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QUANTITATIVE DYNAMICS FOR THE PEDLAR MODEL

Marco CORAZZA (E-mail: corazza@unive.it)
Stefania FUNARI (E-mail: funari@unive.it)

Department of Applied Mathematics
Dorsoduro 3825/E, 30123 Venice

University "Ca Foscari" of Venice

ABSTRACT

This work deals with a dynamical model able to represent the equilibrium of a monopolistic market which is characterized by the presence of a counterfeiter. The formalization of the behaviours of the economic agents (the monopolist, the counterfeiter and the consumers) leads to a first order linear difference equation system, whose solution describes the dynamical equilibrium of the market.

1. INTRODUCTION

Some authoritative international organizations (U.S. International Trade Commission and Comité Colbert) have recently pointed out the increasing influence played by the counterfeiting in several modern economies. In particular, by deceptive counterfeiting we mean the counterfeiting of goods and services in order to fool the consumer on their originality.

Notwithstanding the importance of such a feature, counterfeiting is mainly analyzed in literature by static models (at least to the best of our knowledge). In this work we propose a dynamical model able to represent the equilibrium of this kind of monopolistic market. The formalization of the economic agents' behaviour leads to a linear, finite difference equation, whose solution describes the dynamical equilibrium of the market.

The present work is organized as follows. In Section 2 we describe the dynamic relations arising among the economic agents, that partly refer to the so called *pedlar model* (*Mossetto (1992))*. In Section 3 we present the dynamic system, whereas we discuss the conditions for the convergence to the constant equilibrium level in Section 4. Some final remarks that represent suggestions for a future research are discussed in Section 5.

2. DYNAMIC BEHAVIOUR

We develop a relatively simple model which considers the dynamic relations arising among the economic agents.

Even if the model considers two producers that compete in the market, like the duopoly of Cournot, it incorporates different assumptions about the underlying structure of the market and behaviour of firms.

The economic agents' behaviour may be summarised as follows:

Step 1 - At time $t = t_0 := 0$ the counterfeiter begins to produce and sell counterfeited goods. He satisfies that part of consumers' demand that has not been satisfied by the monopolist.

Step 2 - At time $t = t_0$ consumers, who are unable to distinguish the original trade-mark from the counterfeited one, purchase the total quantity, i.e. the sum of quantities offered by both the monopolist and the counterfeiter.

Step 3 - At time $t = t_1 := t_0 + 1$ the monopolist notices that the market's price at the previous instant was not the right one: therefore he becomes aware of the counterfeiter's presence. In order to maintain his reputation, he carries out costly investments. We may consider investments in marketing strategies, for example advertising expenses in order to spread information on counterfeiter's presence among the potential consumers' population. The monopolist maximises his profit at time t_1, according to his current cost structure.

Step 4 - At time $t = t_1 := t_0 + 1$ consumers react to the reputation investments which have been carried out by the monopolist (usually by diminishing the total demand).

Step 5 - From the instant $t = t_0 := t_1 + 1$ the process restarts from *Step 2*.

2.1 Dynamic demand function

Let us consider the following linear demand function in a counterfeited market:

$$\begin{aligned} p(t) &= a - b \cdot y(t) - b \cdot y_F^o(t) + k \cdot \left[d \cdot y_F^{o,e}(t) \right] \\ &= a - b \cdot \left[y(t) + y_F^o(t) \right] + k \cdot \left[d \cdot y_F^o(t-1) \right], \end{aligned} \tag{1}$$

with
$a > 0$, $b > 0$, $k \in \mathbf{R}$ and $d \geq 0$,

where

$y(l)$	is the demand at time l,
$y_F^o(l)$	is that part of the demand which is satisfied by the counterfeiter, at time l,
k	is the consumers' reaction to the investments on reputation made by the monopolist.
d	is the monopolist's reaction to counterfeiting, in terms of investments on reputation.
$y_F^{o,e}(l)$	is the counterfeited quantity that the monopolist expects at time l.

The above equation takes into account that the economic agents' reaction to counterfeiting is supposed to be linear and that for all time $t < t_0 := 0$ the agents' behaviour coincides with the classical one. We can state the following observations:

Remark 1: investments on reputation linearly depend on monopolist's expectations about counterfeiting, at time t:

$$I(t) = d \cdot y_F^{o,e}(t) \equiv d \cdot y_F^o(t-1)\,. \tag{2}$$

Remark 2: consumers' reaction, in terms of total demand, linearly depends on the monopolist's investments at the same time:

$$k \cdot I(t) = k \cdot (d \cdot y_F^o(t-1))\,. \tag{3}$$

Remark 3: for all $t < t_0 := 0$, given that $y_F^o(t) = 0$ and $y_F^o(t-1) = 0$, equation (1) gives the classical inverse demand function

$$p(t) = a - b \cdot y(t) = a - b \cdot y = p\,. \tag{4}$$

Whereas, at time $t = t_0 := 0$, given that

$$y_F^o(t_0) = y_F^o(0) \neq 0 \;\text{ and }\; y_F^o(t_0 - 1) = y_F^o(-1) = 0 \tag{5}$$

the initial price is

$$p(0) = a - b \cdot (y(0) + y_F^o(0))\,. \tag{6}$$

2.2 The monopolist's behaviour

We assume, as implicitly noted before, that there is only one trade mark owner whose decision problem consists of choosing the level of output that he wishes to sell, in order to maximise his profit. The monopolist maximises profit by setting marginal revenue equal to marginal cost, as usual

$$RM_M(t) = CM_M(t) \tag{7}$$

where

$$RM_M(t) = \frac{\partial\, RT_M(t)}{\partial\, y(t)}, \qquad CM_M(t) = \frac{\partial\, CT_M(t)}{\partial\, y(t)} \tag{8}$$

$RM_M(t)$	is the monopolist's marginal revenue at time t, which is equal to the derivative of monopolist's total revenue $RT_M(t)$,
$CM_M(t)$	is the monopolist's marginal cost at time t, which is equal to the derivative of monopolist's total cost $CT_M(t)$.

Although we do not discuss these issues here, we point out that the monopolist's total cost structure generally differs from that of counterfeiter, even if marginal costs are assumed to be equal, i.e.

$CT_M(t) \neq CT_F(t)$ (in particular $CT_F(t) < CT_M(t)$); $CM_M(t)=CM_F(t)=CM(t)$.

Marginal revenue and marginal cost from a differential increase in the quantity can be expressed as follows (*Corazza-Funari (1996)*):

$$\begin{aligned} RM_M(t) &= a - 2\cdot b\cdot y(t) - b\cdot y_F^o(t) + k\cdot\left[d\cdot y_F^o(t-1)\right] \\ &= a - b\cdot\left[2\cdot y(t) + y_F^o(t)\right] + k\cdot\left[d\cdot y_F^o(t-1)\right] \end{aligned} \tag{9}$$

$$\begin{aligned} CM_M(t) &= c + c'\cdot y(t) + d\cdot y_F^{o,e}(t) \\ &= c + c'\cdot y(t) + d\cdot y_F^o(t-1) \end{aligned} \tag{10}$$

$$0 < c < a$$

From condition (7) we obtain the monopolist's optimal quantity at time *t*, which we denote by $y_M^o(t)$:

$$y_M^o(t) = \frac{a-c}{c'+2\cdot b} - \frac{b}{c'+2\cdot b}\, y_F^o(t) + \frac{d\cdot(k-1)}{c'+2\cdot b}\, y_F^o(t-1). \tag{11}$$

Where $\dfrac{a-c}{c'+2\cdot b} > 0$, $-\dfrac{b}{c'+2\cdot b} < 0$ and $\dfrac{d\cdot(k-1)}{c'+2\cdot b}\begin{cases} <0 & \text{if} \quad k<1 \quad \text{and} \quad d>0 \\ =0 & \text{if} \quad k=1 \quad \text{or} \quad d=0 \\ >0 & \text{if} \quad k>1 \quad \text{and} \quad d>0 \end{cases}$.

Remark 4: we observe that for all $t < t_0 := 0$, equation (11) gives the monopolist's classic optimal quantity

$$y_M^o(t) = \frac{a-c}{c'+2\cdot b} = y_M^o \qquad \forall t < t_0 := 0\,. \tag{12}$$

whereas at time $t = t_0 := 0$, because

$$y_F^o(t_0) = y_F^o(0) \neq 0 \;\text{ and }\; y_F^o(t_0 - 1) = y_F^o(-1) = 0, \tag{13}$$

we observe the following monopolist's initial optimal quantity

$$y_M^o(0) = \frac{a-c}{c'+2\cdot b} - \frac{b}{c'+2\cdot b} y_F^o(0)\,. \tag{14}$$

2.3 The counterfeiter's behaviour

At time *t* counterfeiter's behaviour is assumed to be as follows

$$\begin{aligned} y_F^o(t) &= f\cdot\left[y_S^e(t) - y_M^{o,e}(t)\right] \\ &= f\cdot\left[y_S(t-1) - y_M^o(t-1)\right], \end{aligned} \tag{15}$$

with $0 \leq f \leq 1$,

where

f	is the percentage of the demand which may be satisfied by the monopolist but that is actually satisfied by the counterfeiter.
$y_S^e(l)$	represents the value of the demand that the counterfeiter expects that the monopolist could theoretically satisfy at time *t*.
$y_M^{o,e}(l)$	represents the counterfeiter's expected value about the monopolist's optimal quantity at time l and
$y_S(l)$	represents the socially optimal (competitive) output level.

Remark 5: counterfeiter's quantity at time *t* is assumed to linearly depend on the gap between the counterfeiter's expected value about the socially optimal output level and the value he expects about the monopolist's optimal output level, at same time *t*.

$$y_F^o(t) = f\left[y_S^e(t) - y_M^{o,e}(t)\right]. \tag{16}$$

Moreover, let us assume that

$$y_S^e(t) - y_M^{o,e}(t) = y_S(t-1) - y_M^o(t-1). \tag{17}$$

Some remarks on $y_s(t)$ are necessary: $y_s(t)$ represents the consumers' demand that the monopolist could theoretically satisfy. By denoting by $\Pi[y(l)]$ the monopolist's profit at time l, it follows that:

$$\Pi[y(l)] \begin{array}{lll} < & 0 & \text{se} \quad y(l) > y_S(l) \\ = & 0 & \text{se} \quad y(l) = y_S(l). \\ > & 0 & \text{se} \quad y(l) < y_S(l) \end{array} \tag{18}$$

Given the demand function which has been introduced in section 2.1, it is possible to prove that $y_s(t)$ is the output level such that

$$p(y_S(t)) = CM(y_S(t)). \tag{19}$$

Using equations (1), (10) and condition (19) we find that the socially optimal quantity at time t is given by

$$\begin{aligned} y_S(t) &= \frac{a-c}{c'+b} - \frac{b}{c'+b} y_F^o(t) + \frac{d\cdot(k-1)}{c'+b} y_F^o(t-1) \\ &= \frac{c'+2\cdot b}{c'+b} y_M^o(t) \end{aligned} \tag{20}$$

Finally, by substituting (20) into equation (15) we obtain the following counterfeiter's quantity[1]

[1] Counterfeiter's decisions about quantity do not derive from an optimization problem, as for the monopolist. A counterfeiter's optimizing behaviour might be inserted in the model by choosing a suitable parameter *f*, i.e. by choosing $f \in [0,1]$ as to exogenously maximize his net benefit.

$$y_F^o(t) = f \cdot \left[\frac{c'+2\cdot b}{c'+b} y_M^o(t-1) - y_M^o(t-1) \right] = \frac{f\cdot b}{c'+b} y_M^o(t-1) \qquad (21)$$

3. THE MODEL

From equations (11) and (21) we obtain the following linear difference equation system

$$\begin{cases} y_M^o(t) = \dfrac{a-c}{c'+2\cdot b} - \dfrac{b}{c'+2\cdot b} y_F^o(t) + \dfrac{d\cdot(k-1)}{c'+2\cdot b} y_F^o(t-1) \\ y_F^o(t) = \dfrac{f\cdot b}{c'+b} y_M^o(t-1) \end{cases} \qquad (22)$$

Given the previous first order linear system of two difference equations, we can get the following second order linear difference equation which specify the dynamic of the monopolist's optimal quantity

$$y_M^o(t) - \alpha \cdot y_M^o(t-1) - \beta \cdot y_M^o(t-2) = \gamma, \qquad (23)$$

with $\alpha = -\dfrac{f\cdot b^2}{(c'+b)(c'+2\cdot b)} < 0,$

$$\beta = \frac{f\cdot b\cdot d\cdot(k-1)}{(c'+b)(c'+2\cdot b)} \begin{cases} < 0 & \text{if} \quad k<1 \quad \text{and} \quad d>0 \\ = 0 & \text{if} \quad k=1 \quad \text{or} \quad d=0 \\ > 0 & \text{if} \quad k>1 \quad \text{and} \quad d>0 \end{cases} \qquad (24)$$

and $\gamma = \dfrac{a-c}{c'+2\cdot b} > 0.$

The general solution of (23) is obtained by adding a particular solution of (23) to the general solution of the homogeneous part of the previous equation:

$$y^o_{M,Gen}(t) = y^o_{M,Omo}(t) + y^o_{M,Par}(t) \tag{25}$$

Where

$y^o_{M,Gen}(t)$ represents the general solution of (23),

$y^o_{M,Par}(t)$ represents a particular solution of (23),

$y^o_{M,Omo}(t)$ represents the general solution of the homogeneous part of equation (23), i.e. by setting $\gamma = 0$ in (23):

$$y^o_M(t) - \alpha \cdot y^o_M(t-1) - \beta \cdot y^o_M(t-2) = 0 \tag{26}$$

3.1 Homogeneous solution

Let us begin with the homogeneous equation (26). It can be proved that the general solution $y^o_{M,Omo}(t)$ is given by

$$y^o_{M,Omo}(t) = \begin{cases} c_1\lambda_1^t + c_2\lambda_2^t & \text{if} \quad \lambda_1 \neq \lambda_2 \\ c_1\lambda^t + c_2 t\lambda^t & \text{if} \quad \lambda_1 = \lambda_2 =: \lambda \end{cases} \tag{27}$$

where

$$\begin{aligned} \lambda_{1,2} &= \frac{\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2} \\ &= \frac{-fb^2 \pm \sqrt{f^2 b^4 + 4d(k-1)bf(c'+b)(c'+2b)}}{2(c'+b)(c'+2b)} \end{aligned} \tag{28}$$

with λ_1, λ_2 the two characteristic roots of the quadratic characteristic equation. By the superposition theorem, λ_1 and λ_2 are to be linearly combined each with its arbitrary constant c_1 and c_2, so that the solution of the homogeneous part of (23) is $y^o_{M,Omo}(t) = c_1\lambda_1^t + c_2\lambda_2^t$. On the other hand, if $\lambda_1 = \lambda_2$ (i.e. if. $\Delta = \alpha^2 + 4\beta = 0$), the homogeneous solution is given by $y^o_{M,Omo}(t) = c_1\lambda^t + c_2 t\lambda^t$.

Remark 6: it is possible to prove that

$$\alpha^2 + 4 \cdot \beta =: \Delta \quad \begin{array}{lll} < 0 & \text{if} & k < \tilde{k} \\ = 0 & \text{if} & k = \tilde{k} \\ > 0 & \text{if} & k > \tilde{k} \end{array}, \tag{29}$$

with

$$\tilde{k} := 1 - \frac{f \cdot b^3}{4 \cdot d \cdot (c'+b)(c'+2 \cdot b)} \qquad (\tilde{k} < 1). \tag{30}$$

3.2 Particular solution

It is possible to prove that a particular solution of (23) is the following

$$\begin{aligned} y^o_{M,Par}(t) &= \frac{\gamma}{1-\alpha-\beta} \\ &= \frac{(a-c)(c'+b)}{(c'+2b)(c'+b) + b^2 \cdot f - d \cdot b \cdot f \cdot (k-1)} \end{aligned} \tag{31}$$

Remark 7: it is possible to prove that if $k \neq k^*$, then

$$1 - \alpha - \beta \neq 0$$

If $k < k^*$, then

$$y^o_{M,Par}(t) = y^o_{M,Par} > 0 \tag{32}$$

where

$$k^* := 1 + \frac{(c'+b)(c'+2 \cdot b) + f \cdot b^2}{b \cdot f \cdot d} \qquad (k^* > 1 > \tilde{k}). \tag{33}$$

Remark 8: if there is not counterfeiting ($f = 0$), the solution of (23) is equal to the solution of the classical monopoly, i.e.

$$y^o_{M,Omo}(t) = 0 = y^o_{M,Omo},\; y^o_{M,Par}(t) = \frac{a-c}{c'+2\cdot b} = y^o_{M,Par} \tag{34}$$

$$y^o_{M,Gen}(t) = \frac{a-c}{c'+2\cdot b} = y^o_{M,Gen}. \tag{35}$$

4. STABILITY CONDITIONS

Since the non homogeneous term in (23) is constant the particular solution (31) is given by the steady state value $y^o_{M,E}(t)$, i.e.

$$y^o_{M,E}(t) = y^o_{M,Par}(t), \tag{36}$$

where $y^o_{M,E}(t)$ is such that $y^o_{M,E}(t) = y^o_M(t-1) = y^o_M(t-2) = y^o_M(t)$.

The convergence of the time-path of solution $y^o_M(t)$ to solution $y^o_{M,E}(t)$, the steady-state constant equilibrium level, depends on the nature of the two characteristic roots λ_1 and λ_2. Each characteristic root must be less than unity in absolute value for the path to converge.

We can state the following theorem:

<u>Theorem</u>*: The equilibrium (36) is asymptotically stable if* $\hat{k} < k < k_{\lambda_2}$ *, where*

$$\hat{k} = 1 - \frac{(c'+b)(c'+2\cdot b)}{d\cdot b\cdot f} \qquad (\hat{k} < 1), \tag{37}$$

$$k_{\lambda_2} = 1 + \frac{(c'+b)(c'+2\cdot b) - b^2\cdot f}{d\cdot b\cdot f} \qquad (k_{\lambda_2} > 1). \tag{38}$$

Proof:

- *For* $k < \tilde{k}$ the homogeneous solution is

$$y^o_{M,Omo}(t) = \rho^t(c_1 \cos t\theta + c_2 \operatorname{sen} t\theta),$$

where ρ is given by:

$\rho = \sqrt{\frac{\alpha^2}{4} + \frac{\left|\alpha^2 + 4\beta\right|}{4}}$, with α, β defined as in (24).

Since $\alpha^2 + 4\beta < 0$ for all $k < \tilde{k}$ then $\rho = \sqrt{\frac{d(1-k)bf}{(c'+b)(c'+2b)}}$ and for all $k > \hat{k}$, $\hat{k}$ defined as in (37), we obtain $\rho < 1$ and then equilibrium is asymptotically stable for all $\hat{k} < k < \tilde{k}$.

- For $k = \tilde{k}$ the homogeneous solution is $y^o_{M,Omo}(t) = c_1\lambda^t + c_2 t\lambda^t$, with $\lambda = \frac{-fb^2}{2(c'+b)(c'+2b)}$. Since $-1 < \lambda < 0$ we obtain that the equilibrium is asymptotically stable for $k = \tilde{k}$.
- For $k > \tilde{k}$ the homogeneous solution is $y^o_{M,Omo}(t) = c_1\lambda_1^t + c_2\lambda_2^t$, with λ_1 and λ_2. defined as in (28). Since $|\lambda_1| < 1$ for $k < k^*$ (k* is defined as in (33)) and $|\lambda_2| < 1$ for $k < k_{\lambda_2}$ (k_{λ_2} defined as in (38)), giving that $k_{\lambda_2} < k^*$ we conclude that the equilibrium is asymptotically stable for $\tilde{k} < k < k_{\lambda_2}$.

We notice that

$$\hat{k} < \tilde{k} < 1 < k_{\lambda_2} < k^*.$$

We can summarize the previous results as follows:

$\hat{k}$ $\tilde{k}$ 1 k_{λ_2} k^*

—+————+—————+—————+————+—

$\Delta < 0$ $\Delta = 0$ $\Delta > 0$

+————————————————+

Asymptotic stablity

with $\tilde{k}, k^*, \hat{k}, k_{\lambda_2}$ defined as in (30), (33), (37) and (38).

5. *CONCLUDING REMARKS AND OPEN ITEMS*

In order to obtain

$$y_M^o(t) \geq 0,\ y_F^o(t) \geq 0 \text{ and } p(t) \geq 0 \quad \forall t \geq t_0 = 0$$

we need to define some criterion able to detect a proper range for *k*.

Moreover, it could be interesting to generalize our model by consider time-varying parameters $f(t)$, $d(t)$ and $k(t)$, instead of constant ones, in order to permit to each economic agent to pursue a dynamical policy.

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