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OPEN MAPS AND MAPS ONTO TWO-MANIFOLDS

John David Baildon
State University of New York at Binghamton

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OPEN MAPS AND MAPS ONTO TWO-MANIFOLDS

A Dissertation Presented

By

JOHN DAVID BAILDON

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JOHN DAVID BAILDON

Approved as to style and content by:

Louis F. McAuley

Patricia Tulley McAuley

[illegible]

Dick Wick Hall

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INTRODUCTION

Our primary interest in this thesis is in open maps. We begin by examining open simple (at most two to one) maps and consider when a map may be factored into open simple ones. It is determined that the orbit map of a periodic homeomorphism of period 2^k has this property. Similar results are obtained for homeomorphisms of other periods. In addition, we show that a periodic homeomorphism of period 2^k on an n-manifold which is the identity on a domain must be the identity on the whole manifold, thus extending a result of M. H. A. Newman (10).

In Chapter 2 we also obtain an extension of a theorem of Whyburn (19: X; 7.3), getting a formula relating the degree (order) of a map, the size of the singular set and its inverse, and the Euler characteristics of the domain and range. We consider the case where the domain is a union of 2-manifolds without boundary, intersecting only in arcs, and the range is also a 2-manifold without boundary.

Chapter 3 is concerned with the question, "If $f/_N$ is the restriction of an open map to a compact nodal set (subset having only one boundary point), is there necessarily a compact subset of N mapping onto f(N) so that the restriction of f to this set is open?" A negative answer is provided as are conditions under which such a set will exist.

In Chapter 4, a finite to one open map of Bing's house with two rooms onto a 2-sphere is exhibited. Additional theorems slightly generalize the techniques used to insure that we get an open map.

Chapter 5 contains a collection of examples and counterexamples which are related to work done in the first four chapters.

TABLE OF CONTENTS

CHAPTER I

OPEN SIMPLE MAPS AND PERIODIC HOMEOMORPHISMS

In (1) K. Borsuk and R. Molski studied a class of mappings, called the mappings of finite order. A mapping defined on a space X is said to be of order $\leq k$ if for any y in f(X), $f^{-1}(y)$ contains at most k points. They take particular interest in the mappings of order ≤ 2, which are called simple maps. In this paper we will also require that a simple map have at least one point inverse with cardinality two. Borsuk and Molski provide a number of examples of simple maps and pose several problems concerning them. Among these is the question of whether or not there exists a continuous mapping of finite order which is not a superposition (composition) of a finite number of simple mappings. In (11), K. Sieklucki shows that every such map on a compact, finite dimensional space is such a superposition and gives an example of a map of finite order on an infinite dimensional compact space which cannot be so reduced. J. W. Jaworowski points out in (8) that an open simple map on a compact space induces a continuous involution (and hence a homeomorphism) on the space and vice versa. Thus there is a direct connection between the study of open simple maps and that of homeomorphisms of period two. P. A. Smith has made an extensive study of homeomorphisms of finite period (12,13) which should prove useful to an exhaustive study of open simple maps.

By the singular set B_f of a map f from a space X onto Y we will mean the set of all points in X where f is not locally one to one.

We will show that in the case of an open simple map between manifolds, the singular set is exactly the set of points that have $\{x\} = f^{-1}f(x)$. We first obtain a few preliminary results. Let L denote $\{x : \{x\} = f^{-1}f(x)\}$.

Lemma 1.1: Let f be an open simple map of a compact space X onto Y. If x is in L, then either there exists a neighborhood U of x which is contained in L or x is in B_f.

Proof: If such a U does not exist, then there exists a sequence $\{x_n\}$ converging to x such that $\{x_n\} \neq f^{-1}f(x_n)$. Since f is continuous, $\{f(x_n)\}$ converges to $f(x)$; and since f is open and X is compact, $\{f^{-1}f(x_n)\}$ converges to $f^{-1}f(x) = \{x\}$. Thus any neighborhood of x has points x_n and y_n in it with $f(x_n) = f(y_n)$ and so f is not locally one to one at x, indicating that x is in B_f. //

Lemma 1.2: If f is an open simple map of a compact space X onto Y, then x is in B_f if and only if x is a boundary point of L.

Proof: If x is a boundary point of L, then there exists a sequence of points $\{x_n\}$ contained in the complement of L which converges to x. As in Lemma 1, $\{f^{-1}f(x_n)\}$ converges to $f^{-1}f(x) = \{x\}$ and so f is not a local homeomorphism at x since the cardinality of $f^{-1}f(x_n)$ is two for each n. Consequently, x is in B_f.

If we assume x is in B_f, there will exist sequences $\{y_n\}$ and $\{z_n\}$ converging to x with $f(y_n) = f(z_n)$. Thus any neighborhood V of x has points in the complement of L; but V also contains a point of L, namely x. Thus x is a boundary point of L. //

Lemma 4 is a slight extension of a theorem by M. H. A. Newman (10)

Theorem 1.3: (Newman) If a uniform continuous transformation of period 2 on a locally Euclidean metricized connected n-dimensional space M_n leaves the points of a domain D fixed, it leaves all points of M_n fixed.

Lemma 1.4: If a homeomorphism T of period 2 on an n-manifold M with or without boundary leaves all points of a domain D fixed, then it leaves all points of M fixed.

Proof: The case of a manifold without boundary is covered by Newman's theorem. Let B be the boundary of the manifold M. Consider T restricted to M-B. It is also of period 2, and moreover, M-B is connected and locally Euclidean. Since D is a domain, and hence open, D-B will be open and non-empty since the interior of B is empty. Let D' be a component of D-B. Thus D' is a domain on which $T/_{M-B}$ is fixed, which means that T is fixed on all points of M-B from Theorem 3. If x is in B, then there is a sequence of points $\{x_n\}$ contained in M-B converging to x. By continuity, $\{T(x_n)\}$ will converge to T(x); however, $T(x_n) = x_n$ and so T(x) must be x. Thus T leaves all points of M fixed. //

Corollary 1.5: If a homeomorphism of period 2^k on an n-manifold M with or without boundary leaves all points of a domain D fixed, then it leaves all points of M fixed.

Proof: By Lemma 4, the theorem is true if $k = 1$. Assume it is true up to $k-1$. If f is a homeomorphism of period 2^k which leaves a domain D fixed, then f^2 is a homeomorphism of period 2^{k-1} leaving D fixed, and hence f^2 leaves all of M fixed by our induction assumption. If this is the case then f is a homeomorphism of period 2 and consequently leaves all of M fixed, by Lemma 4. //

We are now ready to show that the singular set is exactly the points which are singleton point inverses.

Lemma 1.6: If f is an open simple map of a compact n-manifold M with or without boundary onto Y then $B_f = L$.

Proof: We first show that L has empty interior by assuming this is not the case and arriving at a contradiction. Suppose D is a domain in L. J. W. Jaworowski (8) has shown that the involution T on X induced by the open simple map f will be continuous and hence a homeomorphism of period 2. The fixed points of T will be the points of L. Since D is a domain on which T is fixed, Lemma 4 implies that T is fixed on all of M. But this would mean that f is one to one on M and hence a homeomorphism, violating our requirement that a simple map have at least one point inverse of cardinality two. This contradiction shows that L must have empty interior. Since this is the case, Lemma 1 indicates that L is contained in B_f.

If x is in B_f, then x is a boundary point of L by Lemma 2. Thus there is a sequence $\{x_n\}$ of points contained in L converging to x. Therefore, by continuity and openness,

$\left\{ f^{-1}f(x_n) \right\}$ converges to $f^{-1}f(x)$ which consequently must be just one point, $\left\{ x \right\}$. Thus x is in L and we have shown containment both ways, so $L = B_f$. //

Remark: Since Church and Hemmingsen (3) have shown that a strongly open map f defined on a locally compact, separable metric space and having the point inverses being sets of isolated points has the property that it is dimension preserving on closed subsets, it follows that the image Y above must be n-dimensional.

In (1), Borsuk and Molski defined a map to be elementary if its domain is metric and if there exists a positive number w so that for every two different points x and x', if $f(x) = f(x')$, then $d(x,x') \geq w$. They then show that every elementary map over a compactum is a superposition of a finite number of simple maps. They also raise the question: "Does there exist a continuous mapping of finite order which is not a superposition of a finite number of simple mappings?" Sieklucki (11) has answered this in part with the following:

Theorem 1.7: (Sieklucki) Every continuous mapping f of order $\leq k$ ($k \geq 2$) defined on a compact n-dimensional space X is a finite superposition of simple mappings.

It thus seems natural to ask:

Question 1.1: Is every open mapping f of order $\leq k$ ($k \geq 2$) defined on a compact n-dimensional space a finite superposition of open simple mappings?

That this is not the case is demonstrated by

Theorem 1.8: If the open map f between 2-manifolds without boundaries is the superposition of n open simple maps, then f is of order 2^n.

Proof: Suppose $f = f_n \circ f_{n-1} \circ \ldots \circ f_1$ where f_i is an open simple map from A_{i-1} onto A_i with $A_0 = A$ and $A_n = B$ (f maps A onto B). From a result of Whyburn (19: X; 4.4), the A_i will all be 2-manifolds. Since a 2-manifold with boundary will not map openly onto one without boundary, and since B has no boundary, the A_i must also have no boundary. Whyburn has also shown (19: X; 6.3) that the set $f_i(L_i) = f_i(B_{f_i})$ is finite since the f_i are light open maps on 2-manifolds. Thus for any open simple map between 2-manifolds without boundary, all but a finite number of point inverses contain exactly two points. We proceed by induction, assuming that any map between 2-manifolds without boundary which is the superposition of n-1 open simple maps has the property that all but a finite number of its point inverses contain exactly 2^{n-1} points. The map f will then have its point inverses containing exactly 2^n points except for the (finite number of) points in B where f_n has singleton point inverses, or the points (finite in number by the induction assumption) which are images of points where $f_{n-1} \circ \ldots \circ f_1$ has point inverses which have less than 2^{n-1} points. Thus the cardinality of $f^{-1}f(y)$ is 2^n except at a finite number of y in B, and so f is 2^n to one.//

Since a three to one map such as $w = z^3$ on the unit sphere could consequently not be a finite superposition of open simple maps, we have obtained a negative answer to Question 1.1.

In a map where such a factoring is possible, we have

Lemma 1.9: If a k to one map between compact 2-manifolds without boundary, A and B, can be factored into open simple maps $f_1, \ldots, f_s$

where f_i maps A_{i-1} onto A_i with $A_o = A$ and $A_s = B$, then $kr - n = \sum_{i=1}^{s} 2^{i-1} r_i$ where r_i is the number of singular points of f_i, r is the cardinality of B_f (i.e., the number of singular points of f), and n is the cardinality of $f^{-1}f(B_f)$.

Proof: As in Theorem 8, the A_i will all be 2-manifolds without boundary. Applying a result of Whyburn (see Theorem 2.1) to the maps f_i, we get $2\chi(A_i) - \chi(A_{i-1}) = 2r_i - n_i$ Where n_i is the cardinality of $f_i^{-1}f_i(B_{f_i})$. But for open simple maps $n_i = r_i$ because $B_{f_i} = \{x : f_i^{-1}f_i(x) = \{x\}\}$ by Lemma 6. Hence $2\chi(A_i) - \chi(A_{i-1}) = r_i$. Theorem 2.1 also gives us $kr - n = k\chi(B) - \chi(A) = 2^s\chi(A_s) - \chi(A_o)$. Thus we can prove our lemma by showing that $\chi(A_o) = 2^s\chi(A_s) - \sum_{i=1}^{s} 2^{i-1} r_i$. If $s = 1$, this is true from our earlier remark on f_i. Proceeding inductively, we assume that $\chi(A_o) = 2^{s-1}\chi(A_{s-1}) - \sum_{i=1}^{s-1} 2^{i-1} r_i$. Since

$2\chi(A_s) - \chi(A_{s-1}) = r_s$, we have $\chi(A_o) = 2^{s-1}(2\chi(A_s) - r_s) - \sum_{i=1}^{s-1} 2^{i-1} r_i = 2^s\chi(A_s) - \sum_{i=1}^{s} 2^{i-1} r_i$. //

As we mentioned earlier, Jaworowski has shown that an open simple map on a compact space induces an involution and vice versa. Thus there is a direct connection between the study of open simple maps and that of homeomorphisms of period 2. In view of Lemma 6, the singular set of an open simple map on a compact space corresponds to the fixed point set of the induced involution. The next theorem indicates that there is also a relationship between homeomorphisms of period 2^k and maps which can be factored into open simple ones. We first prove a necessary lemma.

<u>Lemma 1.10</u>: Let h be a homeomorphism of finite period p defined on a compact space X. Then the orbit map f induced by h is open.

Proof: We will use the following characterization of open maps on compact spaces: "If f is a map on a compact space, then it is open if and only if given any open set U, the set $f^{-1}f(U)$ is also open." Let U be open in X. Then $f^{-1}f(U) = \{ x : f(x)$ is in $f(U)\} = \{x : h^i(x)$ is in U for some i $\}$, since h has finite period. But this is exactly the set $U \cup h(U) \cup \dots \cup h^{p-1}(U)$ which is clearly open since h is a homeomorphism. Therefore f is an open map. //

<u>Theorem 1.11</u>: Let h be a periodic homeomorphism of period 2^k defined on a compact space X. Then the orbit map f associated with h can be written as the superposition of k open simple maps.

Proof: (By induction) If k = 1, then Jaworowski's result shows the orbit map is itself open and simple. We assume the result is true for homeomorphisms of period 2^{k-1}. Let h be of period 2^k. Then h^2 is a homeomorphism of period 2^{k-1} and so the orbit map f_2 of it can be written as the superposition of k-1 open simple maps; $f_2 = g_{k-1} \circ \dots \circ g_1$. Define g_k by $g_k(y) = f(f_2^{-1}(y))$ which maps the orbit space of h^2 onto that of h.

Claim: g_k is an open simple map.

(1) g_k is well defined since for any x, the h^2-orbit of x is contained in the h-orbit of x.

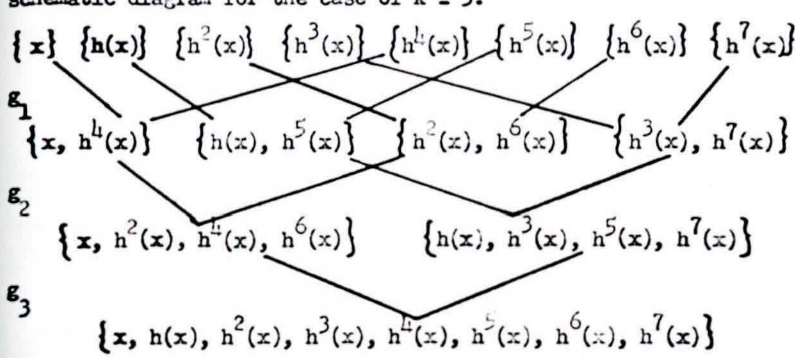
(2) g_k is open since f_2 is continuous and f, being an orbit map of a homeomorphism of finite period, is open by Lemma 10.

(3) g_k is continuous since $g_k^{-1}(U) = f_2(f^{-1}(U))$ will be open if U is open because f is continuous and f_2 is open.

(4) g_k is simple since if y is in the h-orbit of x, $g_k^{-1}(y) = f_2(f^{-1}(y))$ is the set consisting of the h^2-orbit of x unioned with the set consisting of the h^2-orbit of h(x).

Consequently, g_k is open and simple and so $f = g_k \circ g_{k-1} \circ \ldots \circ g_1$.//

Note that at any stage, the map $g_i \circ \ldots \circ g_1$ is the orbit map of $h^{2^{k-i}}$. To illustrate what is happening, consider the following schematic diagram for the case of k = 3:



From top to bottom the rows represent the h^8(identity)-orbits, h^4-orbits, h^2-orbits, and h-orbits of x respectively.

If we turn our attention from open simple maps to open maps of other order, the following generalization of Theorem 11 occurs:

<u>Theorem 1.12</u>: Let h be a periodic homeomorphism of period p^k defined on a compact space X. Then the orbit map f associated with h can be written as the superposition of k open maps of order p.

Proof: (By induction) If k = 1, then the orbit map is an open map by Lemma 10 and obviously has order p. Assume the theorem is true up to k-1. If h is a homeomorphism of period p^k, then h^p

is one of period p^{k-1} and so its orbit map can be factored, by the induction assumption, into k-1 open maps of order p. Thus $f_p = g_{k-1} \circ \cdots \circ g_1$. Define g_k by $g_k(y) = f(f_p^{-1}(y))$ which maps the orbit space of h^p onto that of h.

Claim: g_k is an open map of order p.

(1) g_k is well defined since for any x, the h^p-orbit of x is contained in the h-orbit of x.

(2) g_k is open since f_p is continuous and f is open by Lemma 10 and so $g_k(U) = f(f_p^{-1}(U))$ will be open whenever U is.

(3) g_k is continuous since f is continuous and f_p, being the orbit map of h^p, is open by Lemma 10 and so $g_k^{-1}(U) = f_p(f^{-1}(U))$ will be open whenever U is.

(4) g_k has order p, since if y is the h-orbit of x, then $g_k^{-1}(y)$ is the set consisting of the h^p-orbits of x, h(x), ..., and $h^{p-1}(x)$.

Thus g_k is open and of order p and $f = g_k \circ g_{k-1} \circ \cdots \circ g_1$. //

An even better result along the same lines as Theorems 11 and 12:

<u>Theorem 1.13</u>: Let h be a periodic homeomorphism of period $p = p_1 p_2 \cdots p_k$ defined on a compact space X. Then the orbit map f associated with h can be written as the superposition of k open maps of orders $p_1, p_2, \ldots, p_k$.

Proof: (By induction) The case k = 1 is true since the orbit map f will be open by Lemma 10 and has order p_1. Assume the theorem is true up to k-1 and let h be as specified in the hypothesis. Then h^{p_k} is a homeomorphism of period $p_1 p_2 \cdots p_{k-1}$ and so its orbit map $f_{p_k} = g_{k-1} \circ \cdots \circ g_1$ is the superposition of open maps of orders $p_1, p_2, \ldots, p_{k-1}$. Define g_k by $g_k(y) = f(f_{p_k}^{-1}(y))$. By arguments identical to those in Theorems 11 and 12, we can show that g_k is open and of order p_k and so $f = g_k \circ g_{k-1} \circ \ldots \circ g_1$ satisfies our conclusion. //

We will say that a map f of X onto itself is <u>equivalent to an action by a group</u> G if G is a group (under composition) of homeomorphisms of X onto itself such that for any x in X, the orbit of x, denoted $o(x)$, is equal to $f^{-1}f(x)$. A general question to be asked is

<u>Question 1.2</u>: Under what conditions is a function equivalent to an action by a finite group?

The following theorem indicates a necessary condition for this:

<u>Theorem 1.14</u>: If f is a map of a compact space X onto itself which is equivalent to an action by a finite group G, then f is a finite to one open map.

Proof: Let $G = \{g_i\}_{i=1}^{n}$. Then for each x, $f^{-1}f(x) = \{g_1(x), g_2(x), \ldots, g_n(x)\}$ and so f is n to one. We will show f is open by showing that for any open set U in X, $f^{-1}f(U)$ is also open. This is so because $f^{-1}f(U) = \bigcup_{x \in U} f^{-1}f(x) = \bigcup_{x \in U} o(x) = \{ y \in X : y = g_i(x)$ for some $1 \leq i \leq n$ and some x in $U\} = \bigcup_{i=1}^{n} g_i(U)$ which is open since $g_i(U)$ is open for each i because the g_i are homeomorphisms. Thus f is an open map. //

<u>Theorem 1.15</u>: Let f be a map of a compact space X onto itself which is equivalent to an action by a cyclic group of order $p = p_1 p_2 \ldots p_k$. Then f can be factored into k open maps of orders p_1, $p_2, \ldots,$ and p_k.

Proof: The proof will follow those of Theorems 11, 12, and 13, and will be by induction. If $k = 1$, then by Theorem 14, f is itself an open map of order p_1 in which case the conclusion is satisfied. We assume the result to be true up to k-1. Since the group is cyclic, there is a homeomorphism h in it which is the generator of the group. Let f_{p_k} be the orbit map of h^{p_k}. Then it is equivalent to the cyclic group $G/(h^{p_k})$ of order $p_1 p_2 \cdots p_{k-1}$. Thus f_{p_k} can be factored into maps of orders p_1, $p_2, \ldots, p_{k-1}$ by the induction assumption, so $f_{p_k} = g_{k-1} \circ \cdots \circ g_1$. We define $g_k(y)$ to be $f(f_{p_k}^{-1}(y))$. By the same arguments used in Theorems 11 and 12, g_k will be an open map of order p_k and hence $f = g_k \circ g_{k-1} \circ \cdots \circ g_1$ satisfies the conditions of our conclusion.//

Corollary 1.16: Let f be a map of a compact space X onto itself which is equivalent to an action by a cyclic group of order 2^k. Then f can be factored into k open simple maps.

CHAPTER II

EXTENSIONS OF A THEOREM OF WHYBURN

S. Stoilow (14, 15) began the study of light open maps on manifolds by analyzing them for the case where the domain and range were both regions on a 2-sphere or plane. Whyburn continued this with an extensive study of light open maps on 2-manifolds (19: X), showing, among other things, that they are finite to one and that the image is necessarily a 2-manifold. He also showed that a light open map between 2-manifolds without boundary is locally $w = z^k$ for some k and that one between compact 2-manifolds is simplicial. The result with which we shall be interested establishes a relationship between the degree (order) of the map, the Euler characteristics of the spaces involved, and the cardinalities of the singular set and its inverse. E. E. Floyd (5) has proven a similar appearing formula for periodic homeomorphisms of prime period on certain spaces. Whyburn's result when restricted to the case of a map between compact manifolds without boundary says the following (19: X; 7.3):

<u>Theorem 2.1:</u> (Whyburn) If A and B are compact 2-dimensional manifolds without boundary and $f(A) = B$ is a light open map of degree k (i.e., is k to one), then $k\chi(B) - \chi(A) = kr - n$, where r and n are the numbers of points in Y and $f^{-1}(Y)$, respectively, when Y is the set of all y in B such that $f^{-1}(y)$ contains a point of A at which f is not locally topological (i.e., Y is the image of the singular set of f) and $\chi(Z)$ represents the Euler characteristic of Z.

We can easily obtain an extension of this theorem to n-manifolds without boundary if we limit ourselves to the special case of f being an open simple simplicial map.

Theorem 2.2: If f is an open simple simplicial map between n-manifolds without boundary, A and B, then $2\chi(B) - \chi(A) = \chi(Y)$.

Proof: By Lemma 1.6, Y will be the set of points in B whose inverses consist of exactly one point. Now let K and H be triangulations of A and B making f simplicial. If a relative interior point of a k-simplex is in Y, then this simplex has only one k-simplex mapping onto it and hence the whole k-simplex is in Y. Because of openness all subsimplices of it must also be in Y and hence Y is a subcomplex of H. Let a_i, b_i, and y_i be the number of i-simplices in K, H, and Y, respectively. For each i-simplex in H, there are either one or two i-simplices in K which map onto it. If there is only one, then the simplex is in Y. Thus $a_i = 2b_i - y_i$ or $2b_i - a_i = y_i$. We then get

$$\sum_{i=0}^{n} (-1)^i (2b_i - a_i) = \sum_{i=0}^{n} (-1)^i y_i \text{ or } 2\chi(B) - \chi(A) = \chi(Y). \ //$$

Corollary 2.3: There is no open simple map of a 2-torus with n handles ($n \geq 2$) onto itself.

Proof: The Euler characteristic for such a space is negative, being $-2(n - 1)$, which would force the Euler characteristic of Y, which is just the number of singular points of f, to be negative. Clearly this is impossible. //

Our main effort in this chapter will be to get a formula similar to Whyburn's (although more involved) when f is a light open map onto an oriented 2-manifold from a space consisting of a finite number of oriented 2-manifolds intersecting each other only in a finite number of disjoint arcs. We will show that f will be k to one for some k. For notation, let Y denote the set of all points in B whose inverses contain a point where f/A_i is not locally topological for some i. Let r be the number of points in Y and n the number in $f^{-1}(Y)$. Furthermore, let Q denote $f^{-1}(Y)\cdot(\bigcup_{i\neq j}(A_i\cdot A_j))$, where V·W represents the intersection of the sets V and W. We give the statement of our main theorem now, but defer the proof until the end of the chapter.

Theorem 2.4: Let f be a light open map of $\bigcup_{i=1}^{n}$ onto B, where the A_i and B are oriented 2-manifolds without boundary and $\bigcup_{\substack{i=1 \\ i\neq j}}^{n}(A_i\cdot A_j)$ is a finite union of disjoint arcs. Then

$$k\chi(B) - \sum_{i=1}^{n}\chi(A_i) = kr - n - \sum_{x\in Q} d(x)$$

where $d(x) = -1 + \#\{A_i : x \text{ is in } A_i\}$ (i.e., the number of duplications we get by counting x in each of the manifolds containing it.)

We begin working toward this result by considering a rather restricted case of it.

Theorem 2.5: Suppose f is a k to one open simplicial map of $A_1 \cup A_2$ onto B where A_1, A_2, and B are 2-manifolds without boundary, and $A_1\cdot A_2$ is the union of a finite number of disjoint arcs. Then $k\chi(B) - \chi(A_1) - \chi(A_2) = kr - n - \#(f^{-1}(Y)\cdot A_1\cdot A_2)$.

We will prove this by showing f restricted to each of the A_i maps it openly onto B and then applying Whyburn's result, Theorem 1.

Theorem 2.6: Let H and K be 2-complexes with underlying spaces X and Y which are 2-manifolds without boundary. Let f be a simplicial map of X into Y taking each simplex of H homeomorphically to one of K. Then the set S of all 1-simplices in H where f is not locally one to one (where folding occurs) has the property that S^* has no endpoints, where S^* denotes the set of all points belonging to a member of S.

Proof: Suppose p were an endpoint of S^* and let t_o be the 1-simplex containing p at which f is not locally one to one. Since X is a 2-manifold without boundary, the 2-simplices containing p form a 2-ball. Label the 2-simplices containing p $s_1, s_2, \ldots, s_n$ in clockwise order starting and ending with ones containing t_o. Let t_i denote the 1-simplex that is the common face of s_i and s_{i+1}. Since f is not locally one to one at t_o, s_1 and s_n get mapped to the same 2-simplex in K. Since t_i $(i= 1, 2, \ldots, n-1)$ is not in S, s_i and s_{i+1} always get mapped to different 2-simplices. Because each 1-simplex in K has exactly two 2-simplices containing it and $f(s_1) = f(s_n)$, the simplices s_2 and s_{n-1} must both get mapped to the other 2-simplex which contains $f(t_1) = f(t_{n-1})$; i.e., $f(s_2) = f(s_{n-1})$. The same argument will show that $f(s_3) = f(s_{n-2}), \ldots, f(s_i) = f(s_{n-i+1})$ for $i \leq n/2$. If n is even we get $f(s_{n/2}) = f(s_{n/2+1})$, contrary to our earlier assertion about consecutive simplices. If n is odd then $f(s_{(n-1)/2}) = f(s_{(n+3)/2})$. The 2-simplices $s_{(n-1)/2}$ and $s_{(n+3)/2}$ each abut on $s_{(n+1)/2}$ and so by our construction the faces $t_{(n-1)/2}$ and $t_{(n+1)/2}$ of $s_{(n+1)/2}$ will both be mapped to the same 1-simplex in K, which contradicts f being a simplicial map. Thus whether n is even or odd we are led to a contradiction and hence S^* has no endpoints. //

Corollary 2.7: Let f be a k to one open simplicial map of X onto Y where X is the union of a finite number of 2-manifolds without boundary, A_i, so that the union of their pairwise intersections is at most a finite number of pairwise disjoint arcs and Y is also a 2-manifold without boundary. Then $f/_{A_i}$ maps each A_i openly onto Y.

Proof: Since Y has no boundary and f is open on X, the only 1-simplices where f might not be locally one to one would have to be contained in the intersecting arcs. Restricting our attention to $f_i = f/_{A_i}$, the set S of Theorem 6 must be a finite union of arcs and hence must be empty since it has no endpoints. If p is in the interior of a 2-simplex in H_i, then f being simplicial insures that f(p) is an interior point of the image of any A_i-neighborhood of p. If p is an interior point of a 1-simplex in H_i, then S being empty guarantees that f_i is open there. If p were a vertex and f_i were not open there, then the image of the star neighborhood of p in A_i would have to miss a 2-simplex in the star neighborhood of f(p). Hence the image would have a free edge. But the star neighborhood of p in A_i has no free edges and so S would have to be non-empty which we have shown is not the case. Therefore f_i is open. Whyburn has shown that the light open image of a 2-manifold is itself a 2-manifold (19: X ; 4.4); so $f(A_i)$ is a 2-manifold. Since S is empty, it must be one without boundary and consequently $f(A_i) = Y$ since otherwise, being open and compact, it would produce a separation of the connected space Y. //

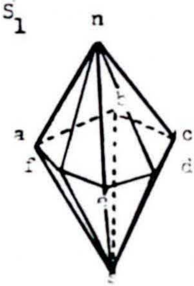
Proof of Theorem 5: Let Y_i be the set of points in B whose inverses contain a point of A_i where f_i is not locally one to one. Let r_i be the number of points in Y_i and n_i be the number in $f_i^{-1}(Y_i)$. Clearly, $Y = Y_1 \cup Y_2 = Y_1' \cup Y_2' \cup (Y_1 \cdot Y_2)$ where $Y_1' = Y_1 - Y_2$ and $Y_2' = Y_2 - Y_1$ and the last union is a disjoint one. Let r_i' and n_i' be the cardinalities of Y_i' and $f_i^{-1}(Y_i')$ and let k_i be the degree of f_i. Whyburn has shown that if f_i is locally one to one at each point of $f_i^{-1}(p)$ then $\# f_i^{-1}(p) = k_i$ (19: X; 6.3). If y is in Y_1', then it has k_2 preimages in A_2 and a certain number of preimages in A_1, with any preimages that are in $A_1 \cdot A_2$ having been counted twice. Hence, in $f^{-1}(Y_1')$ there are $k_2 r_1'$ preimages in A_2, $n_1 - \#(f^{-1}(Y_1 \cdot Y_2) \cdot A_1)$ preimages in A_1 (since n_1 is the number of preimages of Y_1 in A_1 and $\#(f^{-1}(Y_1 \cdot Y_2) \cdot A_1)$ is the number of preimages of $Y_1 \cdot Y_2$ in A_1), and $\#(f^{-1}(Y_1') \cdot A_1 \cdot A_2)$ preimages which are in both. Thus $\# f^{-1}(Y_1') = k_2 r_1' + n_1 - \#(f^{-1}(Y_1 \cdot Y_2) \cdot A_1) - \#(f^{-1}(Y_1') \cdot A_1 \cdot A_2)$ and $\# f^{-1}(Y_2') = k_1 r_2' + n_2 - \#(f^{-1}(Y_1 \cdot Y_2) \cdot A_2) - \#(f^{-1}(Y_2') \cdot A_1 \cdot A_2)$ by a similar argument. Moreover, $\# f^{-1}(Y_1 \cdot Y_2) = \#(f^{-1}(Y_1 \cdot Y_2) \cdot A_1) + \#(f^{-1}(Y_1 \cdot Y_2) \cdot A_2) - \#(f^{-1}(Y_1 \cdot Y_2) \cdot A_1 \cdot A_2)$. Since Y is the disjoint union of Y_1', Y_2', and $Y_1 \cdot Y_2$, the cardinality of $f^{-1}(Y)$ is the sum of these three cardinalities; hence $n = \# f^{-1}(Y) = k_2 r_1' + n_1 - \#(f^{-1}(Y_1') \cdot A_1 \cdot A_2) + k_1 r_2' + n_2 - \#(f^{-1}(Y_2') \cdot A_1 \cdot A_2) - \#(f^{-1}(Y_1 \cdot Y_2) \cdot A_1 \cdot A_2)$. Again because of the disjoint union we have $n = k_2 r_1' + k_1 r_2' + n_1 + n_2 - \#(f^{-1}(Y) \cdot A_1 \cdot A_2)$. But $r_1' = \#(Y_1 - Y_2)$ will be the number, r_1, of points in Y_1 minus the number in $Y_1 \cdot Y_2$ and similarly for r_2'. Therefore $n = k_2 r_1 - k_2(\#(Y_1 \cdot Y_2)) +$

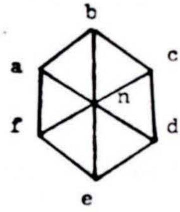
$k_1r_2 - k_1(\#(Y_1 \cdot Y_2)) + n_1 + n_2 - \#(f^{-1}(Y) \cdot A_1 \cdot A_2) = k_2r_1 + k_1r_2 - k(\#(Y_1 \cdot Y_2)) + n_1 + n_2 - \#(f^{-1}(Y) \cdot A_1 \cdot A_2)$, since $k = k_1 + k_2$. For this reason, too, we now get $n = kr_1 - k_1r_1 + kr_2 - k_2r_2 - k(\#(Y_1 \cdot Y_2)) + n_1 + n_2 - \#(f^{-1}(Y) \cdot A_1 \cdot A_2)$. But $r = r_1 + r_2 - \#(Y_1 \cdot Y_2)$, and so $n = kr - k_1r_1 - k_2r_2 + n_1 + n_2 - \#(f^{-1}(Y) \cdot A_1 \cdot A_2)$. Rearranging terms, we have $kr - n = k_1r_1 - n_1 + k_2r_2 - n_2 + \#(f^{-1}(Y) \cdot A_1 \cdot A_2)$. Since f_i is an open map onto B, Theorem 1 is applicable and we get $kr - n = k_1 \chi(B) - \chi(A_1) + k_2 \chi(B) - \chi(A_2) + \#(f^{-1}(Y) \cdot A_1 \cdot A_2)$ or $k\chi(B) - \chi(A_1) - \chi(A_2) = kr - n - \#(f^{-1}(Y) \cdot A_1 \cdot A_2)$, which is what we set out to prove. //

Our method of proof requires us to have f_i be open on each A_i in order to apply Theorem 1. That we need the image B to be a 2-manifold <u>without boundary</u> is evidenced by the following example of an open map from a union of two 2-spheres joined on an arc onto a disc which is not open when restricted to each of the 2-spheres.

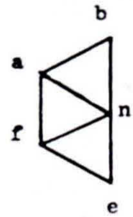
<u>Example 2.1</u>: Let X be the union of two 2-spheres S_1 and S_2 joined on an arc A(nes). We first describe f on S_1. Let S_1 have six points on its equator specified in order, a,b,c,d,e,f. These points together with the north and south poles, n and s, induce a simplicial subdivision of S_1 into twelve 2-simplices. To describe f on S_1, we first reflect the lower hemisphere onto the upper so that sab goes to nab, sbc to nbc, and so on. Now fold first along bne so that c goes

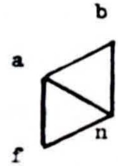
S_1





to a and d goes to f, and then fold again along A(nf) so that e goes to a. The simplicial map this induces is not open at points on the open arc A(nes) since the 2-simplices containing faces ne and es all get mapped to the same 2-simplex naf and the image of ne and se, namely the 1-simplex na, is not a free edge of abnf.





Let S_2 be labelled in the same way let f on S_2 be the map described above followed by a reflection through na so that a simplex in S_1 that went to naf will have its counterpart in S_2 going to nab, and one that went to nab will have its counterpart going to naf. Let S_1 and S_2 have the arc A(nes) in common. Since the 2-simplices in S_1 containing ne and es go to naf, while those in S_2 containing them go to nab, the map f will be open along the arc A(nes), and hence everywhere, even though the restriction of it to S_1 or S_2 is not open. /

Generalizing Theorem 5, we get

<u>Theorem 2.8</u>: Let f be a k to one open simplicial map of $X = \bigcup_{i=1}^{m} A_i$ onto B where B and the A_i are 2-manifolds without boundary and $\bigcup_{i \neq j} (A_i \cdot A_j)$ is a finite number of disjoint arcs. Then $k\,\chi(B) - \sum_{i=1}^{m} \chi(A_i) = kr - n - \sum_{x \in Q} d(x)$ where $d(x) = -1 + \#\{A_i : x \in A_i\}$ and $Q = f^{-1}(Y) \cdot (\bigcup_{i \neq j} (A_i \cdot A_j))$.

Proof: (By induction) From Corollary 7, $f_i = f/_{A_i}$ will map A_i

openly onto B. If $m = 2$, $\sum_{x \in Q} d(x)$ is simply $\#Q = \#(f^{-1}(Y) \cdot A_1 \cdot A_2)$ and so our conclusion holds from Theorem 5. Now suppose the theorem is true up to m-1 and let Y' denote Y on $\bigcup_{i=1}^{m-1} A_i = X'$ with k', r', n', d', and Q' playing corresponding roles. Then $Y = Y' \cup Y_m = (Y'-Y_m) \cup (Y_m-Y') \cup (Y' \cdot Y_m)$ where this last is a disjoint union. Hence <u>(1*)</u> $n = \#f^{-1}(Y) = \#(f^{-1}(Y'-Y_m)) + \#(f^{-1}(Y_m-Y')) + \#(f^{-1}(Y' \cdot Y_m))$. To count the points in $f^{-1}(Y'-Y_m)$ we can count those in A_m, add those in $\bigcup_{i=1}^{m-1} A_i$, and subtract those in both. Since f_m is locally one to one at all points of $f_m^{-1}(Y'-Y_m)$, there will be $k_m(\#(Y'-Y_m))$ such points in A_m. Since there are n' points in $f^{-1}(Y') \cdot \bigcup_{i=1}^{m-1} A_i$, there will be $n' - \#(f^{-1}(Y' \cdot Y_m) \cdot \bigcup_{i=1}^{m-1} A_i)$ points from $f^{-1}(Y'-Y_m)$ in $\bigcup_{i=1}^{m-1} A_i$. Consequently <u>(2*)</u> $\#(f^{-1}(Y'-Y_m)) = k_m(\#(Y'-Y_m)) + n' - \#(f^{-1}(Y' \cdot Y_m) \cdot \bigcup_{i=1}^{m-1} A_i) - \#(f^{-1}(Y'-Y_m) \cdot (\bigcup_{i=1}^{m-1} (A_m \cdot A_i)))$, the last term representing those points in both A_m and X'. Considering $f^{-1}(Y_m-Y')$, it has through similar reasoning, $n_m - \#(f^{-1}(Y_m \cdot Y') \cdot A_m)$ points in A_m and $\#(f^{-1}(Y_m-Y') \cdot \bigcup_{i=1}^{m-1} (A_m \cdot A_i))$ points in both A_m and $\bigcup_{i=1}^{m-1} A_i$. As for the number of points it has in $\bigcup_{i=1}^{m-1} A_i$, since points in Y_m-Y' are not in Y', they have k_i preimages in each of the A_i $(i = 1,2,\ldots,m-1)$. Taking into account the duplications we get from preimages that are in more than one A_i, and the fact that $k' = k_1 + \ldots + k_{m-1}$, we find that

$f^{-1}(Y_m-Y')$ has $k'(\#(Y_m-Y')) - \sum_{x\in R} d'(x)$ points in $\bigcup_{i=1}^{m-1} A_i$ where $R = f^{-1}(Y_m-Y')\cdot \bigcup_{\substack{i\neq j\\ i=1}}^{m-1} (A_i\cdot A_j)$. Thus <u>(3*)</u> $\#(f^{-1}(Y_m-Y')) =$

$k'(\#(Y_m-Y')) - \sum_{x\in R} d'(x) + n_m - \#(f^{-1}(Y_m\cdot Y')\cdot A_m) -$

$\#(f^{-1}(Y_m-Y')\cdot \bigcup_{i=1}^{m-1} (A_i\cdot A_m))$. Similarly, if we write $\#(f^{-1}(Y_m\cdot Y'))$ as the number of points in A_m plus the number in $\bigcup_{i=1}^{m-1} A_i$ minus the number in both, we get <u>(4*)</u> $\#(f^{-1}(Y_m\cdot Y')) = \#(f^{-1}(Y_m\cdot Y')\cdot A_m) +$ $\#(f^{-1}(Y_m\cdot Y')\cdot \bigcup_{i=1}^{m-1} A_i) - \#(f^{-1}(Y_m\cdot Y')\cdot \bigcup_{i=1}^{m-1} (A_m\cdot A_i))$. Combining (1*), (2*), (3*), and (4*), we find that $n = k_m(\#(Y'-Y_m)) + n' +$

$k'(\#(Y_m-Y')) + n_m - \sum_{x\in R} d'(x) \;- \#(f^{-1}(Y)\cdot \bigcup_{i=1}^{m-1} (A_m\cdot A_i))$, the last term coming from the fact that Y is the disjoint union of Y_m-Y', $Y'-Y_m$, and $Y_m\cdot Y'$. Then $n = k_m(r' - \#(Y_m\cdot Y')) + n' + k'(r_m -$ $\#(Y_m\cdot Y')) + n_m - \sum_{x\in R} d'(x) \;- \#(f^{-1}(Y)\cdot \bigcup_{i=1}^{m-1} (A_m\cdot A_i))$. Since $k_m + k' = k$, we have $n = kr' - k'r' - k(\#(Y_m\cdot Y')) + k'(\#(Y_m\cdot Y'))$ $+ n' + kr_m - k_m r_m - k(\#(Y_m\cdot Y')) + k_m(\#(Y_m\cdot Y')) + n_m - \sum_{x\in R} d'(x) -$ $\#(f^{-1}(Y)\cdot \bigcup_{i=1}^{m-1} (A_m\cdot A_i))$. Moreover, since $r = r' - \#(Y'\cdot Y_m)$, we get $n = kr - k'r' + n' - k_m r_m + n_m - \sum_{x\in R} d'(x) - \#(f^{-1}(Y)\cdot \bigcup_{i=1}^{m-1} (A_m\cdot A_i))$. Using Theorem 1 on A_m and the induction assumption, this becomes <u>(5*)</u> $n = kr - k'\chi(B) + \sum_{i=1}^{m-1} \chi(A_i) \;- \sum_{x\in Q'} d'(x) \;- k_m\chi(B) +$

$\chi(A_m) - \sum_{x\in R} d'(x) \;- \#(f^{-1}(Y)\cdot \bigcup_{i=1}^{m-1} (A_m\cdot A_i))$. Since $Q' =$ $f^{-1}(Y')\cdot \bigcup_{\substack{i\neq j\\ j,i=1}}^{m-1} (A_i\cdot A_j)$ and $R = f^{-1}(Y_m-Y')\cdot \bigcup_{\substack{i\neq j\\ j,i=1}}^{m-1} (A_i\cdot A_j)$, they are disjoint and their union is $f^{-1}(Y)\cdot \bigcup_{\substack{i\neq j\\ j,i=1}}^{m-1} (A_i\cdot A_j)$. We make the following

<u>Claim:</u> $\sum_{x\in R} d'(x) + \sum_{x\in Q'} d'(x) + \#(f^{-1}(Y)\cdot \bigcup_{i=1}^{m-1}(A_m\cdot A_i)) = \sum_{x\in Q} d(x).$

Proof: Let x be a point of Q. Since $d(x)$ is one less than the number of A_i $(i\le m)$ containing x and $d'(x)$ is one less than the number for $i \le m-1$, if x is in $Q-A_m$ then $d'(x) = d(x)$ and if x is in $Q\cdot A_m$, then $d(x) = d'(x) + 1$. Thus $\sum_{x\in Q} d(x) = \sum_{x\in Q} d'(x) + \#(Q\cdot A_m)$. But Q' and R together give all points of Q except those which are in A_m and exactly one other A_i. For these points, $d'(x) = 0$ and so $\sum_{x\in Q} d'(x) = \sum_{x\in Q'} d'(x) + \sum_{x\in R} d'(x)$. Consequently, $\sum_{x\in Q} d(x) = \sum_{x\in Q'} d'(x) + \sum_{x\in R} d'(x) + \#(Q\cdot A_m)$. But $Q\cdot A_m = (f^{-1}(Y)\cdot \bigcup_{\substack{i\neq j\\ j,i=1}}^{m}(A_i\cdot A_j))\cdot A_m = f^{-1}(Y)\cdot \bigcup_{i=1}^{m-1}(A_i\cdot A_m)$, proving our claim. /

Combining our claim with (5*) and the fact that $k = k' + k_m$, we get $n = kr - k\chi(B) + \sum_{i=1}^{m} \chi(A_i) - \sum_{x\in Q} d(x)$. Rearranging the terms we get the conclusion of our theorem. //

In order to get a more general result than Theorem 8 by dropping the k to one and the simplicial conditions, we first show that a light open map between spaces of the type we are considering is necessarily finite to one.

<u>Theorem 2.9:</u> Let f be a light open map of $X = \bigcup_{i=1}^{m} A_i$ onto B where B and the A_i are 2-manifolds without boundary and $\bigcup_{\substack{i\neq j\\ j,i=1}}^{m}(A_i\cdot A_j)$ is a finite number of disjoint arcs. Then f is finite to one.

Proof: Let p be a point of B and suppose that $f^{-1}(p)$ is infinite. Since B is a 2-manifold and f is light we can find $2m + 1$ discs, K_i, intersecting pairwise exactly at p, forming a $2m + 1$ petalled flower, $K = \bigcup_{i=1}^{2m+1} K_i$ and such that each component of $f^{-1}(K_i)\cdot A_j$ is contained in a Euclidean neighborhood in A_j. Then $f^{-1}(K)$ is locally connected (19: X; 3.1) and hence has just a finite number of components, each mapping onto K (19:VIII;7.41). Therefore, there is a component C of $f^{-1}(K)$ containing infinitely many points of $f^{-1}(p)$. Being a component of a locally connected set, C is open and so f restricted to it is a light open map onto K. Now $f/_C^{-1}(K_i-p)$ has just a finite number of components, each mapping onto K_i-p and so there is a component of it, R_i, having infinitely many points of $f^{-1}(p)$ in its boundary. We claim that the R_i can be chosen so that the intersection of their boundaries I contains infinitely many points of $f^{-1}(p)$. If we take all possible intersections composed of the boundaries of one of the components of $f/_C^{-1}(K_i-p)$ for each i, then $f^{-1}(p)$ is in the union of these intersections. But there are only finitely many such intersections possible since there are only finitely many such components for each of the i ($i = 1,2,\ldots, 2m+1$). Thus one of the intersections contains infinitely many points of $f^{-1}(p)$, justifying our claim. Each point of I is accessible from each of the regions R_i (i.e., can be joined by an arc to any point in R_i). But in each of the 2-manifolds, we cannot have three regions meeting in more than two points. Consequently, there are a maximum of 2m such regions in X, contradicting our having constructed 2m + 1. Thus $f^{-1}(p)$ must be finite for each p and f is finite to one. //

We wish to show that even if f is not necessarily a simplicial map, $f/_{A_1}$ is still open.

Theorem 2.10: Let f be a light open map of $X = \bigcup_{i=1}^{m} A_i$ onto B where A_i and B are oriented 2-manifolds without boundary and $\bigcup_{\substack{i \neq j \\ j,i=1}}^{m} (A_i \cdot A_j)$ is a finite number of disjoint arcs. Then $f/_{A_i}$ is open.

Proof: We will make use of the following theorem of Titus and Young (16):

Theorem 2.11: (Titus and Young) Let $\Omega = \{ f: M \Rightarrow N : M$ and N are fixed orientable manifolds satisfying (i), (ii), and (iii) below$\}$

(i) For each function f in Ω there is a closed set C_f such that f is sense preserving at each point of $M-C_f$.

(ii) Each function f in Ω is constant on each component of the interior of C_f.

(iii) For each function f in Ω, the set $f(C_f)$ is closed and nowhere dense in N.

Then every function in Ω is quasi-open.

Since light quasi-open maps are open, showing $f/_{A_i}$ is in Ω will prove our theorem. We let $M = A_i$, $N = B$, and $C_f = \bigcup_{\substack{i \neq j \\ i,j=1}}^{m} (A_i \cdot A_j)$ which is a finite collection of arcs and hence closed. Since the interior of C_f is empty, (ii) is satisfied vacuously. Since C_f is compact, $f(C_f)$ is clearly closed. Church and Hemmingsen (3) have shown that an open map defined on a locally compact space and having point inverses consisting of isolated points has the property that it preserves the dimension of closed subsets. Since Theorem 9 shows our point inverses are finite, $f(C_f)$ must be one dimensional and hence is nowhere dense in B, thus satisfying (iii). To demonstrate that f satisfies (i), we make use of the following theorem of Whyburn (19: X; 5.1):

Theorem 2.12: (Whyburn) Let f be a light open map of A onto B where A is any 2-dimensional manifold. For any point q of A there exists a closed 2-cell neighborhood E of q in A and a positive integer k such that if f(q) is a regular (non-boundary) point of B, then on E, f is topologically equivalent to the transformation $w = z^k$ on $|z| \leq 1$.

We will use this with $A = A_i - C_f$ which is an open set, so that f restricted to it is an open (and light) map. Since A is an open subset of a 2-manifold, it is itself one (although non-compact). Because f(A) is an open subset of a manifold without boundary, all its points will be regular points. Hence at each point q of A there is a closed 2-cell neighborhood D of q on which f is topologically equivalent to z^k on $|z| \leq 1$ for some k.

Claim: f is either sense preserving at each point of A or it is sense reversing at each point.

Proof: We will show that the sets of points where f are sense preserving and sense reversing are open, and since A is connected, one or the other is all of A. Let q be a point of A and D be the closed 2-cell described above. Since f is topologically equivalent to z^k on D, there exist homeomorphisms h and h' mapping respectively D and f(D) onto the unit ball B^2 in the plane so that the following diagram commutes:

$$\begin{array}{ccc} D & \xrightarrow{\quad f \quad} & f(D) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ B^2 & \xrightarrow[\quad z^k \quad]{} & B^2 \end{array}$$

If f is sense preserving at q, we may take h and h' both to be orientation preserving. Hence the degree of f at q is k. The diagram indicates then that any other point in the interior of D will have degree 1. If f is sense reversing at q, one of h and h' is orientation reversing. The degree of f at q is then -k and any other point in the interior of D has degree -1. In either case, we have that the sense of f is the same on a neighborhood of q and so $P = \{q: f \text{ is sense preserving at } q\}$ and $R = \{q: f \text{ is sense reversing at } q\}$ are both open. Since f is locally z^k $(k \neq 0)$ for each point of A, these sets cover A. Thus since A is connected, it must equal either P or R, establishing our claim. /

If f is sense preserving on A, then (i) is satisfied and f is in $\mathcal{C}$. If f is sense reversing on A, consider $f' = h''f$ where h" is an orientation reversing homeomorphism of B^2 onto B^2. This will have the same properties we have found for f, except that it will be sense preserving on A. In this case we would conclude that f' is in $\mathcal{C}$ and therefore is open; but then $f = h''^{-1}f'$ will also be open. //

Before getting our major result, we will show that f must map A_1 onto B.

Lemma 2.13: If f is a finite to one open map of a space X onto a 2-manifold without boundary, B, and A is a compact 2-manifold without boundary contained in X having its boundary relative to X containing no simple closed curve, then $f(A) = B$.

Proof: From Whyburn's results (19: X; 4.4), $f(A)$ is a 2-manifold. If it has no boundary then it clearly must equal B. Suppose C is a boundary curve of $f(A)$. Since $A - Bd_X(A)$ is open in X, its image is open in B and thus misses C. Therefore $f^{-1}(C)$ is contained in $Bd_X(A)$. But because of lifting properties of light open maps, the fact that f is finite to one and C is a simple closed curve, we must have $f^{-1}(C)$ containing a simple closed curve, contradicting our hypothesis regarding the boundary of A in X. Thus no such boundary curve C exists and consequently $f(A)$ has no boundary and must therefore be equal to B. //

We are finally ready to give a

Proof of Theorem 2.4: From Theorem 10 and Lemma 13, we know that $f/_{A_i}$ is open and onto B. Hence Whyburn has shown (19: X; 7.2) that we can get subdivisions of A_i and B so that $f/_{A_i}$ is a simplicial map. Taking a common subdivision of B and the induced subdivisions of the A_i's, we get f is a simplicial map from $\bigcup_{i=1}^{m} A_i$ onto B. Our result then follows from Theorem 8. //

CHAPTER III

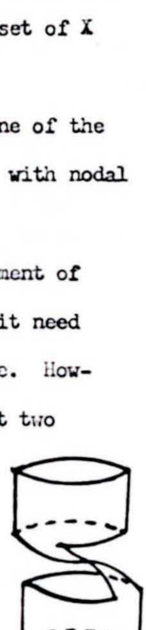
NODAL SETS

In (16) Whyburn introduces the idea of a nodal set and shows that an open mapping restricted to a nodal set has the property that its induced homomorphism maps the first rational Betti group $B^1_R(N)$ onto the first rational Betti group of $f(N)$ provided the first Betti number of the image is finite. A closed subset of a metric space X will be called a <u>nodal set</u> of X if its boundary in X contains at most one point. Such a point will be called a <u>nodal point</u>.

<u>Example 3.1</u>: Let $X = \{(x,y): y = x \text{ or } y = -x \text{ and } 0 \le y \le 1\}$ and $N = X - \{(x,y): y = x \text{ and } \frac{1}{2} < y \le 1\}$. Then N is a nodal set of X with nodal point $(\frac{1}{2},\frac{1}{2})$.

<u>Example 3.2</u>: Let X be two disjoint circles and N be one of the circles plus a point from the other. Then N is a nodal set with nodal point being the isolated point.

While it is always true that the closure of the complement of a nodal set is itself a nodal set, this example shows that it need not have a nodal point, since $\overline{X-N}$ is the whole second circle. However, if X is connected and N is nondegenerate (has at least two points), then a nodal point of N will also be one of $\overline{X-N}$.



<u>Example 3.3</u>: Let X be two "tin cans with tails" as indicated and N be just one can and tail. Note that we can get an open map of this onto a circle by projecting onto the base of the can. However, the

map restricted to the nodal set is not open.

In studying Whyburn's result during the course of a topology seminar at the State University of New York at Binghamton, the question was raised:

Question 3.1: If $f/_N$ is the restriction of an open map to a compact nodal set, is there necessarily a compact subset of N mapping onto f(N) so that f restricted to this set is open?

This is the case in all our examples so far. A negative answer will be provided later in this chapter, as well as some conditions that will assure a positive answer. First we consider a few sufficient conditions for the restriction of an open map to a nodal set to be open. Corollary 2 and Lemmas 4 and 5 will give such conditions.

Lemma 3.1: Let f be a mapping defined on a compact metric space X. If $f^{-1}f(p) = \{p\}$, then f is open at p.

Proof: Let U be an open neighborhood of p and suppose f(U) is not a neighborhood of f(p). Then there is a sequence $\{y_n\}$ converging to f(p) such that $f^{-1}(y_n)$ misses U. Let x_n be an element of $f^{-1}(y_n)$ for each n. Then the sequence $\{x_n\}$ has a convergent subsequence which we can take without loss of generality to be $\{x_n\}$. Let x be the limit of this sequence. By continuity, $\{f(x_n)\}$ converges to f(x). But $\{f(x_n)\} = \{y_n\}$ which converges to f(p), so that f(x) = f(p). Since $f^{-1}f(p) = \{p\}$, we must have x = p. However, since the x_n's all miss U which is an open neighborhood of p, x cannot equal p. This contradiction completes our proof. //

As an immediate consequence of this we have

Corollary 3.2: Let f be an open map of a compact space X onto Y and N be a nodal set of X with nodal point p. If $f_{/N}^{-1}f(p) = \{p\}$ then $f_{/N}$ is open.

Proof: Since N is a nodal set, N-p is open and hence $f_{/N}$ is open at all points of N with the possible exception of p. By our last lemma, $f_{/N}$ is open at p also and hence $f_{/N}$ is open. //

Lemma 3.3: Let f be an open map of a compact space X onto Y and N be a nodal set of X with nodal point p. If $f_{/N}^{-1}f(p)$ contains more than one point, then f(N) has no boundary relative to Y.

Proof: Since N is a nodal set, N-p is open in X and hence f(N-p) is open in Y. But N-p contains a point of $f^{-1}f(p)$ by hypothesis, so f(N-p) = f(N) which is thus open in Y. Since N is compact, f(N) is also closed in Y and hence has no boundary there.//

Lemma 3.4: Let f be an open map of a compact space X onto Y and N be a nodal set of X with nodal point p. If p is not an isolated point of $f_{/N}^{-1}f(p)$, then $f_{/N}$ is open.

Proof: As in Corollary 2, p is the only point where there is any question about openness. Let U be an open neighborhood of p relative to N. Since N is nodal, U-p is open relative to X and hence f(U-p) is open in Y. But since p is not an isolated point of $f_{/N}^{-1}f(p)$, f(U-p) = f(U) which is consequently open. Therefore, $f_{/N}$ is open. //

Combining the contrapositive of Lemma 3 with Corollary 2 yields

Lemma 3.5: Let f be an open map of a compact space X onto Y and N be a nodal set of X with nodal point p. If the boundary of f(N) in Y is non-empty, then $f_{/N}$ is open.

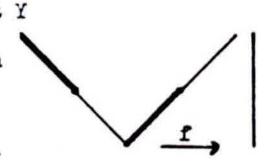
Proof: Since the boundary of $f(N)$ in Y is non-empty, Lemma 3 implies that $f^{-1}_{/N}f(p)$ contains only one point p and hence by Corollary 2, $f_{/N}$ is open. //

Having established these sufficient conditions for $f_{/N}$ to be open and hence for an affirmative answer to our question, we now turn our attention to the situation when $f_{/N}$ is not open. In an unsuccessful effort to show that we could chop open sets away from points where f was not open and eventually get back to a compact set which mapped openly onto $f(N)$, we showed

Lemma 3.6: Let f be an open map of a compact space X onto Y and N be a nodal set of X with nodal point p. If $f_{/N}$ is not open at p, then there exists an open neighborhood U of p such that $f(N-U) = f(N)$.

Proof: Since $f_{/N}$ is not open, Corollary 2 indicates that there is a q in $f^{-1}_{/N}f(p)$ which is different from p. If for all neighborhoods U of p, $f(N-U) \neq f(N)$, we could take $1/n$ neighborhoods of p and get a sequence of points $\{y_n\}$ in $f(N)$ such that $f^{-1}_{/N}(y_n)$ is contained in the $1/n$ neighborhood of p. By picking k so that $1/k$ is less than half the distance between p and q, we have, for all n greater than k, that $f^{-1}_{/N}(y_n)$ misses the $1/k$ neighborhood of q. But $\{y_n\}$ converges to $f(p) = f(q)$, so $f(q)$ is not an interior point of the image of the $1/k$ neighborhood of q. This contradicts the fact that $f_{/N}$ is open at q. Thus there is a neighborhood U of p so that $f(N-U) = f(N)$. //

We recognize that N-U, while compact is not necessarily a nodal set. To continue the process started above, we might hope to consider the set B of points where $f_{/N-U}$ is not open, throw away a neighborhood of it, and still map onto f(N). That this is not the case is illustrated by the following



Example 3.4: Let X be as in Example 1 and let Y be the closed unit interval with f being projection onto the second coordinate. Let C be the compact subset of X consisting of the part of $y = -x$ having $y > \frac{1}{2}$ and the part of $y = x$ having $y \leq \frac{1}{2}$. $f_{/C}$ is open except at $B = \left\{ (-\frac{1}{2},\frac{1}{2}),\ (\frac{1}{2},\frac{1}{2}) \right\}$. But clearly if U is any neighborhood of B then $\frac{1}{2}$ is not in f(C-U) although it is in f(C). So $f(C-U) \neq f(C)$.

Lemma 3.7: Let f be an open map of a compact space X onto Y and N be a nodal set with nodal point p. If $f_{/N}$ is not open at p, then either $\{ p \}$ is open in N or f(p) is a local separating point of f(N).

Proof: Suppose $\{ p \}$ is not an open set in N. Since $f_{/N}$ is not open at p, there is a neighborhood (relative to X) of p, U, so that $f(U \cdot N)$ is not open, and hence $(f^{-1}_{/N}f(p)) \cdot U = \{ p \}$ by an argument like that in Lemma 3.

Claim: U can be chosen so that (a) f(U) is contained in f(N); and (b) f(Bd U) misses f(U).

Proof: (a) is true because $U \cdot f^{-1}f(N)$ is an open neighborhood of p since f(N) is open by Lemma 3.

(b) Let U' denote $U - f^{-1}f(\text{Bd } U)$. Since $f^{-1}f(p)$ is compact and totally disconnected, it is also zero dimensional and so we can choose U so that the boundary of U misses $f^{-1}f(p)$. (7) Thus U' is still a neighborhood of p. Now for any sets, if $R = S-T$, then Bd R is contained in (Bd S) $\cup$ $\overline{T}$. Thus Bd U' is contained in (Bd U) $\cup$ $f^{-1}f$(Bd U), so f(Bd U') is contained in f(Bd U). By our choice of U', f(U') misses f(Bd U) and hence f(Bd U'). /

We will show that f(p) separates f(U) by showing $f(U\cdot N) - f(p)$ is open and closed in $f(U) - f(p)$. Since $f/_N$ is open at all points of N except p, $f(U\cdot N) - f(p)$ is clearly open in $f(U) - f(p)$. To see that $f(U\cdot N) - f(p)$ is closed in $f(U) - f(p)$, let y be a limit point of the former set relative to the latter. There exists a sequence $\{y_n\}$ converging to y and corresponding $\{x_n\}$ in $U\cdot N - p$. By compactness of N, we can assume the sequence $\{x_n\}$ converges, say to x. Since $f(x) = y \neq f(p)$, we know $x \neq p$. Thus x is in $\overline{U}\cdot N - p$. However, x is not in Bd U by (b) since y is in f(U). Thus x is in $U\cdot N$ and hence y is in $f(U\cdot N) - f(p)$. This proves that $f(U\cdot N) - f(p)$ is closed in $f(U) - f(p)$.

Since we are assuming that $\{p\}$ is not open in N, $U\cdot N$ is non-degenerate and hence $f(U\cdot N) - f(p)$ is non-empty. Moreover, the complement of it in $f(U) - f(p)$ is also non-empty since $f(U\cdot N)$ being not open implies $f(U\cdot N) \neq f(U)$. Thus f(p) separates the neighborhood f(U) of it and consequently locally separates f(N), provided $\{p\}$ is not open in N. //

We note that if $\{p\}$ is open in N, then N-p is closed in N and hence compact. Since N is a nodal set, N-p is also an open set in X and so $f/_{N-p}$ is open. Thus in this case we have an affirmative answer to our original question. This is also true if we require that $f(N)$ be either an arc or a simple closed curve and that f have certain properties. We let an arc from x to y be denoted $A(xy)$.

Theorem 3.8: Let f be a light open map defined on a compact space and N be a nodal set of X with nodal point p. If $f(N)$ is an arc $A(ab)$ then there is a compact subset N' of N such that $f_{/N'}$ is an open map onto $f(N)$.

Proof: If $f_{/N}^{-1}f(p) = \{ p \}$, then $f_{/N}$ is open by Corollary 2 and the conclusion of our theorem is true. Otherwise, let p' be in $f_{/N}^{-1}f(p) - p$. If $f(p)$ is an endpoint of $f(N)$, then there is an arc $A(p'r)$ mapping homeomorphically onto $f(N)$ since f is light and open on X. Because p is not in $A(p'r)$ and p' is in N and Bd $N = \{ p \}$, we must have $A(p'r)$ entirely contained in N. Letting $N' = A(p'r)$, the conclusion of our theorem is met. If $f(p)$ is not an endpoint of $f(N)$, we can get arcs $A(sp')$ and $A(p'r)$ mapping homeomorphically onto $A(a\ f(p))$ and $A(f(p)\ b)$, respectively. As above, both these arcs are in N. Consequently, $N' = A(sp') \cup A(p'r)$ maps homeomorphically onto $f(N)$, proving our theorem. //

Theorem 3.9: Let f be an open, finite to one map defined on a compact space X and N be a nodal set of X with nodal point p. If $f(N)$ is a simple closed curve, then there is a compact subset N' of N such that $f_{/N'}$ is an open map onto $f(N)$.

Proof: As in the previous theorem we can choose p_1 in $f_{/N}^{-1}f(p) - p$. Let $q = f(p)$ and pick points a, b, and c in positive rotation from q on $f(N)$. Since f is light and open, there exist arcs

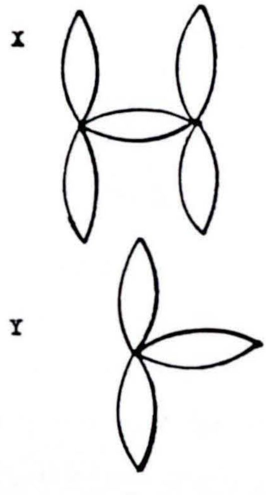
$A(p_1b_1)$ and $A(b_1p_2)$ mapping homeomorphically onto $A(qab)$ and $A(bcq)$ respectively. If $p_1 = p_2$, then $A(p_1b_1) \cup A(b_1p_2)$ is a simple closed curve mapping homeomorphically onto $f(N)$, satisfying our conclusion. If not, then it is an arc $A(p_1b_1p_2)$ which we will denote by X_1. As in Theorem 8, this will be contained in N. If p_2 is different from both p and p_1, we repeat the construction to find $A(p_2b_2) \cup A(b_2p_3)$ which is contained in N. If a point of this other than p_2 hits X_1, let x_o be the first such point in the order from p_2 to p_3. Then letting N' be the arc $A(x_op_2)$ in X_1 unioned with the arc $A(p_2x_o)$ in $A(p_2b_2) \cup A(b_2p_3)$, we have a simple closed curve which maps openly onto $f(N)$. If $p_3 = p_2$, the same is true. If only p_2 hits X_1, and $p_3 \neq p$, we let X_2 be the arc $X_1 \cup A(p_2b_2p_3)$ and continue as before, getting $A(p_3b_3) \cup A(b_3p_4)$. If this hits X_2 in a point other than p_3, we get N' as above. Continuing this process, because f is finite to one, after a finite number of steps we get either an X_{n-1} which hits $A(p_nb_n) \cup A(b_np_{n+1})$ in a point other than p_n in which case we deal with it as before, or a p_n which is equal to p. In this case we let X' be the arc from p_1 to p which we have constructed. We then repeat the construction using the opposite orientation for $f(N)$. That is, we get arcs $A(p_i'b_i')$ and $A(b_i'p_{i+1}')$ mapping homeomorphically onto $A(qcb)$ and $A(baq)$, respectively. As before, we either get X_{n-1}' hitting $A(p_n'b_n') \cup A(b_n'p_{n+1}')$ in a point other than p_n', or we get a p_n' which is equal to p. In the former case, we are done as before. In the latter case, let X" be the arc from $p_1' = p_1$ to p which we have constructed. X' and X" intersect at least in p_1 and p. Let x_o be the first point on X' after p_1 which is also in X". Letting N_1 be the arc $A(p_1x_o)$ in X' unioned with the arc $A(x_op_1)$ in X", we have a simple closed curve which maps openly onto f(N), completing our proof. //

Remark: We do not actually need f to be finite to one in Theorem 9; the existence of one point m in f(N) so that $f/_N^{-1}(m)$ is finite will suffice. This will guarantee that we can stop our arc lifting after a finite number of steps.

Although we have seen that under certain conditions a nodal set must have a compact set in it which maps openly onto the image of the nodal set, this is not always the case as is evidenced by

Example 3.5: There exists an open simple map defined on a compact space X with a nodal set N such that no compact subset of N maps openly onto f(N).

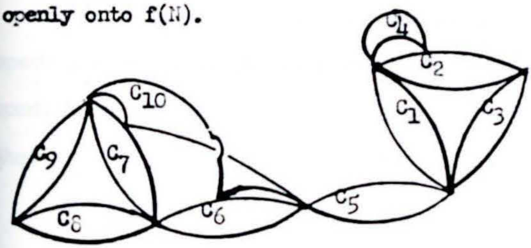
Let X consist of two disjoint figure eights with a simple closed curve hitting them only at their cross points. Let Y be a figure eight plus a simple closed curve hitting it only at its cross point. The map f takes each of the figure eights homeomorphically onto the figure eight in Y and the other simple closed curve onto the simple closed curve in Y by a map equivalent to $w = z^2$. We

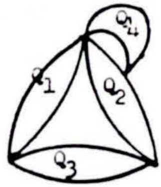


let N be all of X except the top loop of one of the figure eights. The nodal point p is then the cross point of this figure eight. For a compact subset N' of N to map openly onto f(N), it must omit this point. However, to map onto f(N), N' must contain at least part of the joining curve in X and for the map to be open on this compact set of points where it acts like $w = z^2$, it must include all of the curve including p. Thus no such N' could exist. /

In this counterexample and several others, we noted that not only was f(p) a local separating point as Lemma 7 indicated, but it was also a separating point of f(N). Speculation that adding the hypothesis that f(p) does not separate f(N) might lead to an affirmative answer to our question was ended by

<u>Example 3.6</u>: There exists an open, finite to one map f defined on a compact space X with a nodal set N having nodal point p such that f(p) does not separate f(N) and no compact subset of N maps openly onto f(N).





Y and f(N) will consist of four simple closed curves, $Q_1, \ldots, Q_4$, with Q_1 and Q_3 hitting in a point, Q_3 and Q_2 hitting in a point, Q_1, Q_2, and Q_4 hitting in a common point, and Q_2 and Q_4 hitting in another point. X consists of ten simple closed curves, $C_1, \ldots, C_{10}$. C_1, C_2,

C_3, and C_4 form a figure homeomorphic to Y; C_5 meets C_1 and C_3 in their common point as well as hitting C_6 in a point; C_6 also intersects C_7 and C_8 in a common point; C_9 hits each of C_7 and C_8 in a point; finally C_{10} runs from $C_7 \cdot C_9$ to another point in C_7, then to $C_5 \cdot C_6$, to another point in C_6, and back to $C_7 \cdot C_9$. To describe f: C_1, C_5, and C_9 get mapped homeomorphically to Q_1; C_2, C_6, and C_7 go homeomorphically to Q_2; C_3 and C_8 go homeomorphically to Q_3; C_4 goes homeomorphically to Q_4; and C_{10} gets mapped to Q_4 by a mapping equivalent to $w = z^2$. We let $N = C_6 \cup \ldots \cup C_{10}$. By the same type of reasoning as in Example 5, no N' exists. Y contains no separating points, so f(p) does not separate it. Thus this is the example we claimed.

We have shown in this chapter both that the general answer to Question 1 is negative and that there are conditions under which we get an affirmative answer. Summing these latter up, we get

<u>Theorem 3.10</u>: Let f be an open mapping defined on a compact space X having a nodal set N with nodal point p. Any of the following conditions implies that there exists a compact subset N' of N such that $f_{/N'}$ is an open map onto f(N):

(1) $f^{-1}_{/N}f(p) = \{ p \}$

(2) Bd $f(N) \neq \emptyset$

(3) p is not an isolated point of $f^{-1}_{/N}f(p)$

(4) $\{ p \}$ is open in N

(5) f is light and f(N) is an arc

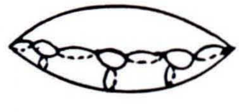
(6) f(N) is a simple closed curve, f is light, and for some m in f(N), $f^{-1}_{/N}(m)$ has finite cardinality.

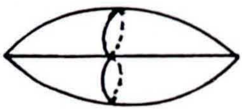
CHAPTER IV

MAPPINGS ONTO TWO-SPHERES

In 1935, Eilenberg (4) raised the question of whether or not an open map between compact spaces X and Y necessarily has the property that it induces a map of the homology groups of X onto those of Y. Whyburn (17) pointed out that the map $w = z^2$ on the unit circle does not induce such a map if the homology groups are taken over the integers. He then asked whether the rational homology groups of X would have to map onto those of Y. In 1936 (18) he showed that the first rational homology group of X would indeed map onto that of Y, provided one of the groups is finitely generated.

The Hopf map (6) from S^3 onto S^2 showed that the homology maps are not necessarily epimorphisms, since the second homology of S^3 is trivial while that of S^2 is not. Additional examples of this type of map were not noted for about 30 years. Bredon (2) was able to exhibit an open at most three to one map of a contractible 2-complex onto S^2. Being contractible, its second homology was trivial and so could not map onto that of S^2. Following this, Roy obtained a somewhat simpler map (see, for example,(9)) of the dunce's hat onto S^2 which thus has the same properties. We will give an example in this chapter of a finite to one open map of Bing's house with two rooms onto S^2. Since Bing's house is contractible, the second homology map will again not be an epimorphism. To illustrate the type of construction we use to get the map on Bing's house, we consider the following example of a map from an n-handled torus onto the two-sphere:





Example 4.1: Consider the n-handled torus T laid out symmetrically in E^3 along the y-axis. The xy-plane hits T in n simple closed curves, each around one of the holes of T. The yz-plane hits T in n + 1 simple closed curves, one between each successive pair of holes and one from each of the end holes to the outside. This divides T into two parts, $T^+ = T \cdot \{x \geq 0\}$ and $T^- = T \cdot \{x \leq 0\}$. We first deform T^+ to a 2-sphere by pinching each of these simple closed curves in to the y-axis (i.e., identify points on them which are symmetric with respect to the y-axis). Let f^+ be the map from T^+ to S^2 induced by this deformation. Notice that it is a homeomorphism on T^+ minus the union of the simple closed curves and is at most two to one on this set. We define the map f from T onto S^2 by

$$f(x,y,z) = \begin{cases} f^+(x,y,z) & \text{if } x \geq 0 \\ f^+(-x,y,-z) & \text{if } x \leq 0 \end{cases}$$

so that points symmetric with respect to the y-axis get mapped to the same point. Geometrically, we think of the map as being composed of two parts, the first a pinching of the torus (the identification) to form two 2-spheres joined on an arc, and the second a rotation of one of the spheres around the arc and onto the second sphere. This rotation is what guarantees that domains hitting both T^+ and T^- will get mapped to open sets.

Example 4.2: We now contruct the map of Bing's house onto S^2. Consider Bing's house with "chimney" in the upper room and "laundry chute" in the lower room, each connected to the outside by a wall (a). Reflect the lower room through its ceiling to get a map onto one room with two "chimneys", each having a lid (b). This is easily seen to be homeomorphic to a double torus containing four discs, one across each of the holes and one inside each of the rings (c). Consider the discs as lieing in the xy and yz planes respectively. As in the previous example, we pinch the double torus to give us two 2-spheres joined on an arc. Since the identification occurs only on the simple closed curves which are the boundaries of the discs, the discs will each get "popped" into a 2-sphere. This gives us four more spheres hitting the previous two in part of their common arc (d). Now rotate as in the first example, at the same time rotating the smaller spheres so that they go inside the larger one (e). Finally, blow out (project) each of the smaller spheres onto the outer one. This gives us our light open map of Bing's house onto S^2, thus demonstrating again that the second homology map induced by a light open map need not be onto, since the second homology group of S^2 is nontrivial, while that of Bing's house is

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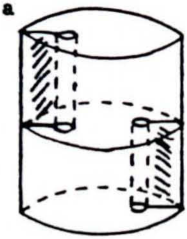

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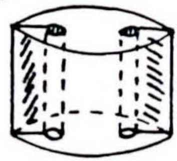

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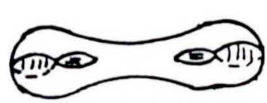

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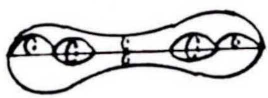

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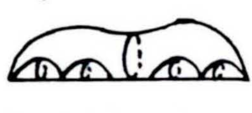

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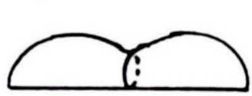


trivial since the space is contractible.

In both of these examples, we first perform a map (pinching) onto a space which is a union of 2-spheres joined along arcs and follow this with another map (rotation) so that the composite is open. The remaining results of this chapter show situations where the second map will exist. Essentially the hypotheses describe a possible result of the pinchings and the conclusion gives the existence of the rotation map which will make the composition open. Clearly the pinching is not in general an open map. The first theorem describes the case where the pinching yields an "arc" of 2-spheres intersecting in arcs:

<u>Theorem 4.1</u>: Let f be a map from a space X onto a space Y contained in E^3 such that

1. Y is a finite union of 2-spheres, $S_1, \ldots, S_n$, where S_i is the union of 2-balls D_{i1} and D_{i2} with $\mathring{D}_{i1}$ contained in $\{(x,y,z): x > 0\}$ and $\mathring{D}_{i2}$ contained in $\{(x,y,z): x < 0\}$ and $D_{i1} \cdot D_{i2}$ is a simple closed curve C_i. Moreover, $S_i \cdot S_j \neq \emptyset$ if and only if $|i-j| \leq 1$ and if $|i-j| = 1$, then $S_i \cdot S_j$ is an arc contained in $C_i \cdot C_j$; and
2. If V is a neighborhood of a point p in X then there exists a neighborhood U of p contained in V so that either
   (a) f(U) is a neighborhood of f(p) in S_i for some $i < n$, or
   (b) there exist i and j such that f(p) is in $S_i \cdot S_j$ and $f(U) \cdot D_{ik}$ and $f(U) \cdot D_{jk}$ are neighborhoods in D_{ik} and D_{jk} where $k = 1$ or 2.

Then there exists a map g from Y onto S^2 so that g o f is an open map of X onto S^2.

Y:

Proof: (By induction on n) If Y is a single 2-sphere, then condition (2b) is not possible and so (2a) will guarantee that f is an open map. Hence we may take g to be the identity map on Y. Now suppose the theorem is true up to n-1 and Y consists of n 2-spheres. Define f' from Y onto $Y' = S_1 \cup S_2 \cup \ldots \cup S_{n-1}$ by leaving f' fixed on Y' and mapping S_n onto S_{n-1} so that the intersecting arc is fixed, D_{n1} goes homeomorphically onto $D_{n-1,2}$ and D_{n2} goes homeomorphically onto $D_{n-1,1}$. Essentially we have rotated S_n around the intersecting arc and onto S_{n-1}.

__Claim:__ f'o f satisfies the conditions of the theorem for n-1.

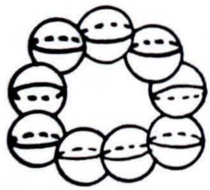
Proof: Y' satisfies part 1 by construction. To show part 2, let p be in X and V be a neighborhood of p. Let U be the neighborhood of p from the hypothesis. If f(p) is not in S_n, let $U' = U - f^{-1}(S_n)$ which will be a neighborhood of p contained in V. Since f' is fixed on Y', $f'f(U') = f(U') = f(U) - S_n$. Thus if (2a) was satisfied originally, f(U) was a neighborhood of f(p) in S_i for $i \leq n-1$ and then $f(U) - S_n$ is also, so that f'o f satisfies (2a). On the other hand, if (2b) was satisfied originally, then there exist i and j (both less than n since f(p) is not in S_n) so that f(p) is in $S_i \cdot S_j$, and $f(U) \cdot D_{ik}$ and $f(U) \cdot D_{jk}$ are neighborhoods in D_{ik} and D_{jk}. As above, $f'f(U') \cdot D_{mk} = f(U) \cdot D_{mk} - S_n$ which is a neighborhood of f(p) in D_{mk} ($m = i,j$) since we are assuming f(p) is not in S_n. Consequently (2b) will hold for Y' if it did for Y, provided f(p) is not in S_n.

If $f(p)$ is in $S_{n-1} \cdot S_n$ and f satisfies (2a), then $f(U)$ is a neighborhood of $f(p)$ in either S_{n-1} or S_n. If the former, then $f'f(U)$ contains $f'(f(U) \cdot S_{n-1}) = f(U) \cdot S_{n-1}$ and consequently is a neighborhood of $f'f(p)$ in S_{n-1}, thus satisfying (2a) for $f' \circ f$. If $f(U)$ is a neighborhood of $f(p)$ in S_n, then $f(U) \cdot S_n$ is, and so $f'(f(U) \cdot S_n)$ which is contained in $f'f(U)$ is a neighborhood of $f'f(p)$ in S_{n-1} since f' maps S_n homeomorphically onto S_{n-1}; again (2a) is satisfied for $f' \circ f$. Now suppose $f(p)$ is in $S_n \cdot S_{n-1}$ and f satisfies (2b). Thus $f(U) \cdot D_{n-1,k}$ and $f(U) \cdot D_{n,k}$ ($k = 1$ or 2) are neighborhoods of $f(p)$ in $D_{n-1,k}$ and $D_{n,k}$ respectively. Since $f'(f(U) \cdot D_{n-1,k}) = f(U) \cdot D_{n-1,k}$, it is a neighborhood of $f'f(p)$ in $D_{n-1,k}$. Because f' maps $D_{n,k}$ homeomorphically onto $D_{n-1,k'}$ ($k' \neq k$ and $k' = 1$ or 2), $f'(f(U) \cdot D_{n,k})$ is a neighborhood of $f'f(p)$ in $D_{n-1,k'}$. Thus $f'(f(U) \cdot D_{n-1,k}) \cup f'(f(U) \cdot D_{n,k})$ is a neighborhood of $f'f(p)$ in S_{n-1} and since it is contained in $f'f(U)$, the latter will be a neighborhood of $f'f(p)$ in S_{n-1}, thus satisfying condition (2a).

Finally, if $f(p)$ is in $S_n - S_{n-1}$, then condition (2b) is impossible and so $f(U)$ is a neighborhood of $f(p)$ in S_n. Because f' maps S_n homeomorphically onto S_{n-1}, $f'f(U)$ is a neighborhood of $f'f(p)$ in S_{n-1} and so (2a) is satisfied. Thus $f' \circ f$ satisfies the hypotheses of the theorem for $n-1$. /

Since Y' consists of just $n-1$ 2-spheres, the induction assumption is applicable and so there exists a map g' from Y' onto S^2 so that $g' \circ (f' \circ f)$ is open. Letting $g = g' \circ f'$, we have proven the theorem for the case where Y consists of n 2-spheres.//

We next consider the possibility of Y being a "circle" of 2-spheres:



Theorem 4.2: Let f be a mapping from a space X onto a subset Y of E^3 such that

1. Y is a finite union of 2-spheres, $S_1, \ldots, S_n$, where S_i is the union of 2-balls D_{i1} and D_{i2} with $\mathring{D}_{i1}$ contained in $\{(x,y,z): x > 0\}$ and $\mathring{D}_{i2}$ contained in $\{(x,y,z): x < 0\}$ and $D_{i1} \cdot D_{i2}$ is a simple closed curve C_i. Moreover, $S_i \cdot S_j \neq \emptyset$ if and only if either $\{i,j\} = \{1,n\}$ or $|i-j| \leq 1$, and in case of the equalities, $S_i \cdot S_j$ is an arc contained in $C_i \cdot C_j$; and
2. if V is a neighborhood of a point p in X then there exists a neighborhood U of p contained in V so that either
   (a) f(U) is a neighborhood of f(p) in S_i for some $i \leq n$, or
   (b) there exist i and j so that f(p) is in $S_i \cdot S_j$ and $f(U) \cdot D_{ik}$ and $f(U) \cdot D_{jk}$ are neighborhoods of f(p) in D_{ik} and D_{jk} where $k = 1$ or 2.

Then there exists a map g from Y onto S^2 so that g o f is an open map of X onto S^2.

Proof: We will first show that we may assume that n is even. If it is not, we will define a map f_1 from Y onto a union of n+1 spheres so that f_1o f has all the desired properties. Essentially all we do to get this map is pinch S_n to form two

2-spheres. Let $Y' = S_1 \cup \ldots \cup S_{n-1} \cup S'_n \cup S'_{n+1}$ have all the properties of part 1 of the hypothesis and define f_1 on $S_1 \cup \ldots \cup S_{n-1}$ to be the identity. The set $C_n - ((S_n \cdot S_1) \cup (S_n \cdot S_{n-1}))$ consists of two arcs. Let p_1 and p_2 be points in each arc and consider arcs $A_1(p_1p_2)$ and $A_2(p_1p_2)$ spanning $\overset{o}{D}_{n1}$ and $\overset{o}{D}_{n2}$ respectively. Then $A_i(p_1p_2)$ separates D_{ni} into two discs D^1_{ni} and D^{n-1}_{ni} where D^j_{ni} contains $S_n \cdot S_j$. Define f_1 on S_n as an extension of the previous definition by mapping $A_i(p_1p_2)$ homeomorphically onto $S'_n \cdot S'_{n+1}$, maintaining the orientation established by the map as already defined on $S_n \cdot S_j$; extend this to take D^1_{ni} homeomorphically onto $D'_{n+1,i}$ and D^{n-1}_{ni} homeomorphically onto D'_{ni}.

<u>Claim</u>: $f_1 \circ f$ satisfies the conditions of the hypothesis.

Proof: Let V be a neighborhood of a point p in X and U be the associated neighborhood from the hypothesis. Suppose U originally satisfied (2a). If $f(U)$ is a neighborhood of $f(p)$ with respect to some S_i for $i \neq n$, then so is $f_1f(U)$ since f_1 is the identity on these S_i. Suppose $f(U)$ is a neighborhood of $f(p)$ in S_n and $f(p)$ is not in either of the $A_i(p_1p_2)$ (where the pinching occurs). If $f(p)$ is in $\overset{o}{D}_{ni}$ then it is in $\overset{o}{D}{}^j_{ni}$ for $j = 1$ or $n-1$ and so $f_1f(U) \cdot D'_{ki}$ is a neighborhood of $f_1f(p)$ in D'_{ki} for $k = n+1$ or n respectively, since f_1 maps the D^j_{ni}'s homeomorphically. If, on the other hand, $f(p)$ is in $D_{n1} \cdot D_{n2}$ then $f(U)$ is a neighborhood of $f(p)$ in both D_{n1} and D_{n2} and consequently in D^j_{n1} and D^j_{n2} for $j = 1$ or $n-1$. But f_1 maps D^j_{ni} homeomorphically so $f_1f(U)$ is a neighborhood of $f_1f(p)$ in D'_{ki} for k = n+1 or n respectively. As a result $f_1f(U)$ is a neighborhood of $f_1f(p)$ in S'_k. Now consider the case where f(p) is in one of the $A_i(p_1p_2)$. Thus f(U) is a neighborhood of f(p) in both D^1_{ni} and D^{n-1}_{ni} and consequently $f_1f(U)$ is a neighborhood of $f_1f(p)$ in both $D'_{n+1,i}$ and D'_{ni}, thus satisfying (2b).

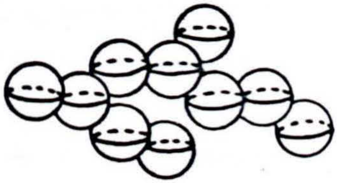
Suppose U originally satisfied (2b). If neither i nor j is equal to n, then $f_1f(U)$ also satisfies (2b), since f_1 will be the identity there. If i = n and j = 1 or n-1, then f(U) is a neighborhood of f(p) in D_{nk} and D_{jk} for k = 1 or 2, and so $f_1f(U)$ is a neighborhood of $f_1f(p)$ in both D'_{nk} and D_{jk}, thus satisfying (2b). /

We have now shown that we may limit our attention to the case where the number of 2-spheres in Y is even, say 2m. We again construct an intermediate map, this time from Y to $Y' = S_1 \cup \ldots \cup S_m$ which will be the identity on Y'. Essentially, we just rotate the "arc" of 2-spheres $S_{m+1} \cup \ldots \cup S_{2m}$ onto the "arc" Y'. Let f' map each of the discs D_{ik} (k=1 or 2; $m+1 \leq i \leq 2m$) homeomorphically onto the disc $D_{2m+1-i,k'}$, where k' = 2 if k = 1 and vice versa and where the joining arcs get mapped to corresponding joining arcs. We wish to show that f'o f satisfies the hypotheses of Theorem 1. Clearly Y' satisfies the requirements on Y. Let a neighborhood of a point p in X be given and let U be the corresponding neighborhood from the hypothesis of this theorem. If f(U) is a neighborhood of f(p) in some S_i, then since f' maps each of the S_i homeomorphically to some S_j, we will have f'f(U) is a neighborhood of $f'f(p)$ in S_j, thus satisfying (2a) of Theorem 1. Now suppose U satisfies (2b) of this theorem and $\{i,j\} \neq \{1,2m\}$ or $\{m,m+1\}$ (the places where the rotation occurs). If i and j are both less than m+1, then $f'f(U) = f(U)$ and so $f' \circ f$ satisfies (2b). If both are greater than m, then $f'f(p)$ is in $S_{i^*} \cdot S_{j^*}$ where $t^* = 2m+1-t$. Thus $f'f(U) \cdot D_{i^*k'} = f'(f(U) \cdot D_{ik})$ and $f'f(U) \cdot D_{j^*k'} = f'(f(U) \cdot D_{jk})$ since f' maps D_{ik} and D_{jk} homeomorphically onto $D_{i^*k'}$ and $D_{j^*k'}$. As a result, (2b) of Theorem 1 is satisfied. Finally, consider what happens if U satisfies (2b) of this theorem and $\{i,j\} = \{1,2m\}$ or $\{m,m+1\}$. In the first case, $f'(f(U) \cdot D_{1k}) = f(U) \cdot D_{1k}$ is a neighborhood of $f(p)$ in D_{1k} and $f'(f(U) \cdot D_{2m,k}) = f'f(U) \cdot D_{1k'}$ is a neighborhood of $f'f(p)$ in $D_{1k'}$ since f' maps $D_{2m,k}$ homeomorphically onto D_{1k}. Thus $f'f(U) \cdot S_1$ contains $f(U) \cdot D_{1k}$ and $f'f(U) \cdot D_{1k'}$, and so is a neighborhood of $f'f(p)$ in S_1, thus satisfying (2a) of Theorem 1. In the same way, $f'f(U) \cdot D_{mk}$ and $f'(f(U) \cdot D_{m+1,k}) = f'f(U) \cdot D_{mk'}$ are neighborhoods of $f'f(p)$ in D_{mk} and $D_{mk'}$, respectively, and so $f'f(U) \cdot S_m$ is a neighborhood of $f'f(p)$ in S_m, again satisfying (2a) of Theorem 1.

Consequently, $f' \circ f$ satisfies the conditions of Theorem 1 and so there exists a map g' from Y' onto S^2 so that $g' \circ f' \circ f$ is an open map from X onto S^2. Thus the map $g' \circ f'$ is the one we need for the conclusion of Theorem 2. //

Our final theorem considers the case where f maps onto a space Y which is a "tree" of 2-spheres embedded nicely in E^3.

Y:



Theorem 4.3: Let f be a map of a space X onto a subset Y of E^3 so that

(1) Y is a finite union of 2-spheres, $S_1, \dots, S_n$, where S_i is the union of 2-balls D_{i1} and D_{i2} with $\mathring{D}_{i1}$ contained in $\{(x,y,z): x > 0\}$ and $\mathring{D}_{i2}$ contained in $\{(x,y,z): x < 0\}$ and $D_{i1} \cdot D_{i2}$ is a simple closed curve C_i. Moreover, if $S_i \cdot S_j \neq \emptyset$ and $i \neq j$, then $S_i \cdot S_j$ will be an arc contained in $C_i \cdot C_j$. In addition, for each $i \neq 1$, there is a unique j less than i so that $S_i \cdot S_j \neq \emptyset$ (this is to prevent a "circle" of 2-spheres from occurring); and

(2) if V is a neighborhood of a point p in X, then there is a neighborhood U contained in V so that either

   (a) f(U) is a neighborhood of f(p) in S_i for some i, or

   (b) there exist i and j so that f(p) is in $S_i \cdot S_j$ and $f(U) \cdot D_{ik}$ and $f(U) \cdot D_{jk}$ are neighborhoods of f(p) in D_{ik} and D_{jk}, respectively, where k = 1 or 2.

Then there exists a map g from Y onto S^2 so that g o f is an open map of X onto S^2.

Proof: (By induction) If n = 1, we are done since (2b) will be impossible and so (2a) will make f itself open. Now suppose the theorem is true up to n-1 and define a map f' from Y onto Y' = $S_1 \cup \dots \cup S_{n-1}$ by making f' be the identity on Y' and map S_n

onto S_m where $S_n \cdot S_m \neq \emptyset$ by taking D_{n1} homeomorphically onto D_{m2} and D_{n2} homeomorphically onto D_{n1} keeping $S_n \cdot S_m$ fixed (i.e., by rotating S_n onto S_m around their common arc).

Claim: f'o f satisfies the hypotheses of the theorem.

Proof: Y' satisfies (1) from its construction, since Y satisfied it. Let V be a neighborhood of a point p in X and let U be the corresponding neighborhood from (2). We wish to show that $f'f(U)$ must satisfy either (2a) or (2b). If $f(U)$ originally satisfied (2a), then $f(U)$ was a neighborhood of $f(p)$ in some S_i and since f' maps each S_i homeomorphically onto an S_j, $f'f(U)$ is a neighborhood of $f'f(p)$ in S_j, satisfying (2a). If $f(U)$ satisfied (2b) where $f(p)$ is in $S_i \cdot S_j$ and both i and j are different from n, then since f' is the identity on Y', $f'(f(U) \cdot D_{ik}) = f'f(U) \cdot D_{ik} = f(U) \cdot D_{ik}$ is a neighborhood of $f'f(p)$ in D_{ik} and similarly $f'f(U) \cdot D_{jk}$ is a neighborhood of $f'f(p)$ in D_{jk}, thus satisfying (2b). Finally, suppose $f(U)$ satisfied (2b) with $f(p)$ being in $S_n \cdot S_m$. Then $f(U) \cdot D_{nk}$ and $f(U) \cdot D_{mk}$ are neighborhoods of $f(p)$ in D_{nk} and D_{mk} respectively, and so $f'(f(U) \cdot D_{nk}) = f'f(U) \cdot D_{mk'}$, and $f'(f(U) \cdot D_{mk}) = f'f(U) \cdot D_{mk}$ are neighborhoods of $f'f(p)$ in $D_{mk'}$ and D_{mk} respectively since f' maps D_{nk} homeomorphically onto $D_{mk'}$ and is the identity on D_{mk}. Therefore $f'f(U)$ will contain these two sets and consequently $f'f(U) \cdot S_m$ is a neighborhood of $f'f(p)$ in S_m, satisfying (2a). This proves our claim. /

As a result of the induction assumption, there is a map g' from Y' onto S^2 so that $g' \circ (f' \circ f)$ is an open map from X onto S^2. To complete the proof of our theorem, we let $g = g' \circ f'$. //

CHAPTER V

EXAMPLES AND COUNTEREXAMPLES

In the course of the preparation of this thesis, a number of aborted proofs and false conjectures occurred. This chapter contains a collection of the examples and counterexamples which resulted. The first two came about in trying to prove in Chapter 2 that f restricted to one of the 2-manifolds was open. It was hoped that this would be the case because the map was open except on a finite number of arcs.

Example 5.1: The extension of a homeomorphism (and hence an open map) to the closure of its domain is not necessarily open.

Let f map the open interval $(0,2\pi)$ into the unit circle by $f(x) = (1,x)$ where the range is described in polar coordinates. This is a homeomorphism onto the unit circle minus the point $(1,0)$. This map can be extended to the closed interval by making $f(0) = f(2\pi) = (1,0)$. The extension is not an open map at endpoints of the interval.

Example 5.2: If f is an open map on a compact space X and U is a subset of X on which $f/_U$ is open, the extension to $\overline{U}$ is not necessarily open.

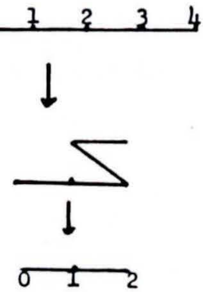
Let f be the map $w = z^2$ of the unit circle onto itself and let U be the lower open semi-circle $\{ |z| = 1 \text{ and } y < 0 \}$. Then $f/_U$ maps U onto the unit circle minus $(1,0)$ homeomorphically by a map that is essentially the same as that in Example 1, and in the identical way its extension to the closed lower semi-circle is not an open map, since it is not open at the endpoints.

Example 5.3: A map may be open relative to both an inverse set and its complement and still not be open on the whole space.

Let X be the closed interval $0 \leq x \leq 4$

and describe f by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 2 \\ 4-x & \text{if } 2 \leq x \leq 3 \\ x-2 & \text{if } 3 \leq x \leq 4 \end{cases}$$



Thus f maps X onto the closed interval from 0 to 2 by being the identity on that set, taking the interval from 2 to 3 onto the one from 1 to 2 by reversing its orientation and mapping the interval from 3 to 4 onto the one from 1 to 2 preserving its orientation. If we let A be the closed interval from 1 to 2 in Y, then $f^{-1}(A)$ is the closed interval from 1 to 4 in X. Then $f/_{f^{-1}(A)}$ is an open map relative to its image A and $f/_{X-f^{-1}(A)}$ is also an open map (in fact a homeomorphism) of the half open interval from 0 to 1 onto itself. However, the map f itself is not open at the point $x = 3$ since the open interval $(2\frac{1}{2}, 3\frac{1}{2})$ gets mapped to the half open interval from 2 to $3\frac{1}{2}$ which includes 2.

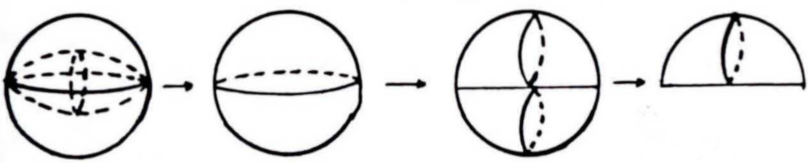
Remark: Of course, if a map is strongly open on both an inverse set and the complement of it, then the map is necessarily open on the whole space.

In deriving the formula in Chapter 2 similar to that of Whyburn, we showed that f restricted to each of the 2-manifolds was an open map and then used Whyburn's result. The following example shows that this method would not work if we had assumed that our manifolds intersected in simple closed curves rather than in arcs.

Example 5.4: A space X consisting of two 2-spheres joined on a simple closed curve may be sent by an open map f onto a 2-sphere in such a way that f restricted to one of the 2-spheres is not an open map and yet both map onto the image sphere.

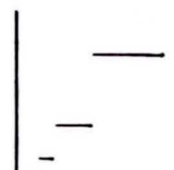
Consider X as being composed of the 2-spheres A_1 and A_2 joined at their equators, with A_1 lieing inside A_2. Project each of the hemispheres of the inner sphere A_1 onto the top hemisphere of A_2. Note that this satisfies all the conditions we desire except that of having the image of A_1 be a 2-sphere. To add this, we pinch the sphere A_2 along its equator to form two 2-spheres joined on an arc and now rotate the bottom 2-sphere around the arc and onto the top sphere. Since the top hemisphere is thus mapped to a 2-sphere and A_1 had been mapped onto this top hemisphere, the image of A_1 is now a sphere also. Clearly the mapping restricted to A_1 is not open at most points of its equator since open discs get mapped to half open discs due to the folding occurring there. However, because these are the only points where it is not open and because f restricted to A_2 is open everywhere including these points, the map f on X will be open.

f:



$f/_{A_1}$:

In order to use Titus' and Young's theorem (16) to show f was open on each of the 2-manifolds, it was necessary to know that the image of the joining arcs was nowhere dense in f(X). Before taking note of the result of Church and Hemmingsen (3), an attempt was made to show this by showing that for any map between compact spaces, if we took an arc in the image starting at a point q, one of its components containing a point q' of $f^{-1}(q)$ must contain an arc starting at q'. If this had been the case, we could have looked at the map restricted to the joining arcs, taken a many spoked wheel with center q (if the image of the joining arcs had been 2-dimensional) and shown that these "semi-lifted" back to too many arcs in the domain because the map was finite to one. This would have shown that the image of the joining arcs could not be 2-dimensional. However, the following example shows that the lemma suggested above is not true.



Example 5.5: There is a map f from a compact space X onto an arc A having a point q so that each component of X containing a point of $f^{-1}(q)$ is contained in $f^{-1}(q)$ and thus will not map onto an arc.

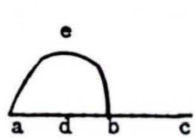
Let the space X be the set in the plane $\{(0,0)\} \cup \bigcup_{n=1}^{\infty} A_n$ where $A_n = \{(x, 1/2^n): 1/2^{n+1} \leq x \leq 1/2^n\}$. Let Y be the closed interval on the x-axis from 0 to $\frac{1}{2}$ and f be the vertical projection of X onto Y. Let q = (0,0) in Y. Then $f^{-1}(q) = \{(0,0)\}$ in X and the component of X containing (0,0) is the singleton $\{(0,0)\}$. Thus we do not even have the partial lifting property for arcs which we were seeking.

Another counterexample to a proof attempt along similar lines is

Example 5.6: There is a map from an arc B onto a compact space Y containing an arc C so that no component of $f^{-1}(C)$ maps onto C.

Let f map the arc B onto the space Y which is a "reclining P" according to the schema shown at the right. The inverse of the horizontal arc C = A(adbc) will consist of the two components A(adbda) and A(bc), neither of which maps onto C.

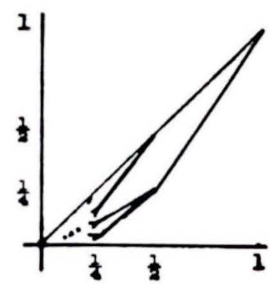
$\overline{\text{a d b d a e b c}}$



In trying to show that the map in Chapter 2 was k to one for some k, it was thought for a while that the fact that it was finite to one and X was compact should give us the result. The next example shows this is not the case.

Example 5.7: A finite to one open map on a compact space X need not be k to one.

Let $X = \{(0,0)\} \cup \bigcup_{n=0}^{\infty} A_n$ where A_n is the set of all lines in the plane from points of the form $(1/2^n, k/4^n)$ to points of the form $(1/2^{n+1}, 2k/4^{n+1})$ or $(1/2^{n+1}, (2k-1)/4^{n+1})$ where k runs between 1 and 2^n. Let f be

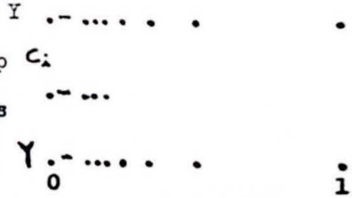


the projection map of X onto the unit interval Y on the x-axis. If a point in Y is in the half open interval from $1/2^n$ to, but not including $1/2^{n-1}$, then it has 2^n preimages in X. Hence f is clearly finite to one but not k to one since 2^n grows without bound.

While working on the nodal set problem, an attempt was made to use intersections of sets to produced the desired compact set on which the map was open. During the course of this work, the following example was noted.

Example 5.8: There exists a map f of a compact space X onto a space Y and a nested sequence $\{C_i\}$ of subsets of X so that $f/_{C_i}$ is an open map of C_i onto Y, but f restricted to the intersection of the C_i's is not an open map.

Let $X = C_1$ where $C_i = \{(1/n, 2)$: n is a positive integer$\} \cup \{(1/n, 1): n \geq i\} \cup \{(0,1), (0,2)\}$. In addition let Y be the set $\{(1/n, 0)\}_{n=1}^{\infty} \cup \{(0,0)\}$ and f map onto Y by the projection $f(x,j) = (x,0)$. This map will be open on each of the C_i, but if C is the intersection of the C_i's, then $C = \{(1/n,2)$: n is a positive integer$\} \cup \{(0,1), (0,2)\}$ and f is not open on C since it is not open at the point $(0,1)$.

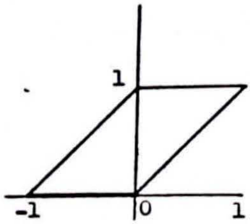


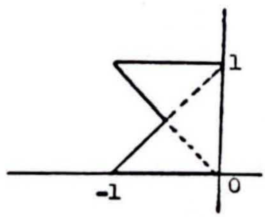
The map $f/_{A_1}$ of Example 4 is a simple map between two 2-manifolds without boundary which is not open. We next show the existence of such a map between 2-manifolds with boundary.

Example 5.9: There exists a simple map between 2-manifolds with boundary which is not open.

Let X be the parallelogram in the plane with vertices (1,1), (0,1), (-1,0), and (0,0) and let Y be the pentagon with vertices (-1,1), (0,1), (-1,0), (0,0), and $(-\frac{1}{2},\frac{1}{2})$. Let the map f be described by

$$f(x,y) = \begin{cases} (x,y) & \text{if } x \leq 0 \\ (-x,y) & \text{if } x \geq 0 \end{cases}$$





so that the map is the identity on the triangle with vertices (-1,0), (0,0), and (0,1) and maps the triangle with vertices (0,0), (0,1), and (1,1) linearly onto the one with vertices (0,0), (0,1), and (-1,1). The map is clearly a simple one, but it is not open at points on the half open line segments from $(-\frac{1}{2},\frac{1}{2})$ to but not including (0,1) and from $(\frac{1}{2},\frac{1}{2})$ to but not including (0,0).

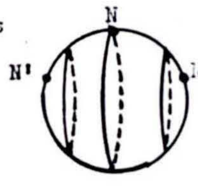
In Chapter 1, our work with orbit maps produced some false speculations about the nature of point inverses. The first was that if the map was one to one on the inverse of the image of the singular set then this in some sense divided the space up nicely so that any other point inverses would have the same number of points.

Example 5.10: There is a three to one map from a space X onto a 2-sphere which is one to one on the singular set, but is not exactly three to one off it.

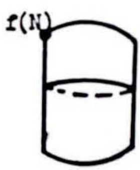
Let the space X be that of Example 4, the two 2-spheres joined along their equators, and let the map f be the one that projects the inside sphere up onto the top hemisphere of the outer one and leaves the outer sphere fixed. The map is one to one on the singular set, which is the common equator. However, point inverses of points in the upper hemisphere of Y consist of three points, while those of points in the lower hemisphere have just one point.

A second false speculation on point inverses came about from the fact that the order of a subgroup will divide the order of the group containing it and the question of whether every map which factors into open simple maps is equivalent to an action by a finite group. The thought was that it might be the case that the order of any point inverse of such a map might have to divide the order of the map. However,

Example 5.11: There is a four to one map between 2-spheres which factors into open simple maps and yet has a point inverse consisting of three points.



Consider the sphere with three longitudinal circles as indicated. Let the map f be the composition of two open simple maps, the first of which we get by pinching on the middle circle and then rotating one of the resulting sphere onto the other so that the two outside circles have the same image. The second map will be the same type with the pinching being done on the image of these outer circles. A point f(n) which is the

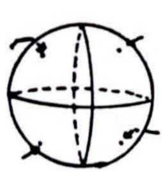


composite image of the north pole of the middle circle will have its inverse image consisting of three points, one on each of the longitudinal circles.

Finally, we give an example of a map of one of the types we were considering in Chapter 1.

Example 5.12: A map which factors into open simple maps may be equivalent to an action by a finite group.

Let f be the map obtained by first pinching and rotating the 2-sphere about a great circle and then doing the same about the equator of the result. Thus f is the composition of two open simple maps. As indicated, $f^{-1}f(x,y,z)$ will be $\{(x,y,z), (-x,-y,z), (-x,y,-z), (x,-y,-z)\}$. Consider the group of homeomorphisms of the sphere onto itself, $\{e, a, b, ab\}$ where e is the identity, a is rotation 180° in the xy-plane, b is rotation 180° in the xz-plane, and hence ab is rotation 180° in the yz-plane. Thus $e(x,y,z) = (x,y,z)$; $a(x,y,z) = (-x,-y,z)$; $b(x,y,z) = (-x,y,-z)$; and $ab(x,y,z) = a(-x,y,-z) = (x,-y,-z)$; and so the orbit of (x,y,z) is $f^{-1}f(x,y,z)$, showing that f is equivalent to an action by this group.



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