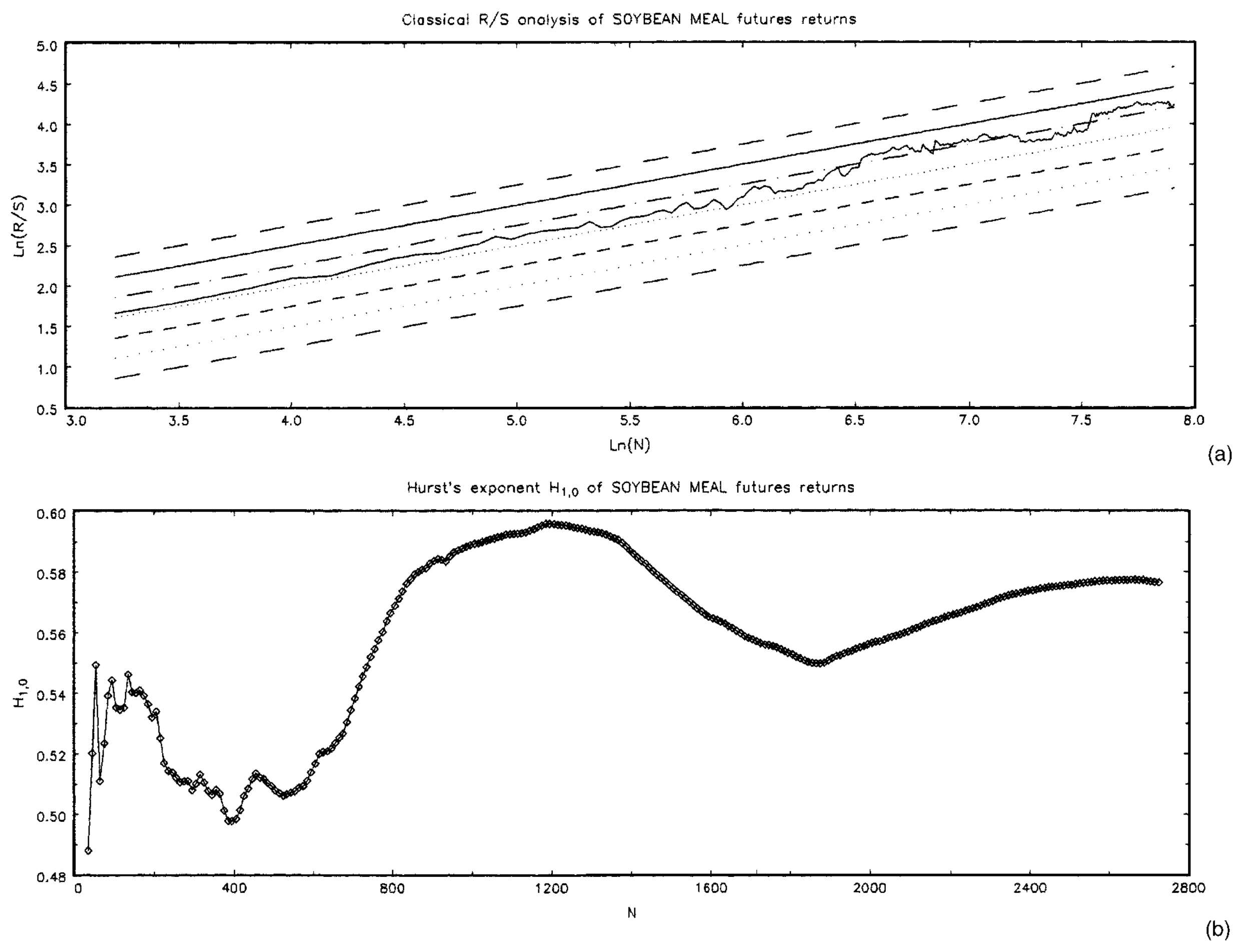


FIGURE 9
(a) Classical *R/S* analysis of soybeans futures returns. (b) Hurst's exponent $H_{1,0}$ of soybeans futures returns.

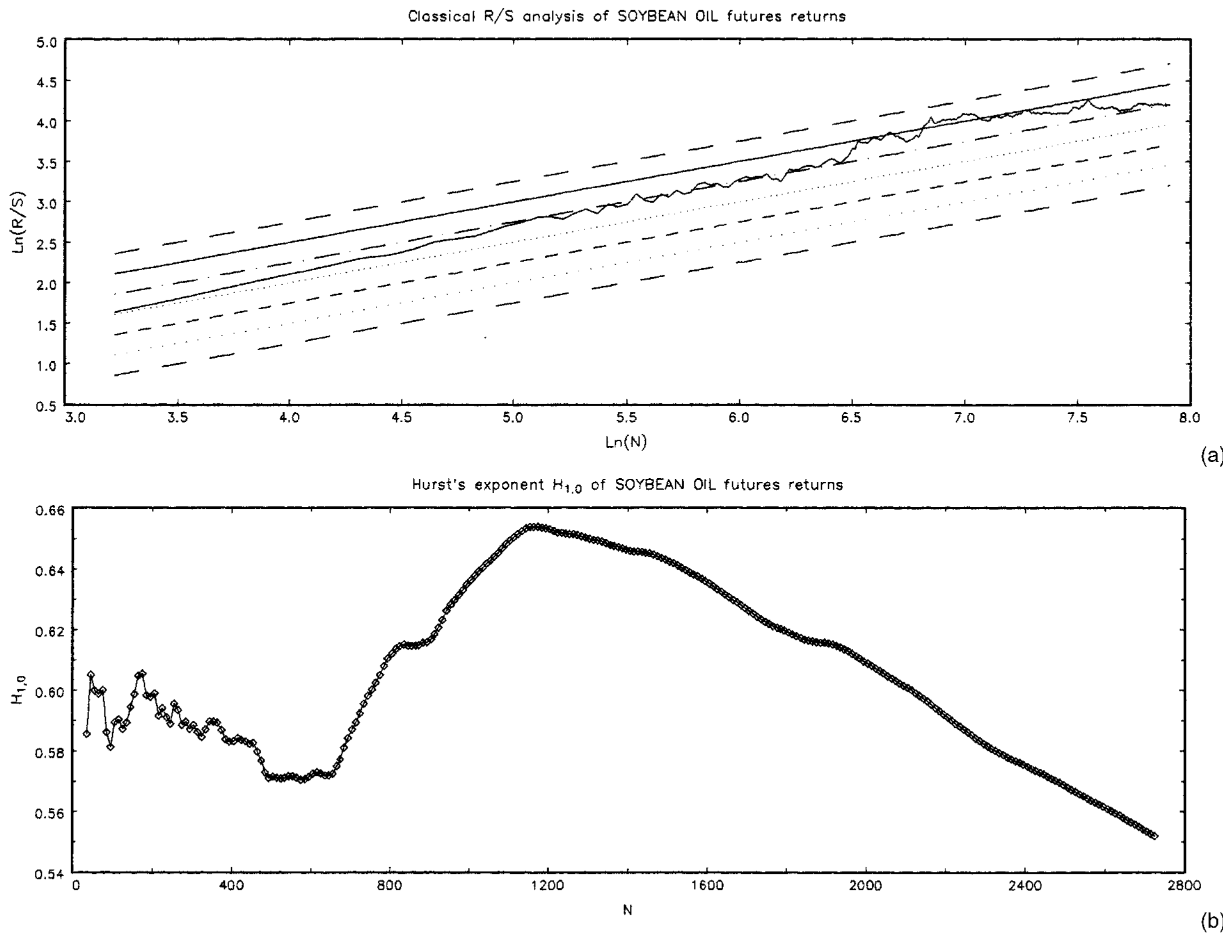
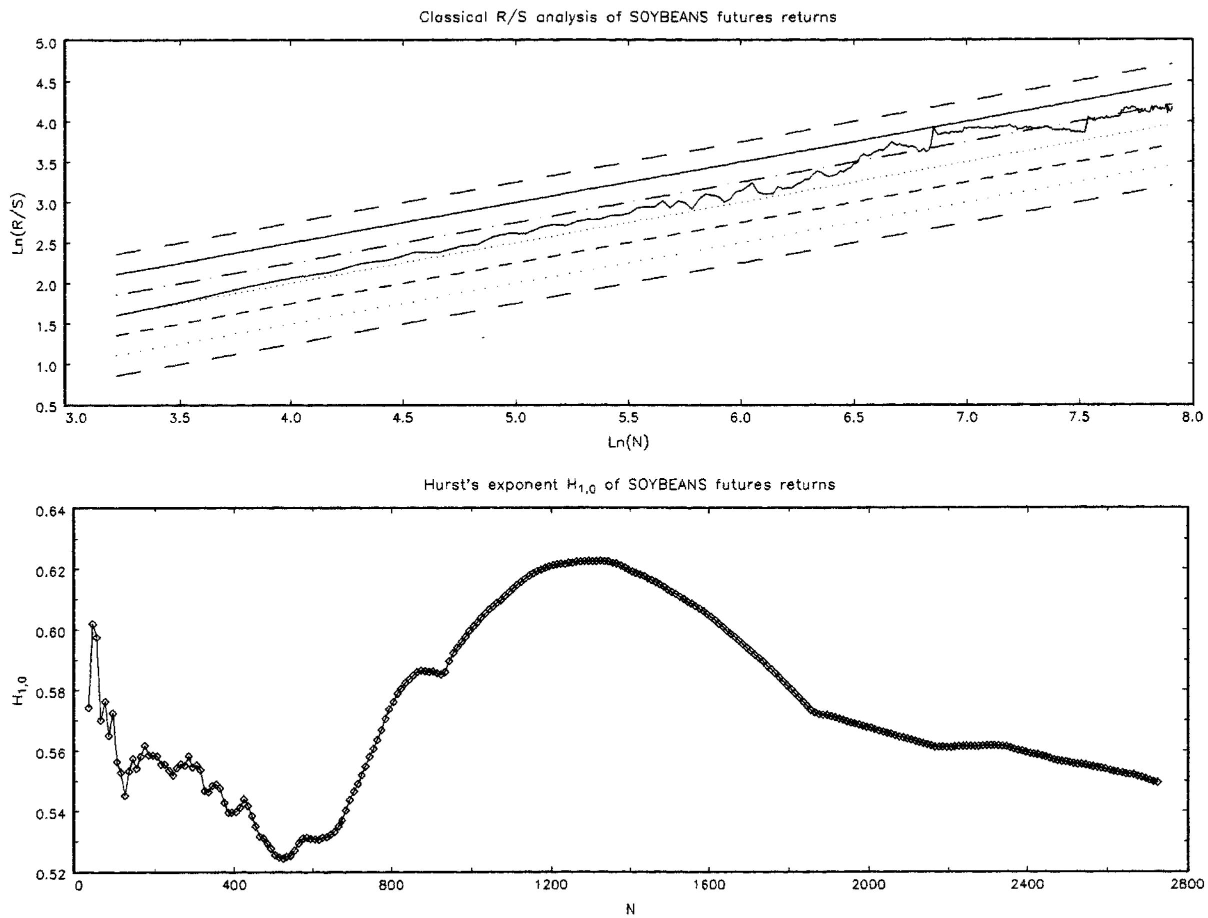


FIGURE 10
(a) Classical R/S analysis of soybean meal futures returns. (b) Hurst's exponent $H_{1,0}$ of soybean meal futures returns.



(a)

(b)

FIGURE 11

(a) Classical *R*/*S* analysis of soybean oil futures returns. (b) Hurst's exponent $H_{1,0}$ of soybean oil futures returns.

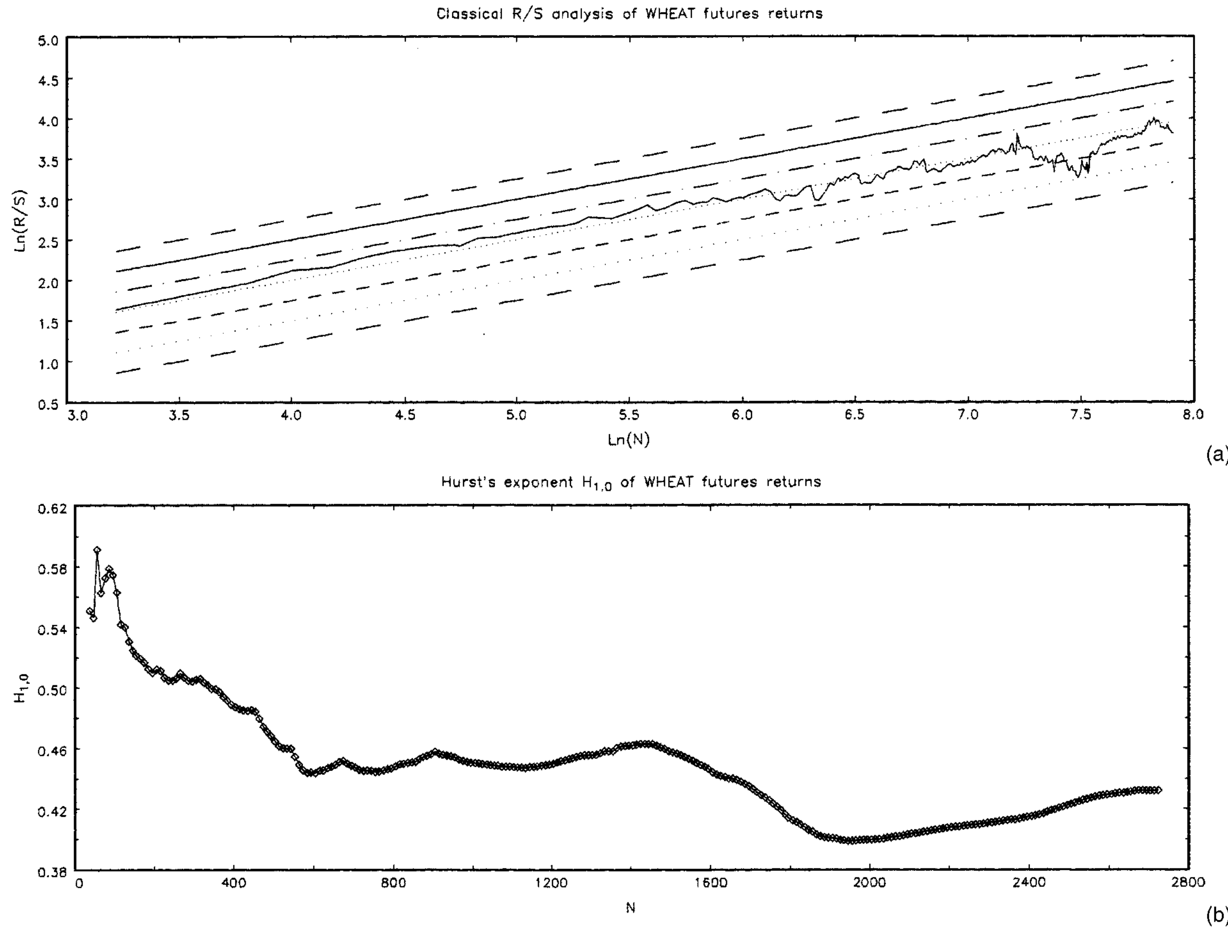
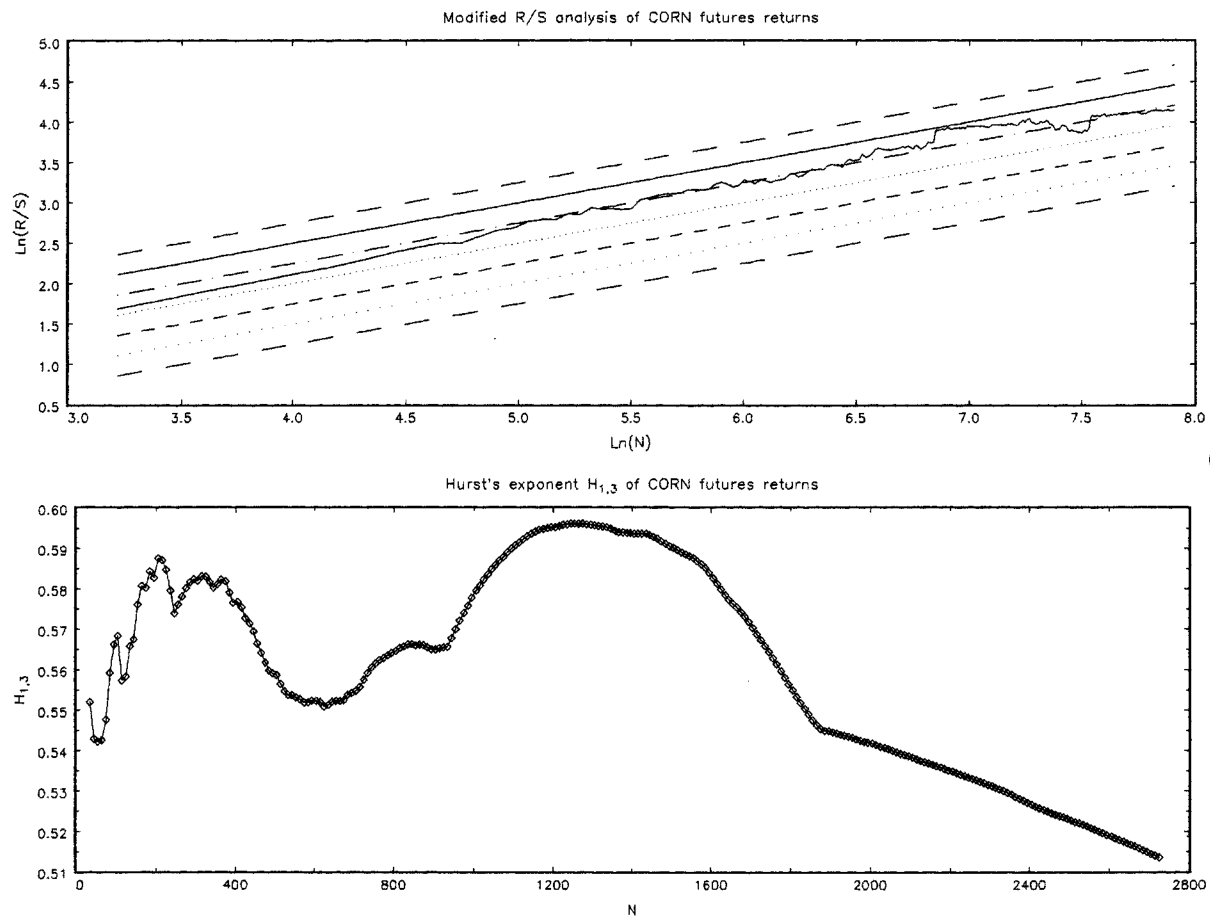


FIGURE 12
(a) Classical *R/S* analysis of wheat futures returns. (b) Hurst's exponent $H_{1,0}$ of wheat futures returns.



(a)

(b)

FIGURE 13

(a) Modified *R/S* analysis of soybean oil futures returns. (b) Hurst's exponent $H_{1,3}$ of soybean oil futures returns.

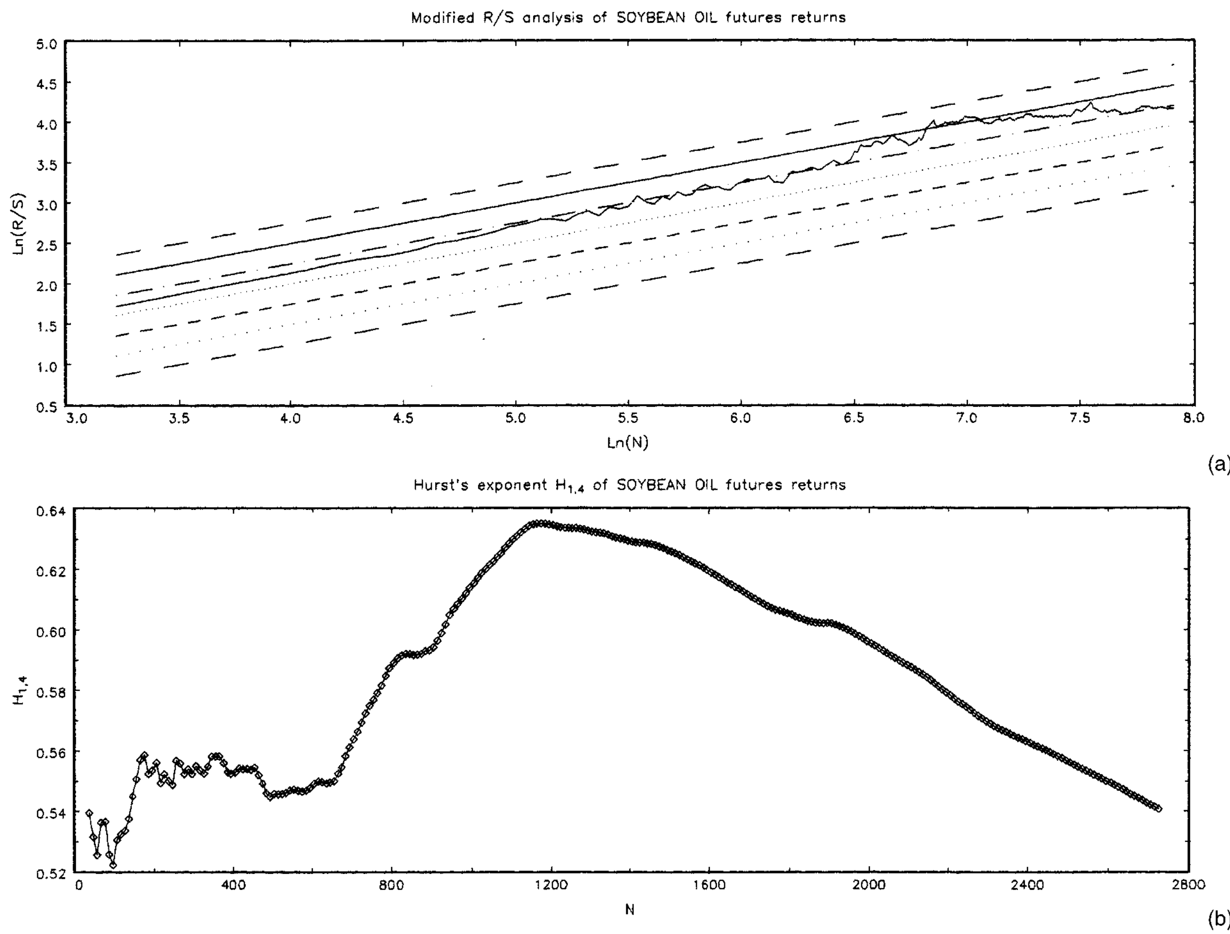


FIGURE 14
(a) Modified *R/S* analysis of soybean oil futures returns. (b) Hurst's exponent $H_{1,4}$ of soybean oil futures returns.

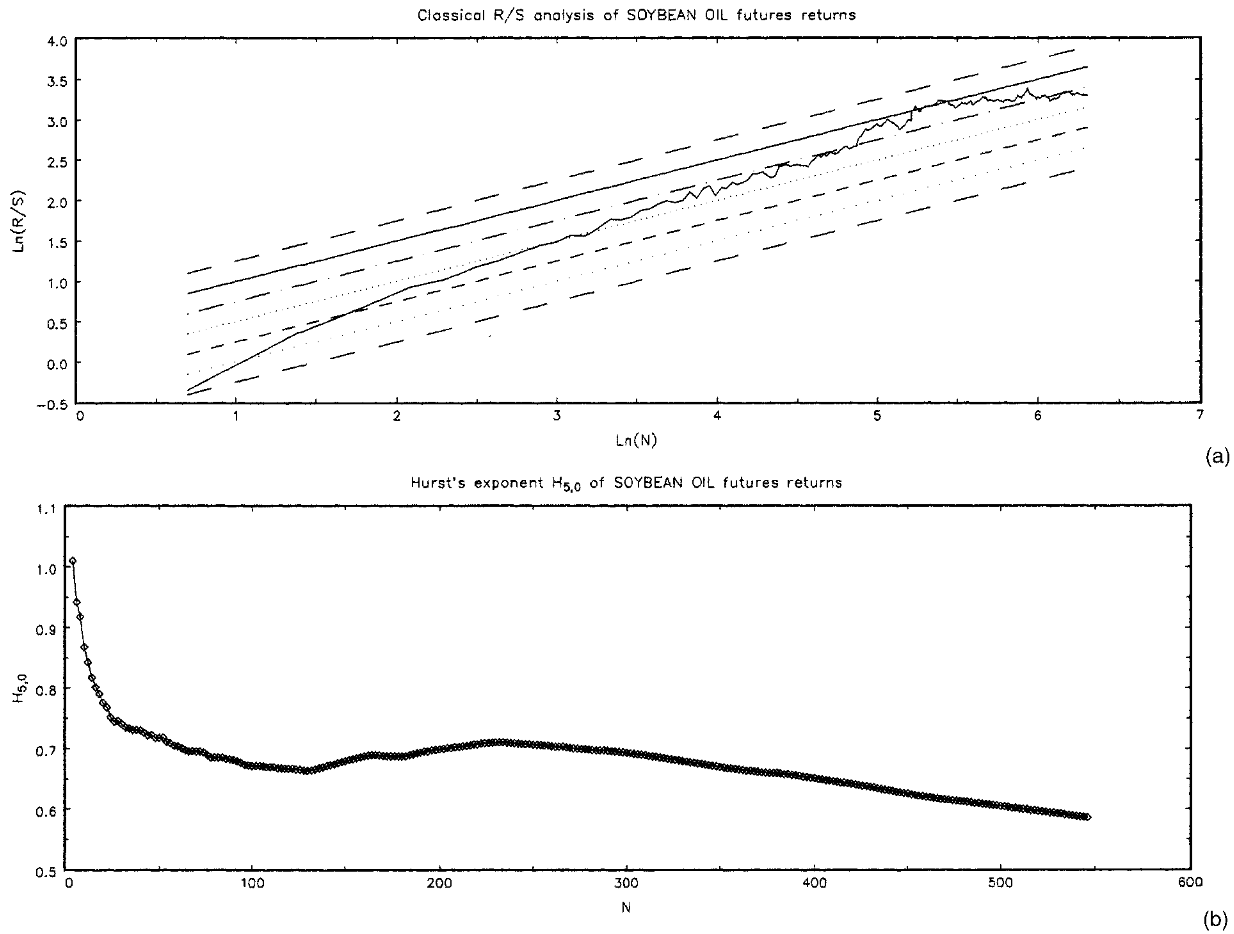


FIGURE 15
(a) Classical *R/S* analysis of soybean oil futures returns. (b) Hurst's exponent $H_{5,0}$ of soybean oil futures returns.

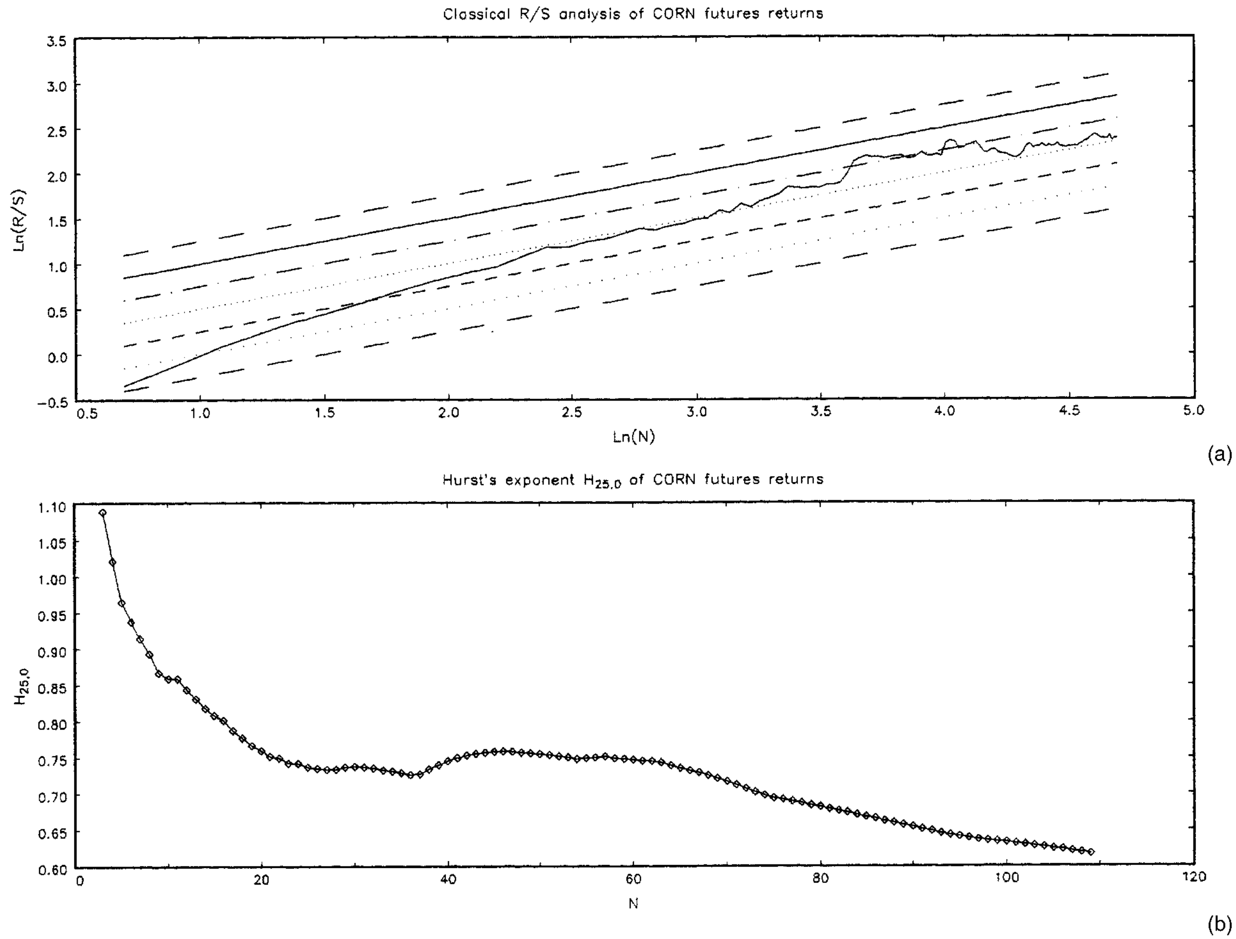


FIGURE 16
(a) Classical *R/S* analysis of soybean oil futures returns. (b) Hurst's exponent $H_{25,0}$ of soybean oil futures returns.

for corn, oats, soybeans, soybean meal, and soybean oil and detects negative dependence [i.e., $H \in (0,0.5)$] for wheat. This can be explained in this way. For the positive dependence case, the ability of economic agents to make optimal decisions under uncertainty is easier because buy and hold strategies yield returns in the same direction. However, because of the irregular arrival of new important information to the market, persistent return movements can at times reverse direction suddenly, and decisions are more difficult at turning points. For the negative dependence case, because of to some exogenous structural sociopolitical changes, the arrival of contrasting information to the market can induce the economic agents to frequently change the rule-governed behavior.

The values of the MOP associated to the daily returns time series go from 950 (i.e., about less than 4 years) to 1350 (i.e., about more than 5 years) for the positive dependence case and go from 1800 (i.e., about more than 7 years) to 1950 (i.e., about more than 8 years) for the negative dependence case, assuming both independence ($q = 0$) and dependence. Notice that for time lags greater than MOP, the underlying stochastic processes lose their long-term memory and the corresponding daily returns become long-term independent.

The results of the graphical analysis for both weekly (i.e., $n = 5$) and monthly (i.e., $n = 25$) returns time series (see Table V) are qualitatively the same for both $H_{n,0}$ estimates and MOP values with the exception of wheat.

The results of the statistical analysis when it rejects the null hypothesis of no long-term dependence for the daily, weekly, and monthly returns time series (see Tables IV and V) confirm the results obtained from the graphical analysis for both $H_{n,q}$ estimates and MOP values.

Because $1/H_{n,q} \in [1.25,1.89]$, with $H_{n,q} \in [0.5,1]$ (see Tables IV and V), it is verified empirically that the dimension of the underlying process probability distributions are not integer. It is another confirmation of the fractal nature of such processes.

Recall that for FBM the Hurst exponent is also equal to the statistical self-similarity parameter, K; therefore the property of invariability by time-scale change for the analyzed time series is confirmed, proving again the relationship (12) (see Table VI). In particular, this result holds better for daily returns than for weekly or monthly ones.

CONCLUDING REMARKS

It is not enough to reject randomness and market efficiency hypotheses. To make scientific progress, alternatives to randomness must be specified.

This article offers an answer to the question: if asset returns do not follow random walk what are they? Using daily agricultural futures data, this study finds that returns are fractal.

What does it mean to say that returns are fractal? Returns are fractal if they are characterized by properties such as fine structure, local and global irregularities, self-similarity, and noninteger dimension. Such fractal processes generalize the well-known random walks and martingales of financial economics.

To support this claim that agricultural futures returns are fractal, three pieces of statistical evidence are presented. Tests are conducted that reject the hypothesis that returns are normally distributed. Then, the four parameters of the Pareto–Lévy stable distribution are estimated. This distribution generalizes the special case of the normal distribution. With the use of certain mathematical facts, it is found that the estimates of the four parameters are consistent with the conjecture that the stochastic process generating the returns is fractal.

The second set of tests uses the classical rescaled range analysis by computing the Hurst exponent. The third test is an extension of the second using a recent modification proposed by Lo (1991). With the use of both these tests, evidence is found that returns are fractal.

What are the implications of these findings? Suppose that all financial returns (not only the agricultural futures studied in this article are fractal. This would imply that financial returns behave in ways that are more general than random walks. Put differently, random walks are only a very special case of general fractal processes. Technically, this means that although a fractal process may have a Hurst exponent that ranges theoretically over (0, 1) set, the random walk is only one special case when the Hurst exponent receives the value 0.5. This means that market efficiency is a special theory and not a general theory; it holds sometimes but not always. In other words, the empirical evidence that efficiency holds in some cases and does not hold in others is now consistent with the evidence that returns are fractal. Obviously, much more research is needed to confirm or reject the fractal behavior of returns for nonagricultural futures.

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