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ELECTROMAGNETIC FIELDS

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Abstract–Qualitative behaviour of time average power flow in electromagnetic fields can be studied by observing the critical points of the Poynting vector field, **S**. In order to analyze the behaviour of the flow lines of a plane Poynting vector field in the neighbourhood of a critical point, the **S** field is expanded in a Taylor series. Using this expansion, critical points can be classified according to their order and degeneracy. A formula for the index of rotation of the **S** field at a critical point is derived. The behaviour of the transverse electric or magnetic field component in the neighbourhood of the critical point is also studied. Lowest order critical points are always nondegenerate and they have interesting properties with regards to polarization and energy distribution. Examples involving linearly polarized system of interfering plane and/or cylindrical waves are given to show the critical points. The behaviour of flow lines is illustrated in these examples.

1. INTRODUCTION

Despite the controversy about its interpretation, the Poynting vector has been successfully used to describe the flow of electromagnetic power. Intuitively the power flow is visualized just as a fluid flow. However for fluid flow a differential equation can be directly written down but in the case of electromagnetic power flow the situation is different. Maxwell's equations have to be solved for the electric and magnetic fields and then Poynting vector is calculated, which is a quadratic quantity. Study of power flow in an electromagnetic problem is really the study of Poynting vector fields. These fields exhibit some interesting structure. A very interesting example is given in [1] where electromagnetic power flows around a diffracting half plane making vortices in front of the half plane. What other structures are possible as electromagnetic power flows through space? In this paper an attempt will be made to classify the structures that are present in a source free Poynting vector field. A qualitative study of Poynting vector fields can help in understanding the power flow in electromagnetic fields without completely solving for electric and magnetic fields at all points in the region of interest.

2. FLOW LINES AND CRITICAL POINTS

Throughout this paper only time harmonic electromagnetic fields will be considered. The time dependence $\exp(-i\omega t)$ will be assumed for the field quantities $\mathbf{E}$ and $\mathbf{H}$. This factor will be suppressed. The term Poynting vector will be used to imply the time average Poynting vector which is defined as

$$\mathbf{S}(\mathbf{r}) = \frac{1}{2}\Re\{\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})\}, \tag{1}$$

where $\mathbf{E}$ is phasor electric field $\mathbf{H}^*$ is complex conjugate of the phasor magnetic field and $\Re$ denotes the real part of the expression. In this paper attention will be focused on regions of space which do not contain sources or sinks of electromagnetic power. In most of the practical problems the generation or loss of power occurs only in bounded regions of space. The bounded regions containing sources or sinks can be eliminated from consideration without great loss of generality. The Poynting's theorem for time average Poynting vector field in regions devoid of sources or sinks is given as

$$\oint_S \mathbf{S} \cdot \mathbf{e}_n da = 0. \tag{2}$$

This may be stated in differential form as $\nabla \cdot \mathbf{S} = 0$. The study of time average flow of electromagnetic energy is now reduced to the study of lines of **S**, which may also be called flow lines. Due to the restrictions imposed above, **S** is a solenoidal vector field and its flow lines are continuous in the regions of interest.

In principle, the lines of flow can be obtained by solving the following differential equation

$$\frac{dx}{S_x} = \frac{dy}{S_y} = \frac{dz}{S_z}, \tag{3}$$

where S_x, S_y, and S_z are cartesian components of **S**. Solution of (3) is by no means a trivial matter. In a lot of cases the qualitative behaviour of flow lines alone provides sufficient amount of information. In order to facilitate any further discussion on the subject a number of terms will have to be defined first. Consider a point P and its neighbourhood where $\mathbf{S} \neq 0$. The direction cosines of the flow line through P are well defined, hence a unique line segment can be drawn through P. Such a point is called an *ordinary point.* On the other hand if there exists a point P such that the Poynting vector at P is zero and does not vanish in the neighbourhood of P then such a point is called an *isolated critical point* of the Poynting vector field. It is the presence of critical points which drastically modifies the behaviour of flow lines. It is evident that the direction cosines of a line through P cannot be uniquely defined. Therefore a line of flow through a critical point, if indeed there is any, may not unique. Since the critical points play a major role in determining the pattern of flow lines, an attempt will be made in this paper to understand the behaviour of the Poynting vector field in the neighbourhood of its critical points.

3. PLANE POYNTING VECTOR FIELDS

In general the Poynting vector field is three dimensional and its flow lines are space curves. It is difficult to visualize, represent and study these space curves in much detail except in a few simple cases. For this reason the problem will be simplified by assuming that the electric and the magnetic fields are two dimensional. By two dimensional it is meant that the field vectors are independent of z-coordinate in a cartesian coordinate system. It can be easily shown that two dimensional electromagnetic fields are completely specified in terms of z-directed components of $\mathbf{E}$ and $\mathbf{H}$. Specifically, the total electromagnetic field can be written down as

$$\mathbf{E} = E_z\mathbf{e}_z + \frac{i}{\omega\epsilon}\nabla H_z \times \mathbf{e}_z, \tag{4a}$$

$$\mathbf{H} = H_z\mathbf{e}_z - \frac{i}{\omega\mu}\nabla E_z \times \mathbf{e}_z, \tag{4b}$$

where E_z and H_z are functions of x and y only. The Poynting vector for this field can be expressed using (1) as

$$\begin{aligned}\mathbf{S} &= \frac{1}{2\omega\mu}\Re\{iE_z\nabla E_z^*\} + \frac{1}{2\omega\epsilon}\Re\{iH_z\nabla H_z^*\} + \frac{1}{\omega^2\mu\epsilon}\Re\{\nabla E_z^* \times \nabla H_z\} \\ &= \mathbf{S}_e + \mathbf{S}_h + \mathbf{S}_{eh}.\end{aligned} \tag{5}$$

It follows from (5) that the partial vector fields $\mathbf{S}_e$ and $\mathbf{S}_h$ are entirely transverse to $\mathbf{e}_z$ while $\mathbf{S}_{eh}$ is parallel to $\mathbf{e}_z$. Another simplifying assumption will be made at this point to make sure that the Poynting vector field is plane field. It will be assumed that the partial Poynting vector field $\mathbf{S}_{eh}$ of an electromagnetic field under consideration is identically zero. The flow lines of such plane fields are directed plane curves which can be easily visualized and sketched on paper. It is also observed from (5) that the components $\mathbf{S}_e$ and $\mathbf{S}_h$ are completely specified by E_z and H_z respectively. Therefore the electromagnetic field can be partitioned in an E-polarized field and an H-polarized field. The E-polarized field is given as

$$\mathbf{E}_1 = E_z\mathbf{e}_z, \tag{6a}$$

$$\mathbf{H}_1 = \frac{-i}{\omega\mu}\nabla E_z \times \mathbf{e}_z, \tag{6b}$$

while the H-polarized field is given as

$$\mathbf{H}_2 = H_z\mathbf{e}_z, \tag{7a}$$

$$\mathbf{E}_2 = \frac{i}{\omega\epsilon}\nabla H_z \times \mathbf{e}_z. \tag{7b}$$

The Poynting vectors of E-polarized and H-polarized fields are $\mathbf{S}_e$ and $\mathbf{S}_h$ respectively. The total power flux density is just the sum of the power flux densities of both the polarizations for the case of plane Poynting vector fields.

In this paper the critical points of the Poynting vector fields $\mathbf{S}_e$ generated by E-polarized fields given by (6) will be studied. The case for H-polarized fields given by (7), which generate the Poynting vector field $\mathbf{S}_h$ is analogous. This is not a seriously limiting restriction. There is a whole class of problems in the electromagnetic field theory in which the fields are either E-polarized of H-polarized. For example scattering by cylindrical objects or by long edges is solved separately for each polarization. when an electromagnetic field is a sum of fields due to both the polarization then the critical points of the total Poynting vector field do not have a simple correspondence with the critical points of partial fields $\mathbf{S}_e$ and $\mathbf{S}_h$. It is clear from (5) that the total Poynting vector field will also have a critical point where both $\mathbf{S}_e$ and $\mathbf{S}_h$ have a critical point. There may be additional critical points at isolated points where $\mathbf{S}_e$ and $\mathbf{S}_h$ cancel each other.

A plane Poynting vector field is also obtained if the flow lines of $\mathbf{S}_e + \mathbf{S}_h$ are parallel straight lines and the partial field $\mathbf{S}_{eh}$ does not vanish identically. In this case a cartesian coordinate system can be chosen such that the total Poynting vector field has the following representation,

$$\mathbf{S} = S_y(x)\mathbf{e}_y + S_z(x,y)\mathbf{e}_z. \tag{8}$$

It is immediately observed that the points where $\mathbf{S}$ field vanishes are not isolated in the y-z plane. Hence such plane Poynting vector fields do not possess isolated critical points and therefore these fields will not be considered any further.

Consider an E-polarized electromagnetic field. This field is completely described in terms of a complex scalar function of two variables namely the coordinates x and y. This function can chosen as z directed component of the electric field. Let this function be expressed in terms of its real and imaginary parts or in terms of its amplitude and phase, i.e.,

$$E_z(x, y) = R(x, y) + iI(x, y) = A(x, y)\exp\{i\phi(x, y)\}. \tag{9}$$

The function R, I, A, and ϕ are real valued functions. In physically realizable problems the electric field is well behaved. Therefore it is reasonable to assume that the functions R, I, and A are continuous and differentiable. It should be noted that at the points where the amplitude A goes to zero the phase ϕ may become ambiguous. At such points the representation of the electric field in terms of its real and imaginary parts is unique and hence more convenient. The Poynting vector of this field is calculated using (1) and (6) as

$$\mathbf{S} = \frac{1}{2\omega\mu}(R\nabla I - I\nabla R) = \frac{1}{2\omega\mu}A^2\nabla\phi. \tag{10}$$

4. INDEX OF ROTATION

An important property of the critical points of plane vector fields is their index of rotation. The index of rotation γ is defined as follows. Let there be a closed curve C which encloses only one critical point and there are no critical points on C itself. When the curve C is traversed once in the positive direction, i.e., counterclockwise, the angle between the vector at the moving point on C and a fixed direction changes. If the total change in this angle is $2n\pi$ then the index of rotation of the critical point is defined to be n. It can easily shown that n can only take integral values. For example the two vector fields shown in Figures 1(a) and 1(b) have indices of rotation, $\gamma = 1$ and $\gamma = -2$ respectively. The dots denote the location of the critical points in this figure and subsequent figures.

It should be noted that the index γ is independent of the choice of curve C. The index of a critical point can be measured using the geometrical method employed for its definition or it can be calculated analytically with the help of Poincaré's formula,

$$\gamma = \frac{1}{2\pi}\oint_C \frac{S_x dS_y - S_y dS_x}{S_x^2 + S_y^2}, \tag{11}$$

where S_x and S_y are cartesian components of vector field $\mathbf{S}(\mathbf{r})$. In most of the cases the integration indicated in (11) is very complicated.

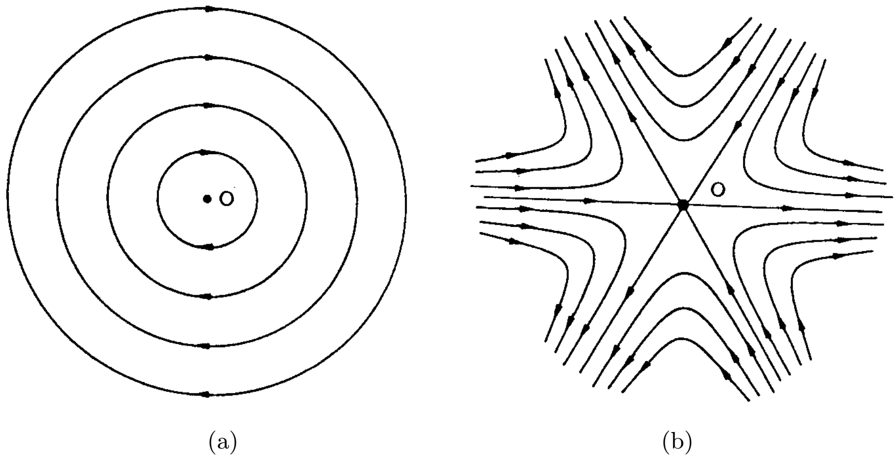


Figure 1. Critical point is located at O. (a) The index of rotation is 1. (b) The index of rotation is -2.

In such cases it is advantageous to use the geometric method. Index of rotation is a very important concept. A number of properties of the plane vector fields can be derived with the help of this concept. Two important theorems on this subject will now be stated without proof. These theorems will be utilized later on in this paper. For a proof of these theorems and a comprehensive discussion on this topic the reader is referred to the excellent work of Krasnoseĺskiy et al. [2].

Theorem 1. *If a closed curve C has a finite number of critical points in its interior each with its own index of rotation γ_i then the total index of rotation γ on C is the sum of the individual indices of rotation of the critical points, i.e.,*

$$\gamma = \sum_i \gamma_i. \tag{12}$$

This theorem provides the means to calculate the index of rotation on any given curve without critical points.

Theorem 2. *Two vector fields have an identical index of rotation on a closed curve C if and only if they are homotopic to each other.*

A vector field $\mathbf{S}_p$ is homotopic to another vector field $\mathbf{S}_q$ on a closed curve if the field $\mathbf{S}_p$ can be continuously transformed into the field $\mathbf{S}_q$ without going through a null vector, i.e., a critical point. If a vector field **S** can be partitioned as

$$\mathbf{S} = \mathbf{S}_p + \mathbf{S}_q \tag{13a}$$

such that

$$|\mathbf{S}_p| > |\mathbf{S}_q|. \tag{13b}$$

then the component vector field $\mathbf{S}_p$ is called the principal part of **S** field. Krasnoselśkiy et al. [2] go on to prove that a vector field is homotopic to its principal part. This theorem provides an easy method to calculate the index of rotation if a simple principal part can be extracted from the total field. This method will be utilized in Section 7 to calculate the index of rotation of a critical point.

5. TAYLOR SERIES EXPANSION AT CRITICAL POINTS

In regions of space devoid of charges or currents the electric field satisfies homogeneous Helmholtz's equation. Therefore the real and imaginary parts of E_z also satisfy homogeneous Helmholtz's equation, i.e.,

$$\nabla^2 R + k^2 R = 0 \tag{14}$$

$$\nabla^2 I + k^2 I = 0, \tag{15}$$

where $k = \omega\sqrt{\mu\epsilon}$. The behaviour of the Poynting vector at any particular point can now be studied. For this purpose let the origin of the coordinate system be placed at the point of interest. Circular cylindrical coordinate system with radial coordinate ρ and angular coordinate θ has been employed here. This choice facilitates the ensuing analysis. The solutions to the above differential Equations (14) and (15) can be expanded in a series of eigenfunctions which are valid on a disk $\rho < \rho_0$. It is well known that such eigenfunction expansion for Helmholtz's equation in two dimensions are given as

$$R(\rho, \theta) = \sum_{j=0}^{\infty} A_j J_j(k\rho)\cos(j\theta + \alpha_j) \tag{16}$$

and

$$I(\rho, \theta) = \sum_{l=0}^{\infty} B_l J_l(k\rho)\cos(l\theta + \beta_l), \tag{17}$$

where J_j and J_l are Bessel functions of order j and l respectively. A_j, B_l, α_j, and β_l are constants of the eigenfunction expansion. These constants can be evaluated for a given problem. Here they are completely arbitrary.

The Poynting vector can now be calculated using (10) and it is given as

$$\mathbf{S} = \frac{1}{2\eta}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} A_j B_l \Bigg\{ \cos(j\theta+\alpha_j)\cos(l\theta+\beta_l)$$
$$\left\{J_j(k\rho)J_l'(k\rho) - J_l(k\rho)J_j'(k\rho)\right\}\mathbf{e}_\rho + \frac{1}{k\rho}J_j(k\rho)J_l(k\rho)$$
$$\left\{j\sin(j\theta+\alpha_j)\cos(l\theta+\beta_l) - l\cos(j\theta+\alpha_j)\sin(l\theta+\beta_l)\right\}\mathbf{e}_\theta\Bigg\}. \quad (18)$$

The primes denote differentiation with respect to the argument. The constant η is the impedance of the medium defined as $\eta = \sqrt{\mu/\epsilon}$. Since the Poynting vector field will be investigated in a small neighbourhood around the origin, the Bessel functions $J_n(k\rho)$ may be written as

$$J_n(k\rho) = \frac{1}{2^n n!}(k\rho)^n + O(k\rho)^{n+1}, \quad (19)$$

where $O(k\rho)^{n+1}$ implies that rest of the terms are of the order equal to or greater than $n+1$. Therefore in a small neighbourhood around the origin the components of the Poynting vector can be written down as

$$S_\rho = \frac{1}{2\eta}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} A_j B_l \cos(j\theta+\alpha_j)\cos(l\theta+\beta_l)$$
$$\left\{\frac{l-j}{2^{j+l}j!l!}(k\rho)^{j+l-1} + O(k\rho)^{j+l+1}\right\}, \quad (20a)$$

$$S_\theta = \frac{1}{2\eta}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} A_j B_l \left\{j\sin(j\theta+\alpha_j)\cos(l\theta+\beta_l)\right.$$
$$\left. - l\cos(j\theta+\alpha_j)\sin(l\theta+\beta_l)\right\}$$
$$\left\{\frac{1}{2^{j+l}j!l!}(k\rho)^{j+l-1} + O(k\rho)^{j+l+1}\right\}. \quad (20b)$$

This may be recognized as the Taylor series expansion of the Poynting vector field around the origin. It is useful to extract the leading term of the above Taylor series. If the expansion coefficients A_j are zero for all $j < m$ and the coefficients B_l vanish for all $l < n$ then the leading non zero terms in (20) may be concisely written as

$$S_\rho = \frac{-C_{mn}}{m+n}(k\rho)^{m+n-1}v'(\theta) + O(k\rho)^{m+n}, \tag{21a}$$
$$S_\theta = C_{mn}(k\rho)^{m+n-1}v(\theta) + O(k\rho)^{m+n}, \tag{21b}$$

where

$$C_{mn} = \frac{A_m B_n}{2^{m+n+1}n!m!\eta}, \tag{22}$$
$$\begin{aligned} v(\theta) &= m\sin(m\theta+\alpha_m)\cos(n\theta+\beta_n) \\ &\quad - n\cos(m\theta+\alpha_m)\sin(n\theta+\beta_n). \end{aligned} \tag{23}$$

Throughout this paper the variables m and n will be used to denote the leading terms in the eigenfunction expansion of R and I respectively. It is observed from Equation (21) that the origin is a critical point if and only if $m + n \geq 2$.

A critical point will be called *non-degenerate* if the function $v(\theta)$ has no roots or only simple roots. This condition implies that the coefficients α_m and β_n should be such that the set of simultaneous equations

$$\cos(m\theta + \alpha_m) = 0, \tag{24a}$$
$$\cos(n\theta + \beta_n) = 0. \tag{24b}$$

has no real solutions. Otherwise the critical point will be referred to as a *degenerate* critical point. If the critical point is non degenerate then the truncated Poynting vector field

$$\mathbf{S}_1 = C_{mn}(k\rho)^{m+n-1}\left\{-\frac{v'(\theta)}{m+n}\mathbf{e}_\rho + v(\theta)\mathbf{e}_\theta\right\} \tag{25}$$

also has an isolated critical point at the origin. If the critical point is degenerate then $\mathbf{S}_1 = 0$ for $\theta = \theta_i$, where θ_i are solutions to the set of simultaneous equations (24). If there is at least one value of θ which satisfies (24) then there are exactly two times $\gcd(m, n)$ other such values of θ. The behaviour of the flow lines in a small angular sector around θ_i cannot be described correctly by truncated Poynting vector field $\mathbf{S}_1$ alone. For this purpose higher order terms in (20) should be taken into account.

It may be noted that if $m = n$ and $\alpha_m = \beta_n$ then the leading terms of R and I are proportional and the resulting $\mathbf{S}_1$ is identically zero. It may appear that this critical point is a degenerate critical point but this is not the case. In Section 9 it will be shown how to handle this situation by a simple transformation of eigenfunction expansions (16) and (17). At this point it is assumed that if $m = n$ then $\alpha_m \neq \beta_n$. This is not a restrictive assumption. Critical points of any type will not be left out of the classification scheme because of this assumption.

6. NON DEGENERATE CRITICAL POINTS

In this section the behaviour of flow lines in the neighbourhood of a non degenerate critical point will be studied. For this purpose flow lines of the truncated Poynting vector field $\mathbf{S}_1$ will be studied first. The differential equation for the flow lines given by (3) can be modified for the case of two dimensional Poynting vector fields. This modified equation in cylindrical polar coordinates is given as

$$\frac{d\rho}{d\theta} = \rho \frac{S_\rho}{S_\theta}. \tag{26}$$

Substituting the components of $\mathbf{S}_1$ in (26) gives the required differential equation for the flow lines of truncated Poynting vector field, i.e.,

$$\frac{d\rho}{d\theta} = -\frac{1}{m+n}\rho \frac{v'(\theta)}{v(\theta)}. \tag{27}$$

Integration of (27) yields,

$$\rho = K\left(|v(\theta)|\right)^{-1/(m+n)}, \tag{28}$$

where K is the constant of integration. Equation (28) gives a family of curves which coincides with the flow lines of field $\mathbf{S}_1$. The directions of flow can be attached to these curves by inspection of (25).

The behaviour of flow lines is closely connected with the behaviour of function $f(\theta)$ defined as,

$$f(\theta) = |v(\theta)|. \tag{29}$$

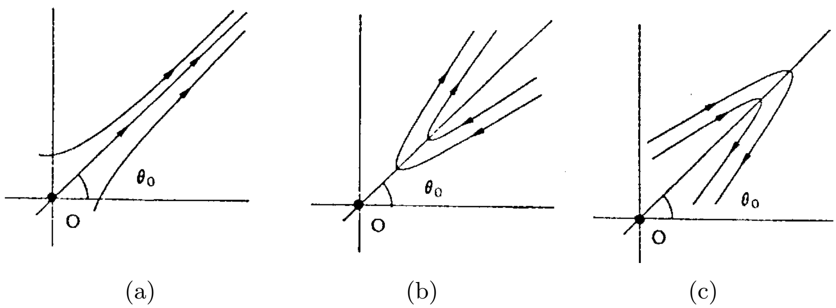


Figure 2. Possible behaviours of flow lines in a small sector containing the direction θ_0 such that (a) $f(\theta_0)$ is zero, (b) $f(\theta_0)$ is a local maximum, and (c) $f(\theta_0)$ is a local minimum.

Consider the case when $m = n$ then the function $f(\theta)$ is a constant and

$$\rho = K\left(m|\sin(\alpha_m - \beta_m)|\right)^{-1/2m} \tag{30}$$

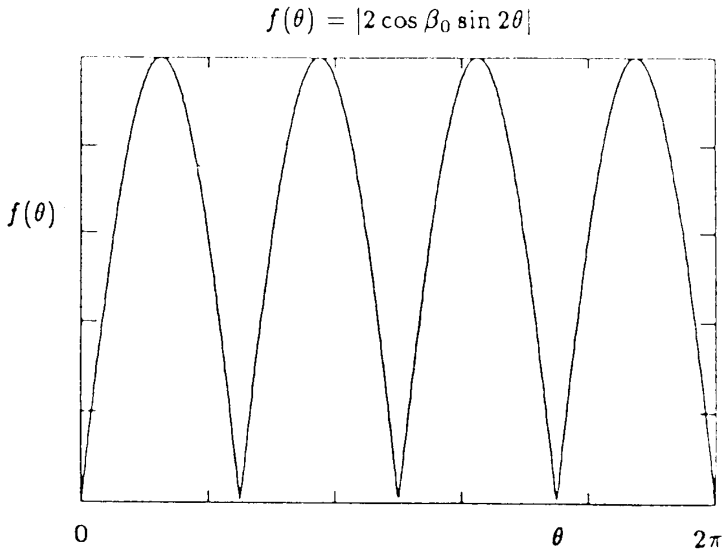
gives a family of concentric circles centered on the origin. Therefore in this case the flow lines form closed loops around the critical point. Such a critical point is designated as a *center point.*

In the case when $m \neq n$, the function $f(\theta)$ has some real roots in $[0, 2\pi)$. If θ_0 is one of the roots. Then the flow line through a point (ρ_0, θ_0) is a ray $\theta = \theta_0$ as depicted in Figure 2(a). If $f(\theta_0)$ is a local maximum then the flow lines cut across the ray $\theta = \theta_0$ orthogonally as shown in Figure 2(b). If $f(\theta_0)$ is a local minimum different from zero then the flow lines cut across the ray $\theta = \theta_0$ orthogonally and Figure 2(c) is representative of this behaviour. The direction of flow lines in Figure 2 has been put arbitrarily and figures with reversed directions are equally valid. The directions θ_0 for which $f(\theta_0) = 0$ are called *critical directions* and the flow lines in or out of the critical point are called *critical rays.*

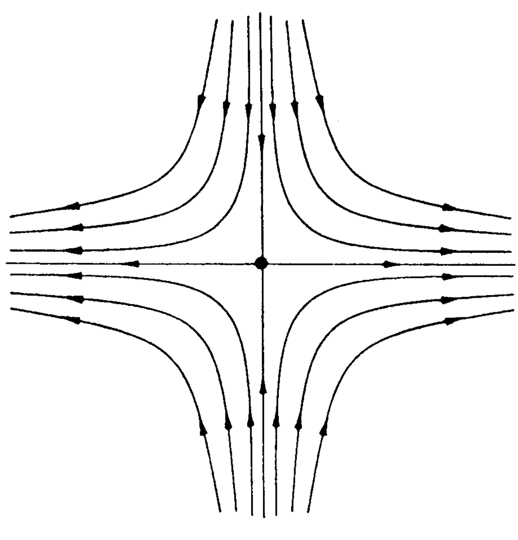
It can be easily shown that the function $f(\theta)$ has $2\max(m, n)$ maxima and they are located at the roots of $\cos(m\theta + \alpha_m) = 0$ if $\max(m, n) = m$ or at the roots of $\cos(n\theta + \beta_n) = 0$ if $\max(m, n) = n$. It can also be shown that $f(\theta)$ possesses $2\min(m, n)$ local minima different from zero. These minima are located at the roots of $\cos(m\theta + \alpha_m) = 0$ if $\min(m, n) = m$ or at the roots of $\cos(n\theta + \beta_n) = 0$ if $\min(m, n) = n$. Since $f(\theta)$ is a continuous function it may be deduced from the above discussion that it has exactly $2|m - n|$ simple roots. These roots lie between two extrema of $f(\theta)$ both of which are a local maximum. In short, a non degenerate critical point and the flow lines in its neighbourhood are completely described by four constants m, n, α_m, and β_n.

In the case when $\min(m,n) = 0$ the flow lines around the critical point except for their orientation in the x-y plane are characterized by one number alone, i.e., $\max(m,n)$. Such a point is called a *p-sectored saddle point*, where $p = 2\max(m,n)$. In particular if $p = 4$ it is simply called a *saddle point* and if $p = 6$ then it is referred to in literature as a *monkey saddle*. So much for the nomenclature. In general a critical point can be roughly characterized by the quantity $|m-n|$ alone because its flow lines are homeomorphic to the flow lines of a p-sectored saddle point, where $p = 2|m-n|$. That is to say there are $2|m-n|$ rays coming into or going out of the critical point in critical directions. A circle centered on the critical points is divided by these rays into $2|m-n|$ sectors. In each of these sectors the flow lines form a family of curves which are homeomorphic to one branch of a hyperbola. For the sake of illustration the flow lines in the neighbourhood of two types of critical points are given in Figures 3 and 4. In Figure 3(a) the function $f(\theta)$ is shown for the case when $m = 2$, $n = 0$, and $\alpha_2 = 0$. The corresponding flow lines are sketched in Figure 3(b). In Figure 4(a) the function $f(\theta)$ is shown for a critical point whose parameters are $m = 3$, $n = 1$, $\alpha_3 = \pi/2$, and $\beta_1 = 0$. In Figure 4(b) the flow lines near this critical point are shown.

Now consider a non degenerate critical point whose Poynting vector field is given by (20). It can be shown using Forster's [3] results that the flow lines of the Poynting vector field in this case will behave essentially as the flow lines of truncated Poynting vector field except in the neighbourhood of a center point. According to Forster a center point of the truncated field may be either a center point of a focal point of the original vector field. A critical point is designated as a *focal point* when all the flow lines in its neighbourhood either emerge from or tend to it as shown in Figure 5. Forster's results are general and apply to any plane vector field. But the Poynting vector field is solenoidal and hence field lines cannot emerge from or tend to a single point. Therefore the possibility of a focal Point in the Poynting vector fields has to be ruled out. Hence it is concluded that the behaviour of flow lines in the neighbourhood of a non degenerate critical point is similar to the behaviour of the flow lines of the corresponding truncated field at that point.

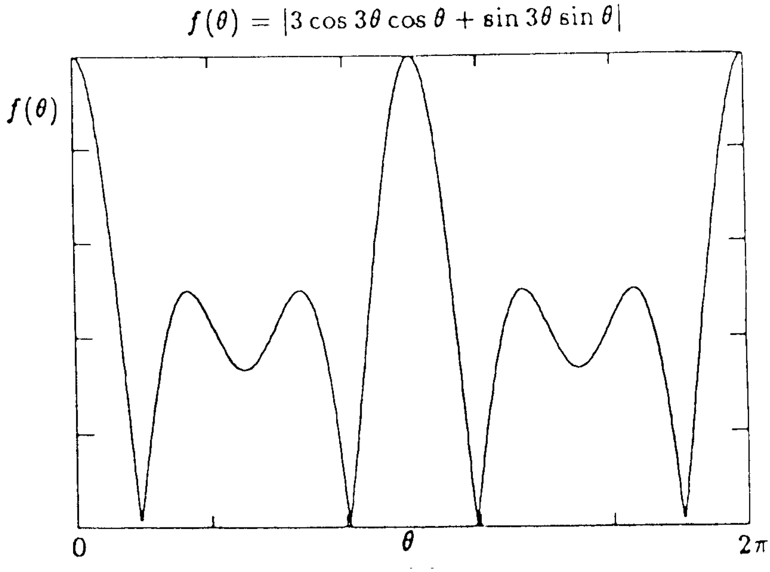


(a)

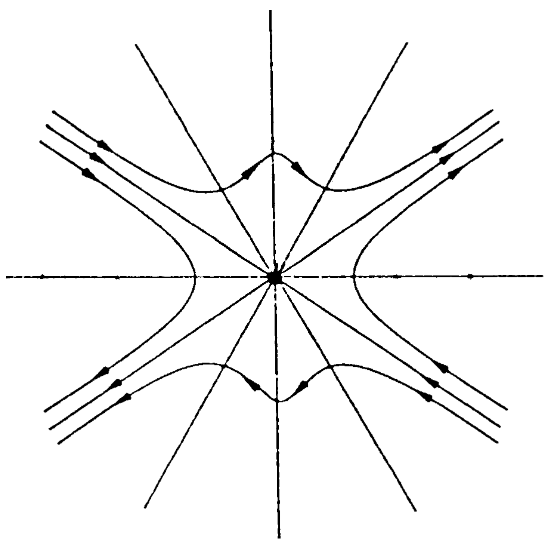


(b)

Figure 3. (a) Plot of the angular function $f(\theta)$ for a critical point whose parameters are $m = 2$, $n = 0$, and $\alpha_2 = 0$. (b) Sketch of the flow line in the neighbourhood of this critical point.

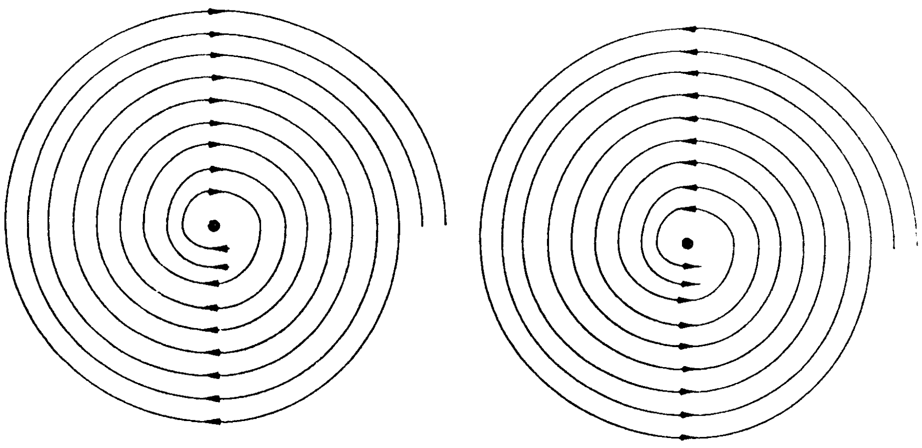


(a)



(b)

Figure 4. (a) Plot of the angular function $f(\theta)$ for a critical point whose parameters are $m = 3$, $n = 1$, $\alpha_3 = \pi/2$, and $\beta_1 = 0$. (b) Sketch of the flow line in the neighbourhood of this critical point.



(a) (b)

Figure 5. Lines of flow near a focal point. (a) Focal point behaving as a source. (b) Focal point behaving as a sink.

7. INDEX OF ROTATION AT NON DEGENERATE CRITICAL POINTS

The index of rotation of a non degenerate critical point will now be determined. It has been shown that the index of rotation of a critical point is identical to the index of rotation of the corresponding critical point of the truncated Poynting vector field [4]. The index of rotation γ can be calculated with the help of Poincaré's formula which can rewritten in terms of cylindrical polar components of the Poynting vector field as

$$\gamma = 1 + \frac{1}{2\pi}\int_0^{2\pi} \frac{S_\rho(dS_\theta/d\theta) - S_\theta(dS_\rho/d\theta)}{S_\rho^2 + S_\theta^2} d\theta. \tag{31}$$

Unit circle is the path chosen to calculate γ, i.e., $k\rho = 1$ in (25). Direct application of (31) leads to a very complicated integral hence the concept of homotopy will be employed to compute γ. It can be readily verified that the truncated Poynting vector field may be written as

$$\mathbf{S}_1 = \mathbf{S}_p + \mathbf{S}_q, \tag{32}$$

where

$$\mathbf{S}_p = C_{mn}(k\rho)^{m+n-1}\Bigg\{(n-m)\cos\{(n-m)\theta+\beta_n-\alpha_m\}\mathbf{e}_\rho$$

$$-(n+m)\sin\{(n-m)\theta+\beta_n-\alpha_m\}\mathbf{e}_\theta\Bigg\}, \tag{32a}$$

$$\mathbf{S}_q = C_{mn}(k\rho)^{m+n-1}\Bigg\{(m-n)\sin(m\theta+\alpha_m)\sin(n\theta+\beta_n)\mathbf{e}_\rho$$

$$+\{m\sin(n\theta+\beta_n)\cos(m\theta+\alpha_m)$$

$$-n\cos(n\theta+\beta_n)\sin(m\theta+\alpha_m)\}\mathbf{e}_\theta\Bigg\}. \tag{32b}$$

It can easily be shown that

$$|\mathbf{S}_p| > |\mathbf{S}_q|.$$

Hence $\mathbf{S}_p$ represents the principal part of $\mathbf{S}_1$. Therefore the index of rotation will now be calculated with the help of theorem 2. Using the cylindrical polar components of $\mathbf{S}_p$ in (31) the index γ is given as

$$\gamma = 1 - \frac{1}{2\pi}\int_0^{2\pi}\frac{(m+n)(m-n)^2}{m^2+n^2-2mn\cos\{2(m-n)\theta+2\alpha_m-2\beta_n\}}d\theta. \tag{33}$$

Techniques of complex variables are used to evaluate the above integral which after reduction yields the simple result for the index of rotation

$$\gamma = 1 - |m-n|. \tag{34}$$

The index of rotation of the critical point was established using purely analytical techniques. In light of the behaviour of flow lines around the critical point deduced in Section 6, the index of rotation can also be calculated from purely geometrical arguments. Following Nemytskii and Stepanov [4] a critical direction will be called a critical direction of *type II* if the flow lines, in a small sector containing no other critical direction, behave as depicted in Figure 6(a). A sector is labeled as a *hyperbolic domain* if it is bounded by a pair of successive critical

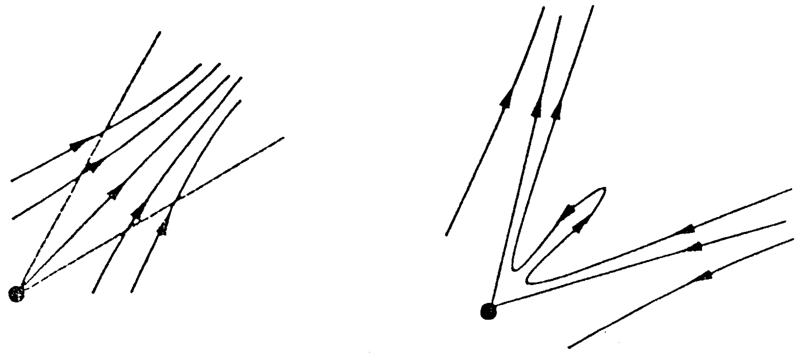


Figure 6. (a) Flow lines around a critical direction of type II. (b) Flow lines within a hyperbolic domain.

directions. A possible behaviour of flow lines in a hyperbolic domain is sketched in Figure 6(b) as an example.

If a circular region around the critical point can be completely divided into n_h hyperbolic domains then according to a theorem by Poincaré (see for example [5]) the index of rotation of the critical point is given by

$$\gamma = 1 - \frac{n_h}{2}. \tag{35}$$

Referring back to Section 6 it is immediately apparent that the region around the critical point can be completely divided into $2|m-n|$ hyperbolic domains. Therefore the index computed by this method is identical to (34). The method of counting the number of hyperbolic domains to compute the index of rotation will be useful in the discussion of degenerate critical points.

8. DEGENERATE CRITICAL POINTS

This section will be devoted to the study of degenerate critical points. The constants α_m and β_n have to satisfy exactly the set of simultaneous Equations (24) for some values of θ. In most of the cases this is not a probable scenario yet the case of degenerate critical points will be taken up for the sake of completeness.

Let θ_0 be a root of the set of simultaneous Equations (24) then the truncated Poynting vector field may be expanded in a Taylor series around θ_0. Keeping only the leading terms in this expansion, $\mathbf{S}_1$ may be written as

$$\mathbf{S}_1 = \pm mn(n-m)C_{mn}(k\rho)^{m+n-1}\left\{(\theta-\theta_0)^2\mathbf{e}_\rho - \frac{m+n}{3}(\theta-\theta_0)^3\mathbf{e}_\theta\right\}. \tag{36}$$

The $\pm$ sign in front of (36) is of no practical significance and will depend on which of the roots of system (24) is under consideration. According to Nemytskii and Stepanov [4] the direction θ_0 remains a critical direction if

$$\lim_{\rho\to 0}\lim_{\theta\to\theta_0}\frac{S_\theta}{S_\rho} = 0, \tag{37}$$

where S_ρ and S_θ are the components of the total Poynting vector field. Using the Taylor series expansion (36) in the expression (21) for the components of $\mathbf{S}$, it is easy to show that the condition enunciated in (37) is satisfied. Therefore all the directions θ_0 which are the roots of system of Equations (24) remain critical directions for the untruncated Poynting vector field.

To study the behaviour of the flow lines in a small sector $|\theta-\theta_0| < \delta$ and $\rho < \rho_0$, higher order terms in the expansion (21) will have to be taken into account. The next higher order term is of the order $m+n$ in (21). Since the divergence of the Poynting vector is zero term by term in expansion (20), the $(m+n)$th term of $\mathbf{S}$ can be written as

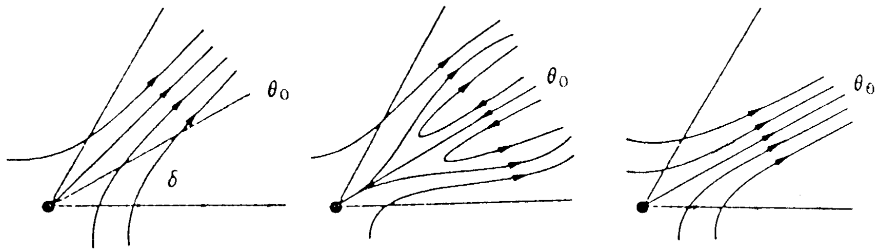
$$\mathbf{S} = (k\rho)^{m+n}\left\{\frac{-1}{m+n}g'(\theta)\mathbf{e}_\rho + g(\theta)\mathbf{e}_\theta\right\} \tag{38}$$

in the sector $|\theta-\theta_0| < \delta$ and $\rho < \rho_0$, where δ and ρ_0 are small constants. The form of the function $g(\theta)$ is not important for this discussion except that it is continuous and differentiable which is guaranteed from (20). There are two cases to be considered.

The first case arises when $g(\theta_0) \neq 0$. In this case the function $g(\theta)$ may be approximated as a constant in the sector under consideration. Let g_0 be such a constant. Then the Poynting vector in this sector may be approximately written as

$$\mathbf{S} = \widetilde{C}(k\rho)^{m+n-1}\left\{(\theta-\theta_0)^2\mathbf{e}_\rho - \frac{m+n}{3}(\theta-\theta_0)^3\mathbf{e}_\theta\right\} + g_0(k\rho)^{m+n}\mathbf{e}_\theta, \tag{39}$$

where $\widetilde{C}$ is a new constant obtained by lumping the constants in (36). In the sector under consideration the flow lines have the behaviour



(a) (b) (c)

Figure 7. Behaviour of the flow lines of **S** field in the neighbourhood of a critical direction of the truncated Poynting vector field. There are three cases, (a) the constant $g_0 \neq 0$, (b) when $g_1\widetilde{C} > 0$, (c) when $g_1\widetilde{C} < 0$.

indicated in Figure 7(a) according to expression (39). It is assumed that the Poynting vector filed given by (39) is sufficient to characterize the flow lines of the untruncated Poynting vector filed. It is also seen from Figure 7(a) that the critical direction is of type II.

The second case arises when $g(\theta_0) = 0$ but $g'(\theta_0) \neq 0$. In this case $g(\theta)$ be approximated as a linear function in the sector under consideration, i.e.,

$$g(\theta) = g_1\{\theta - \theta_0\} \qquad \text{in} \qquad |\theta - \theta_0| < \delta,\ \rho < \rho_0, \tag{40}$$

where g_1 is a constant. Then the Poynting vector, up to the leading term is given as

$$\begin{aligned}\mathbf{S} = &\widetilde{C}(k\rho)^{m+n-1}\left\{(\theta-\theta_0)^2\mathbf{e}_\rho - \frac{m+n}{3}(\theta-\theta_0)^3\mathbf{e}_\theta\right\}\\ &+ g_1(k\rho)^{m+n}\left\{\frac{-1}{m+n}\mathbf{e}_\rho + (\theta-\theta_0)\mathbf{e}_\theta\right\}.\end{aligned} \tag{41}$$

Expression (41) is utilized to investigate the behaviour of the flow lines in a given sector. If $g_1\widetilde{C} > 0$ then the qualitative behaviour of the flow lines of **S** is given in Figure 7(b). It is observed that two new hyperbolic domains have been created in this case. If $g_1\widetilde{C} < 0$ then the behaviour of flow lines is indicated in Figure 7(c). In this case it may be noticed that the critical direction is of type II and no new hyperbolic domains has been created. Once again it is assumed that the behaviour of the Poynting vector filed is completely characterized by the approximate field given in (41).

In sectors around the critical point which do not include a critical direction, the behaviour of flow lines is described by the truncated Poynting vector field. This behaviour will depend on the function $f(\theta)$ which has been discussed in Section 6. If it happens that both the constants, g_0 and g_1 are zero, i.e., $g(\theta_0) = g'(\theta_0) = 0$ then the next term of higher order in expression (20) should be taken into account. This term will have a representation similar to expression (38) except that the power of $(k\rho)$ will change from $m+n$ to an appropriate greater number. The above method of analysis can once again be applied. This procedure may be repeatedly applied till a function $g(\theta)$ is found such that $g(\theta_0) \neq 0$ or $g'(\theta_0) \neq 0$.

The index of rotation of a degenerate critical point cannot be calculated from a knowledge of m and n of the truncated Poynting vector field only. The behaviour of the flow lines in each of the $2\mathrm{gcd}(m,n)$ sectors has to be known to compute the index, γ. With the help of following straightforward geometrical argument the index of rotation can be computed. If there are j_1, j_2, and j_3 sectors with the behaviour of their flow lines as depicted in Figures 7(a), 7(b), and 7(c) respectively, then

$$j_1 + j_2 + j_3 = 2\mathrm{gcd}(m,n). \tag{42}$$

Since the number of hyperbolic sectors increases by $2j_2$, i.e., only in the case when $g_1\widetilde{C} > 0$, the index of rotation is given as

$$\gamma = 1 - |m-n| - j_2. \tag{43}$$

This result follows directly from Equation (35).

9. CRITICAL POINTS DUE TO ARBITRARY ELECTRIC FIELD

In Section 5 it was observed that when $m = n$ and $\alpha_m = \beta_n$, the critical point is not a degenerate critical point. In this section this statement will be explained. In the previous discussions all possible behaviours of the flow lines of the Poynting vector field have been classified. This classification has been in terms of real and imaginary parts of the electric field, namely, the functions R and I. The eigenfunction expansions of these functions, given in Equations (16) and (17), will be termed as canonical forms. The following question will now be posed. Given an arbitrary electric field, how can one characterize the behaviour of flow lines in the neighbourhood of a critical point using only m and n? This question is crucial because two electric fields differing by an arbitrary phase constant give rise to the same Poynting vector field although they have different real and imaginary parts. To illustrate this point, consider an arbitrary electric field $\widetilde{E}_z$ such that

$$\widetilde{E}_z = E_z \exp(i\phi_0), \tag{44}$$

where E_z is canonical electric field and ϕ_0 is a constant. The real and imaginary parts of this electric field in terms of canonical real and imaginary parts given as

$$\widetilde{R} = R\cos\phi_0 - I\sin\phi_0, \tag{45a}$$

$$\widetilde{I} = R\sin\phi_0 + I\cos\phi_0. \tag{45b}$$

If the origin is a critical point then the canonical eigenfunction expansions can be written as

$$R = \sum_{j=m}^{\infty} A_j J_j(k\rho)\cos(j\theta + \alpha_j), \tag{46a}$$

$$I = \sum_{l=n}^{\infty} B_l J_l(k\rho)\cos(l\theta + \beta_l). \tag{46b}$$

If the analysis of the critical point is made, on the basis of leading term, in the expressions for $\widetilde{R}$ and $\widetilde{I}$, one encounters an apparent discrepancy. It is evident from Equations (45) and (46) that the leading terms of $\widetilde{R}$ and $\widetilde{I}$ are of the same order, which is the minimum of m and n. Thus one may conclude that the critical point is a center point. Such a conclusion would be in error because m and n are not equal in general. An analysis of canonical electric field will yield a different result. To resolve this discrepancy, without the loss of generality suppose that $m \le n$. Substitution of Equation (46) in (45) results in

$$\widetilde{R} = \sum_{j=m}^{n-1} A_j J_j(k\rho)\cos\phi_0\cos(j\theta + \alpha_j) + \sum_{j=n}^{\infty} J_j(k\rho) \cdot \left\{ A_j\cos\phi_0\cos(j\theta + \alpha_j) - B_j\sin\phi_0\cos(j\theta + \beta_j) \right\}, \tag{47a}$$

$$\widetilde{I} = \sum_{j=m}^{n-1} A_j J_j(k\rho) \sin\phi_0 \cos(j\theta + \alpha_j) + \sum_{j=n}^{\infty} J_j(k\rho) \cdot \left\{ A_j \sin\phi_0 \cos(j\theta + \alpha_j) + B_j \cos\phi_0 \cos(j\theta + \beta_j) \right\}. \quad (47b)$$

If $m = n$ then the first summation in expressions (47a) and (47b) disappears. The leading terms in the expansions of $\widetilde{R}$ and $\widetilde{I}$ are of the same order. These terms are not proportional to each other. In this case the lines of flow of the Poynting vector field form concentric circles and the origin is a center point.

On the other hand if $m \neq n$ then $\widetilde{R}$ and $\widetilde{I}$ are of equal order. Their leading terms are proportional. The phase constant ϕ_0 can be calculated from the leading terms as

$$\phi_0 = \tan^{-1} \left\{ \frac{\text{Leading term of } \widetilde{I}}{\text{Leading term of } \widetilde{R}} \right\}. \quad (48)$$

Using this value of ϕ_0 it is possible to calculate the canonical form of the electric field as

$$R = \widetilde{R} \cos\phi_0 + \widetilde{I} \sin\phi_0, \quad (49a)$$

$$I = -\widetilde{R} \sin\phi_0 + \widetilde{I} \cos\phi_0. \quad (49b)$$

With the help of m and n of the canonical electric field derived above it will be possible to classify the behaviour of the flow lines. There is no need to calculate the Poynting vector field. The assumption that $m \leq n$ forces the leading term of R to be of order m. This is not restrictive as far as classification of the critical points is concerned. This assumption works because the qualitative behaviour of the flow lines depends only on the quantities $m + n$ and $|m - n|$, which are symmetrical in m and n.

10. MAGNETIC FIELD AT CRITICAL POINTS

In the case of E-polarized electromagnetic fields given by Equation (6), the magnetic field is completely in the x-y plane. In general this magnetic field is elliptically polarized and can be written down as

$$\mathbf{H} = \mathbf{H}_R + i\mathbf{H}_I = \frac{1}{\omega\mu}(\nabla I \times \mathbf{e}_z - i\nabla R \times \mathbf{e}_z). \quad (50)$$

It is of interest to investigate the magnetic field in the neighbourhood of a critical point of the Poynting vector field. The Poynting vector field is approximated by the truncated Poynting vector field $\mathbf{S}_1$ which in turn is calculated using the leading terms of R and I. The magnetic field can be approximately calculated with the help of the same leading terms. The fields near the origin are of particular interest, therefore, the Bessel functions can be approximated as in Equation (19). Thus the leading terms of real and imaginary parts of the magnetic field are given as

$$\mathbf{H}_R = \frac{-nB_n}{\eta 2^n n!}(k\rho)^{n-1}\left\{\sin(n\theta+\beta_n)\mathbf{e}_\rho + \cos(n\theta+\beta_n)\mathbf{e}_\theta\right\}, \tag{51a}$$

$$\mathbf{H}_I = \frac{mA_m}{\eta 2^m m!}(k\rho)^{m-1}\left\{\sin(m\theta+\alpha_m)\mathbf{e}_\rho + \cos(m\theta+\alpha_m)\mathbf{e}_\theta\right\}. \tag{51b}$$

The major and minor diameters of the polarization ellipse are designated as d_1 and d_2 respectively. They are calculated to be

$$d_1 = \sqrt{\frac{1}{2}\left\{\mathbf{H}\cdot\mathbf{H}^* + \sqrt{(\mathbf{H}\cdot\mathbf{H}^*)^2 - 4(|\mathbf{H}_R\times\mathbf{H}_I|)^2}\right\}}, \tag{52a}$$

$$d_2 = \sqrt{\frac{1}{2}\left\{\mathbf{H}\cdot\mathbf{H}^* - \sqrt{(\mathbf{H}\cdot\mathbf{H}^*)^2 - 4(|\mathbf{H}_R\times\mathbf{H}_I|)^2}\right\}}. \tag{52b}$$

The magnetic field is linearly polarized when $d_2 = 0$. The direction of rotation of the $\mathbf{H}$ vector can be deduced as follows

$$\mathbf{H}_R\times\mathbf{H}_I\cdot\mathbf{e}_z \begin{cases} >0 & \text{counterclockwise,} \\ =0 & \text{linear,} \\ <0 & \text{clockwise.} \end{cases} \tag{53}$$

It can be easily shown that

$$\nabla\times\mathbf{S} = \omega\mu\mathbf{H}_R\times\mathbf{H}_I. \tag{54}$$

Therefore information about the polarization of the magnetic field can also be gleaned from the curl of the Poynting vector. In the present case Equation (53) implies that

$$mn\sin\left((m-n)\theta+\alpha_m-\beta_n\right) \begin{cases} >0 & \text{counterclockwise,} \\ =0 & \text{linear,} \\ <0 & \text{clockwise.} \end{cases} \tag{55}$$

If $m = n$, then the magnetic field is elliptically polarized in the same sense everywhere in the neighbourhood of the critical point, which is a center point. If the condition $m \neq n \neq 0$ holds, then the region around the critical point is divided into $2|m - n|$ equiangular sectors. The polarization is linear on the boundaries of these sectors. The sense of rotation is alternately clockwise and anticlockwise in the contiguous sectors. It may be recalled that the critical point, for which $m \neq n$, is isomorphic to a saddle point with $2|m - n|$ hyperbolic sectors.

As a result of above discussion it becomes clear that the morphology of flow lines at a critical point may be determined from a knowledge of the sense of rotation of the magnetic field on a small circle around the critical point. For example, suppose, there is a critical point. The sense of rotation of the magnetic field is measured on a circle of small radius, centered on the critical point. It is found that the sense is clockwise on q number of arcs interleaved with q number of arcs on which sense of rotation is anticlockwise. It may then be concluded that critical point under investigation has a flow line structure which is isomorphic to the flow line structure in the neighbourhood of a $2q$-sectored saddle point. As a bonus one may also calculate the index of rotation of the critical point from such a measurement using expression

$$\gamma = 1 - q. \tag{56}$$

If either $m = 0$ or $n = 0$ then the magnetic field is linearly polarized in the vicinity of the critical point.

11. ELEMENTARY CRITICAL POINTS

The order of a critical point is defined to be the degree of the leading term in the Taylor series expansion of the Poynting vector field given by expression (20). This Taylor series is expanded about the critical points. Therefore in terms of the notation employed in this paper, the order of a critical point is $(m + n - 1)$. The lowest possible order of any critical point is one. The critical points of order one are called *elementary* critical points. For elementary critical points $m + n = 2$. There are two distinct cases. In the first case m and n are equal while in the second case they are not equal. Elementary critical points are the subject of discussion in this section and both the cases will be dealt separately.

The first case arises when $m = n = 1$. This is the case when the lines of the Poynting vector field are known to circulate the critical

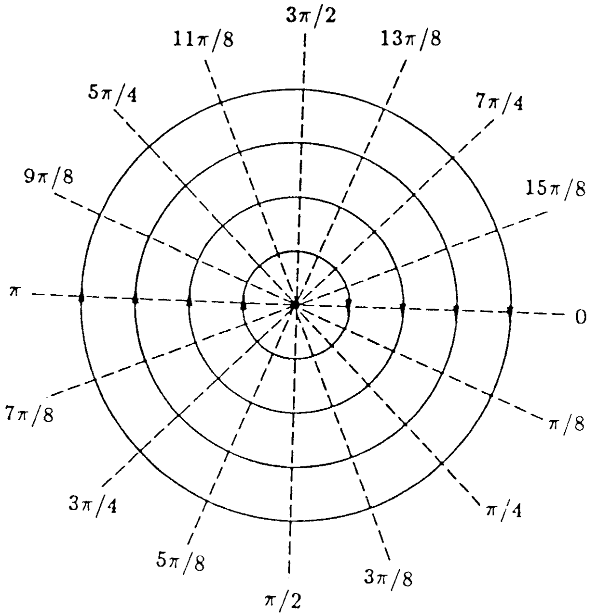


Figure 8. The lines of flow and the lines of constant phase in the neighbourhood of an elementary center point. The phase is undefined at the center.

point, which is also called a center point. The lowest order critical point of this type will be labeled as an elementary center point. It follows from Section 5 that the magnitude of the electric field is zero at this critical point. The phase of the electric field is undefined at this type of critical point. Lines of constant phase emanate radially outward from the critical point and they are sketched in Figure 8. The magnetic field at the critical point is not zero. It can be obtained from Equation (51) by letting m and n to be equal to one, i.e.,

$$\mathbf{H} = \frac{1}{2\eta}\left\{(-B_1 \sin\beta_1 + iA_1 \sin\alpha_1)\mathbf{e}_x + (-B_1 \cos\beta_1 + iA_1 \cos\alpha_1)\mathbf{e}_y\right\}. \tag{57}$$

The magnetic field is elliptically polarized and the expression for the polarization ellipse is given as

$$\begin{aligned}&\left(A_1^2 \cos^2\alpha_1 + B_1^2 \cos^2\beta_1\right) H_x^2 + \left(A_1^2 \sin^2\alpha_1 + B_1^2 \sin^2\beta_1\right) H_y^2 \\ &- \left(A_1^2 \sin 2\alpha_1 + B_1^2 \sin 2\beta_1\right) H_x H_y = 4\eta^2 A_1^2 B_1^2 \sin^2(\alpha_1 - \beta_1).\end{aligned} \tag{58}$$

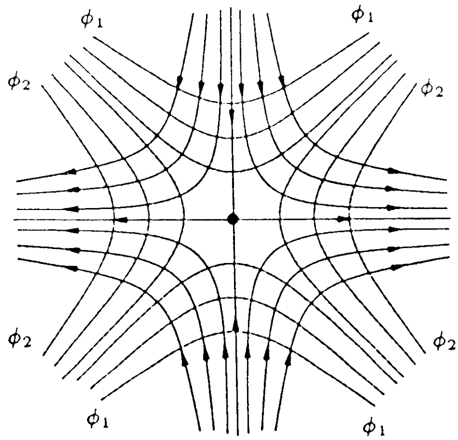


Figure 9. The lines of flow and the lines of constant phase in the neighbourhood of an elementary saddle point.

The electric field energy density, W_e, is zero at the critical point and it increases in the neighbourhood. The contours of constant energy density are ellipses and are given by

$$\begin{aligned}W_e = \frac{\epsilon}{4}\bigg\{ &\left(A_1^2 \cos^2 \alpha_1 + B_1^2 \cos^2 \beta_1\right)(kx)^2 \\ &+ \left(A_1^2 \sin^2 \alpha_1 + B_1^2 \sin^2 \beta_1\right)(ky)^2 \\ &- \left(A_1^2 \sin 2\alpha_1 + B_1^2 \sin 2\beta_1\right)(kx)(ky)\bigg\}. \end{aligned} \tag{59}$$

Comparison of Equations (58) and (59) brings out an interesting fact that the polarization ellipse is similar to the contours of constant electric field energy density. Therefore if the polarization ellipse is measured at one point near an elementary center point, the rate of increase of W_e in different directions can be predicted.

The second case of elementary critical point arises when $m = 2$ and $n = 0$ or vice versa. Without loss of generality it will be assumed that $m > n$. In this case the lines of flow form a family of rectangular hyperbolae. The lines of constant phase which are orthogonal to the lines of flow also form a family of rectangular hyperbolae. Such a critical point is called an elementary saddle point. The flow lines near an elementary saddle point are depicted in Figure 9. The polarization of the magnetic field is linear in the neighbourhood of this critical point. Neither the electric field nor the magnetic field is zero at the elementary saddle point. These fields are in time quadrature at this type of critical point and this is the reason why the Poynting vector is zero.

It may be verified that the elementary critical points are always non degenerate. This verification can be carried out by substituting appropriate values for m and n in Equation (23) and checking the conditions stipulated in Equation (24). In general Taylor series coefficients are non-zero more often than zero. Hence one would expect to find elementary critical points more often. Therefore at this juncture it is appropriate to discuss ways and means to detect the elementary critical points and determine their nature. For this purpose consider a long wire made of a good conductor. Let the diameter of this wire be much smaller than the wavelength. Place the wire parallel to the z-axis. It will experience a force proportional the strength of the Poynting vector field and in the same direction. At the critical point the Poynting vector field is zero and hence at this point no net time average force will be exerted on the wire. Therefore in principle one can locate the critical points by moving a wire in the electromagnetic field and measuring force on it. If the electromagnetic field is E-polarized then the current induced in the wire will be proportional to E_z. Therefore at a center point no current will be induced in the wire. At the saddle point the electric field is not zero hence current will be induced in the wire but no average force will be exerted on it. Thus the induced current will distinguish a saddle point from a center point. Another method to distinguish between these two types of critical points is to construct a device similar to vane. It is constructed with the help of two long straight wires of diameters which are small in comparison with the wavelength. The wires are connected together by a piece of light insulating material such as plastic. A sketch of this contraption appears in Figure 10(a). This device is capable of measuring a torque about its axis due to the radiation pressure. If the axis is placed in such a way that it coincides with a center point then a net torque will be exerted on the device and it will rotate. On the other hand if the axis is placed on the saddle point the devise will tend to align itself with one of the arms of the saddle point because in this configuration the net force and the net torque on it are zero. The behaviour of this device is illustrated in Figure 10. It may be noted that the device described above is sensitive to the curl of the Poynting vector field which

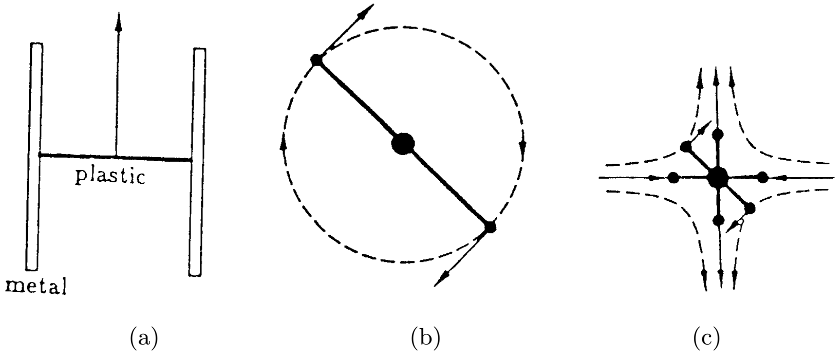


Figure 10. (a) A device which may be used to distinguish a center point from a saddle point. (b) The behaviour of the device at a center point and (c) a saddle point.

is finite and non zero for the center point and zero for the saddle point. At this point it may be noted that Latmiral's [6] suggestion on using a tuned dipole as a true sensor of the curl of **S** is not workable in the case of plane Poynting vector fields. The reason is that once the dipole is aligned with E_z and charges are induced on its ends there is no transverse part of the electric field to act on these charges. Thus there will be no rotation of the dipole.

12. WAVE INTERFERENCE: EXAMPLE OF CRITICAL POINTS

In the previous discussions it was assumed that isolated critical points of the Poynting vector field exist. No examples were given. The existence of these points will now be demonstrated in a fairly simple situation. It is well known that the Poynting vector of a standing wave is zero everywhere. Therefore it is logical to look for critical points in the problems in which the electromagnetic field is a slight variation of a standing wave field. Consider the interference of three linearly polarized plane waves traveling in the directions which make angles $0°$, $180°$ and $\theta_\circ$ with the x-axis. The amplitudes of their respective electric fields are $\frac{1}{2}$, $\frac{1}{2}$ and α. Thus the total electric field is as

$$\mathbf{E} = \left\{ \frac{1}{2} \exp(ikx) + \frac{1}{2} \exp(-ikx) + \alpha \exp(ikx \cos\theta_\circ + iky \sin\theta_\circ) \right\} \mathbf{e}_z. \tag{60}$$

The first two waves alone would form a standing wave. The case

$\theta_\circ = 90°$ was considered by Braunbek [7] to point out the existence of vortices in the Poynting vector field. It will be shown below that critical points in this case exist only if $0 < \alpha \leq 1$. If $\theta_\circ = 90°$, then the Poynting vector is given as

$$\mathbf{S} = \frac{\alpha}{2\eta}\left\{\sin(kx)\sin(ky)\mathbf{e}_x + (\alpha + \cos(kx)\cos(ky))\mathbf{e}_y\right\}. \tag{61}$$

This Poynting vector field does not possess any critical points if $|\alpha| > 1$. If $\alpha = 0$ then the Poynting vector is zero everywhere and all the points are critical points but they are not isolated. If the range of α is restricted such that $0 < |\alpha| \leq 1$ then there are four sets of points where the $\mathbf{S}$ field has isolated critical points. Let these sets be denoted by P_1, P_2, P_3, and P_4. These sets are enumerated as

$$P_1 : \left\{(kx, ky)|kx = 2q_1\pi \pm \cos^{-1}\alpha,\ ky = (2q_2+1)\pi\right\}, \tag{62a}$$

$$P_2 : \left\{(kx, ky)|kx = (2q_1+1)\pi \pm \cos^{-1}\alpha,\ ky = 2q_2\pi\right\}, \tag{62b}$$

$$P_3 : \left\{(kx, ky)|kx = (2q_1+1)\pi,\ ky = 2q_2\pi \pm \cos^{-1}\alpha\right\}, \tag{62c}$$

$$P_4 : \left\{(kx, ky)|kx = 2q_1\pi,\ ky = (2q_2+1)\pi \pm \cos^{-1}\alpha\right\}, \tag{62d}$$

where q_1 and q_2 are any integers and the range of $\cos^{-1}\alpha$ is restricted to $[0, \pi]$. If $0 < |\alpha| < 1$ then all the critical points are found to be elementary critical points. Such an analysis also yields the information that points corresponding to the sets P_1 and P_2 are elementary center points and the points corresponding to the sets P_3 and P_4 are elementary saddle points. If $\alpha = 1$, the set P_1 is identical to the P_4 and the set P_2 is identical to the set P_3. If $\alpha = -1$ then the set P_1 coincides with the set P_3 and the set P_2 coincides with set P_4. In both of the above cases the critical points are not elementary. These are critical points of order 2 with the indices $m = 2$ and $n = 1$.

An explicit expression for the lines of flow of the Poynting vector field can be obtained by the integration of differential Equation (3). This expression is given by

$$\sin(kx)\cos(ky) + \alpha(kx) = \text{constant}. \tag{63}$$

Sketches of the flow lines for the two cases $0 < |\alpha| < 1$ and $|\alpha| = 1$, are given in Figures 11(a) and 11(b) respectively. A cursory glance at these figures is enough to show that they are qualitatively different.

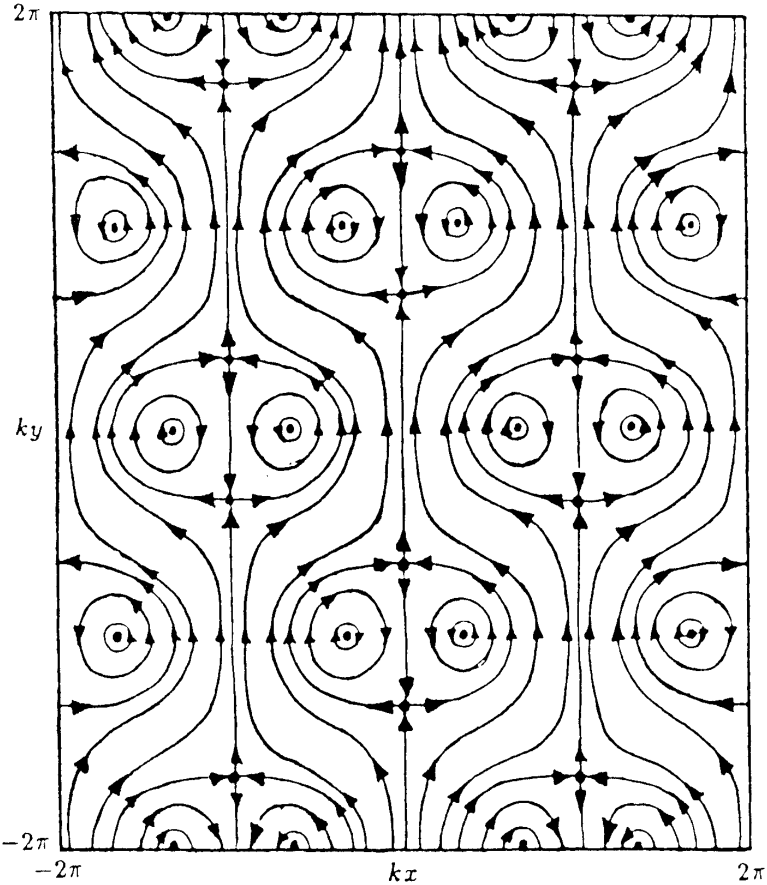


Figure 11. (a) Flow lines of the Poynting vector field for the case of three wave interference when $0 < |\alpha| < 1$.

Another example is that of a plane wave interfering with a circularly cylindrical wave which also gives rise to critical points. For the sake of definiteness consider a line source of electrical current located at the origin of the coordinate system. This line source radiates linearly polarized circularly cylindrical waves. In addition suppose there is a linearly polarized plane wave traveling the direction of the negative y-axis. The total electric field may be written as

$$\mathbf{E} = \left\{ \exp(-iky) + bH_0^{(1)}(k\rho) \right\} \mathbf{e}_z, \tag{64}$$

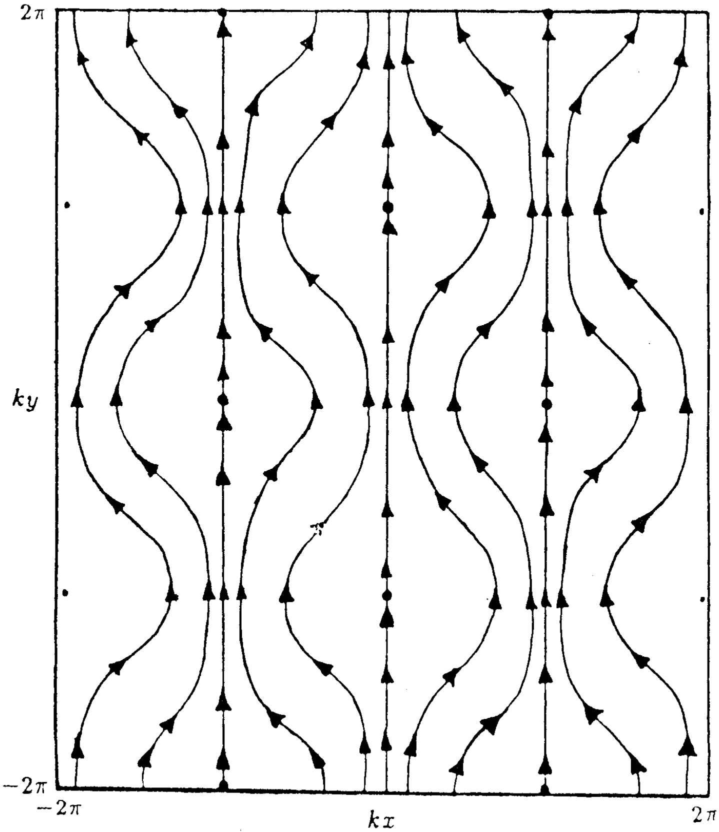
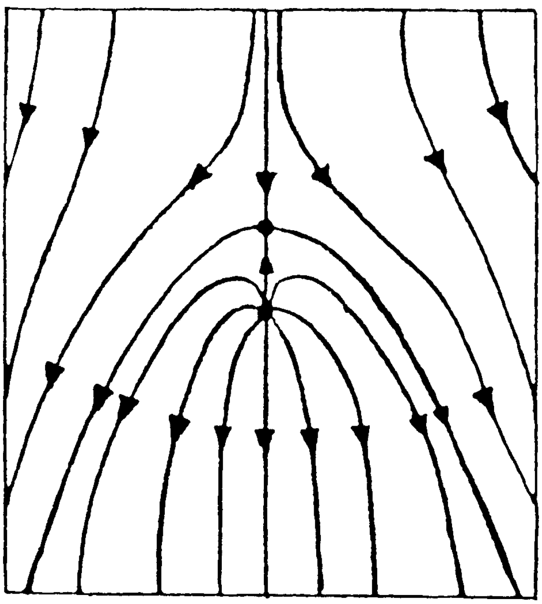


Figure 11. (b) Flow lines of the Poynting vector field for the case of three wave interference when $|\alpha| = 1$.

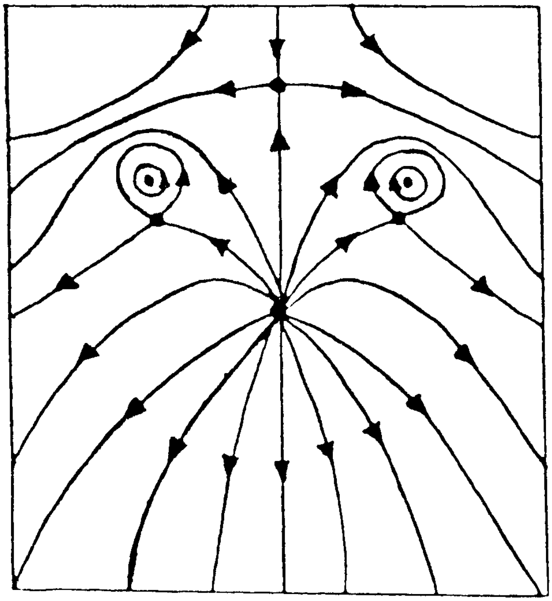
where b is the strength of the cylindrical wave and $H_0^{(1)}$ is Hankel function of the first kind. The strength of cylindrical wave may be changed by changing the amount of current flowing in the line source. When $b = 0$ the plane wave alone does not give rise to any critical points. For an arbitrarily small but non zero b an elementary saddle point appears on the positive y-axis. As the value of b is increased the location of the saddle point moves up alone the y-axis. A sketch of the flow lines of the Poynting vector field for this situation is given in Figure 12(a). The index of rotation of the source point is $+1$ because all the flow lines emanate radially outward from the line source. Therefore the total index of rotation in the **S** field remains zero. This can be verified by calculating the rotation on any closed curve which contains both the line source and the elementary saddle point. There are no other critical points of the **S** filed. The reason for this is that the amplitude of the cylindrical wave falls with the distance as $1/\sqrt{k\rho}$ and far from the source it is completely swamped by the field of the plane wave. The ensuing result is that all the flow lines from the source point are eventually directed towards the negative y-axis.

As the strength of the cylindrical wave is increased the location of the elementary saddle point moves up on the y-axis. Above a certain value of b two elementary center and saddle points appear as indicated in Figure 12(b). The total index of rotation still remains zero. As b increased further more pairs of critical points are created. Each of those pairs consists of an elementary center and saddle point.



(a)

Figure 12. Critical points resulting from the interference of a plane wave and cylindrical wave. (a) When the amplitude of cylindrical wave is small. (b) When the amplitude of cylindrical wave is increased beyond a certain value.



(b)

Figure 12. Continued.

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