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On marker set distance Laplacian eigenvalues in graphs.

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Abstract

In our previous paper, we had introduced the marker set distance matrix and its eigenvalues. In this paper, we extend them naturally to the Laplacian eigenvalues. To define the Laplacian, we have defined the distance degree sequence of the marker set in the graph. Here we have considered the study of the Laplacian matrix, its characteristic polynomial and related results.

Keywords

Marker set of a graph, M-set distance matrix, M-set distance Laplacian, characteristic polynomial, eigenvalues.

AMS Subject Classification

Primary 05C12,05C50,92E10.

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Contents

1. Introduction

Cluster identification is an important operation usually done in fields like electrical network connections, using algorithms depending on matrix manipulations. Graph spectral methods are widely used in non-bonded clusters in protein structures [17], [18]. The main algorithm of [18], the clustering procedure depends on eigenvalues and eigenvector components of weighted adjacency matrix. In fact, the second smallest eigenvalue of the Laplacian matrix and its associated vector components yield the clustering of vertices in the graph. Hence a matrix different from the adjacency matrix and distance matrix of a graph that can cater to the clustering distances was introduced in the form of the marker set distance matrix and consequent energy parameters in [12].

We recall some of the definitions for ready reference.

2. Preliminaries

Definition 2.1. *[12] Let $G=(V,E)$ be a simple connected graph of order p. Let $M \subseteq V(G)$ be a nonempty marker set of G. We define M-set distance between two vertices v_i and v_j as $d_{ij} = |d(v_i,M) - d(v_j,M)|$.*

Here $d(v_i,M) = \min\{d(v_i,w) : w \in M\}$.

The $p \times p$ matrix $D_M(G) = [d_{ij}]$ is called the M-set distance matrix of the marker set M in the graph G.

Definition 2.2. *[12] The M-set eccentricity of a vertex v of G, denoted by $e_M(v)$ is defined as the maximum of all the M-set distances of v.*

Definition 2.3. *[12] The M-set diameter of a graph G with respect to a marker set M is denoted by $diam_M(G)$ and is defined as the maximum of all the M-set eccentricities of the vertices of G.*

We now give the general structure of a M-set distance matrix taken from [12] Let G be a simple connected graph of order p and M be a marker set with $|M| = k$ and $diam_M(G) = m$. Let k_i be the number of vertices of G at M-distance i $(1 \le i \le m)$ so that $k + \Sigma_{i=1}^{m} k_i = p$. The M-distance matrix is given by $D_M(G)$

$$= \begin{bmatrix} 0_{k\times k} & 1_{k\times k_1} & \dots & m_{k\times k_m} \\ 1_{k_1\times k} & 0_{k_1\times k_1} & \dots & (m-1)_{k_1\times k_m} \\ 2_{k_2\times k} & 1_{k_2\times k_1} & \dots & (m-2)_{k_2\times k_m} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ m_{k_m\times k} & (m-1)_{k_m\times k_1} & \dots & 0_{k_m\times k_m} \end{bmatrix}.$$

We call the above matrix as the standard form of marker set distance matrix. We recall the definition of the distance degree sequence of the M-set of a graph G.

Definition 2.4. *Given a simple connected graph G and a marker set M of G, the distance degree sequence denoted by $DDS_G(M)$ can be defined as*

$DDS_G(M) = (k_0, k_1, k_2, \ldots, k_n)$ *written in a non-decreasing order where k_i is the number of vertices of G at distance i from M where, $0 \leq i \leq m$ and $m = diam_M(G)$.*

The first entry of the sequence is the number of vertices of G which are at distance 0 from M, so is always $|M|$.

Example 2.5. *Consider a graph G (Figure 1). Let the marker set be $M_1 = \{u_1, u_2\}$. Then, $DDS_G(M_1) = (2, 1, 2)$. Let $M_2 = \{u_1, u_4\}$. Then, $DDS_G(M_2) = (2, 3)$.*

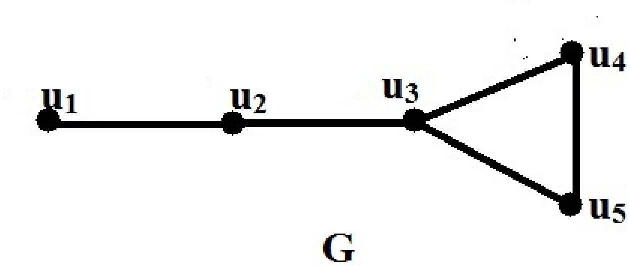

Figure 1

Remark 2.6. *If the marker set M is a dominating set with $|M| = k$ for a graph G of order p then, $DDS_G(M) = (k, (p - k))$.*

To define the Laplacian matrix derived from the marker set distance matrix, we now make use of the marker set distance degree sequence.

3. Marker set distance Laplacian

Definition 3.1. *Let $G = (V, E)$ be a simple connected graph of order p. Let $M \subseteq V(G)$ be a non empty marker set of G. The M-set distance Laplacian is defined as $L_M(G) = D_M(G) - diag[DDS_G(M)]$, where $D_M(G)$ is the marker set distance matrix in the standard form.*

The $p \times p$ matrix $L_M(G) = [a_{ij}]$ is called the *M-set distance laplacian matrix* of the marker set M in the graph G.

If $M = V(G)$ then $D_M(G) = [0]_{p \times p}$ and $L_M(G)$ is also a $p \times p$ matrix with all the entries 0 except for the first principal diagonal element which is $-p$ since $DDS_G(M) = (p)$. Unless specified we always consider marker set M such that $M \neq \Phi$ and $M \neq V(G)$. It is easy to get the M-set distance Laplacian matrix of a graph G for any marker set M. In the following section we give an algorithmic construction for the same.

Algorithm

Let G be a graph of order p. Let $|M| = k$ and let $DDS_G(M) = (k_i / k_i =$ number of vertices of G at distance i from M where $0 \leq i \leq m)$ and $m = diam_M(G)$.
Step 1: Write the M-set distance matrix of the graph G.
Step 2: Rewrite the matrix with the rows corresponding to the vertices in M as the first k-rows, the rows corresponding to vertices which are at distance 1 from the M-set as the next k_1 rows, the rows corresponding to vertices which are at distance 2 from the M-set as the next k_1 rows and so on upto the rows corresponding to vertices at distance m from the M-set to get the M-set distance matrix in the standard form.
Step 3: Sum this matrix with $-diag[DDS_G(M)]$ to get the required M-set Laplacian matrix of G.

Marker Set Distance Laplacian Energy

The characteristic polynomial of $L_M(G)$ is defined as $f(G : M, \lambda) = det(\lambda I - L_M(G))$, the roots of which are assumed to be in non-increasing order and are called the M-distance Laplacian eigen values of M in G. The *distance Laplacian marker set energy* (DLMSE in short) of M in G is defined as

$$L\varepsilon_M(G) = \sum_{i=1}^{p} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the M-distance Laplacian eigen values.
Since $L_M(G)$ is a real symmetric matrix, the M-set Laplacian eigen values are real. Thus $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$.

Example 1 : The M-set Laplacian matrix of the graph in Figure 1 is as follows.
Let $M_1 = \{u_1, u_4\}$ be a marker set of the graph G. In this case it is clear that $M_1(G)$ is a dominating set of the graph G. $DDS_G(M_1) = (2, 3)$. The M-set Laplacian matrix is written as

$$L_{M_1}(G) = \begin{bmatrix} -2 & 0 & 1 & 1 & 1 \\ 0 & -3 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of the M_1-set distance Laplacian matrix is
$f(G : M_1, \lambda) = det(\lambda I - L_{M_1}(G)) = \lambda^5 + 5\lambda^4 - 15\lambda^2$ and the M_1 -set Laplacian eigen values are $\lambda_1 = 1.5171, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = -2.4036, \lambda_5 = -4.1135$.
The M_1 - set Laplacian energy $= L\varepsilon_{M_1}(G) = \Sigma_{i=1}^{5} |\lambda_i| = 8.0341$.

For $M_2 = \{u_4, u_5\}$ another marker set (a non-dominating set) of the graph G, $DDS_G(M_2) = (2, 1, 1, 1)$.

$$L_{M_2}(G) = \begin{bmatrix} -2 & 0 & 1 & 2 & 3 \\ 0 & -1 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 & 2 \\ 2 & 2 & 1 & -1 & 1 \\ 3 & 3 & 2 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of the M_2-set distance Laplacian matrix is
$f(G:M_2,\lambda) = det(\lambda I - L_{M_2}(G) = \lambda^5 + 5\lambda^4 - 25\lambda^3 - 164\lambda^2 - 280\lambda + 153$ and the M_2 -set eigen values are $\lambda_1 = 5.7896, \lambda_2 = -1.4030, \lambda_3 = -1.5501, \lambda_4 = -2.1289, \lambda_5 = -5.7076$.
The M_2 - set Laplacian energy $= L\varepsilon_{M_2}(G) = \Sigma_{i=1}^{5}|\lambda_i| = 16.5792$.
For another marker set $M_3 = \{u_1, u_2\}$ (a non-dominating set) of the graph G, $DDS_G(M_3) = (2,1,2)$.

$$L_{M_3}(G) = \begin{bmatrix} -2 & 0 & 1 & 2 & 2 \\ 0 & -1 & 1 & 2 & 2 \\ 1 & 1 & -2 & 1 & 1 \\ 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of the M_3-set distance Laplacian matrix is
$f(G:M_3,\lambda) = det(\lambda I - L_{M_3}(G) = \lambda^5 + 5\lambda^4 - 12\lambda^3 - 77\lambda^2 - 76\lambda$ and the M_3 -set laplacian eigen values are $\lambda_1 = 4, \lambda_2 = 0, \lambda_3 = -1.4679, \lambda_4 = -2.6527, \lambda_5 = -4.8794$.
The M_3 - set Laplacian energy $= L\varepsilon_{M_3}(G) = \Sigma_{i=1}^{5}|\lambda_i| = 13$.
For $M_4 = \{u_2, u_4\}$ another marker set (a dominating set) of the graph G, $DDS_G(M) = (2,3)$.

$$L_{M_4}(G) = \begin{bmatrix} -2 & 0 & 1 & 1 & 1 \\ 0 & -3 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

It can be seen that $L_{M_4}(G) = L_{M_1}(G)$. The characteristic polynomial of the M_4-set distance Laplacian matrix is equal to that of M_1 and so has the same eigenvalues. Also, the M_4 - set Laplacian energy $= L\varepsilon_{M_4}(G) = \Sigma_{i=1}^{5}|\lambda_i| = 8.0341$.

Definition 3.2. *Two marker sets M_1 and M_2 of a graph G are said to be Laplacian equienergetic if the DLMSEs of G corresponding to the two sets are equal.*

Remark 3.3. *In Example 1, the marker sets M_1 and M_4 are equienergetic.*

From the Example 1, the following conclusions can be drawn.

1. Two marker sets M_1 and M_2 which are minimum dominating sets are Laplacian equienergetic

2. The DMSLE of a graph G with respect to a marker set M_1 is greater than that with respect to M_2 where $|M_1| = |M_2|$ whenever
$diam_{M_1}(G) > diam_{M_2}(G)$.

In general, the characteristic polynomial of a square matrix A of order n can be written as $\Delta(A) = t^n - S_1 t^{n-1} + S_2 t^{n-2} - S_3 t^{n-3} + \ldots + (-1)^n S_n$. In case of a graph of order p and its distance laplacian matrix of a marker set M being $L_M(G)$, the characteristic polynomial can be written as
$f(G:M,\lambda) = \Delta(L_M(G) = \lambda^p - S_1\lambda^{p-1} + S_2\lambda^{p-2} - \ldots + (-1)^p S_p$.
It is clear from [13] that $(-1)^i S_i = \Sigma M_{L_M(i)}$, where $M_{L_{M(i)}}$ are the principal minors of $L_M(G)$ with order i. (Minors whose diagonal elements belong to the main diagonal of $L_M(G)$).
$S_0 = 1$ and $S_1 = traceL_M(G) = -p$.
Also, we can derive the following lemma.

Lemma 3.4. *If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the M-distance Laplacian eigenvalues of the set M in G, then $\sum_{i=1}^{p} \lambda_i^2 = 2S_2 + p^2$.*

Proof. We have $\sum_{i=1}^{p} \lambda_i^2$
$=$ trace of$L_M^2(G)$
$= \sum_{i=1}^{p}\sum_{j=1}^{p} a_{ij}a_{ji}$
$= \sum_{i=1}^{p}\sum_{j=1}^{p} a_{ij}^2$(since $\quad a_{ij} = a_{ji}$)
$= 2\sum_{i=1}^{p-1}\sum_{j>i}^{p} a_{ij}^2 + \sum_{i=1}^{p} a_{ii}^2$
$= 2\sum_{i=1}^{p-1}\sum_{j>i}^{p} a_{ij}^2 + \sum_{i=1}^{p} k_i^2$(since $\quad a_{ii} = k_i$)
$= 2\sum_{i=1}^{p-1}\sum_{j>i}^{p} a_{ij}^2 + p^2 - 2\sum_{i=1}^{p-1}\sum_{j>i}^{p} k_i k_j$(since $\Sigma_{i=1}^{p} k_i = p$)
$= 2\sum_{i=1}^{p-1}\sum_{j>i}^{p} [a_{ij}^2 - k_i k_j] + p^2$
$= 2 \times$ (sum of all principal minors of order 2 of $L_M(G)) + p^2$
$= 2S_2 + p^2$.

□

Lemma 3.5. *For a complete graph of order p and any marker set M, zero is a M-set Laplacian eigenvalue of multiplicity $(p-3)$ or $(p-4)$.*

Proof. In a complete graph K_p the M-set distance matrix is a $(0,1)$ matrix as maximum M -set distance is 1 for any set M. Let $|M| = k$. Also, $DDS_G(M) = (k,(p-k))$. The M-set distance matrix of K_p is of the form

$$D_M(K_p) = \begin{bmatrix} 0_{k\times k} & 1_{k\times(p-k)} \\ 1_{(p-k)\times k} & 0_{(p-k)\times(p-k)} \end{bmatrix}$$

$L_M(G) = D_M(G) - diag[DDS_G(M)]$. Now $L_M(K_p) = (a_{ij})_{p\times p}$ can be different from $D_M(K_p)$ in only two places a_{11} and a_{22}. $a_{11} = -k, a_{22} = -(p-k)$.
For $k = 1$, the first row has first entry as $-k$ and the rest of the entries as 1. The second row has the first entry 1, second entry $-(p-k)$ and the rest of the entries zero. The third row to p^{th} row has the first entry 1 and the rest of the entries zero. Hence for $k = 1$, the number of distinct rows is 3 and so, the number of nonzero eigenvalues is 3. This implies that zero is an eigenvalue of multiplicity 3.
For $k = 2$, the first row has first entry as $-k$, second entry zero and the rest of the entries as 1, the second row has the first entry 0, second entry $-(p-k)$ and the rest of the entries zero and the third row to p^{th} row has the first two entries 1 and the rest of the entries zero. Hence for $k = 2$, the number of distinct rows is 3 and so, the number of nonzero eigenvalues is 3. This implies that zero is an eigenvalue of multiplicity 3.

For $k \geq 3$, the first row has first entry $-k$, second entry to k^{th} entry zero and the rest of the entries 1. The second row has first entry zero, the second entry $-(p-k)$, third entry to k^{th} entry zero and the rest of the entries 1. The third row to k^{th} row has the first k entries zero and the rest of the entries 1. The $(k+1)^{th}$ row to p^{th} row have the first k entries 1 and the rest of the entries zero. So, there are only 4 distinct rows. Hence, there can be only 4 nonzero eigenvalues and so zero is an eigenvalue of multiplicity $(p-4)$.
Therefore, zero is an eigenvalue of multiplicity $(p-3)$ or $(p-4)$.

□

Lemma 3.6. *If G is a simple connected graph with a marker set M then*

$$\sqrt{2S_2+p^2} \leq L\varepsilon_M(G) \leq \sqrt{p(2S_2+p^2)}.$$

Proof. Consider the Cauchy - Schwartz inequality,

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right).$$

Choosing $a_i = 1$ and $b_i = |\lambda_i|$ we get,

$$\left(\sum_{i=1}^{p} |\lambda_i|\right)^2 \leq p\sum_{i=1}^{p} \lambda_i^2 \quad \text{from which we get}$$

$$L\varepsilon_M^2(G) \leq p(2S_2+p^2) \quad \text{(by Lemma 1.1)}$$

which in turn implies

$$L\varepsilon_M(G) \leq \sqrt{p(2S_2+p^2)} \qquad \text{(eqn 1)}.$$

Now

$$L\varepsilon_M^2(G) = \left(\sum_{i=1}^{p} |\lambda_i|\right)^2 \geq \sum |\lambda_i|^2 = 2S_2+p^2.$$

$$L\varepsilon_M(G) \geq \sqrt{2S_2+p^2} \qquad \text{(eqn 2)}.$$

From eqn 1 and eqn 2, we get the result.

□

Next, we consider equienergetic marker sets.

Lemma 3.7. *For any simple connected graph G, two minimum dominating sets M_1 and M_2 are Laplacian equienergetic.*

Proof. We know from [12] that $D_{M_1}(G) = D_{M_2}(G)$ when both of them are in the standard form. Also, $DDS_G(M_1) = DDS_G(M_2) = (k,(p-k))$. This implies that, $L_{M_1}(G) = L_{M_2}(G)$. Both $L_{M_1}(G)$ and $L_{M_2}(G)$ have the same eigenvalues. Therefore, M_1 and M_2 are Laplacian equienergetic. □

Theorem 3.8. *Let G be a simple connected graph on p vertices and M be a dominating marker set of G with $|M| = k$. Then the characteristic polynomial of the M-set Laplacian is $\lambda^p - S_1\lambda^{p-1} + S_2\lambda^{p-2} - S_3\lambda^{p-3} + S_4\lambda^{p-4}$ where $S_1 = -p$, $S_2 = 0$,*

$$S_3 = \begin{cases} (p-1)(p-2), & \text{when} \quad k=1 \\ (k-1)p(p-k), & \text{when} \quad k \geq 2 \end{cases}$$

$$S_4 = \begin{cases} 0, & \text{when} \quad k \leq 2 \\ -k(k-1)(p-k)^2, & \text{when} \quad k \geq 3. \end{cases}$$

Proof. Let G be a simple connected graph and M be a dominating set. Then, the M-set distance matrix is a $(0,1)$ matrix as maximum M -set distance is 1. Let $|M| = k$. Also, $DDS_G(M) = (k,(p-k))$. The M-set distance matrix of G is of the form

$$D_M(G) = \begin{bmatrix} 0_{k\times k} & 1_{k\times(p-k)} \\ 1_{(p-k)\times k} & 0_{(p-k)\times(p-k)} \end{bmatrix}$$

$L_M(G) = D_M(G) - diag[DDS_G(M)]$. Now $L_M(G) = (a_{ij})_{p\times p}$ can be different from $D_M(G)$ in only two places a_{11} and a_{22}. $a_{11} = -p$, $a_{22} = -(p-k)$. Every subset of the principal diagonal gives a principal minor. Now, $S_1 = -k + [-(p-k)] = -p$. Also, it is evident that $L_M(G)$ has $2k(p-k)$ entries as 1 and $a_{11} = -k$, $a_{22} = -(p-k)$. It follows that there are $k(p-k)$ number of principal minors of order 2 each having value -1 and one principal minor of value $k(p-k)$, i.e. $S_2 = 0$.
The values of S_3 and S_4 are found for different values of k as follows:
For $k = 1$, it is obvious that the only non-zero principal minors of order 3 are those which have the first and second row and one of the rows from 3^{rd} row to p^{th} row. That is, the sum of all nonzero principal minors of order 3 is equal to $(p-2) \times \begin{vmatrix} -1 & 1 & 1 \\ 0 & -(p-1) & 0 \\ 1 & 0 & 0 \end{vmatrix} = (p-2)(p-1)$.
Hence $S_3 = (p-1)(p-2)$. Minors of order more than 3 have atleast one zero row or two like rows. In both the cases, their value is zero. So, $S_i = 0$ for $i \geq 4$. Therefore, the characteristic polynomial is $\lambda^p + p\lambda^{(p-1)} - (p-1)(p-2)\lambda^{(p-3)}$.
For $k = 2$, the only nonzero minors of order 3 are those with the first row, second row and one of the rows from 3^{rd} row to p^{th} row. That is, the sum of all nonzero principal minors of order 3 is equal to $(p-2) \times \begin{vmatrix} -2 & 0 & 1 \\ 0 & -(p-2) & 1 \\ 1 & 1 & 0 \end{vmatrix} = (p-2)p$.
Hence $S_3 = p(p-2)$. Also, minors of order more than 3 have atleast one zero row or two like rows. In both the cases, their value is zero. That is, $S_i = 0$ for $i \geq 4$.
For $k \geq 3$, the nonzero principal minors of order 3 are the following:

1) The 3×3 minors with first two rows and one of the rows from $k+1^{th}$ row to p^{th} row $\begin{vmatrix} -k & 0 & 1 \\ 0 & -(p-k) & 1 \\ 1 & 1 & 0 \end{vmatrix}$.

2) The 3×3 minors with first row, one of the rows from 3^{rd} row to k^{th} row as the 2^{nd} row and one of the rows from $(k+1)^{th}$ to p^{th} as the third row $\begin{vmatrix} -k & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$.

3) The 3×3 minors with the second row, one of the rows from 3^{rd} row to k^{th} row as the 2^{nd} row and one of the rows from $(k+1)^{th}$ to p^{th} as the third row are $\begin{vmatrix} -(p-k) & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$.

The sum of all nonzero principal minors of order 3 is equal to $S_3 = (p-k)p + k(k-2)(p-k) + (k-2)(p-k)^2 = (k-1)p(p-k)$. The 4×4 nonzero principal minors are those which have the first row, second row, one of the rows from third row to the k^{th} row and one of the rows from $(k+1)^{th}$ row to the p^{th} row. That is $\begin{vmatrix} -p & 0 & 0 & 1 \\ 0 & -(p-k) & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$. The sum of all nonzero principal minors of order 4 is equal to $S_4 = -(k-2)p(p-k)^2$. Since, any principal minor of order more than 4 has either two equal rows or two zero rows its value is equal to zero. Hence $S_i = 0$ for $i \geq 5$. $\square$

Corollary 3.9. *For a complete graph of order p and any marker set M, with $|M| = k$, the characteristic polynomial of the M-set Laplacian is $\lambda^p - S_1\lambda^{p-1} + S_2\lambda^{p-2} - S_3\lambda^{p-3} + S_4\lambda^{p-4}$ where $S_1 = -p$, $S_2 = 0$,*

$$S_3 = \begin{cases} (p-1)(p-2), & when \quad k = 1 \\ (k-1)p(p-k), & when \quad k \geq 2 \end{cases}$$

$$S_4 = \begin{cases} 0, & when \quad k \leq 2 \\ -k(k-1)(p-k)^2, & when \quad k \geq 3. \end{cases}$$

Now, we give the condition for a polynomial to be realized as the characteristic polynomial of a M-set distance Laplacian matrix of a graph.

Theorem 3.10. *The set of polynomials of the form $x^p + px^{p-1} - (p-1)(p-2)x^{(p-3)}$ where p is a positive integer, can be realized as the characteristic polynomial of a M-set distance Laplacian matrix of a simple connected graph G on p vertices.*

Proof. Let $P(x) = x^p + px^{p-1} - (p-1)(p-2))x^{p-3}$ be the given polynomial. Then, this is the characteristic polynomial of the M-set distance Laplacian matrix $L_M(G) = D_M(G) - diag[1,(p-1)]$ where $D_M(G) = \begin{bmatrix} 0_{1\times1} & 1_{1\times(p-1)} \\ 1_{(p-1)\times1} & 0_{(p-1)\times(p-1)} \end{bmatrix}$ The corresponding graph G can be obtained by sequentially joining a graph M on 1 vertex with $\overline{K_{p-1}}$.
i.e.

$$G = \langle M \rangle + \overline{K_{p-1}}.$$

Now, G with marker set M has the marker set distance Laplacian characteristic polynomial $P(x)$. $\square$

An algorithm can be written for the above theorem. An example is as given below.

Example 3.11. *Let $P(x) = x^6 + 6x^5 - 20x^3$. Then $P(x)$ can be written as $x^6 + 6x^5 - (6-1)(6-2)x^5$. Take a single vertex graph M.*
Join this sequentially with $\overline{K_5}$ to get a graph G (Figure 2).

$$G = \langle M \rangle + \overline{K_5}$$

G is the graph with marker set M, whose M-set distance Laplacian matrix is $L_M(G) = D_M(G) - diag[1,5]$ where $D_M(G) = \begin{bmatrix} 0_{1\times1} & 1_{1\times5} \\ 1_{5\times1} & 0_{5\times5} \end{bmatrix}$ and the corresponding characteristic polynomial is $x^6 + 6x^5 - 20x^3$.

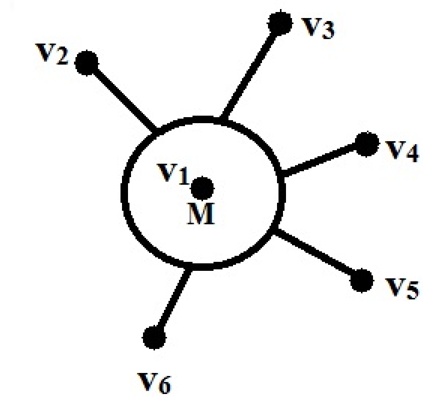

Figure 2

Theorem 3.12. *The set of polynomials of the form $x^p + px^{p-1} - (k-1)p(p-k)x^{p-3} - k(k-1)(p-k)^2x^{p-4}$ where p and k are positive integers with $k \leq p$ can be realized as the characteristic polynomial of a M-set distance Laplacian matrix of a simple connected graph G on p vertices.*

Proof. Let $P(x) = x^p + px^{p-1} - (k-1)p(p-k)x^{p-3} - k(k-2)(p-k)^2x^{p-4}$ be the given polynomial. Then, this is the characteristic polynomial of the M-set distance Laplacian matrix $L_M(G) = D_M(G) - diag[k,(p-k)]$ where $D_M(G) = \begin{bmatrix} 0_{k\times k} & 1_{k\times(p-k)} \\ 1_{(p-k)\times k} & 0_{(p-k)\times(p-k)}. \end{bmatrix}$ The corresponding graph G can be obtained by sequentially joining a graph M on k vertices with $\overline{K_{p-k}}$.
i.e.

$$G = \langle M \rangle + \overline{K_{p-k}}.$$

Now, G with marker set M has the marker set distance Laplacian characteristic polynomial $P(x)$. $\square$

Example 3.13. *Let $P(x) = x^6 + 6x^5 - 36x^3 - 27x^2$. Then $P(x)$ can be written as $x^6 + 6x^5 - (3-1)6(6-3)x^3 - 3(3-2)(6-3)^2x^2$. Take a graph M on 3 vertices. Join this sequentially with $\overline{K_3}$ to get a graph G (Figure 3).*

$$G = \langle M \rangle + \overline{K_3}$$

G is the graph with marker set M, whose M-set distance Laplacian matrix is $L_M(G) = D_M(G) - diag[3,3]$ where $D_M(G) =$

$\begin{bmatrix} 0_{3\times3} & 1_{3\times3} \\ 1_{3\times3} & 0_{3\times3} \end{bmatrix}$ *and the corresponding characteristic polynomial is* $x^6+6x^5-36x^3-27x^2$.

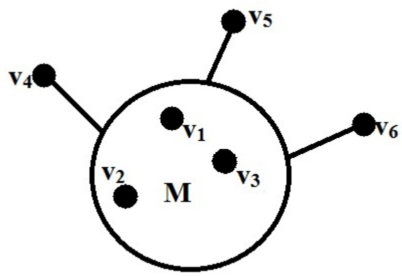

Figure 3

Conclusion

Although, we have initiated the study of marker set Laplacian eigenvalues in graphs in this paper, there are many related results which can be obtained. Many cases are being proved for the natural extensions.

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