Economics & Complexity

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on Public, Financial, Globalisation and Social Economics

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Inquiries on the Applications of Multidimensional Stochastic Processes to Financial Investments

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Abstract: This paper presents a selection model for investments in financial securities whose prices follow an Itô multidimensional stochastic process. The optimal investment shares are obtained by maximizing the expected utility of consumption at one future date, subject to the level of attained wealth. This problem is solved by applying dynamic programming techniques and requires solving a non linear partial differential equation in order to obtain the optimal shares in explicit form. We formulate this equation for more general processes than as in Merton (1973), and then we study possible methods to reach its solution for some special cases. In these situations, we show some simple examples, verify the empirical applicability of the model, and discuss upon the methodology of empirical applications.

Keywords: Portfolio Management, Investment, Stochastic Control, Merton Model, Italian Stock Market.

1 Introduction

This work presents a model for investment in financial activities following the definition given by Merton [7][8]. In his articles he suggests some rules for allocating wealth between current consumption and financial investment. Such models are explicitly solved for a special type of stochastic process concerning the price of financial activities: the geometric Brownian motion. For more general processes, the possibility of obtaining solutions depends on the ability to solve, either analytically or numerically, the partial differential equation (PDE) associated to the Bellman equation of the underlying problem of stochastic optimal control.

In Section. 2, the above mentioned Bellman equation will be proposed for a very general stochastic process concerning security prices and for a subgroup of HARA utility functions. Section. 3, instead, will show, for specific cases of the process describing the motion of security prices, how the assumption of zero consumption (joined to the selected particular utility function) can make it possible to obtain solutions starting from a first-order differential equation system rather than from a partial differential equation. In the cases examined the system will, then, be solved in order to reach the analytical solutions of the original investment problem.

In the following Sections the theoretical models illustrated will be applied to the study of some securities of the Italian stock market, in order to reach two main objectives:

I - to give examples of the solution of investment problems through the suggested models;

II - by applying more general models to verify if it is possible to obtain investment performances superior to the classical geometrical Brownian motion (at least for the considered securities and the chosen period).

2 Model formulation

We make the following assumptions:

perfect markets: no transaction costs and taxes, freely available information, perfect divisibility of securities, trading in continuous time, identical borrowing and lending rate, no short-sale restrictions;

n risky and one $(n+1)$-st. riskless securities;

the investor is interested in consumption only at the end of the time horizon;

price dynamics for securities is described by the following system of stochastic differential equations (SDEs)

$$\begin{cases} dP = P\mu\,(P,X,t)\,dt + PS\,(P,X,t)\,dz \\ dP_{n+1} = P_{n+1}r\,dt \\ dX = f\,(X,t)\,dt + G\,(X,t)\,dq \end{cases} \tag{1}$$

dP, $\mu(P,X,t)$ and dz are vectors in R^n corresponding respectively to the instantaneous price differential, the expected return in the time unit and to the differential of n-dimensional standard Brownian motion;

P and S are n-dimensional diagonal matrices, corresponding to the security prices and the standard deviation of returns in the time unit;

r is the instantaneous rate of return of the riskless security;

X is a vector of R^m containing factors influencing the vector μ and the matrix S;

$f\,(X,t)$ is a vector of R^n expressing the expected variation of the X process in the time unit;

$G\,(X,t)$ is a diagonal matrix whose m values are the standard deviations in the above-said processes;

dq is a vector in R^m corresponding to the differential of m-dimensional standard Brownian motion.

Also:

Ω is an n-dimensional not singular square matrix; s_i are the elements of the diagonal of S; $\Omega \equiv [s_i\,\rho_{ij}\,s_j]$ with $i,j = 1,...,n$ and $\rho_{ij}\,dt = dz_i\,dz_j$;

V is an m-dimensional not singular square matrix, g_i are the elements of the diagonal of G; $V \equiv [g_i\,v_{ij}\,g_j]$ with $i,j = 1,...,m$ and $v_{ij}\ dt = dq_i\,dq_j$;

Γ is a $m \times n$ full rank matrix; $\Gamma \equiv [g_i\,\eta_{ij}\,s_j]$ with $i = 1,...,m$; $j = 1,...,n$ and $\eta_{ij}\,dt = dq_i\,dz_j$;

Ω is the variance-covariance matrix among the security return rates, V the instantaneous variance-covariance matrix of the variations between the variables of X and Γ expresses the instantaneous covariances between the variations of X and the security return rates.

Assuming that the decision maker's wealth derives only from investment in the $n+1$ securities, its evolution can be expressed by the following stochastic differential equation [7]

$$dW = W\omega^T (\mu - r1)\, dt + rW dt + W\omega^T S dz \tag{2}$$

with initial condition $W(0) = W_0$,

where W is the wealth, 1 the vector with all elements equal to one, and ω is a vector of R^n containing the weights of the investment in the n risky securities. The weight of the riskless security, $(n+1)$-st., is computed residually being $\sum_{i=1}^{n+1} \omega_i = 1$. We introduce the objective function[3][6][8]:

$$J(W_0, P_0, X_0, 0) \equiv \max_{\omega} E_0 \left[\int_0^T U(C_s, s)\; ds + B(W_T, T) \right] \tag{3}$$

with the constraint (2), and where the utility functions U and B are supposed to be concave and not decreasing.

Since we are concerned with investment, rather than consumption problems, we set $C_t = 0$ on $[0, T[$. It follows that $C_T = W_T$ and $U = B$, i.e. the wealth is cumulated to be completely consumed at the final time.

(3) can so be reformulated:

$$J(W_0, P_0, X_0, 0) \equiv \max_{\omega} E_0 \left(U(C_T, T)\right) \tag{4}$$

By applying Bellman's dynamic programming method [1][5][7] we can obtain the following functional Φ, whose maximum with respect to the vector ω must be equal to zero:

$$\max_{\omega} \left[\Phi(W, P, X, t, \omega)\right] = 0, \tag{5}$$

where

$\Phi \equiv J_W\, W \omega^T\, (\mu - r1) + J_W\, rW + J_t + J_P^T\, P\mu + J_X^T\, f + \frac{1}{2} J_{WW} W^2 \omega^T \Omega \omega + J_{WP}^T W P \Omega \omega + W J_{WX}^T \Gamma \omega + \frac{1}{2}\,(P J_{PP}) \circ (\Omega P) + \frac{1}{2} J_{XX} \circ V + (P J_{PX}) \circ \Gamma^T$

and where the subscripts in function J denote the variables with respect to which the partial derivation is carried out and the symbol "$\circ$" indicates the sum of the componentwise product of the two matrices.

By setting the first-order conditions on Φ, we obtain:

$$\omega = -\frac{J_W}{W J_{WW}} \Omega^{-1} (\mu - r1) - \frac{1}{W J_{WW}} P J_{WP} - \frac{1}{W J_{WW}} \Omega^{-1} \Gamma^T J_{WX}. \quad (6)$$

Since $Hessian\,(\Phi) = J_{WW} W^2 \Omega$, where Ω is positive definite by hypothesis, then the condition $J_{WW} < 0$ assures the strictly concavity of Φ and it is a sufficient condition for the existence of only one maximum point.

As before said the utility function chosen for the final consumption is a HARA function and, precisely, of iso-elastic type: $U\,(C) = C^\gamma/\gamma$. For these utility functions J can be expressed as the product of two functions [3][9]:

$$J\,(W, P, X, t) = Q\,(P, X, t)\, W^\gamma/\gamma \quad (7)$$

Using this remark and substituting (6) in (5), we obtain:

$$Q_t + Q r \gamma + Q_P^T P \mu + Q_X^T f + \frac{1}{2}\, dz^T S P^T Q_{PP}\, P S\, dz + \frac{1}{2} dq^T G Q_{XX} G\, dq +$$

$$+ dz^T\, P S Q_{PX}\, G\, dq - \frac{\gamma}{\gamma - 1} Q_P^T P (\mu - r1) - \frac{\gamma}{\gamma - 1} Q_X^T \Gamma \Omega^{-1} (\mu - r1) -$$

$$- \frac{\gamma}{Q(\gamma - 1)} Q_X^T \Gamma P Q_P - \frac{\gamma}{2Q(\gamma - 1)} Q_P^T P \Omega P Q_P -$$

$$- \frac{\gamma}{2Q(\gamma - 1)} Q_X^T \Gamma \Omega^{-1} \Gamma^T Q_X - \frac{Q\gamma}{2(\gamma - 1)} (\mu - r1)^T \Omega^{-1} (\mu - r1) = 0 \quad (8)$$

with final condition $Q(P, X, T) = 1$.

After obtaining the function Q, it will be possible to reach explicit solutions by substituting Q in (7) e consequently J in (6).

The existence of solutions of (8) depends on qualification of system (1), that is, on the type of stochastic process adopted for security prices and on the parameter γ of the utility function. As concerns the cases examined, conditions will be discussed presently.

3 Alternative formulations for the stochastic prices modelling process

3.1 Geometric Brownian motion (model 1)

In the case considered here the parameters of the system of differential equation describing the prices dynamic, i.e. μ and S, are constant [3][6][7][9]. The optimal weights are therefore constant with respect to wealth and time:

$$\omega = \frac{1}{1-\gamma}\Omega^{-1}(\mu - r1). \tag{9}$$

3.2 Ornstein-Uhlenbeck process for the riskless rate of return r (model 2)

In this first example we assume that r is not constant, but is a stochastic process satisfying the equation $dr = \beta(\xi - r)dt + gdq$.

Therefore the $(n+1)$-st. security is only locally riskless [8]. The prices system is

$$\begin{cases} dP = P\mu dt + PSdz \\ dP_{n+1} = P_{n+1}rdt \\ dr = \beta(\xi - r)dt + gdq \end{cases} \tag{10}$$

(8) is in our case:

$$Q_t + \left[\gamma r + \frac{\gamma}{2(1-\gamma)}\left(r^2 A_{11} - 2rA_{12} + A_{13}\right)\right] Q+ \tag{11}$$

$$+\left[\beta\left(\xi-r\right)+\frac{\gamma}{1-y}\left(A_{22}-rA_{23}\right)\right]Q_r+\frac{\gamma}{2(1-\gamma)}\frac{A_3(Q_r)^2}{Q}+\frac{1}{2}g^2Q_{rr}=0$$

with final condition $Q(r,T)=1$ and where (Γ is in our case an n-dimensional vector)

$$\begin{array}{l}A_{11}\equiv 1^T\Omega^{-1}1; \quad A_{12}\equiv\mu^T\Omega^{-1}1; \quad A_{13}\equiv\mu^T\Omega^{-1}\mu;\\ A_{22}\equiv\mu^T\Omega^{-1}\Gamma; \quad A_{23}\equiv 1^T\Omega^{-1}\Gamma;\\ A_3 \equiv\Gamma^T\Omega^{-1}\Gamma\,.\end{array}$$

We look for a solution of (11) among functions of the kind

$$Q(r,t)=\exp\Big(A\left(\tau\right)+rB\left(\tau\right)+r^2C\left(\tau\right)\Big)$$

where $\tau\equiv T-t$ and the functions A, B and $C\in C^1$.

Then (11) is equivalent to the following first-order system of differential equations:

$$\left\{\begin{array}{lcl}A'(\tau) & = & h+f\,B(\tau)+\frac{1}{2}aB^2(\tau)+g^2C(\tau)\\ B'(\tau) & = & d+b\,B(\tau)+2\,k\,C(\tau)+2aB(\tau)C(\tau)\\ C'(\tau) & = & c+2bC(\tau)+2aC^2(\tau)\end{array}\right. \tag{12}$$

with initial condition $A(0)=B(0)=C(0)=0$, where

$$a\equiv\frac{\gamma A_3}{1-\gamma}+g^2; \;\; b\equiv-\beta-\frac{\gamma A_{23}}{1-\gamma}; \;\; c\equiv\frac{\gamma A_{11}}{2(1-\gamma)};$$

$$d\equiv\gamma-\frac{\gamma A_{12}}{1-\gamma}; \;\; k\equiv\beta\xi+\frac{\gamma A_{22}}{1-\gamma}; \;\; h\equiv\frac{\gamma A_{13}}{2(1-\gamma)}.$$

Equation (6) expressing the optimal investment solutions can be reformulated as follows, considering the form selected for function $Q\left(r,t\right)$:

$$\omega=\frac{1}{1-\gamma}\Omega^{-1}(\mu-r1)+\frac{B\left(\tau\right)+2rC\left(\tau\right)}{1-\gamma}\Omega^{-1}\Gamma. \tag{13}$$

From (7) we obtain that the sufficient condition for the existence of only one maximum point, $J_{WW} < 0$, is met, for the functions Q considered, if $\gamma < 1$; which restrains our analysis only to risk-averse investors, as we have set $U(C) = C^\gamma/\gamma$.

To spot the optimal investment, it is therefore sufficient to obtain the functions $B(\tau)$ and $C(\tau)$ resulting from the system given by the two last equations in (12), being these independent from the first one.

The last equation of system (12) is a Riccati equation with constant coefficients, its particular solution being the constant y dependent on the values of the parameters of the stochastic process and on the value γ chosen for the utility function. The explicit solution for function $C(\tau)$ is:

$$C(\tau) = \frac{-y(2ay+b)}{ay + (b+ay)\exp\left[-2(2ay+b)\tau\right]} + y. \tag{14}$$

If we reconsider the second equation of system (12) and substitute the solution (14) obtained for $C(\tau)$, we obtain a linear non-homogeneous differential equation whose solution is:

$$\begin{aligned} B(\tau) \quad = \quad & \frac{\exp[-(2ay+b)\tau]}{ay+(b+ay)\exp[-2(2ay+b)\tau]} \cdot \\ & \cdot \left[\frac{day - 2fay^2 - 2fby}{2ay+b} \{\exp[(2ay+b)\tau] - 1\} - \right. \\ & \left. - \frac{(ay+b)(2fy+d)}{2ay+b} \{\exp[-(2ay+b)\tau] - 1\} \right]. \end{aligned} \tag{15}$$

By substituting the functions thus obtained in (13), we can reach the explicit optimal solutions for the $n+1$ securities.

3.3 Affine dependence of expected return of risky securities on an external variable (model 3)

We assume that the vector of the expected returns affinely depends on an external variable X, i. e., $\mu = \alpha X + \lambda$, and that the matrix S is constant. If X follows an Ornstein-Uhlenbeck process, the price system is:

$$\begin{cases} dP = P\left(\alpha X + \lambda\right)dt + PSdz \\ dP_{n+1} = P_{n+1} r\, dt \\ dX = \beta(\xi - X)dt + g dq \end{cases} \tag{16}$$

(8) consequently becomes

$$\begin{aligned} Q_t + \left[\gamma r + \frac{\gamma}{2(1-\gamma)}\left(X^2 A_{11} + 2XA_{12} + A_{13}\right)\right] Q + & \\ + \left[\beta(\xi - X) + \frac{\gamma}{1-\gamma}(XA_{22} + A_{23})\right] Q_X + & \\ + \frac{\gamma}{2(1-\gamma)}\frac{A_3\, Q_X^2}{Q} + \frac{1}{2} g^2 Q_{XX} & = 0 \end{aligned} \tag{17}$$

with final condition $Q\left(X,T\right) = 1$, and where

$$A_{11} \equiv \alpha^T \Omega^{-1}\alpha; \quad A_{12} \equiv \alpha^T \Omega^{-1}\left(\lambda - r1\right); \quad A_{13} \equiv \left(\lambda - r1\right)^T \Omega^{-1}\left(\lambda - r1\right);$$
$$A_{22} \equiv \alpha^T \Omega^{-1}\Gamma; \;\; A_{23} \equiv (\lambda - r1)^T \Omega^{-1}\Gamma; \;\; A_3 \equiv \Gamma^T \Omega^{-1}\Gamma\,.$$

Proceeding in the same way as in the previous section and choosing a solution of (17) among functions of the kind

$$Q(X,t) = \exp\left[A(\tau) + XB(\tau) + X^2 C(\tau)\right]$$

where $\tau \equiv T - t$; we reach a new formulation of the system (12), where the parameters a, c, d are defined as above, while b, k, h are as follows:

$$b \equiv -\beta - \frac{\gamma A_{22}}{1-\gamma}; \;\; k \equiv \beta\xi + \frac{\gamma A_{23}}{1-\gamma}; \;\; h \equiv \gamma r + \frac{\gamma A_{13}}{2\left(1-\gamma\right)}.$$

Let us consider again equation (6) expressing the optimal values of the weights of investments and whose form is:

$$\omega = \frac{1}{1-\gamma}\Omega^{-1}(\alpha X + \lambda - r1) + \frac{B\left(\tau\right) + 2XC\left(\tau\right)}{1-\gamma}\Omega^{-1}\Gamma\,. \tag{18}$$

Here, too, it will be sufficient to find the solutions $B(\tau)$ and $C(\tau)$ to analytically solve our investment problem. Such functions have formally identical solutions to the ones obtained in the previous section.

3.4 Applicability conditions

The possibility of solving system (12) depends on the possibility of solving the last equation of the system, namely the Riccati equation with constant coefficients.

The necessary condition for the solution is that the discriminant of the quadratic polynomial associated with the equation be positive, which, in the parameter terms of the system, means

$$b^2 - 2ac \geq 0 \tag{19}$$

where a, b, c , let us remember it, are functions of the parameters characterizing the initial system (10) and of the parameter γ of the utility function.

Assuming the parameters of system (10) as known, the values of γ must satisfy the following inequality:

$$\begin{aligned} \gamma^2 \left(A_{23}^2 - A_3 \, A_{11} - 2\beta A_{23} + g^2 A_{11} + \beta^2 \right) + \\ + \gamma \left(2\beta A_{23} - g^2 A_{11} - 2\beta^2 \right) + \beta^2 \quad \geq \quad 0. \end{aligned} \tag{20}$$

The equation thus found corresponds to that of a parabola in γ, and it will be fully determined when the parameters of the stochastic differential equation system (10) are defined. So it will be possible to find the class of investors for which the problem can be solved. Given the analogy in the problem formalization, (20) is valid both for model 2 and for model 3.

3.5 Optimal weights and mutual funds

The equation describing the optimal weights (13) puts into evidence an inverse-ratio relation between the fraction of wealth invested by a decision maker in risky securities and his relating risk-aversion. In fact, we can see that, for the chosen utility function, the measure of relating Arrow-Pratt risk-aversion is strictly positive for the considered class of investors.

For models 2 and 3 we can extend the mutual-fund theorem introduced in Merton [7]. In fact, the equation (13) can be reformulated as

$$\omega = \alpha_F(\gamma, r) f(r) + \alpha_H(\gamma, r, \tau) h \tag{21}$$

where

$$\alpha_F(\gamma, r, \tau) \equiv \frac{1^T \Omega^{-1} (\mu - r1)}{1 - \gamma} \qquad \alpha_H(\gamma, r, \tau) \equiv \frac{[B(\tau) + 2rC(\tau)] 1^T \Omega^{-1} \Gamma}{1 - \gamma}$$

$$f(r) \equiv \frac{\Omega^{-1} (\mu - r1)}{1^T \Omega^{-1} (\mu - r1)} \qquad h \equiv \frac{\Omega^{-1} \Gamma}{1^T \Omega^{-1} \Gamma}$$

and where $f(r)^T i = h^T i = 1$, and consequently $\omega_{n+1} = 1 - \alpha_F(\gamma, r) - \alpha_H(\gamma, r, \tau)$.

In a similar way to what has been discussed in Merton, it can be said that the optimal choice, among the $n+1$ original securities, is equivalent to the choice among three mutual funds. The portfolios are made up of the $n+1$ dimensional vectors (22), whose weights are independent of the investor's preferences and consequently defined only with respect to the parameters of system (10):

$$v_1^T \equiv \left[f(r) \vdots 0 \right] \qquad v_2^T \equiv \left[h^T \vdots 0 \right] \qquad v_3^T \equiv \left[0 \vdots 1 \right] \tag{22}$$

where v_3 invests only in the riskless security, while v_1 and v_2 invest only in risky securities.

If F and H are the unit price of v_1 and v_2 respectively, then we obtain that their differentials are:

$$dF = F_{\mu_F}(r)\, dt + F_{\sigma_F}(r)\, dz_F\,, \qquad dH = H_{\mu_H} dt + H_{\sigma_H}\, dz_H \tag{23}$$

where

$$\begin{array}{lllllll}
\mu_F(r) & \equiv & f(r)^T\mu & \qquad & \mu_H & \equiv & h^T\mu \\
\sigma_F^2(r) & \equiv & f(r)^T\Omega f(r) & & \sigma_H^2 & \equiv & h^T\Omega h \\
dz_F & \equiv & \frac{f(r)^T Sdz}{\sigma_F} & & dz_H & \equiv & \frac{h^T Sdz}{\sigma_H}.
\end{array}$$

The weights of fund v_1 vary following the interest rate; this fund has the function of guaranteeing a diversification of the investment. The price F is not lognormally distributed like the price of the initial securities (unlike in Merton [7] [9]), its drift and diffusion coefficients being dependent on r.

The price H of fund v_2 follows a geometric Brownian motion as direct consequence of the constant volatility hypothesis of the process dr, which makes constant the variance-covariance matrix Γ between the rate variations and the security returns.

The function of this portfolio is to guarantee a hedging against the effects of the variations of the interest rate level. The return of this portfolio has its maximum correlation with the variations of rate r among all the returns of the portfolios that can be built with the n risky activities.

Reconsidering the investment weights in each fund $\alpha_F(\gamma, r)$ and $\alpha_H(\gamma, r, \tau)$, we write the partial derivatives with respect to the rate r

$$\frac{\partial\, \alpha_F(\gamma, r)}{\partial r} = -\frac{1^T\Omega^{-1}1}{1-\gamma} < 0 \tag{24}$$

$$\frac{\partial \alpha_H(\gamma, r, \tau)}{\partial r} = \frac{2C(\tau)1^T\Omega^{-1}1}{1-\gamma}.$$

In the first weight we can observe an inverse relation between the return rate of the security $n+1$ and the weight of the fund v_1, while the behaviour of the second weight is not to be determined a priori, its sign not being univocally determined. However, the following relation:

$$\lim_{\tau\to 0} \alpha_H(\gamma, r, \tau) = 0 \tag{25}$$

is valid, that is, the necessity of hedging against the variations of the rate r disappears as the application period draws to its end.

4 Parameters estimation

To apply the models we have presented it is necessary to obtain, in a preliminary way, an estimation of the parameters that the model assumes as known, namely the parameters characterizing the initial SDE system. More precisely, it is necessary to obtain, for the n risky securities, the estimation of the vector μ and the matrix S; as well as the estimation of the parameters β, ξ and g relating to process r in the case of model 2, and process X in the case of model 3.

For this purpose, consider the following general linear SDE system of the first order

$$dY = \Theta Y(t)\, dt + \Psi\, dt + S\, dz \tag{26}$$

where the parameters are expressed by the square matrix Θ, the vector Ψ and the diagonal matrix S. By integration of system (26) between $t - \Delta t$ and t we obtain

$$\int_{t-\Delta t}^{t} dY = \int_{t-\Delta t}^{t} \Theta Y(\theta)\, d\theta + \int_{t-\Delta t}^{t} \Psi d\theta + \int_{t-\Delta t}^{t} S\ dz(\theta). \tag{27}$$

which, if Δt is sufficiently small, can be approximated using the trapezoidal rule, here exposed for the general function $\delta(t)$:

$$\int_{t-\Delta t}^{t} \delta(s) ds \cong \frac{1}{2}\Delta t\left[\delta(t) + \delta(t-\Delta t)\right]. \tag{28}$$

Relation (27) can be re-written as

$$Y(t) - Y(t-\Delta t) = \frac{1}{2}\Delta t\, \Theta\left[Y(t) - Y(t-\Delta t)\right] + \Psi\, \Delta t + \epsilon(t) \tag{29}$$

with $\epsilon(t) \approx N(0, \Sigma\Delta t)$, where Σ is the variance-covariance matrix of the processes Y, and zero correlation for time increments different from Δt.

In this way we obtain a SDE system not difficult to estimate. So, in the case that the SDE system utilized with the model hypotheses can be brought back to (26), the described procedure can be utilized to obtain the estimation of the parameters[1].

As an example, we refer to the system (16) used for model 3, with the assumption of the existence of only one risky security ($n = 1$)

$$\left\{\begin{array}{lcl} dP_1 & = & P_1(\alpha X + \lambda)\, dt + P_1 s\, dz \\ dP_2 & = & P_2\, r\, dt \\ dX & = & \beta(\xi - X)\, dt + g\, dq \end{array}\right. \tag{30}$$

and perform the transformation

$$\begin{array}{lcl} Y_1 & \equiv & \ln(P_1) \\ Y_2 & \equiv & \ln(P_2) \\ X & \equiv & X. \end{array} \tag{31}$$

The differential of process (31) is, according to the Itô lemma, the following

$$\begin{array}{lcl} dY_1 & = & \left(\lambda - \frac{1}{2}s^2\right) dt + \alpha X\, dt + s\, dz \\ dY_2 & = & r\, dt \\ dX & = & \beta\xi\, dt - \beta X dt + g\, dq \end{array} \tag{32}$$

which is of the type (26).

Approximating system (32) in accordance with what explained, after some steps, we reach

$$\begin{array}{lcl} Y_1(t) & = & \eta_1 + \eta_2\left[X(t) + X(t - \Delta t)\right] + \phi(t) \\ X(t) + X(t - \Delta t) & = & \eta_3 X(t - \Delta t) + \eta_4 + \upsilon(t) \end{array} \tag{33}$$

where, from estimation of $\eta_1 ... \eta_4$, $var(\phi)$, $var(\upsilon)$, $cov(\phi, \upsilon)$ it is possible to obtain the parameters α, λ, s, β, ξ, g, $cov(dz, dq)$ of the original model (30).

[1]It would be possible to obtain estimations of the parameters starting from finite difference systems "stochastically equivalent" to continuous ones, as the former are satisfied by every set of equal width observations generated by the latter. However the suggested approximation limits the error to a term of the third order with respect to Δt [2] [11][12].

5 An application the some Italian securities

5.1 Preliminary statement

The model is continuous, so, in order to be coherent with the used method, any time interval here considered has to be very small with respect to the time unit, which in our case is one year, whether such interval is the one used for the parameters estimation or the one regarding the time interval of portfolio revision. Generally it is easy to collect daily data, and the daily interval is sufficiently small with respect to the year. Unfortunately the daily interval is not advisable for revision purposes because of the very great incidence of transaction costs. In this case an acceptable compromise may be to choose a weekly interval, which appears to be a good trade-off between theoretical requirements and practical feasibility.

5.2 The application

In this section we compare the performances of the above models using weekly data of three Italian securities: SIP, BANCA COMMERCIALE ITALIANA (BCI) and MEDIOBANCA (MDB), starting from October, 12th 1990 and applying the models to the year 1993.

We assume to rebalance weekly the portfolio, which is made up of one of the above securities and a riskless security with a yearly rate of return equal to 12% in models 1 and 3. The risk-aversion parameter in the utility function (γ) is fixed to -0.8, and the initial wealth to 1. Whereas in model 2 we use for r the three-month time series of Italian BOT. In model 3 we identify the variable X with the rate of return of the Comit index of Milan stock exchange.

The graphs and table report the weekly courses of the historical series used and some standard statistics on the stocks (Figures 1, 2 and 3).

The following graphs (Figure 4) show the weekly weights of the SIP

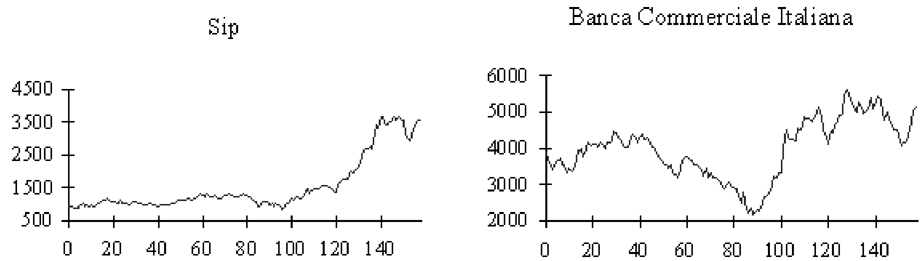

Figure 1: Sip and BCI prices

security in the year 1993 resulting from the three models considered and the wealth obtained from the investment in that security and in the locally riskless security. Taking again into consideration (21), we can see that the difference between the optimal investment weights in the two models is given by the existence, in model 2, of the edging fund h, generated in consequence of the randomness of the rate r (in the case of an investment in only one risky security, h is equal to $f(r)$, that is to the fund that guarantees the optimal diversification of the investment, because both funds contain only the risky security). The presence of the fund h involves, in model 2, a shifting of wealth from the riskless security towards the risky one, which makes it possible to better exploit the positive behaviour of the security in the first periods of application.

	SIP	**BCI**	**MDB**
Average (year)	0.429	0.090	0.117
Variance (year)	0.122	0.114	0.153
Normality test	13.324	13.426	109.058
Autocorr. test (D.W.)	1.854	1.846	1.981

Figure 2: MDB prices and stock statistics

In the case of model 3, we have assumed that the added variable X directly bears on the return and consequently on the weight invested in fund f. The greater volatility of X goes together with a much more variable behaviour of the invested quota than for the other two models, and this is obviously reflected on the state of the accumulated wealth.

The evolution of the wealth is satisfactory enough, but in the last weeks, it shows a sharp decrease, only later recovered, with a consistent loss of the previous gain. This fact suggests to introduce a stop loss rule, in order to suspend the model in case the return should go below a fixed level.

The return of the risky security, at the initial time, is positive and higher than the one of rate r. This fact involves, for model 1 and 2, a positive exposure on that security (see equation (9) and (13)), which brings the wealth onto a good level, owing to the rising trend during most of the period. This situation is quite favourable to the use of these models, but it is also very particular.

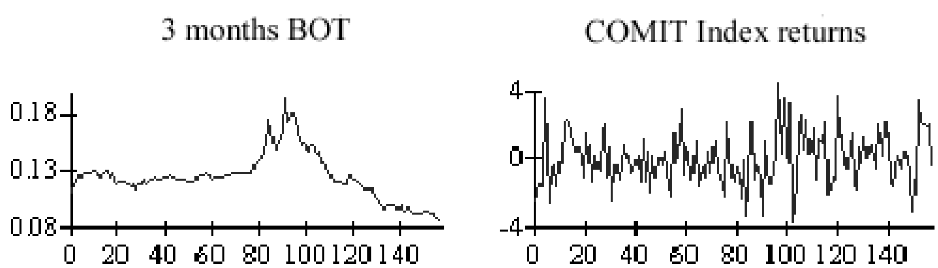

Figure 3: BOT and COMIT Index returns

Also, the weights conform but slowly to the changed trend conditions, even if an overlapping technique[2] is applied to the estimation sample data. This observation acquires greater significance if the models are

[2]This technique consists in inserting the new data as they become available into the sampling used for estimations, while eliminating the oldest ones. By so doing, the sample length remains unchanged. It must be noticed that the use of this technique

applied to short periods, for instance a year. As a matter of fact, if, in the long run, it may appear less arbitrary to accept a stability assumption of the expected return for some asset, and this is a requirement of the geometric Brownian motion, for shorter periods it is convenient to let the weights conform to external signals (i.e. the variable X) that give information about trend changing.

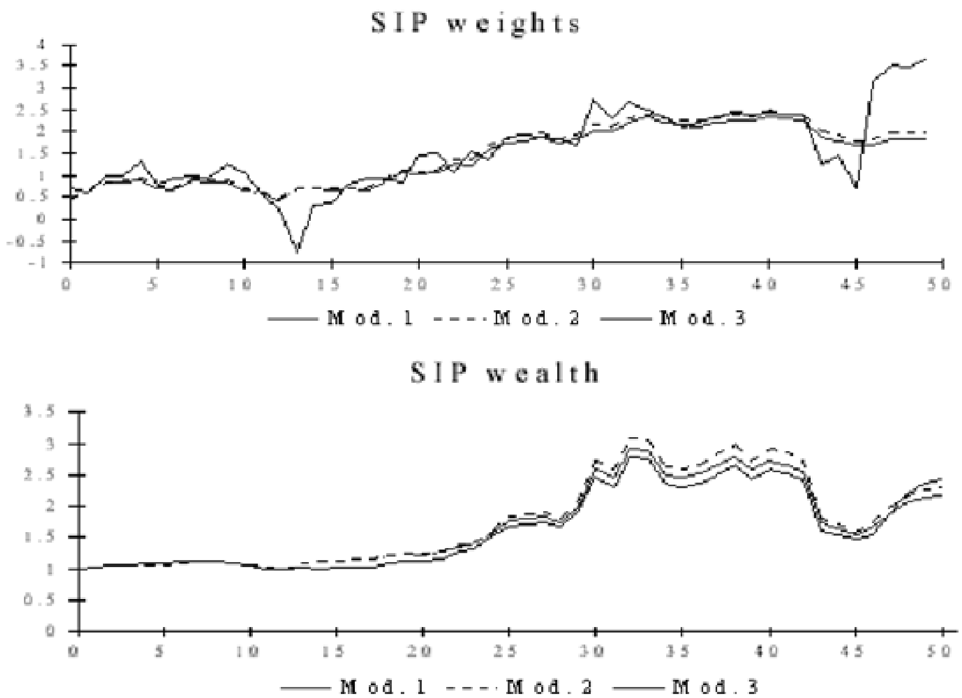

Figure 4: SIP weights and wealth

One may wonder, then, why model 3 shows a less positive performance. First of all we can see that the greater variability of the weights

would be in contrast with a correct application of the Merton model (model 1), since the geometric Brownian motion hypothesis implies a return expectation stability in the long run.

is not favourable in periods of regular courses, i. e., in a time when the security course largely reflects the one assumed by the geometric Brownian motion, the greater stability of the weights it suggests is a sure advantage. Let us also analyze the mean values (remember the use of the data overlapping) of the regression coefficients of the system used for parameters estimation in model 3:

$$\begin{aligned} \ln[P(t)/P(t-\Delta t)] &= 0.00784 + 0.0052\,[X(t) + X(t-\Delta t)] + st.\,noise \\ X(t) + X(t-\Delta t) &= -0.000148 + 1.1064\,X(t-\Delta t) + st.\,noise \end{aligned} \tag{34}$$

We can anticipate that the coefficient which depends on the explicative variable in the first equation of system (34), is very low if compared with the one obtained for the other two considered securities. The first addendum in the right side of the equation (18) (which, in model 3, is not constant but dependent on the explicative variable) will not bring a high differential advantage in the computation of the weights with respect to the two first models. However the weights will differentiate themselves as it occurs in model 2, because of the second addendum of the equations (13) and (18).

As to the other two risky securities, which the models have been applied to (Figure. 5 and Figure 6), the said slowness in conforming of the weights and the up-and-down behaviour of the risky security return involves a less satisfactory remuneration of the invested capital than for the first security. Model 2, moreover, being more exposed in the risky security, owing to the previously discussed hedging necessity, achieves an even inferior performance.

The case is different for model 3, which, though showing a high variability of its weights, ensures interesting capital gains, more regular for the Comit security.

As anticipated, we report the results of the linear regression used in the model 3:

Banca Commerciale Italiana

$$\begin{aligned} \ln[P(t)/P(t-\Delta t)] &= 0.0019 + 0.1163\,[X(t) + X(t-\Delta t)] + st.\,noise \\ X(t) + X(t-\Delta t) &= -0.000148 + 1.1064\,X(t-\Delta t) + st.\,noise \end{aligned} \tag{35}$$

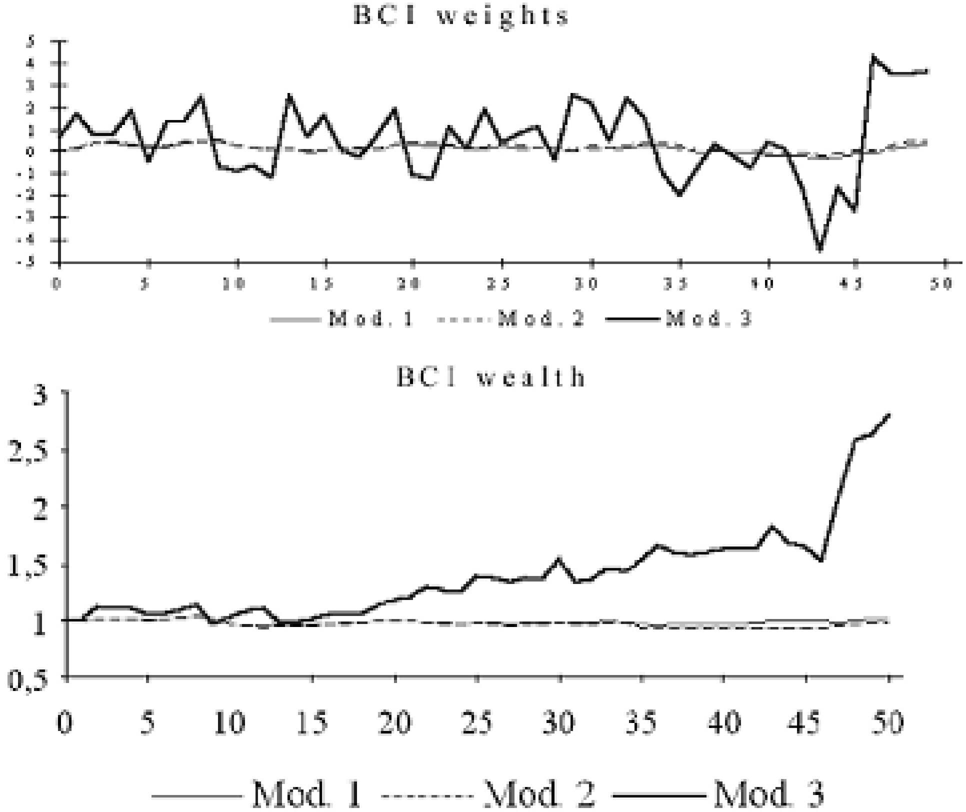

Figure 5: BCI weights and wealth

$$
\begin{array}{rcl}
 & & \mathit{Mediobanca} \\
\ln[P(t)/P(t-\Delta t)] & = & 0.0026 + 0.17\,[X(t) + X(t-\Delta t)] + st.\,noise \\
X(t) + X(t-\Delta t) & = & -0.000148 + 1.1064\,X(t-\Delta t) + st.\,noise.
\end{array}
\quad (36)
$$

Notice that, with respect to the Sip security, there is a net increase of the parameter dependent on X in the first equation of the systems (34) (36). This fact suggests a relation between the results of the application and the dependence of the price process on the chosen external variable.

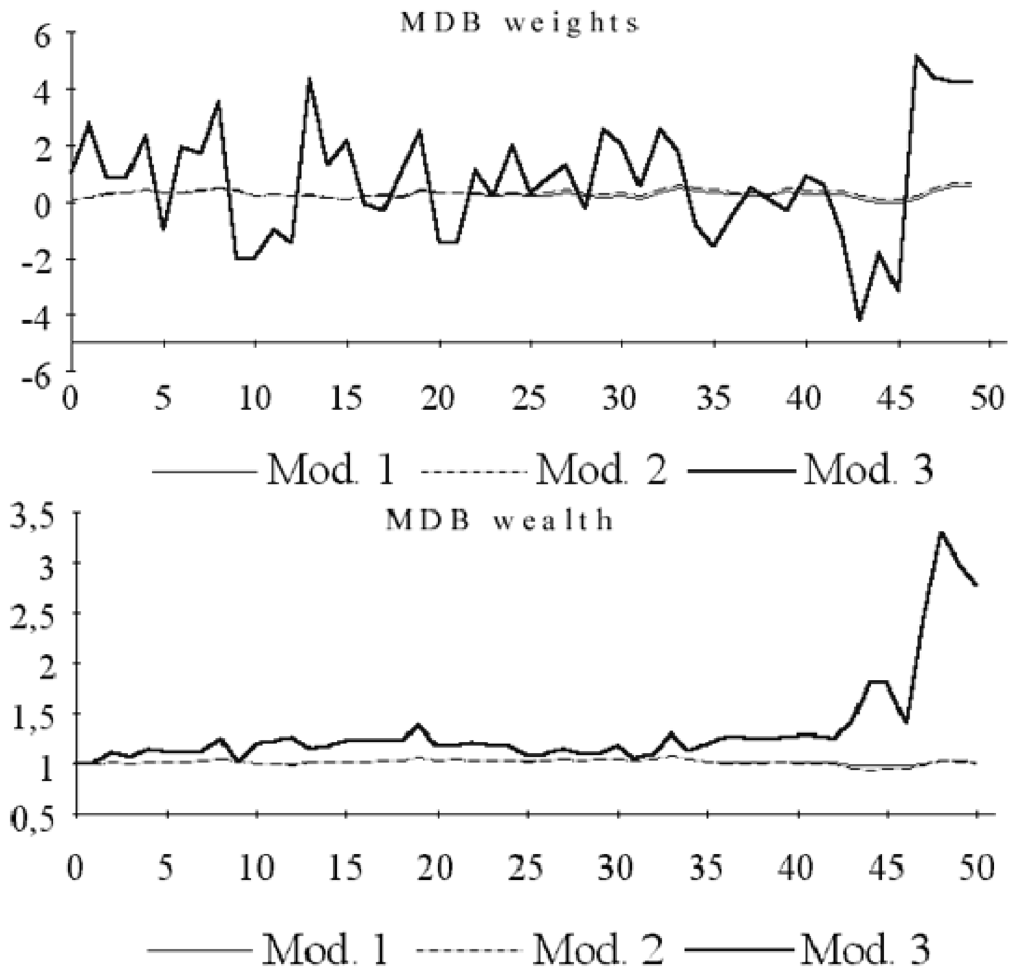

Figure 6: MDB weights and wealth

6 Some conclusions

In the first part of this paper we have given analytical solutions to investment problems for models in which the evolution in time of the prices of the considered securities can be referred back to stochastic dynamic processes different from the classic geometric Brownian motion.

In the remaining sections we have proposed some explanations of the results achieved and we have illustrated a method for the estimations of the parameters of the stochastic process utilized, that being the first indispensable step to proceed to the practical application of the models. Also,we have undertaken to verify if, with respect to the traditional model, the models we used can ensure advantages in the return obtainable from the investment.

So we have wanted to conclude this work with a rapid presentation of the results of a simplified implementation of the models taken into consideration, in order mainly to conclude this ideal applicative itinerary of the general model proposed in section 2. All that has led us to reach analytical solutions for two stochastic processes different from the GBM (which is far, anyway, from exhausting the possible hypotheses on price processes) and to discuss some empirical evidence attained by the practical implementation of the models.

The specification of the two stochastic processes different from the GBM and the formulation of explicit solutions aim at exemplifying the way of utilizing the general model. These developments are consequently not intended to present models necessarily more performing with respect to the GBM.

The short discussion of the application has been done on the comparison of the performances of the models, those being, in our opinion, the main index for evaluating the correctness of the hypotheses introduced. However, we have not neglected to link those results with the variability of the investment quotas and consequently with the incidence of transaction costs on the portfolio revision frequency.

For the selected securities and the used time period, advantages have been evident only for the model admitting a dependence of the expected value on a given variable, though the use of such a model is not free from criticisms; investment weights with a volatility as great as that recorded here appear to hardly be acceptable in practice.

For securities that exhibit a non-negligible dependence on the chosen explicative variable, we have, however, reached interesting results, that should motivate more extensive applications, with reference to the return obtained also during periods when the trend of returns deviates from past history.

Finally we can conclude saying that, although the main purpose of this article was to propose an exemplification of the practical applicability of the general model suggested, these first empirical evidences allow us to affirm that also simple extensions of the basic stochastic process (GBM) consent to increase the possibility of remunerating the invested capital in a promising way.

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