# Local Search Solution of a Museum Advertising Problem via the Analysis of Linear Optimal Control Problems

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**Abstract**

We consider a local search algorithm for the museum visitors flow problem and propose some variants of it in order to reduce its complexity. A new definition of neighbourhood of an admissible solution, permits us to exploit the piecewise linearity of the museum visitors flow problem. Most iterations of the resulting algorithm require to solve finite sets of one–dimensional nonlinear programming problems and each of these problems has, as objective function, the sum of the optimal value functions of two linear optimal control problems. Two first improvement variants of the local search algorithm are also presented.

**Key Words:** local search, optimal control, mathematical programming, museum, advertising

## Introduction

The museum visitors flow problem is the problem of determining the possible optimal advertising policies for a museum institution which seeks the maximum net benefit from a temporary exhibition. Visitors are supposed to spread both positive and negative information about the event, where an unfavorable message is a consequence of observing a congested exhibition. The museum visitors flow problem was analyzed in our previous work (see Funari and Viscolani (1997)). It is formulated as a nonlinear, nondifferentiable optimal control problem and local search techniques are proposed in order to determine possible optimal advertising strategies. The final result is the *cycle based maximum depth algorithm,* a local search algorithm which exploits the special *cycle structure* of the feasible solutions of the museum visitors flow problem. The algorithm involves the solution of a sequence of optimal control subproblems which concern the behaviour of the system in special time subintervals. The result is still unsatisfactory from a computational viewpoint, as the optimal control subproblems are still nondifferentiable.

In this paper we propose a heuristic in order to reduce the complexity of the cycle based maximum depth algorithm. We first define the *cycle preserving* neighbourhood structure on the set of admissible solutions. The neighbourhoods that we consider are proper subsets of the neighbourhoods which are fundamental in the cycle based maximum depth algorithm. Then we analyze the consequences on the corresponding local search algorithm and observe that most iterations of the algorithm require to solve finite sets of one–dimensional nonlinear programming problems. Each nonlinear programming problem has, as objective function, the sum of the optimal value functions of two linear optimal control problems. Finally, we discuss two *first improvement* variants of the local search algorithm in order to further reduce its complexity. The paper is organized as follows. In Section 1 we introduce the museum visitors flow problem and in Section 2 we present the cycle based maximum depth algorithm. In Section 3 we modify the general cycle based algorithm by using a new neighbourhood structure. In Section 4 we discuss the two families of linear optimal control problems which are relevant with reference to the new neighbourhoods. In Section 5 we show how the problem of the cycle preserving local improvement procedure, which is the essential part of the steepest descent algorithm, can be formulated as a one–dimensional nonlinear programming problem. Finally we present in Section 6 two first improvement variants of the local search algorithm.

## 1. The museum visitors flow problem

The museum visitors flow problem ($MVF$) is the following nondifferentiable optimal control problem:

$$
\begin{aligned}
\text{maximize} \quad & J = \int_0^T [\alpha y(t) - k\max\{0, y(t) - \overline{y}\} - v(t)]dt, && (1)\\
\text{subject to} \quad & \dot{x}(t) = y(t)(1 - \mathbb{I}(y(t) - \overline{y})), && (2.1)\\
& \dot{y}(t) = -\gamma \max\{0, y(t) - \overline{y}\} + a_x x(t) - a_z z(t) + bv(t), && (2.2)\\
& \dot{z}(t) = y(t)\mathbb{I}(y(t) - \overline{y}), && (2.3)\\
\text{and to} \quad & x(0) = 0, && (3.1)\\
& y(0) = y_0, \quad y(T) \geq 0, && (3.2)\\
& z(0) = 0, && (3.3)\\
& v(t) \in [0, \overline{v}], && (4)\\
& T \in [0, \overline{T}], && (5)
\end{aligned}
$$

where

$$
\mathbb{I}(y - \overline{y}) = \begin{cases} 0, & \text{if} \quad y < \overline{y}, \\ 1, & \text{if} \quad y \geq \overline{y}, \end{cases} \qquad (6)
$$

and the following conditions are supposed to hold for the parameters:

$$
\begin{aligned}
& 0 < y_0 < \overline{y}, && (7)\\
& \gamma > 0, \quad a_x > 0, \quad a_z > 0, \quad b > 0, \quad \alpha > 0, \quad k \geq 0. && (8)
\end{aligned}
$$

We restrict our attention to the solutions $(x(t), y(t), z(t), v(t), T)$ such that

i) $v(t)$ is piecewise continuous,

ii) $\dot{y}(t^*-) \neq 0$ at all $t^*$ such that $y(t^*) = \overline{y}$,

iii) if $y(t^*) = \overline{y}$ then there exists $\epsilon > 0$ such that

either $y(t) < \overline{y},\ t \in (t^* - \epsilon, t^*)$ and $\dot{y}(t^*+) \geq \dot{y}(t^*-)$,

or $y(t) > \overline{y},\ t \in (t^* - \epsilon, t^*)$, and $\dot{y}(t^*+) \leq \dot{y}(t^*-)$,

iv) the set $\{t \in [0, T] \mid y(t) = \overline{y}\}$ is finite.

Solutions of this kind are, for instance, those which are determined by the constant control functions.

The meaning of the symbols is the following:

$T$, the final time, which is the end time of the exhibition, $0 \leq T \leq \overline{T}$;

$\overline{T}$, the least upper bound of the feasible final times, $\overline{T} > 0$;

$y(t)$, the visitors attendance rate at time $t$;

$\overline{y}$, the congestion threshold, $\overline{y} > 0$;

$x(t)$, the cumulative number of satisfied visitors at time $t$;

$z(t)$, the cumulative number of unsatisfied visitors at time $t$;

$v(t)$, the advertising expenditure rate at time $t$;

$\overline{v}$, the maximum advertising expenditure rate, $\overline{v} > 0$,

$J$, the total benefit of the museum in the interval $[0, T]$.

Let $\xi = (x(t), y(t), z(t), v(t), T)$ be a feasible solution to the museum visitors flow problem, which satisfies the conditions (i)–(iv). We say that the system is in *normal regime* at time $t$, if $y(t) < \overline{y}$, and that it is in *congested regime* at time $t$, if $y(t) \geq \overline{y}$. We define the sequence $t_0, t_1, \ldots, t_n$, of *transition times* associated with $\xi$ as follows:

$$t_0 = 0, \tag{9}$$

for all $i \geq 0$, if $t_i$ has been defined and $t_i < T$, then let

$$Y_i = \begin{cases} \{t \in (t_i, T) \mid y(t) \geq \overline{y}\}, & \text{if } i \text{ is even,} \\ \{t \in (t_i, T) \mid y(t) < \overline{y}\}, & \text{if } i \text{ is odd,} \end{cases} \tag{10}$$

and

$$\begin{cases} t_{i+1} = \inf Y_i, & \text{if } Y_i \neq \emptyset, \\ n = i + 1 \text{ and } t_n = T, & \text{if } Y_i = \emptyset. \end{cases} \tag{11}$$

We call $i$th *epoch* the time interval

$$e_i = [t_{i-1}, t_i], \qquad i = 1, .., n, \tag{12}$$

in which the system is observed staying either in normal regime ($i$ odd, in view of (7)) or in congested regime ($i$ even). We call $i$th *cycle* the time interval

$$c_i = e_i \cup e_{i+1} = [t_{i-1}, t_{i+1}], \qquad i = 1, .., n-1. \tag{13}$$

Moreover, we call $c_i$ a *normal/congested cycle* if $i$ is odd and we call $c_i$ a *congested/normal cycle* if $i$ is even. Finally, we denote also by $e_l(\xi)$ the last epoch of $\xi$, i.e. the epoch with upper bound $T$, and denote by $\mathcal{C}(\xi)$ the set of the cycles of $\xi$.

## 2. The cycle based maximum depth algorithm

The concepts of cycle and epoch of an admissible solution to the museum visitors flow problem and Bellman's optimality principle have been used, in Funari and Viscolani (1997), in order to formulate a special local search algorithm (see Anderson (1996), Papadimitriou and Steiglitz (1982), Pirlot (1996)) for the solution of the museum visitors flow problem. This is the *cycle based maximum depth algorithm,* which is the steepest descent algorithm associated with a special *cycle based* neighbourhood structure (see Verhoeven and Aarts (1995)).

**Definition** Let $\Sigma$ be the set of all admissible solutions to the $MVF$ problem, i.e. the solutions which satisfy the motion equations (2), the assumptions i)–iv) of Section 2, the state conditions (3), the control constraint (4) and the time constraint (5). For each admissible solution $\xi = (x, y, z, v, T) \in \Sigma$, let $F(\xi) = \mathcal{C}(\xi) \cup \{e_l(\xi)\}$ be the set of intervals which are either cycles or the last epoch of $\xi$. We define the *cycle based neighbourhood* $N(\xi)$ of the solution $\xi \in \Sigma$ as the subset of $\Sigma$ such that if $\eta = (x', y', z', v', T') \in N(\xi)$, then $T' \geq t'$ and there exists $I \in F(\xi)$ such that

$$\eta|_{[0,\min\{T,T'\}]\setminus I} \;=\; \xi|_{[0,\min\{T,T'\}]\setminus I},$$

if $\sup I < T$, then $T' = T$.

While executing the algorithm, let $\xi \in \Sigma$ be the current admissible solution and let $I \in F(\xi)$; we call $I$ an *improvable* interval of $\xi$ as long as we do not know whether $\xi$ is optimal among all the solutions $\eta = (x', y', z', v', T') \in N(\xi)$

such that $\eta|_{[0,\min\{T,T'\}]\setminus I} = \xi|_{[0,\min\{T,T'\}]\setminus I}$. We write $\eta \succ \xi$ to state that "$\eta$ is better than $\xi$", i.e. that the value of the objective functional associated to $\eta$ is greater than the value of the objective functional associated to $\xi$.

The iterative process of moving from the current admissible solution to the best solution in its neighbourhood is based on the construction of a list of improvable intervals, which is updated during the executon of the algorithm.

*Cycle based maximum depth algorithm*

Step 0: construct a feasible solution $\xi^0 = (x^0, y^0, z^0, v^0, T^0)$;
$L \leftarrow F(\xi^0)$; (list of improvable intervals)
$n \leftarrow 0$;
Step 1: **while** $L \neq \emptyset$
**do begin**
$\eta^n \leftarrow \xi^n$;
Step 2: **for all** $I \in L$
**do begin**
2.1: let $\eta = (x, y, z, v, T) \in N(\xi^n)$ maximize the functional $J$, subject to $\eta|_{[0,\min\{T^n,T\}]\setminus I} = \xi^n|_{[0,\min\{T^n,T\}]\setminus I}$;
2.2: **if** $\eta \succ \eta^n$
**then** $\eta^n \leftarrow \eta$; $I' \leftarrow I$;
**else** $L \leftarrow L \setminus \{I\}$;
**end**; (Step 2)
Step 3: **if** $\eta^n \succ \xi^n$ (i.e. if $\eta^n$ has been modified)
**then begin**
$\xi^{n+1} \leftarrow \eta^n$;
$L \leftarrow L \setminus A^-(I', \xi^n) \cup A^+(I', \xi^{n+1})$;
$n \leftarrow n + 1$;
**end**; (Step 3)
**end**; (Step 1)

At Step 3 of the algorithm, the set $A^+(I', \xi^{n+1})$ is the set of the new improvable intervals which have to be considered in the next iteration, which is

$$A^+(I',\xi^{n+1}) \;=\; \{I \subseteq [0,T^{n+1}] \mid I \text{ non–final cycle of } \xi^{n+1} \text{ and } I \cap I' \text{ epoch of } \xi^{n+1}\}. \tag{14}$$

the set $A^-(I',\xi^n)$ is the set of the deleted improvable intervals which have not to be considered in the next iteration, which is

$$A^-(I',\xi^n) \;=\; \{I'\} \cup \{I \mid I \text{ cycle of } \xi^n \text{ and } I \cap I' \text{ epoch of } \xi^n\}. \tag{15}$$

We will refer to the couple of Steps 2 and 3 as to the *local improvement procedure*. We observe that Step 2.1 may not be well defined as the optimization problem might not have any optimal solution. Moreover, even if there exists a function $\eta \in N(\xi^n)$ which maximizes the functional $J$, Step 2.1 requires to find optimal solutions to a set of control problems which have the same characteristics of nonlinearity and nondifferentiability as the original problem.

## 3. Cycle preserving neighbourhood structure

In order to reduce the complexity of Step 2 of the cycle based maximum depth algorithm, we define a special subset of the cycle based neighbourhood $N(\xi)$ of a feasible solution $\xi$, in order to restrict the search for better solutions. This is equivalent to define a special first improvement variant of the algorithm. The basic idea is to focus on the feasible solutions which preserve the cycle structure of the interval under consideration.

**Definition** Let the set $\Sigma$ and the mapping $F : \Sigma \to \mathcal{P}(\Sigma)$ be as in the Section 3. We define the *cycle preserving neighbourhood* $N'(\xi)$ of the solution $\xi \in \Sigma$ as the subset of $\Sigma$ such that if $\eta = (x', y', z', v', T') \in N'(\xi)$, then there exists $I \in F(\xi)$ such that

$\eta|_{[0,\min\{T,T'\}]\setminus I} \;=\; \xi|_{[0,\min\{T,T'\}]\setminus I}$,
if $\sup I < T$, then $T' = T$
and $\eta$ has at most one transition time internal to $I$.

Of course, now we call $I \in F(\xi)$ an *improvable* interval of $\xi$ as long as we do not know whether $\xi$ is optimal among all the solutions $\eta \in N'(\xi)$ such that $\eta|_{[0,\min\{T,T'\}]\setminus I} \;=\; \xi|_{[0,\min\{T,T'\}]\setminus I}$. In terms of the general algorithm we have to redefine Step 2.1 in agreement with the following reasoning.

If $I = [t', t''] = [t', T^n]$, i.e. if $\sup I = T^n$, then $I$ is either the final cycle or the final epoch of $\xi^n$ and in that case the problem of the new Step 2.1 is the same as in the general algorithm. More explicitly, if $t'' = T^n$, then the problem of Step 2.1 is the following subproblem on the interval $[t', \overline{T}]$, which we denote by $MVF_{\xi^n,[t',\overline{T}]}$:

$$\begin{aligned}
\text{maximize} \quad & \int_{t'}^{\tau} [\alpha y(t) - k \max\{0, y(t) - \overline{y}\} - v(t)]dt, && (16)\\
\text{subject to} \quad & (2.1),\ (2.2),\ (2.3),\ (4), \\
\text{and to} \quad & (x, y, z)(t') = (x, y, z)^n(t'), && (17)\\
& \tau \in [t', \overline{T}]. && (18)
\end{aligned}$$

We observe that the solution to the problem $MVF_{\xi^n,[t',\overline{T}]}$ may as well present one cycle in the interval $[t', \tau]$ as none or more than one.

On the contrary, if $I = [t', t''] \neq [t', T^n]$, i.e. if $\sup I < T^n$, then $I$ is a cycle of $\xi^n$ (not the final one) and the problem of Step 2.1 is changed into the following *cycle preserving subproblem* on the cycle $I$, which we denote by $MVF^{cp}_{\xi^n,I}$:

$$\begin{aligned}
\text{maximize} \quad & \int_{t'}^{t''} [\alpha y(t) - k \max\{0, y(t) - \overline{y}\} - v(t)]dt, && (19)\\
\text{subject to} \quad & (2.1),\ (2.2),\ (2.3),\ (4),\ (17), \\
\text{to} \quad & (x, y, z)(t'') = (x, y, z)^n(t''), && (20)
\end{aligned}$$

and to the further constraint that there exists $t^* \in I = [t', t'']$ such that

i) if $I$ is a normal/congested cycle then

$$0 < y(t) < \overline{y}, \quad \text{for } t' < t < t^*, \qquad \text{and} \qquad y(t) \geq \overline{y}, \text{ for } t \geq t^*; \qquad (21)$$

ii) if $I$ is a congested/normal cycle then

$$y(t) \geq \overline{y}, \quad \text{for } t \leq t^*, \qquad \text{and} \qquad 0 < y(t) < \overline{y}, \quad \text{for } t^* < t < t''. \qquad (22)$$

Hence, in place of Step 2.1 of the cycle based maximum depth algorithm we have to use the following:

Step 2.1$'$: **if** $I = [t', t'']$ and $t'' = T^n$

**then** let $\eta_I$ be an optimal solution to problem $MVF_{\xi^n,[t',\overline{T}]}$ and let $T'$ be the final time of $\eta_I$ ($T' \in [t', \overline{T}]$)

**else** let $\eta_I$ be an optimal solution to problem $MVF^{cp}_{\xi^n,I}$ and let $T'$ be the final time of $\eta_I$ ($T' = t''$);

$$\eta(t) = \xi^n(t), \quad t \in [0, t'] \cup [t'', T^n];$$
$$\eta(t) = \eta_I(t), \quad t \in [t', T'];$$

After executing the local improvement procedure for a cycle $I$ of $\xi^n$, with $\sup I < T^n$, the number of transition times associated to the improved admissible solution $\xi^{n+1}$ is the same as that associated to the previous admissible solution $\xi^n$.

We observe also that the solution $\eta$ which is found after executing Step 2.1$'$ might not satisfy the conditions i)–iv) of Section 1.

## 4. Constant regime subproblems

Each admissible solution to the museum visitors flow problem satisfies linear motion equations in all time intervals which are subsets of epochs of the admissible solution. Furthermore, also the objective functional is linear in the same intervals. Then the restriction of an optimal solution to the $MVF$ problem in a time interval, which is a subset of an epoch of the solution, is an optimal solution to a special linear control problem. Therefore we are led to define two special interval subproblems.

Let the time $t' \in [0, \overline{T})$ and the interval $I'' \subseteq (t', \overline{T}]$ be given, and let us consider the restrictions of the $MVF$ problem to the intervals $[t', t'']$, $t'' \in I''$, under the constraint that the system regime does not change.

First, let $x', z' \in [0, +\infty)$, $X'' \subseteq (x', +\infty)$, $y' \in (0, \overline{y}]$ and $Y'' \subseteq [0, \overline{y}]$ be given and let the required regime be normal, then we define the *normal regime subproblem* $P^N(t', I'', x', y', z', X'', Y'')$:

$$\text{maximize} \quad \int_{t'}^{t''} [\alpha y(t) - v(t)]dt, \tag{23}$$

$$\text{subject to} \quad \dot{x}(t) = y(t), \tag{24}$$

$$\dot{y}(t) = a_x x(t) - a_z z' + bv(t), \tag{25.1}$$

$$0 < y(t) < \overline{y}, \quad \text{for } t' < t < t'', \tag{25.2}$$

$$\text{and to} \quad x(t') = x', \quad x(t'') \in X'', \tag{26.1}$$

$$y(t') = y', \quad y(t'') \in Y'', \tag{26.2}$$

$$v(t) \in [0, \overline{v}], \quad t'' \in I''. \tag{27}$$

Alternatively, let $x', z' \in [0, +\infty)$, $y' \in [\overline{y}, +\infty)$, $Y'' \subseteq [\overline{y}, +\infty)$, and $Z'' \subseteq (z', +\infty)$ be given and let the required regime be congested, then we define the *congested regime subproblem* $P^C(t', I'', x', y', z', Y'', Z'')$:

$$\text{maximize} \quad \int_{t'}^{t''} [\alpha y(t) - k(y(t) - \overline{y}) - v(t)]dt, \tag{28}$$

$$\text{subject to} \quad \dot{y}(t) = -\gamma(y(t) - \overline{y}) + a_x x' - a_z z(t) + bv(t), \tag{29.1}$$

$$y(t) \geq \overline{y}, \tag{29.2}$$

$$\dot{z}(t) = y(t), \tag{30}$$

$$z(t') = z', \quad z(t'') \in Z'', \tag{31}$$

and to (26.2) and (27) as above.

Both problems $P^N$ and $P^C$ are linear optimal control problems and the conditions of Pontryagin's maximum principle are also sufficient for optimality, as a consequence of Mangasarian's sufficiency theorem (see Seierstad and Sydsaeter (1987)). We denote by $F^N(t', I'', x', y', z', X'', Y'')$ and $F^C(t', I'', x', y', z', Y'', Z'')$ the optimal values of the objective functionals of the normal regime and congested regime subproblems. The following two theorems state some interesting qualitative results concerning the possible optimal solutions to problems $P^N$ and $P^C$. The proofs of the theorems are presented in the Appendix, together with the Pontryagin's maximum principle necessary conditions.

**Theorem 1** If there exists an admissible solution $(x(t), y(t), v(t))$ to problem $P^N$, then $P^N$ has an optimal control function which takes only the values 0 and $\overline{v}$. Moreover such control function has at most 2 discontinuities.

**Theorem 2** If there exists an admissible solution $(y(t), z(t), v(t))$ to problem $P^C$, then $P^C$ has an optimal piecewise constant control function, which takes only the values 0 and $\overline{v}$. Moreover, if $\gamma^2 > 4a_z$, then such control function has at most 2 discontinuities.

## 5. Cycle preserving subproblems

Here we consider the cycle preserving subproblem $MVF^{cp}_{\xi^n,I}$, which has been introduced in Section 3, i.e. the problem of maximizing the functional $J$ restricted to the cycle $I$ and subject to the additional constraint that the cycle structure is preserved. In particular we discuss how the cycle preserving subproblem can be reformulated as a one–dimensional nonlinear programming problem, once we assume to know the optimal value functions of the linear optimal control problems $P^N$ and $P^C$.

Let $I = [t', t''] \in F(\xi)$, with $t'' < T$, be an arbitrary nonfinal cycle of the current solution $\xi = (x, y, z, v, T)$, let $t^* \in I$ be the (internal) transition time of the optimal solution to the cycle preserving subproblem $MVF^{cp}_{\xi,I}$ and let $(x^*, y^*, z^*)(t^*)$ be the state at time $t^*$ in the same optimal solution.

If $I$ is a normal/congested cycle, then $(x^*, y^*, z^*)(t^*) = (x^*(t''), \overline{y}, z^*(t'))$ is the state at time $t^*$ and $t^*$ is an optimal solution of the following nonlinear programming problem:

$$\begin{aligned} \text{maximize} \quad & L^N(t) + R^C(t), \\ \text{subject to:} \quad & t' \leq t \leq t'', \end{aligned}$$

where

$$\begin{aligned} L^N(t) &= F^N(t', t, x(t'), y(t'), z(t'), x(t''), \overline{y}), \\ R^C(t) &= F^C(t, t'', x(t''), \overline{y}, z(t'), \overline{y}, z(t'')). \end{aligned}$$

We observe that in $F^N$ the value $y(t')$ can be substituted by $\overline{y}$ for all cycles $I = [t', t'']$ with the exception of the first one (i.e. when $t' = 0$).

The result follows from Bellman's optimality principle, which states that, given the value $(x^*(t^*), \overline{y}, z^*(t^*))$ of the state at time $t^*$, both the portions of an optimal path in the intervals $[t', t^*]$ and $[t^*, t'']$ are optimal. The same argument proves the following proposition in the alternative case of a congested/normal cycle, when the state $(x^*(t'), \overline{y}, z^*(t''))$ at time $t^*$ is given.

If $I$ is a congested/normal cycle, then $(x^*, y^*, z^*)(t^*) = (x^*(t'), \overline{y}, z^*(t''))$ is the state at time $t^*$ and $t^*$ is an optimal solution of the following nonlinear programming problem:

$$\begin{aligned} \text{maximize} \quad & L^C(t) + R^N(t), \\ \text{subject to:} \quad & t' \le t \le t'', \end{aligned}$$

where

$$\begin{aligned} L^C(t) &= F^C(t', t, x(t'), \overline{y}, z(t'), \overline{y}, z(t'')), \\ R^N(t) &= F^N(t, t'', x(t'), \overline{y}, z(t''), x(t''), \overline{y}). \end{aligned}$$

## 6. First improvement rules

Instead of using the *best improvement pivoting rule*, which requires to move at each step from the current solution $\xi$ to the best solution in its cycle based neighbourhood $N(\xi)$, we may use a *first improvement pivoting rule* and move to one solution in $N(\xi)$ which presents an improved value of the objective functional, which is not necessarily the best one. Using a first improvement pivoting rule with reference to the cycle based neighbourhood structure is equivalent to using the best improvement pivoting rule with reference to a suitably defined neighbourhood structure whose elements are subsets of the neighbourhoods $N(\xi)$, $\xi \in \Sigma$. We present here two modified versions of the general algorithm which implement first improvement pivoting rules. In both algorithms, a crucial role is played by the list of improvable intervals of the current solution.

*First improvement rule – Version I: "Move to the best solution with reference to a minimum subset of improvable intervals"*

Step 0: construct $\xi^0 = (x^0, y^0, z^0, v^0, T^0) \in \Sigma$; $L \leftarrow F(\xi^0)$; $n \leftarrow 0$;
Step 1: **while** $L \neq \emptyset$
        **do begin**
            improved $\leftarrow$ false;
Step 2:     **while** (not improved and $L \neq \emptyset$)

**do begin**

choose an interval $I = [t', t''] \in L$;

**if** $t'' = T^n$

**then** let $\eta_I$ be an optimal solution to $MVF_{\xi^n,[t',\overline{T}]}$

and let $T'$ be the final time of $\eta_I$ $(T' \in [t', \overline{T}])$;

**else** let $\eta_I$ be an optimal solution to $MVF^{cp}_{\xi^n,I}$

and let $T'$ be the final time of $\eta_I$ $(T' = t'')$;

**if** $\eta_I \succ \xi^n|_I$

**then begin**

$\eta(t) = \xi^n(t), \quad t \in [0, t'] \cup [t'', T^n]$;

$\eta(t) = \eta_I(t), \quad t \in [t', T']$;

improved $\leftarrow$ true;

**end**;

**else** $L \leftarrow L \setminus \{I\}$;

**end;** (Step 2)

Step 3: **if** improved

**then begin**

$\xi^{n+1} \leftarrow \eta$;

$L \leftarrow L \setminus A^-(I', \xi^n) \cup A^+(I', \xi^{n+1})$;

$n \leftarrow n + 1$;

**end**; (Step 3)

**end**; (Step 1)

In the above algorithm, the list of improvable intervals of the current solution is explored until the best solution with reference to an improvable interval results better than the current solution.

In the following algorithm, the list of improvable intervals of the current solution is explored until an admissible solution is found, with reference to an improvable interval, which is better than the current solution.

*First improvement rule – Version II: "Move to a better solution with reference to a minimum subset of improvable intervals"*

Step 0: construct $\xi^0 = (x^0, y^0, z^0, v^0, T^0) \in \Sigma$; $L \leftarrow F(\xi^0)$; $n \leftarrow 0$;

Step 1: **while** $L \neq \emptyset$

**do begin**

improved $\leftarrow$ false;

Step 2: **while** (not improved and $L \neq \emptyset$)

**do begin**

choose an interval $I = [t', t''] \in L$;

**if** $t'' = T^n$

**then** $P \leftarrow MVF_{\xi^n,[t',\overline{T}]}$;

**else** $P \leftarrow MVF^{cp}_{\xi^n,I}$;

**if** there exists an admissible solution $\eta_I$ to $P$ such that $\eta_I \succ \xi^n|I$

**then begin**

let $T'$ be the final time of $\eta_I$;

$\eta(t) = \xi^n(t), \quad t \in [0, t'] \cup [t'', T^n]$;

$\eta(t) = \eta_I(t), \quad t \in [t', T']$;

improved $\leftarrow$ true;

**end**;

**else** $L \leftarrow L \setminus \{I\}$;

**end;** (Step 2)

Step 3: **if** improved

**then begin**

$\xi^{n+1} \leftarrow \eta$;

$L \leftarrow L \setminus A^-(I', \xi^n) \cup A^+(I', \xi^{n+1})$;

$n \leftarrow n + 1$;

**end**; (Step 3)

**end**; (Step 1)

It is important to notice that the way of choosing the interval $I$ within the list $L$, at Step 2 in both algorithms, may affect the final solution to the museum visitors flow problem. Then we should analyze different choice strategies in order to single out a strategy which is not restrictive unnecessarily. A special attention should be devoted to random choice strategies, as pointed out in Anderson (1996).

In order to provide some elementary examples of a choice rule, let us order the intervals of $L \neq \emptyset$ according to the order $\leq$ on the set of their infimums, i.e. let us say "$I$ precedes $I'$" if and only if $\inf I \;\leq\; \inf I'$. In fact the relation "precedes" is an order on $L$ and $L$ is a finite set, therefore there exist a first element and a last element of $L$.

Three deterministic choice rules are:

a) choose the first element of $L$;

b) choose the last element of $L$;

c) choose the last cycle of $L$, if $L$ has at least one cycle, otherwise choose the last interval of $L$.

On the other hand, if $\{X_n\}_{n\geq 1}$ is a sequence of random variables, where $X_n$ takes values in $\{1, 2, ..., n\}$ with a fixed probability distribution, then a random choice rule is given by:

d) if $|L| = n$, then choose the $X_n$–th element of $L$.

## 7. Conclusions

In this paper we have developed a local search algorithm in order to find an optimal solution to the museum visitors flow problem. The key issue of our analysis has been a new definition of neighbourhood of an admissible solution, which permits us to restrict the attention to a special subset of feasible solutions. As a consequence of the neighbourhood definition we have obtained some local search algorithms which exploit the feature of the museum visitors flow problem of being piecewise linear. Then the improvement procedure of the best–improvement algorithm has been formulated as a nonlinear programming problem, whose definition requires to know the optimal solutions of two linear control problems.

We have also proposed two first–improvement versions of the previous algorithm. In particular the improvement procedure of one of this requires to find an admissible solution to the nonlinear programming problem instead of an optimal one.

In both first–improvement versions of the algorithm the way of choosing an object from the list of the improvable intervals has an important role in determining the final solution. Here we have suggested some different choice strategies and further analysis is needed to compare their efficiency.

Another open question concerns the initialization process. It seems reasonable to presume that the piecewise linearity property of the problem may be used also to this purpose.

## 8. Appendix

*Proof of Theorem 1*

Let us consider the normal regime subproblem with fixed end time, i.e. the problem $P^N(t', t'', x', y', z', x'', y'')$.

The existence of an optimal solution $(x^*(t), y^*(t), v^*(t))$ to problem $P^N$ follows from the Filippov–Cesari theorem (see Seierstad and Sydsaeter (1987), p.132–133).

From the Pontryagin Maximum Principle we have the following necessary conditions for optimality, which do not take into account the pure constraint (25.2) on the state variable $y$ . The Hamiltonian function is

$$H(x, y, p_0, p_1, p_2, v, t) \;=\; p_0(\alpha y - v) + p_1 y + p_2(a_x x - a_z z' + bv).$$

If $(x^*, y^*, v^*)$ is an optimal solution to problem $P^N$, then there exists a constant $p_0 \in \{0, 1\}$ and a continuous and piecewise continuously differentiable function $(p_1(t), p_2(t))$, such that, after defining $p(t) = (p_0, p_1(t), p_2(t))$, the following conditions hold:

i)

$$p(t) \neq 0, \qquad \text{for all} \;\; t \in [t', t''], \tag{32}$$

ii)

$$\dot{p}(t) = -p(t)A, \quad v.e. \tag{33}$$

where

$$A = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 1 \\ 0 & a_x & 0 \end{pmatrix}, \tag{34}$$

iii) $v^*(t)$ maximizes $H(x(t), y(t), p(t), v, t)$ subject to $v \in [0, \overline{v}]$, that is the optimal control satisfies the condition

$$v^*(t) = \begin{cases} 0, & \text{if} \quad \sigma(t) < 0, \\ \overline{v}, & \text{if} \quad \sigma(t) > 0, \end{cases} \tag{35}$$

where the switching function $\sigma(t)$ is given by

$$\sigma(t) = p(t)B, \tag{36}$$

with $B = (-1, 0, b)^T$.

Let us assume that $\sigma(t) = 0$, for all $t \in [\tau_1, \tau_2]$, for some nondegenerate interval $[\tau_1, \tau_2] \subseteq [t', t'']$. Then we obtain the following identity

$$(p(t), \dot{p}(t), \ddot{p}(t))B = 0, \quad t \in [\tau_1, \tau_2], \tag{37}$$

which can be written, by using (33), as follows

$$p(t)[B, AB, A^2B] = 0, \quad t \in [\tau_1, \tau_2]. \tag{38}$$

Equation (38) has only the trivial solution $p(t) = 0$, $t \in [\tau_1, \tau_2]$, because the vectors $B$, $AB$ and $A^2B$ are linearly independent, and this contradicts condition (32).

Finally the number of discontinuities of an optimal control $v^*(t)$ is less than or equal to the number of solutions to equation $\sigma(t) = p(t)B = 0$. Now, $p(t)$ satisfies the adjoint equation (33) and the matrix $A$, as defined in (34), has three real and distinct eigenvalues, $\lambda_0 = 0$, $\lambda_1 = \sqrt{a_x}$, $\lambda_2 = -\sqrt{a_x}$; hence the switching function $\sigma(t)$ has the form

$$\sigma(t) = c_0 + c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \tag{39}$$

and, using the same argument as in Leitmann (1981), p.216, we conclude that it has at most 2 zeros. ■

*Proof of Theorem 2*

Let us consider the congested regime subproblem with fixed end time, i.e. the problem $P^C(t', t'', x', y', z', y'', z'')$.

The existence of an optimal solution $(y^*(t), z^*(t), v^*(t))$ to problem $P^C$ follows from the Filippov–Cesari theorem. From the Pontryagin Maximum Principle we have the following necessary conditions for optimality, which do not consider the pure state constraint (29.2) on $y(t)$. Let the Hamiltonian function be

$$H(y,z,p_0,p_2,p_3,v,t) \;= p_0[(\alpha-k)y+k\overline{y}-v] \;+\; + p_2[-\gamma(y-\overline{y})+ \\ + a_x x' - a_z z + bv] + \; p_3 y \ .$$

If $(y^*, z^*, v^*)$ is an optimal solution to problem $P^C$, then there exists a constant $p_0 \in \{0,1\}$ and a continuous and piecewise continuously differentiable function $(p_2(t), p_3(t))$, such that, after defining $p(t) = (p_0, p_2(t), p_3(t))$, the conditions (32), (33) hold, with the matrix $A$ given by

$$A = \begin{pmatrix} 0 & \alpha-k & 0 \\ 0 & -\gamma & -a_z \\ 0 & 1 & 0 \end{pmatrix}. \tag{40}$$

Moreover the optimal control function maximizes $H(y(t), z(t), p(t), v, t)$ subject to $v \in [0, \overline{v}]$, that is $v^*(t)$ satisfies the condition (35), where the switching function $\sigma(t)$ is given by (36) and the vector $B$ is given by

$$B = \begin{pmatrix} -1 \\ b \\ 0 \end{pmatrix}. \tag{41}$$

Finally, we observe that the optimal control function $v^*(t)$ has at most 2 discontinuities, because, if $\gamma^2 > 4a_z$, then the matrix $A$, as defined in (40), has 3 real eigenvalues: $\lambda_0 = 0$, $\lambda_1 = (-\gamma + \sqrt{\gamma^2 - 4a_z})/2$, $\lambda_2 = (-\gamma - \sqrt{\gamma^2 - 4a_z})/2$, where $\lambda_1 \neq \lambda_2$. Hence the switching function $\sigma(t)$ has the same form as in (39) and we conclude that it has at most 2 zeros. ∎

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