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Irreversibility and Symmetry Principles in Quantum Information

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Abstract

Symmetry principles have played a crucial role in the development of modern physics and underpin our most fundamental theories of nature. The present thesis is concerned with the analysis of symmetries in the context of quantum information theory. Whenever a quantum mechanical system interacts with its environment it is subjected to decoherence, a process that is typically irreversible and most generally treated abstractly using the concept of quantum operations. If there is an underlying symmetry principle, additional structures emerge.

The central question we address is, *What are the consequences of global or local gauge symmetry on the structure of many-body quantum processes?* This leads us to a diagrammatic framework of decomposing quantum operations into terms that respond to the symmetry principle in particular ways and respect the causal structures involved. We present two core applications. First, we address the interplay between irreversibility and repeatable use of coherent resources under symmetry constraints. Second, we give an information-theoretic perspective on gauging globally symmetric dynamics to a local symmetry applicable even in the presence of irreversibility and thus it goes beyond the usual Lagrangian formulation. Finally we analyse the departure from conservation laws under symmetric dynamics subject to decoherence.

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Contents

- 1 Introduction** **12**
- 1.1 Symmetries in physics 12
- 1.2 Outline 14
- 1.3 Quantum reference frames 15
- 1.4 Quantum resource theory of asymmetry 17
- 1.5 Mathematical Preliminaries 19
 - 1.5.1 Overview of group representations 19
 - 1.5.2 Haar measure on compact groups 22
 - 1.5.3 Matrix coefficients of irreducible representations 22
 - 1.5.4 Irreducible tensor operators 23
 - 1.5.5 Modes of asymmetry 24
 - 1.5.6 Quantum channels: transformations under symmetry principles 25
 - 1.5.7 Clebsch-Gordan coefficients 28
 - 1.5.8 Example: SU(2) 30
 - 1.5.9 Homogeneous spaces 34
 - 1.5.10 The Hilbert space $\mathcal{L}^2(G)$ and Peter-Weyl theorem 35
 - 1.5.11 Spherical harmonics 36

1.5.12	Generalised spherical harmonics	37
2	A framework for quantum processes under symmetry principles	39
2.1	Process modes – a diagrammatic representation	40
2.1.1	Irreducible tensor superoperators	40
2.1.2	Process modes as a natural symmetry adapted basis for $\mathcal{S}(\mathcal{H}, \mathcal{K})$	44
2.2	Simulation of open quantum systems and the process orbit of a channel	51
2.2.1	Process orbit and relative alignments	52
2.3	A polar-decomposition for quantum channels	56
2.3.1	Case I: $SU(2)$ symmetry	57
2.3.2	General case of compact Lie group symmetry	64
2.4	Symmetric quantum channels on bipartite systems	67
2.4.1	Structure theorem for symmetric channels	67
2.4.2	Diagrammatic representation of symmetric channels on bipartite systems	69
2.4.3	Example: symmetric channels on two qubits under $SU(2)$ symmetry	71
3	Irreversibility in the symmetry breaking degrees of freedom	83
3.1	Irreversibility with respect to a fixed processing task	84
3.1.1	Repeatable use of quantum resources for simulation protocols	85
3.1.2	Repeatable use of resources and connections with quantum-classical channel from the reference system into the multiple simulations	88
3.2	Demystifying Åberg’s catalytic coherence	92
3.3	Repeated use of symmetry breaking resources: a process mode analysis	97
3.3.1	Asymptotic reference frames	97
3.3.2	Arbitrary repeatable use of coherence resources	98

3.3.3	Discussion: Are non-commutative symmetry breaking resources reusable?	101
4	How to gauge general quantum channels?	103
4.1	From global to local gauge symmetries	104
4.1.1	What does gauging mean?	104
4.1.2	Global vs local symmetry for many body systems	105
4.2	Gauging beyond Lagrangian formulations	106
4.2.1	I. Inclusion of background reference frame systems	110
4.2.2	II. Dynamics on the reference frame system	112
4.2.3	III. Gauging globally symmetric processes to local symmetry	114
4.3	Gauge fixing	116
4.4	Examples of gauge theories revisited: a quantum information theory perspective	119
4.4.1	U(1) symmetry	119
4.4.2	Unitary dynamics for a lattice gauge theory	121
4.4.3	Gauging quantum states	123
4.5	Connections between resource theories and Gauss's law	125
4.6	Further discussion	127
5	Beyond Noether's Theorem	129
5.1	Noether's charges and symmetric dynamics	130
5.2	Quantifying deviations from conservation laws	132
5.3	Quantifying openness of a quantum system	134
5.3.1	Definitions and properties	135
5.3.2	Symmetry constraints	137

5.4	Maximal deviations from conservation laws for spin systems	139
5.4.1	Simplex representation of symmetric channels with multiplicity-free irreducible decomposition of the operator space	140
5.4.2	Liouville representation of extremal SU(2)-symmetric channels	143
5.4.3	Unitarity of SU(2)-symmetric dynamics	144
5.4.4	Maximal spin reversal under symmetric dynamics	146
5.4.5	Maximal deviation for spin angular momentum conservation	149
5.5	Trade-off relations between unitarity and deviation from conservation laws . . .	150
5.5.1	Dynamics that approximates symmetric unitary have approximate conservation laws	151
5.5.2	When do conservation laws hold (beyond unitary dynamics)?	152
5.5.3	Spin systems: approximate deviations from conservation laws lead to approximate symmetric unitary dynamics	154
5.6	Supplementary Results	155
5.6.1	Extremal irreducible SU(2)-symmetric channels	155
5.6.2	Simplex representation of irreducibly symmetric extremal channels – A proof of Theorem 5.6.2	158
5.6.3	Maximal reversal of spin polarization for spin j systems	160
5.6.4	Maximal spin polarisation reversal under symmetric operations on spin systems $j_1 \rightarrow j_2$	161
6	Conclusion	164
	Bibliography	168

List of Figures

1.1	Pictorial overview of the thesis.	14
2.1	Process Mode Diagram	46
2.2	Directionality of process modes	51
2.3	Examples of process orbits.	54
2.4	Axial channels and resource demands.	63
2.5	Diagrammatic representation of symmetric superoperators.	70
2.6	The set of 2-qubit $SU(2)$ -diagrams.	72
2.7	The set of induced channels on A via a globally symmetric channel on AB	76
2.8	$SU(2)$ -symmetric, relational channels on 2-qubits.	79
2.9	T-state transformations.	80
2.10	Diagrammatic decomposition of $SU(2)$ symmetric unitary channel	82
3.1	Illustration of a 2-repeatable protocol on system \mathcal{H}_B	87
3.2	Circuit of symmetric operations \mathcal{W} that induce identical local simulations.	89
3.3	Aberg's protocol as broadcasting of reference frame data	96
4.1	Lattice associated with multipartite systems undergoing a globally symmetric dynamics.	107

4.2	3-symmetric channels.	109
4.3	Lattice gauge theory	123
5.1	Symmetric operations vs charge conserving operations.	131
5.2	Spin polarization of an environment.	139
5.3	SU(2)-invariant qutrit channels.	143
5.4	Maximal spin reversal under symmetric dynamics for 1/2-spin particle and j-spin particle.	148

Notation

In the following we summarise some commonly used notation throughout the thesis. Some of these are standard, while others are particular to the present work:

\mathcal{H}, \mathcal{K}		Hilbert spaces
$\mathcal{B}(\mathcal{H})$		C^* -algebra of bounded operators on \mathcal{H}
$\mathcal{S}(\mathcal{H}, \mathcal{K})$	Superoperator space (i.e linear maps between $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$)	
G, H		Groups
\mathcal{U}		Unitary representation acting on operators
\mathfrak{U}		Unitary representation acting on superoperators
$J[\mathcal{E}]$		The Choi operator of \mathcal{E}
$L(\mathcal{E})$		The Liouville representation matrix of \mathcal{E}
λ		Labels an irreducible representation of the group
\hat{G}		Set of non-equivalent irreducible representations of G
$\text{Irrep}(\mathcal{H}, \mathcal{K})$	Set of irreducible representations in the decomposition of $\mathcal{S}(\mathcal{H}, \mathcal{K})$	
$v_{kj}^\lambda(g)$		Matrix coefficients for the λ -irrep

Chapter 1

Introduction

1.1 Symmetries in physics

From impressive architectural feats to the fundamental forces of nature and the structure of DNA itself, the legacy of humankind employs symmetry and structure in every step of the way to *semper prorsum*. While for the ancient civilisations of Greece or Egypt for instance symmetry meant proportionality and harmony, the role symmetry takes in modern science is much more precise and rests on the mathematical notion of a group. It is directly related to the idea of invariance of a mathematical (e.g laws of motion) or physical (e.g crystals) object, under some set of transformations. These transformations themselves are subject to rules such that they can be composed and un-done thus respecting the composition and inverse conditions for a group.

When it comes to the role of symmetry in physics, there has been an important paradigmatic shift at the beginning of the twentieth century. The emphasis changed from deriving the symmetries of a system of interest from its laws of motion to a more central stage whereas the laws of nature were instead derived from the laws of invariance. At least since the time of Galileo, physicists have assumed that space is homogeneous, isotropic and that the arrow of time flows uniformly, all of which assume an implicit manifestation of symmetry. Einstein's theory of relativity brings symmetry at the forefront, and the laws of invariance are promoted to the status of postulate in the principle of relativity alongside the invariance of the speed of light in vacuum for all inertial observers. Previously, Poincaré realised that translations in time and space, rotations and boosts form a group structure – the group of isometries of the Minkowski space. Special relativity postulates that the laws of physics must be invariant under these global

space-time transformations, and thus brings about a new perspective on symmetries in physics. Without a doubt this shift was also to a large extent influenced by Noether's theorems. Exactly a century ago in 1918, Emmy Noether published "Invariante Variationsprobleme" [1]. It concerns physical theories that can be cast in terms of an action integral. For such systems, any continuous global symmetry of the action gives rise to conserved charges and the converse also holds. This establishes a fundamental link between conservation laws and symmetries. A second theorem underlines the differences when the global symmetry group is a subgroup of local symmetries.

In quantum mechanics, symmetry principles are no longer associated solely with global continuous space-time transformations; there are *internal* symmetries that dictate the spin of particle, *discrete* symmetries associated with indistinguishability of particles, parity and charge. The developments of quantum mechanics are tightly intertwined with the evolution of the role of symmetries in physics and even with the abstract mathematics that underpins it.

What does it mean for a quantum system described by Hilbert space \mathcal{H} to carry a particular type of symmetry? Mathematically it is often the case that we ascribe a priori to \mathcal{H} a particular type of symmetry given by a unitary representation $U(g)$ of some group G . This means that for any element of the group $g \in G$ there is a unitary transformation $U(g)$ acting on the Hilbert space in such a way that collectively all of these transformations have the same algebraic properties as the group. A given group can have many types of representations that act on a system and each of them corresponds to a different physical scenario. Whether the system we are interested in carries one type of representation or another is a statement about which of its physical properties we are interested in. Therefore on top of the Hilbert space constraints we put extra structure provided by the symmetries.

In reality the symmetry constraints may arise from either a particular type of dynamics that is invariant under the symmetries of the group, from a superselection rule (lack of a reference frame) [2] or imposed by the geometry of the physical system, which is the case in crystallography or quantum chemistry [3]. Other times the origin of the symmetries involved is deeper and points to fundamental laws of physics [4]. For completion, note that continuous symmetries are associated with Lie groups while discrete symmetries (like point-symmetry) are associated with discrete groups.

In this thesis we make no assumption on the origin of the symmetry constraints and quite often the starting point will be an existent symmetry principle present for the physical system of

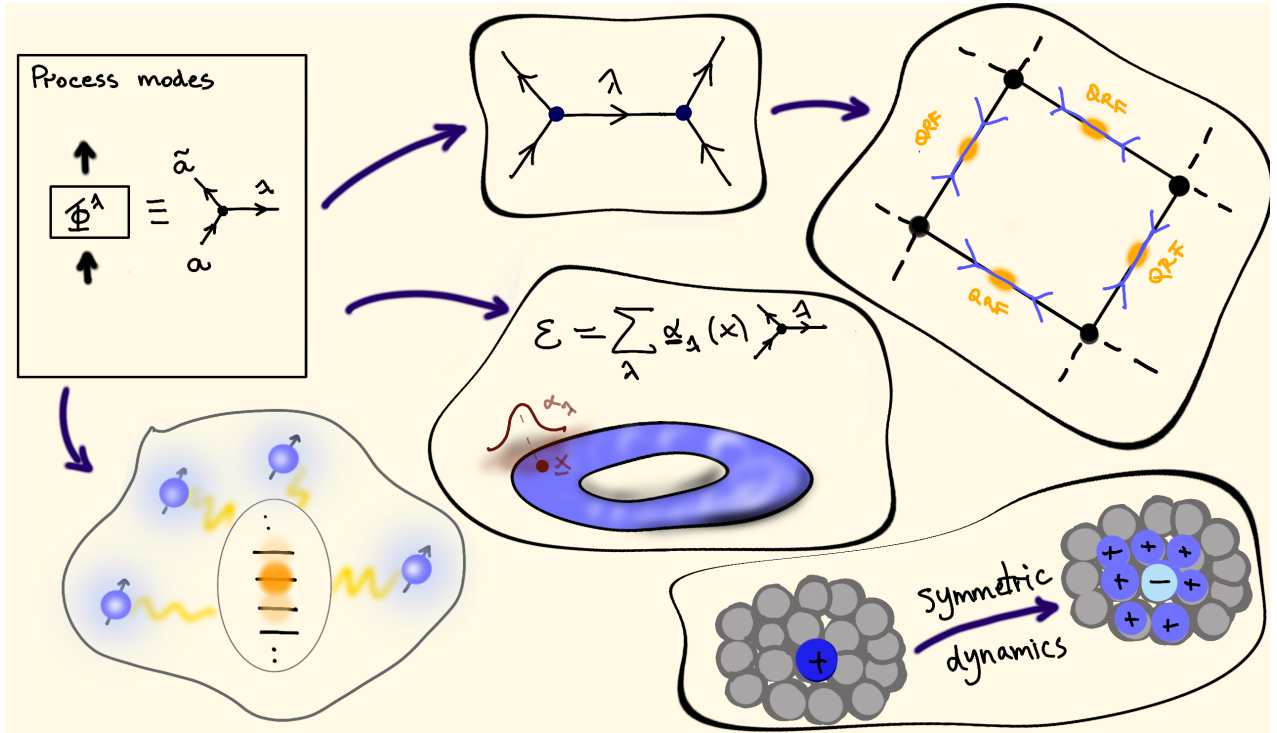


Figure 1.1: Pictorial overview of the thesis.

interest. To clarify, we will call a symmetry principle any fixed representation of a group acting on the physical system. In contrast, objects that remain invariant under a symmetry principle will be called symmetric e.g symmetric states, symmetric operations.

1.2 Outline

The results in this thesis are based on two different projects:

- Global and local gauge symmetries beyond Lagrangian formulations [5]
- Maximal deviations from Noether's charge conservation laws under symmetric dynamics (*in preparation*¹).

The current chapter gives a brief overview of mathematical tools related to representation theory and quantum channels. Chapter 2 gives a full treatment of the framework of process modes, introduced in [5]. The same work provides two different applications of the framework. First

¹Joint work with Kamil Korzekwa and David Jennings

chapter 3 expands on the role of process modes to analyse a quantum processing task relevant to quantum thermodynamic settings (and extensions to other resource-theoretic formulations). Second, chapter 4 gives an information-theoretic procedure to gauge general quantum processes from a global to a local symmetry. Finally, chapter 5 contains (part of) the author's contribution to the second project on characterising the trade-off between decoherence and deviations from conserved quantities under a general (non-unitary) symmetric process.

1.3 Quantum reference frames

In classical physics, the reference frames are macroscopic: rulers, gyroscopes, clocks. All of these are suitable to describe large classical systems, and different observers will generally have access to a common reference frame such as the Earth or stars to synchronise their physical descriptions. Within this regime observations of the systems involved have no effect on such classical frames.

Reference frames are physical objects themselves. These should have the same nature as the systems of interest and the measurement apparatus. When dealing with microscopic systems the reference frames involved can be treated as quantum mechanical objects. Quantum reference frames (QRFs) are additional quantum systems that transform non-trivially under a symmetry group which encodes relational information about a particular physical property of interest. An ideal mathematical realisation is given by the space of complex valued functions on a group $\mathcal{L}^2(G)$, which is considered to be the classical limit of a quantum reference frame because it encodes group elements into perfectly distinguishable states.

There have been several situations in quantum optics, superconductivity, Bose-Einstein condensates, where not taking the reference frames under explicit considerations has led to ambiguities in the description of quantum states exhibiting superpositions of particle number eigenstates [2]. For example, a laser generates a coherent state given by a superposition of eigenstates of the photon number; however a quantised treatment of the dipole source as in [6] leads to a reduced state of the electromagnetic field given by an incoherent mixture of the photon number eigenstates. It turns out these are not inconsistent descriptions but rather treat different degrees of freedom [2].

Any useful mathematical representation of quantum states or observables involves a description

relative to a particular basis. In order to describe the spin up $|\uparrow\rangle$ or spin down $|\downarrow\rangle$ states prepared by a Stern-Gerlach apparatus it is essential to specify a particular z -direction. A different laboratory can align the apparatus along another direction. It is in this sense that a quantum state includes both an intrinsic description of the quantum system involved and extrinsic information relative to an external reference frame.

In the quantum optics example above, the first case implicitly treats the phase reference frame externally with respect to which the state of the laser field has a fixed phase. The second case explicitly treats the quantum phase reference frame internally and while the bipartite state of the laser and QRF contains extrinsic information relative to an external reference frame, the reduced state of the laser field does not have a fixed phase with respect to the internal QRF subsystem.

Earlier ideas from Eddington [7] already hinted at the necessity of considering the uncertainty features of reference frames when reaching small scales where quantum mechanical effects are unavoidable. Quantum reference frames started gaining attention with the seminal papers by Aharonov and Susskind [8,9] and later Aharonov and Kaufherr [10] established a core result linking the lack of a shared reference frame with the superselection rules. Later, an operational perspective emerged in [11] by studying the transformations between QRFs associated with space-time.

In recent years, the study of QRFs found a natural framework in the context of quantum information theory [2, 12]. In quantum communication tasks one can make the distinction between unspeakable and speakable information whereas the way in which information is encoded matters for the former and does not for the latter. In particular, reference frames carry unspeakable information. Two distant agents Alice and Bob, without access to any common Cartesian reference cannot for instance communicate a direction using bit strings alone. Instead they must encode the directional information into a particular degree of freedom that can break rotational invariance.

Following [2], we look at the scenario where two agents Alice and Bob do not share a common reference frame that encodes symmetry-breaking degrees of freedom corresponding to the symmetry group G . For example, the role of a cartesian reference frame that specifies the z -direction is to break rotational invariance (associated to $G = SO(3)$). Suppose that Alice prepares state $\rho_A \in \mathcal{B}(\mathcal{H})$. Bob will describe such a state by $U(g)\rho_A U(g)^\dagger$ where $U(g)$ is a mapping that "rotates" Alice's local reference frame to be aligned with Bob's for some group element $g \in G$.

A lack of common reference frame means that the group parameter g describing the relative alignment of the two local frames is completely unknown. Therefore, from Bob's perspective if Alice sends him state ρ he can only describe the system with respect to his reference frame as a uniform probabilistic mixture of states $U(g)\rho U(g)^\dagger$ over *all* group elements $g \in G$. Hence Bob will not be able to distinguish the state ρ_A that Alice sends him from a state $\mathcal{G}(\rho_A)$ given by:

$$\mathcal{G}(\rho_A) = \int_G U(g)\rho U(g)^\dagger dg. \quad (1.1)$$

The state $\mathcal{G}(\rho_A)$ is symmetric such that $[U(h), \mathcal{G}(\rho)] = 0$ for all $h \in G$ and any state ρ . Therefore the absence of a shared reference frame gives rise to a symmetry constraint on what states Alice can prepare. Any operation or measurement will also be subject to this constraint.

However this restriction can be lifted if Alice sends Bob a token of her reference frame. In principle, such an additional quantum system will allow Bob to estimate the element g that for instance rotates Alice's definition of the z -direction to his own. Alignment of the two reference frames may also be partially achieved even when the token represents a bounded quantum system. Note however that even without establishing a shared reference frame, quantum information can still be transmitted between Alice and Bob if it is encoded in the relational degrees of freedom in a multipartite system.

Quantum reference frames have been considered in diverse contexts such as quantum thermodynamics [13,14], quantum cryptography [2], resource theories [12] and recently quantum information in a relativistic setting [15,16]. They also play an essential role in the relational description of quantum theory [17] and in [18] Rovelli argued that a quantum treatment of reference frames may be required to define physical operators for quantum gravity.

1.4 Quantum resource theory of asymmetry

Strongly motivated by quantum computation and communication, it has become standard to think of quantum phenomena in terms of "resources". As per usual it bears the connotation that some tasks are more useful than others and thus at its very core it is a relative or relational type of description. Resources are tautologically scarce. A *quantum resource* describes a physical property that cannot be readily associated with a hermitian observable. They are characterised in relation to freely available states or operations; a setting in which the presence of a quantum

resource is required in order to perform specific physical transformations of processing tasks which would otherwise be inaccessible. Entanglement is the archetypal example [19] – an entangled state is one which is not separable, and the amount of entanglement does not increase under local operations and classical communication (LOCC). There is no hermitian observable that describes entanglement but it is required to perform for instance quantum computations or quantum teleportation. In a resource-theoretic formulation the free operations are LOCC and the free states are separable states. Entangled states are resources that cannot be created using free operations acting on free states alone.

A quantum resource theory consists of a set of free states \mathcal{F} , a restricted set of operations and resource states \mathcal{R} , which are those not in \mathcal{F} .

The resource theory approach provides a cohesive general framework [20, 21] to analyse a variety of physical phenomena such as athermality [22–24], coherence [25–27], non-gaussianity in quantum optics [28, 29], magic states in quantum computation [30, 31]. Moreover, seemingly disconnected resource theories are found to share many common structural features [20, 32, 33].

Of interest for the present work is the resource theory of asymmetry [34–38], which has emerged from the study of quantum reference frames [2, 12]. It aims to conceptually characterise symmetry breaking properties of states. The free states are symmetric states and the free operations are symmetric operations. More precisely, given a group G acting on a system \mathcal{H} via the (unitary) representation U then the symmetric states $\rho \in \mathcal{F}$ remain invariant under the group action that is they satisfy $U(g)\rho U(g)^\dagger = \rho$ for all $g \in G$. The resource states are those that break this symmetry. The symmetric operations \mathcal{E} on system \mathcal{H} are the ones that commute with the group action such that $U(g)\mathcal{E}(\rho)U(g)^\dagger = \mathcal{E}(U(g)\rho U(g)^\dagger)$ holds for all $g \in G$ and $\rho \in \mathcal{B}(\mathcal{H})$.

In [35, 36] Marvian and Spekkens introduce a formalism for quantifying the symmetry-breaking degrees of freedom of a state in terms of *modes of asymmetry*. We will review some of the related technical details in Section 1.5.5. Their analysis leads in [34] to introduce novel information-theoretic measures of quantifying the asymmetry of a state. In particular it illustrates that for mixed states in an otherwise isolated system undergoing a unitary symmetric evolution, conservation laws of Noether’s charges do not capture all the consequences of this invariance.

By adopting a resource-theoretic perspective with these asymmetry modes, it is possible to encode symmetry principles in general processes, and to quantify their effects in a rigorous manner. This has provided a range of novel insights; in particular the structure has recently

provided a natural, explanatory framework for quantum thermodynamics [23,39–53]. It has been used to prove that no formalism based solely on free energy functions can fully describe coherence in the thermodynamics of extreme quantum regimes [45]. The modes perspective also makes explicit general upper and lower bounds for quantum coherence [54,55].

1.5 Mathematical Preliminaries

The role of this section is to set the scene for the rest of this thesis by emphasizing some key results from quantum information and representation theory of groups and algebras. However, it is not meant to be an exhaustive exposition on either of these topics. We assume familiarity with the common quantum information theory concepts, as found in the introductory textbook of Nielsen and Chuang [56] and with basic group theory techniques as in [57]. Instead we focus on setting notation for the key concepts used later on, and address some issues that are either typically overlooked in the literature or presented from a different perspective than we need here. For more in-depth results on representation theory we refer to [58], the excellent classical monograph [59,60] thoroughly treats special functions, while for irreducible tensor operators [61] gives an intuitive but extensive overview of this topic alone.

1.5.1 Overview of group representations

We will consider only compact² groups typically denoted by G . Some of the results presented may well be expanded beyond, but there would be significant technical challenges to do so.

A representation U of the (compact) group G on a vector space V is a (continuous) map that takes every group element $g \rightarrow U(g)$ to an operator in $GL(V)$, the space of general linear maps on V , in a way that preserves the group structure: $U(gh) = U(g)U(h)$ for all $g, h \in G$ and $U(e) = \mathbb{I}$ where e is the identity group element.

The character χ of a given representation U is a complex valued function acting on the group given by $\chi(g) := \text{Tr}(U(g))$.

Typically we are interested in the group action on a physical system as represented by a Hilbert

²A topological group is compact if it compact as a topological space. This also includes finite groups with the discrete topology.

space \mathcal{H} . For a Hilbert space we can identify $GL(\mathcal{H})$ with $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on \mathcal{H} .

In addition, unless otherwise stated we will consider *unitary* representations, for which $U(g) \in \mathcal{B}(\mathcal{H})$ is a unitary operator $U(g)U(g)^\dagger = U(g)^\dagger U(g) = \mathbb{I}$ for all $g \in G$. This is not a restriction because for compact groups any representation is equivalent to a unitary representation.

Definition 1.5.1. *A representation $V : G \rightarrow \mathcal{B}(\mathcal{H})$ is irreducible if and only if there is no subspace $S \subset \mathcal{H}$ of \mathcal{H} (other than 0 and itself) which remains invariant under V i.e $V(g)(S) \subset S$ for all $g \in G$.*

Irreducible representations form the "building blocks" of any representation. In a more informal way, \mathcal{H} carries an irreducible representation (for short, irrep) of G if equivalently for any $|\psi\rangle \in \mathcal{H}$ and orthogonal $|\phi\rangle \in \mathcal{H}$ with $\langle\psi|\phi\rangle = 0$ there exists a group element $g \in G$ such that $\langle\psi|V(g)|\phi\rangle \neq 0$.

In particular, all irreducible representations of a compact group are finite dimensional and any unitary representation can be decomposed into a direct sum of orthogonal irreducible representations.

Lemma 1.5.2. *Let U be a unitary representation of a compact group G on a finite dimensional system \mathcal{H} . Then \mathcal{H} can be decomposed into a direct sum of orthogonal irreducible subspaces:*

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_n$$

and with respect to this, the unitary $U(g)$ has a block-diagonal form for every $g \in G$:

$$U = V_1 \oplus V_2 \oplus \dots \oplus V_n \tag{1.2}$$

where each $V_i : G \rightarrow \mathcal{B}(\mathcal{H}_i)$ is an irreducible unitary representation of G .

Two representations U_1, U_2 of G on spaces $\mathcal{H}_1, \mathcal{H}_2$ are equivalent if there is a G -invariant linear map $\mathcal{I} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ i.e it commutes with the group action $U_2(g)\mathcal{I} = \mathcal{I}U_1(g)$ for all $g \in G$. Any such map is called an *intertwiner* between representations³ \mathcal{H}_1 and \mathcal{H}_2 . Schur's lemma states that the space of intertwiners between irreducible representations is one-dimensional.

³Aligned with much of the literature, in this work as well we make a slight abuse of terminology in calling the representation spaces simply representation. It will be (hopefully) obvious from the context if we refer to the representation itself or the underlying representation space.

Lemma 1.5.3. Schur's lemma *Let \mathcal{H}_1 and \mathcal{H}_2 carry irreducible representations V_1 and V_2 of some group G . Then given an operator $O : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ that commutes with the group action $OV_1(g) = V_2(g)O$ for all $g \in G$ then O is either a multiple of the identity $O \propto \mathbb{I}$ if the irreps are isomorphic $V_1(g) = V_2(g)$ and $\mathcal{H}_1 = \mathcal{H}_2$ or the zero operator otherwise.*

We will denote by \hat{G} the set of inequivalent irreducible representations of G . Each of them will be associated with a unique label λ .

The decomposition in equation (1.2), may contain for instance many equivalent representations. The number of such distinct irreps in decomposing U is called the multiplicity of said irrep in U . Equivalently:

$$U \cong \bigoplus_{\lambda \in \hat{G}} m_\lambda V^\lambda \quad (1.3)$$

where $m_\lambda \in \mathbb{N}$ is the multiplicity label corresponding to the λ -irrep given by V^λ and $m_\lambda V^\lambda = \underbrace{V^\lambda \oplus \dots \oplus V^\lambda}_{m_\lambda \text{ times}} = V_\lambda \otimes \mathbb{I}_{m_\lambda}$ denotes an orthogonal direct sum of m_λ equivalent irreps.

Lemma 1.5.4. *Any unitary representation U of a compact group G has a unique decomposition into isotypical components:*

$$U = \bigoplus_{\lambda \in \hat{G}} U^\lambda$$

where $U^\lambda \cong m_\lambda V^\lambda$ is called the λ -irrep isotypical component.

Equivalently, for any representation the carrier space \mathcal{H} admits a coarse graining into a fixed structure that depends only on the symmetry properties. The decomposition into fully irreducible components is a much finer grained structure which does not admit a unique orthogonal decomposition. In the latter case the additional freedom comes from choosing a basis for the space of all intertwiners between \mathcal{H} and any fixed λ -irrep.

For Lie groups, their irreducible representations give rise to irreducible representation of the Lie algebra. One may characterise every finite dimensional irreducible representation of G uniquely by a vector of highest weights. A weight of a representation of a Lie algebra \mathfrak{g} is a vector given by the eigenvalues of all elements in the maximal commutative (Cartan) subalgebra of \mathfrak{g} . One can introduce a partial order on the set of weights given by $\lambda \succeq \mu$ if $\lambda - \mu$ is a linear combination of the positive weights of the adjoint representation. A detailed discussion of this topic may be found in [58].

1.5.2 Haar measure on compact groups

One of the many nice features of compact groups is that they allow for an essentially unique measure invariant under left (and right) translations. This allows to define an integral for (measurable) complex-valued functions $f : G \rightarrow \mathbb{C}$ on the group given by:

$$\int_G f(g) dg \tag{1.4}$$

where the measure is normalised such that $\int_G dg = 1$.

This is invariant under left and right translation, meaning that:

$$\int_G f(hg) dg = \int_G f(g) dg \quad \text{and} \quad \int_G f(gh) dg = \int_G f(g) dg \tag{1.5}$$

hold for any $h \in G$.

1.5.3 Matrix coefficients of irreducible representations

For any $\lambda \in \hat{G}$ with unitary irreducible representation V^λ , its matrix coefficients with respect to a fixed orthonormal basis $\{|i\rangle\}_{i=1}^{\dim(\lambda)}$ for the carrier space are given by:

$$v_{ij}^\lambda(g) := \langle j | V^\lambda(g) | i \rangle. \tag{1.6}$$

These can be viewed as complex valued functions on the group $v_{ij}^\lambda : G \rightarrow \mathbb{C}$. Moreover they satisfy several properties, particularly Schur orthogonality which we will use extensively.

Lemma 1.5.5. *Let G be a compact group. Then the following hold:*

1. For any $\lambda, \mu \in \hat{G}$

$$\int_G v_{ij}^\lambda(g) (v_{mn}^\mu(g))^* dg = \frac{\delta_{\mu, \lambda} \delta_{im} \delta_{jn}}{\dim(\lambda)}$$

2. The dual λ^* of an irreducible representation λ is irreducible and

$$v_{mn}^{\lambda^*}(g) = (v_{mn}^\lambda(g))^*$$

1.5.4 Irreducible tensor operators

The presence of a symmetry on the Hilbert space \mathcal{H} , as described via a unitary representation U , implies a particular structure. This will naturally also manifest itself in higher algebraic constructs such as the operator space $\mathcal{B}(\mathcal{H})$ or superoperator spaces that are built on \mathcal{H} . The symmetry is lifted to such objects in a way that is compatible with their algebraic structure.

The notion of *irreducible tensor operators* illustrates the structure that emerges from the symmetry lifted to the space of operators $\mathcal{B}(\mathcal{H})$.

Definition 1.5.6. *Let G be a compact group with U a unitary representation on \mathcal{H} . Then for any irrep $\lambda \in \hat{G}$ the set of operators $\{T_k^\lambda\}_{k=1}^{\dim(\lambda)}$ in $\mathcal{B}(\mathcal{H})$ are called irreducible tensor operators if they transform under the group action as:*

$$U(g)T_k^\lambda U(g)^\dagger = \sum_{k'} v_{k'k}^\lambda(g) T_{k'}^\lambda \quad (1.7)$$

for all vector label k and $g \in G$.

As representation spaces for the group G we have that $\mathcal{B}(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}^*$ are isomorphic, where the dual space \mathcal{H}^* carries the dual representation U^* . As Hilbert spaces, \mathcal{H} and \mathcal{H}^* are isomorphic so we can view $\mathcal{B}(\mathcal{H})$ as two copies of \mathcal{H} with the tensor product representation $U \otimes U^*$. In this sense, the irreducible decomposition of $U \otimes U^*$ will be the same as the decomposition of \mathcal{U} . Moreover every irreducible tensor operator set $\{T_k^\lambda\}_{k=1}^{\dim(\lambda)} \subset \mathcal{B}(\mathcal{H})$ gives an orthogonal basis for a λ -irrep subspace of $\mathcal{B}(\mathcal{H})$. Indeed, from Schur orthogonality of matrix coefficients it follows that any irreducible tensor operators $\{T_k^\lambda\}$ and $\{T_j^\mu\}$ transforming under the λ -irrep and μ -irrep respectively are orthogonal with respect to the Hilbert-Schmidt inner product:

$$\text{Tr}((T_k^\lambda)^\dagger T_j^\mu) = N_\lambda \delta_{\lambda,\mu} \delta_{kj} \quad (1.8)$$

where N is a normalisation factor independent on the vector component. Typically, for convenience we may assume that $N = 1$.

The irreducible tensor operators give the decomposition of $\mathcal{B}(\mathcal{H})$ into irreducible subspaces such that:

$$\mathcal{B}(\mathcal{H}) = \bigoplus_{\lambda, m_\lambda} \{T_k^\lambda : 1 \leq k \leq \dim(\lambda)\} \quad (1.9)$$

where the direct sum is over all irrep λ in $U \otimes U^*$ of multiplicity m_λ . Conversely, any orthogonal

basis for each irreducible subspace of $\mathcal{B}(\mathcal{H})$ must transform according to equation 1.7 so that it forms a set of irreducible tensor operators.

For the case of compact Lie groups, there is an alternative definition of irreducible tensor operators in terms of the generators $\{J_1, \dots, J_n\}$ of the Lie algebra representation on $\mathcal{B}(\mathcal{H})$. Then the set of operators $\{T_k^\lambda\}_{k=1}^{\dim(\lambda)}$ in $\mathcal{B}(\mathcal{H})$ transform as:

$$[J_m, T_k^\lambda] = \sum_{k'} \langle \lambda, k | J_m | \lambda, k' \rangle T_{k'}^\lambda \quad (1.10)$$

for all generators J_m and all vector labels k , where the $\langle \lambda, k | J_m | \lambda, k' \rangle$ are given by the matrix coefficients of the generators with respect to an eigenstate basis $|\lambda, k\rangle$ corresponding to weight vectors.

For more details on this topic we refer to [61].

1.5.5 Modes of asymmetry

We give a brief overview of the work of Marvian and Spekkens [35, 36] that formulates the resource theory of asymmetry in order to quantify the symmetry-breaking properties of a quantum state in terms of its irreducible components.

In the resource theory of asymmetry there is an inherent assumption of an underlying symmetry principle on the Hilbert space \mathcal{H} given by the unitary representation U of G . Specifically, one can decompose a density matrix into $\rho = \sum_{\lambda, k} \rho_k^\lambda$, where the operators ρ_k^λ form *irreducible tensor operators* under the group action, namely they transform according to $\mathcal{U}_g(\rho_k^\lambda) = \sum_j v_{kj}^\lambda(g) \rho_k^\lambda$ with $v^\lambda(g)_{kj}$ being the matrix components of λ -irrep of the group G . Each term ρ_k^λ is called an *asymmetry mode*.

A simple example is provided by quantum coherence in a quantum system S between eigenstates of the number operator $N = \sum_{n \geq 0} n |n\rangle \langle n|$. The system S has a $U(1)$ group action $U(\theta) := e^{i\theta N}$ corresponding to phase shifts. An operator X is a mode of coherence if it transforms irreducibly under the group. For $U(1)$ this simply means that $U(\theta) X U(\theta)^\dagger = e^{ik\theta} X$ for some integer $k \in \mathbb{Z}$. A coherent state such as $\rho = |\psi\rangle \langle \psi|$ with $|\psi\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$ has two non-zero off-diagonal terms $|a\rangle \langle b|$ and $|b\rangle \langle a|$. Under the group action

$$|a\rangle \langle b| \rightarrow e^{i\theta N} |a\rangle \langle b| e^{-i\theta N} = e^{i(a-b)\theta} |a\rangle \langle b| \quad (1.11)$$

and therefore is a mode of coherence with $k = (a - b)$, while $|b\rangle\langle a|$ is a $k = (b - a)$ mode. In contrast the diagonal terms $|a\rangle\langle a|$ and $|b\rangle\langle b|$ are each $k = 0$ modes.

1.5.6 Quantum channels: transformations under symmetry principles

Quantum channels describe the dynamical evolution of quantum systems, in a general way. The formalism includes system-environment interaction and admits several equivalent representations. A quantum channel (or quantum operation) is a completely positive trace preserving linear map $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ taking density matrices $\rho \in \mathcal{B}(\mathcal{H})$ to density matrices $\mathcal{E}(\rho) \in \mathcal{B}(\mathcal{K})$.

We denote by $\mathcal{S}(\mathcal{H}, \mathcal{K})$ the set of all linear operators from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$, often referred to as superoperators.

A major point of focus of this thesis is the study of quantum channels under symmetry principles. As previously described, a symmetry principle involves specifying the action of a compact group G on the input and output systems described by Hilbert spaces \mathcal{H} and \mathcal{K} respectively. Consider U and U' to be unitary representations of G acting on \mathcal{H} and \mathcal{K} . This lifts to a representation of G acting on the space of bounded operators, in such a way that it respects its algebraic structure. Therefore under such a symmetry transformation any state $\rho \in \mathcal{B}(\mathcal{H})$ is mapped to:

$$\mathcal{U}_g(\rho) := U(g)\rho U(g)^\dagger \quad (1.12)$$

and similarly for system \mathcal{K} . We say ρ is a symmetric state if it remains invariant under the above group representation so that $U(g)\rho U(g)^\dagger = \rho$ for all $g \in G$

In the same way the symmetry transformation lifts to an action on the space of superoperators such that under the group action of G , any linear map (and in particular quantum channel) $\mathcal{E} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ transforms as:

$$\mathfrak{U}_g[\mathcal{E}] := \mathcal{U}'_g \circ \mathcal{E} \circ \mathcal{U}_g^\dagger \quad (1.13)$$

for any $g \in G$.

Definition 1.5.7. A quantum channel $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ is symmetric (or G -covariant) if it is invariant under the group action, that is $\mathfrak{U}_g[\mathcal{E}] = \mathcal{E}$ for all $g \in G$. Equivalently, for all $\rho \in \mathcal{B}(\mathcal{H})$ and all $g \in G$:

$$U'(g)\mathcal{E}(\rho)U'(g)^\dagger = \mathcal{E}(U(g)\rho U(g)^\dagger).$$

Kraus representation

For any quantum channel $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$ there is a set of so-called Kraus operators $A_i : \mathcal{H} \longrightarrow \mathcal{K}$ such that:

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger \quad (1.14)$$

where $\sum_i A_i^\dagger A_i = \mathbb{I}$ in order to satisfy the trace-preserving condition.

In [12] it was shown that if \mathcal{E} is a symmetric channel then it admits a Kraus decomposition $\{A_{j,m,\alpha}\}$ labelled by the j -irrep with multiplicity α in the tensor product representation $U' \otimes U^*$ of $\mathcal{K} \otimes \mathcal{H}^*$ such that under the induced group action on the space of linear operations from \mathcal{H} to \mathcal{K} it transforms as an irreducible tensor operator:

$$U'(g)A_{j,m,\alpha}U(g)^\dagger = \sum_{m'} v_{m'm}^j(g)A_{j,m',\alpha}. \quad (1.15)$$

Stinespring dilation

For any quantum channel $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$ there exists a Hilbert space \mathcal{H}_E representing the environment and an isometry $V : \mathcal{H} \longrightarrow \mathcal{K} \otimes \mathcal{H}_E$ such that:

$$\mathcal{E}(\rho) = \text{Tr}_E(V\rho V^\dagger). \quad (1.16)$$

If \mathcal{E} is a symmetric channel then in addition the isometry V itself is symmetric such that $VU(g) = U'(g) \otimes U_E(g)V$ for all $g \in G$, where U_E is a representation of G acting on the environment space \mathcal{H}_E .

The Stinespring dilation is unique only up to a partial isometry on the environment system and the minimum dimension for \mathcal{H}_E gives the minimal Stinespring dilation, which is determined up to a local unitary on \mathcal{H}_E .

Choi-Jamiołkowski representation

To every linear map $\mathcal{E} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ we can *uniquely* associate an operator $J[\mathcal{E}] : \mathcal{K} \otimes \mathcal{H} \longrightarrow \mathcal{K} \otimes \mathcal{H}$, its Choi matrix, defined via:

$$J[\mathcal{E}] := (\mathcal{E} \otimes id) \left(\sum_{i=1}^d |i\rangle|i\rangle\langle j|\langle j| \right) \quad (1.17)$$

where $\{|i\rangle\}_{i=1}^d$ is an orthonormal basis for the d -dimensional input Hilbert space \mathcal{H} . Conversely, in terms of the Choi matrix, the channel is given by $\mathcal{E}(\rho) = \text{Tr}_{\mathcal{H}}(J[\mathcal{E}](\rho^T \otimes \mathbb{I}))$.

The superoperator \mathcal{E} is completely positive if and only if its Choi matrix is positive $J[\mathcal{E}] > 0$. Moreover if \mathcal{E} is trace-preserving then $\text{Tr}_{\mathcal{K}}(J[\mathcal{E}]) = \mathbb{I}_{\mathcal{H}}$. Therefore $\mathcal{E} \longrightarrow J[\mathcal{E}]$ gives an isomorphism between quantum channels and positive operators in $\mathcal{B}(\mathcal{K} \otimes \mathcal{H})$ with identity as the reduced operator on the input system.

Under a symmetry principle, the group action of G on quantum channel transforms \mathcal{E} into $\mathcal{U}_g[\mathcal{E}]$. This induces a group action on Choi operators such that $J[\mathcal{E}] \longrightarrow J[\mathcal{U}_g[\mathcal{E}]]$. One can easily check this corresponds to the tensor product representation $U' \otimes U^*$ on $\mathcal{K} \otimes \mathcal{H}$ such that the transformed Choi operator under the group element $g \in G$ is:

$$J[\mathcal{E}] \longrightarrow U'(g) \otimes U^*(g) J[\mathcal{E}] U'(g)^\dagger \otimes U(g)^T. \quad (1.18)$$

Liouville representation

Let $\{T_i\}_{i=1}^{d^2}$ be an operator basis for $\mathcal{B}(\mathcal{H})$ with d the dimension of \mathcal{H} , such that it is orthonormal with respect to the Hilbert-Schmidt inner product defined by $\langle A, B \rangle = \text{Tr}(A^\dagger B)$ for $A, B \in \mathcal{B}(\mathcal{H})$. Then the Liouville representation of an operator A is given by the unique column vector $|A\rangle\rangle$ in \mathbb{C}^{d^2} with entries given by $\text{Tr}(T_i^\dagger A)$ the inner product of each of the basis operators with A .

The Liouville representation of a superoperator $\mathcal{E} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ is a d^2 by $d_{\mathcal{K}}^2$ matrix $L(\mathcal{E})$, with $d_{\mathcal{K}}$ the dimension of \mathcal{K} , such that for any $A \in \mathcal{B}(\mathcal{H})$

$$L(\mathcal{E})|A\rangle\rangle = |\mathcal{E}(A)\rangle\rangle. \quad (1.19)$$

The above relation uniquely defines the Liouville representation of a superoperator once orthonormal basis $\{T_j\}_{j=1}^{d^2}$ and $\{S_i\}_{i=1}^{d_{\mathcal{K}}^2}$ for $\mathcal{B}(\mathcal{H})$ respectively $\mathcal{B}(\mathcal{K})$ have been fixed. Moreover the

entries of the matrix $L(\mathcal{E})$ are given by:

$$L(\mathcal{E})_{ij} = \langle \langle S_i | L(\mathcal{E}) | T_j \rangle \rangle = \text{Tr}(S_i^\dagger \mathcal{E}(T_j)) \quad (1.20)$$

In the Liouville representation composition of maps (channels) becomes multiplication of matrices such that:

$$L(\mathcal{E} \circ \mathcal{F}) = L(\mathcal{E})L(\mathcal{F}). \quad (1.21)$$

The Liouville representation of a unitary channel $\mathcal{V}(\cdot) = V(\cdot)V^\dagger$ is $L(\mathcal{V}) = V \otimes V^*$. From these two results we can deduce how the Liouville representation of a channel will transform under a group representation. We get that:

$$L(\mathcal{U}_g[\mathcal{E}]) = L[\mathcal{U}'_g]L[\mathcal{E}]L[\mathcal{U}_g^\dagger] = U'(g) \otimes U'(g)^* L[\mathcal{E}] U(g)^\dagger \otimes U(g)^T \quad (1.22)$$

for any $g \in G$ and any $\mathcal{E} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$

1.5.7 Clebsch-Gordan coefficients

In physics, Clebsch-Gordan coefficients appear when coupling spin angular momentum in quantum mechanics where the relevant symmetry group is $G = SO(3)$. Conceptually, what follows is just a generalisation to an arbitrary compact group G .

Let V^μ and V^ν be two irreducible representations of the compact group G and assume these are realised on the spaces \mathcal{H}^μ and \mathcal{H}^ν respectively where μ and ν label the particular irreducible representation (and may for instance correspond to highest weight vectors). The tensor product representation $\mathcal{H}^\mu \otimes \mathcal{H}^\nu$ decomposes into irreducible components:

$$\mathcal{H}^\mu \otimes \mathcal{H}^\nu \cong \bigoplus_{\lambda \in \hat{G}} m_\lambda \mathcal{H}^\lambda \quad (1.23)$$

where m_λ is the multiplicity of the λ -irrep in the above. This implies that the product of representations $V^\mu \otimes V^\nu$ is unitarily equivalent to a block decomposition where each block is an irreducible representation of the group. One can write that for all $g \in G$

$$C^\dagger(V^\mu(g) \otimes V^\nu(g))C = \bigoplus_{\lambda} m_\lambda V^\lambda(g) \quad (1.24)$$

for some unitary matrix C which represents nothing more than a change of basis in $\mathcal{H}^\mu \otimes \mathcal{H}^\nu$ from the tensor product basis to a basis that achieves the block-diagonal decomposition. The entries of this matrix are what we call the Clebsch Gordan coefficients.

Let $|e_k^\mu\rangle$ and $|e_k^\nu\rangle$ be basis for \mathcal{H}^μ and \mathcal{H}^ν respectively and $|e_k^{\lambda,\alpha}\rangle \in \mathcal{H}^\mu \otimes \mathcal{H}^\nu$ a basis for the λ -irreducible component labelled by multiplicity α in the above decomposition. The Clebsch-Gordan coefficients depend on a particular such choice of basis for \mathcal{H}^μ , \mathcal{H}^ν and $\mathcal{H}^\mu \otimes \mathcal{H}^\nu$:

$$|e_k^{\lambda,\alpha}\rangle = \sum_{m,n} \langle \mu, m; \nu, n | \lambda, k \rangle |e_m^\mu\rangle |e_n^\nu\rangle \quad (1.25)$$

where the coefficients $\langle \mu, m; \nu, n | \lambda, k \rangle$ represent entries for the unitary matrix C and for simplicity⁴ we denote both the irrep and the multiplicity label in the decomposition by the single label $\lambda = (\lambda, \alpha)$.

Explicitly equation (1.24) that defines Clebsch-Gordan coefficients can be cast in terms of matrix coefficients:

$$v_{k'k}^\lambda(g) = \sum_{m,m',n,n'} \langle \lambda, k | \mu, m; \nu, n \rangle v_{mm'}^\mu(g) v_{nn'}^\nu(g) \langle \mu, m'; \nu, n' | \lambda, k' \rangle. \quad (1.26)$$

This expresses a duality between matrix coefficients and Clebsch-Gordan coefficients, which is detailed in [59]. Moreover, unitarity of C implies the following conditions hold:

$$\delta_{kk'} \delta_{\lambda,\lambda'} = \sum_{m,n} \langle \lambda, k | \mu, m; \nu, n \rangle \langle \mu, m; \nu, n | \lambda', k' \rangle \quad (1.27)$$

$$\delta_{m,m'} \delta_{n,n'} = \sum_{\lambda,k} \langle \mu, m; \nu, n | \lambda, k \rangle \langle \lambda, k | \mu, m; \nu, n \rangle. \quad (1.28)$$

Using the above orthogonality conditions, or equivalently that C is a unitary matrix then equation (1.26) transforms into the following useful relation:

$$\sum_{k'} \langle \mu, m; \nu, n | \lambda, k' \rangle v_{k'k}^\lambda(g) = \sum_{m',n'} v_{mm'}^\mu(g) v_{nn'}^\nu(g) \langle \mu, m'; \nu, n' | \lambda, k \rangle \quad (1.29)$$

which simply expresses $(V^\mu(g) \otimes V^\nu(g))C = C(\bigoplus_\lambda m_\lambda V^\lambda(g))$ at the level of matrix entries.

This illustrates that Clebsch-Gordan coefficients are fundamentally linked with a particular

⁴This is no longer necessary for $SU(2)$, since every tensor product of irreps has a multiplicity-free decomposition. In such case we simply write $\langle \mu, m; \nu, n | \lambda, k \rangle$ for the Clebsch-Gordan coefficients.

choice of matrix coefficients for the irreducible representations. However the defining relation above does not fully fix these coefficients so that there is freedom in choosing an overall phase. This ensures that one can find real Clebsch-Gordan coefficients and for $SU(2)$ we will employ the Condon-Shortley choice of phase, which is defined in such a way as to give a simple isomorphism between irreducible representations and their dual [61].

On computing the Clebsch-Gordan coefficients: a complexity aside

The Clebsch-Gordan problem: Given a compact Lie algebra, and highest weights μ, ν and λ compute the multiplicity m_λ of V^λ in the irrep decomposition of the tensor product $V^\mu \otimes V^\nu$.

It was recently proven in [62] that this problem is generally $\#P$ -complete. However, for Lie algebras with fixed rank there is a polynomial time algorithm (in the dimensions of the weights) computing these multiplicities [63].

A (simplified) version of the problem is to determine if $m_\lambda > 0$ for a given a triplet of highest weights (μ, ν, λ) . This decision problem has a polynomial time algorithm which was proved in [64] and bears connections with the question of P vs NP via the program in geometric complexity theory put forward by Mulmuley and Sohoni [64].

Moreover [65] presents a simple algorithm to compute the actual Clebsch-Gordan coefficients numerically for $SU(n)$ and $SL_n(\mathbb{C})$.

1.5.8 Example: $SU(2)$

We illustrate the main mathematical concepts reviewed so far in this chapter for the compact Lie group $G = SU(2)$.

Irreducible representations for $SU(2)$ and matrix coefficients

$SU(2)$ is the group of 2×2 matrices with determinant one and can be characterised by:

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}. \quad (1.30)$$

It is a compact simply-connected Lie group whose irreducible representations have a simple characterisation. They are labelled by integer or half-integer numbers λ – these are the highest weight vectors that correspond to the $d = 2\lambda + 1$ -dimensional irrep. Each λ -irrep can be realised on the d -dimensional vector space of homogeneous polynomials of degree $d - 1$ in two real variables z_1 and z_2 with complex coefficients. This is given by:

$$H^d := \text{span}\{m_k^d(z_1, z_2) = \frac{z_1^k z_2^{d-1-k}}{\sqrt{k!(d-1-k)!}} : k \in \{0, \dots, d-1\}\} \quad (1.31)$$

where $\{m_k^d\}_{k=0}^d$ form an orthonormal basis of homogeneous polynomials with respect to the inner product $\langle P, Q \rangle = \partial(P)Q^*$ for any $P, Q \in H^d$ where $\partial(P)$ is an operator whose action is given by substituting z_i^k with $\frac{\partial^k}{\partial z_i^k}$ for $i = 1$ or 2 . More specifically we can write any polynomials P and Q in a unique way $P = \sum_k p_k z_1^k z_2^{d-1-k}$ and $Q = \sum_k q_k z_1^k z_2^{d-1-k}$ for some arbitrary complex coefficients. Then we can see the above does indeed define an inner product because $\langle P, Q \rangle = \sum_k k!(d-1-k)! p_k q_k^*$.

Any element of H^d will be a homogeneous polynomial $f(z_1, z_2) = \sum a_k m_k^d$ for some arbitrary complex coefficients a_k . $SU(2)$ acts irreducibly on H^d via the group action:

$$g \cdot f(z_1, z_2) = f(g^T(z_1, z_2)) = f(\alpha z_1 + \beta z_2, -\beta^* z_1 + \alpha^* z_2) \quad (1.32)$$

for any group element $g = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$. This explicitly gives the irreducible representation $V^\lambda : SU(2) \rightarrow GL(H^d)$ with $V^\lambda(g)[f(z_1, z_2)] = f(\alpha z_1 + \beta z_2, -\beta^* z_1 + \alpha^* z_2)$.

This allows to compute the corresponding matrix coefficients $v_{kk'}^\lambda$ with respect to the orthonormal basis of homogeneous monomials. In particular:

$$V^\lambda(g)[m_k^d(z_1, z_2)] = \frac{(\alpha z_1 + \beta z_2)^k (-\beta^* z_1 + \alpha^* z_2)^{d-1-k}}{\sqrt{k!(d-1-k)!}}. \quad (1.33)$$

A binomial expansion of the RHS would allow to express the rotated monomial function with respect to the basis chosen for H^d . The corresponding coefficients will give the matrix coefficients of the λ -irrep. For completeness we give the full form:

$$v_{kk'}^\lambda(g) = \frac{\sqrt{k'!(d-1-k')!}}{\sqrt{k!(d-1-k)!}} \sum_{m=0}^{\min(k,k')} \binom{k}{m} \binom{d-1-k}{k'-m} \alpha^m \beta^{k-m} (-\beta^*)^{k'-m} (\alpha^*)^{d-1-k-k'+m} \quad (1.34)$$

where $k, k' \in \{0, 1, \dots, d-1\}$. These are also related to Wigner matrices that express the representations of $SO(3)$ in terms of the Euler rotation angles.

Clebsch-Gordan coefficients

For $SU(2)$, the Clebsch-Gordan coefficients have closed analytical formulas. As we have seen, these depend on a choice of basis for the irreducible representations, and the typically Clebsch-Gordan coefficients employed in this thesis are computed with respect to irreps given in terms of equation 1.34.

For $SU(2)$ every irreducible representation is isomorphic to its own dual. Moreover the Condon-Shortley choice of phase explicitly gives this isomorphism such that the dual basis for the j^* irrep are related to the basis for the j irrep via:

$$|j^*, m\rangle = (-1)^{j-m} |j, -m\rangle. \quad (1.35)$$

The decomposition of tensor products of irreps in $SU(2)$ is multiplicity free such that:

$$j_1 \otimes j_2 = j_1 + j_2 \oplus j_1 + j_2 - 1 \oplus \dots \oplus |j_1 - j_2| \quad (1.36)$$

for all j_1, j_2 irreps.

Irreducible tensor operators for $SU(2)$

Let \mathcal{H} carry a j -irrep of $SU(2)$ and choose $\{|n\rangle\}_{n=0}^{2j}$ to be the computational basis on the $d = 2j + 1$ dimensional space \mathcal{H} . Given such a symmetry, how does one construct a set of irreducible tensor operators that decomposes $\mathcal{B}(\mathcal{H})$ into irreps? Since $j \otimes j^* = 0 \oplus 1 \oplus \dots \oplus 2j$, then for every J -irrep in this decomposition we can build the *standard irreducible tensor operators* $\{T_k^J\}_{k=-J}^J$ for any J -irrep such that:

$$T_k^J := \sum_{m,n} (-1)^{n+j} \langle j, m; j, n | J, k \rangle |m+j\rangle \langle -n+j|. \quad (1.37)$$

By construction, the above transforms under the adjoint action of $U : SU(2) \longrightarrow \mathcal{B}(\mathcal{H})$:

$$U(g)T_k^J U^\dagger(g) = \sum_{k'} v_{k'k}^J(g) T_{k'}^J \quad (1.38)$$

for any $g \in SU(2)$ and any irrep J in the decomposition of $j \otimes j^*$ and all vector components k . The matrix coefficients $v_{kk'}^J(g)$ are necessarily related with the choice of Clebsch-Gordon coefficients as explained in section 1.5.7.

Moreover the Condon-Shortley choice of phase ensures that these operators satisfy the following relation:

$$(T_k^\mu)^\dagger = (-1)^{J-k} T_{-k}^\mu. \quad (1.39)$$

This is a very convenient relation because it will allow us to rotate this standard basis to a complete orthonormal basis of *hermitian* irreducible tensor operators. For each J and positive k this unitary rotation can be described by

$$\begin{aligned} T_{-k}^J &\longrightarrow \sigma_{-k}^J := \frac{T_{-k}^J + (-1)^k T_k^J}{\sqrt{2}} \\ T_k^J &\longrightarrow \sigma_k^J := \frac{i(T_{-k}^J - T_k^J)}{\sqrt{2}} \end{aligned} \quad (1.40)$$

and whenever $k = 0$ we take $\sigma_0^J = T_0^J$. Conveniently we can package this into a column vector of operators with entries corresponding to each vector component $\mathbf{T}^J := (T_{-J}^J, \dots, T_J^J)^T$ so that the defining transformation under the group action takes the form of $\mathcal{U}_g(\mathbf{T}^J) = V^J(g)\mathbf{T}^J$. Similarly, if R denotes the unitary matrix implementing the above basis rotation then we can write $\boldsymbol{\sigma}^J = R\mathbf{T}^J$. The operators $\boldsymbol{\sigma}^J$ themselves transform irreducibly under the adjoint group action \mathcal{U}_g . However they correspond to a *different* choice of basis so the irreducible matrix coefficients will reflect just that:

$$\mathcal{U}_g(\boldsymbol{\sigma}^J) = R v^J(g) R^\dagger \boldsymbol{\sigma}^J. \quad (1.41)$$

Example 1.5.8. One qubit spin system under $SU(2)$ symmetry principle *In this case the qubit system $\mathcal{H} \cong \mathbb{C}^2$ carries the fundamental $j = 1/2$ irrep of $SU(2)$. Thus $\mathcal{B}(\mathcal{H})$ decomposes into a 1-irrep subspace and a trivial 0-irrep subspace. As above with respect to the standard computational basis we get*

the irreducible tensor operators that span these irreducible subspaces:

$$\begin{aligned} T_{-1}^1 &= \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} & T_0^1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ T_1^1 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & T^0 &= \frac{1}{\sqrt{2}} \mathbb{I}. \end{aligned} \quad (1.42)$$

These will transform according to the 1-irrep of $SU(2)$. The corresponding matrix coefficients, that are associated with the standard choice of Clebsch-Gordan coefficients are easily computed using action of $SU(2)$ on homogeneous polynomials of degree two $H^2 := \text{span}_{\mathbb{C}}\{z_1^2/\sqrt{2}, z_1z_2, z_2^2/\sqrt{2}\}$. For every group element $g = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$ a column vector of the basis $\mathbf{z} = (z_1^2/\sqrt{2}, z_1z_2, z_2^2/\sqrt{2})^T$ will transform according to the the 1-irrep matrix of $SU(2)$, $\mathbf{z} \longrightarrow V^1(g)\mathbf{z}$ where:

$$V^1(g) = \begin{pmatrix} \alpha^2 & \sqrt{2}\alpha\beta & \beta^2 \\ -\sqrt{2}\alpha\beta^* & (|\alpha|^2 - |\beta|^2) & \sqrt{2}\alpha^*\beta \\ (\beta^*)^2 & -\sqrt{2}\alpha^*\beta^* & (\alpha^*)^2 \end{pmatrix} \quad (1.43)$$

Moreover we can also obtain a complete set of orthonormal hermitian ITOs for $\mathcal{B}(\mathcal{H})$ as explained in the previous section. In this particular case we recover the Pauli matrices. Although the Pauli matrices form an orthogonal basis for the 1-irrep subspace of $\mathcal{B}(\mathcal{H})$, they will transform according to different matrix coefficients given by Rv^1R^\dagger where explicitly:

$$\sigma^1 = R\mathbf{T}^1 := \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \end{pmatrix} \mathbf{T}^1. \quad (1.44)$$

1.5.9 Homogeneous spaces

A *homogeneous space* is a topological space that has a transitive group action.

Suppose that G acts on a manifold \mathcal{M} by $\cdot : G \times \mathcal{M} \longrightarrow \mathcal{M}$, then under this action every point $\mathbf{x} \in \mathcal{M}$ is send to a point in its orbit $Orb(\mathbf{x}) := \{g \cdot \mathbf{x} : g \in G\}$. This splits \mathcal{M} into a disjoint union of orbits. G acts transitively on \mathcal{M} if for all $\mathbf{x}, \mathbf{y} \in \mathcal{M}$ there exists a group element $g \in G$ such that $\mathbf{y} = g \cdot \mathbf{x}$. In such a case there is a single orbit, the full homogeneous space \mathcal{M} .

Let H be a subgroup of G . The quotient G/H is the set of all left cosets of H in G and consists of the set of equivalence classes $\{gH : g \in G\}$ such that $g \sim g'$ iff $g = g'h$ for some $h \in H$ and thus $gH = g'H$. The group G acts transitively on G/H so that there is a one to one correspondence between homogeneous spaces and quotient spaces.

In particular if \mathcal{M} is a homogeneous space and H is the stabilizer subgroup of a point $\mathbf{x} \in \mathcal{M}$ such that $H = \{g \in G : g \cdot \mathbf{x} = \mathbf{x}\}$, then the following defines an invariant measure on $\mathcal{M} = G/H$:

$$\int_{\mathcal{M}} d\mathbf{x} \int_H f(gh)dh = \int_G f(g)dg \quad (1.45)$$

where dg and dh are the Haar measures on the compact groups G and H respectively.

1.5.10 The Hilbert space $\mathcal{L}^2(G)$ and Peter-Weyl theorem

For a compact group G , any continuous complex valued functions on the group are also square integrable. We denote by $\mathcal{L}^2(G) = \{f : G \rightarrow \mathbb{C} : \int f(g)f(g)^*dg \leq \infty\}$ the space of square integrable functions on the group. There is a natural group action of G on $\mathcal{L}^2(G)$ called the left regular representation and given by left shifts:

$$g \cdot f(g') = f(g^{-1}g') \quad (1.46)$$

for all $g, g' \in G$. The action induces the left regular representation on $\mathcal{L}^2(G)$ which is reducible.

Theorem 1.5.9. Peter-Weyl theorem

Let G be a compact group then:

- i) The set of all matrix coefficients $v_{ij}^\lambda : G \rightarrow \mathbb{C}$ for $\lambda \in \hat{G}$ and i, j is dense in $C(G)$ the space of continuous complex-valued functions on G and also in $\mathcal{L}^2(G)$.
- ii) Under the left regular representation, the space $\mathcal{L}^2(G)$ decomposes into a direct sum of every irreducible representation of G with multiplicity equal to its dimension.

The Peter-Weyl theorem implies that the set $\{\sqrt{d_\lambda}v_{ij}^\lambda : \lambda \in \hat{G}, 1 \leq i, j \leq d_\lambda\}$ forms an orthonormal basis for $\mathcal{L}^2(G)$ and moreover the matrix coefficients for the λ -irrep will span the λ -irrep isotypical component in the decomposition of $\mathcal{L}^2(G)$. Therefore any function $f \in \mathcal{L}^2(G)$

can be expanded into a generalised Fourier series with respect to this basis:

$$f(g) = \sum_{\lambda \in \hat{G}} \sum_{i,j} c_{ij}^\lambda v_{i,j}^\lambda(g) \quad (1.47)$$

for some coefficients defined by $c_{ij}^\lambda = d_\lambda \int_G f(g) (v_{ij}^\lambda(g))^* dg$.

1.5.11 Spherical harmonics

Spherical harmonics are special functions on a sphere, have wide applicability in mathematics, physics, chemistry and their structure combines different elements of differential equations, group theory, harmonic analysis and geometry.

Typically spherical harmonics appear as eigenfunctions of the Laplacian operator and form a basis for the space of complex valued functions on the unit sphere $\mathcal{L}^2(S^2, \mathbb{C})$. In polar coordinates, every point in the sphere $x \in S^2$ corresponds to angles (θ, ϕ) with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The spherical harmonics are joint eigenfunctions of the total angular momentum operator \mathbf{L}^2 and the generator of rotations around the z axis L_z :

$$\mathbf{L}^2 Y_m^l(\theta, \phi) = l(l+1) Y_m^l(\theta, \phi) \quad \text{and} \quad L_z Y_m^l(\theta, \phi) = m Y_m^l(\theta, \phi) \quad (1.48)$$

for any θ, ϕ where $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$ and L_x, L_y, L_z are components of the orbital angular momentum operator $ih(\mathbf{x} \times \nabla)$. The raising and lowering operators $L_\pm = L_x \pm iL_y$ and L_z form generators of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. The label l ranges over all positive integers and m is an integer such that $-l \leq m \leq l$. Explicitly:

$$Y_m^l(\theta, \phi) = (-1)^m P_{lm}(\cos \theta) e^{im\phi} \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} \quad (1.49)$$

where $P_{lm}(x)$ are Legendre functions, with

$$P_{lm}(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m P_l(x)}{dx^m} \quad (1.50)$$

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l (x^2-1)^l}{dx^l}. \quad (1.51)$$

Moreover they are orthonormal $\int_{S^2} Y_m^l Y_{m'}^{l'} d\Omega = \delta_{ll'} \delta_{mm'}$, with integration given by Haar measure over the sphere S^2 . The spherical harmonics defined above satisfy $(Y_m^l)^* = (-1)^m Y_{-m}^l$ so

they are chosen to satisfy the Condon-Shortley phase convention.

The spherical harmonics form a complete orthonormal basis for $\mathcal{L}^2(S^2, \mathbb{C})$. However one can view the S^2 as a homogeneous space $SO(3)/SO(2)$, so that there is a natural action of the rotation group. For every integer l the set of spherical harmonics $\{Y_m^l : -l \leq m \leq l\}$ form an orthonormal basis for the $2l + 1$ dimensional irreducible representation of $SO(3)$. Therefore, under the left regular representation of $SO(3)$, the spherical harmonics transform irreducibly:

$$g \cdot Y_m^l(\theta, \phi) = \sum_{m'} v_{mm'}^l(g) Y_{m'}^l(\theta, \phi) \quad (1.52)$$

for all $g \in SO(3)$ with $v_{mm'}^l$ matrix coefficients of the l -irrep of $SO(3)$. In the above, for any point $\mathbf{x} \in S^2$ with polar coordinates θ, ϕ , the group action is $g \cdot Y_m^l(\mathbf{x}) = Y_m^l(g^{-1} \cdot \mathbf{x})$, where $g^{-1} \cdot \mathbf{x}$ corresponds to the point on S^2 resulting from point \mathbf{x} under a rotation by $g^{-1} \in SO(3)$.

In particular, the space $\mathcal{L}^2(S^2, \mathbb{C})$ decomposes into a direct sum of irreducible subspaces containing all irreps of $SO(3)$ (or equivalently the odd dimensional irreps of $SU(2)$), each appearing with multiplicity one. This property shows how one can extend the definition of spherical harmonics to generalised spherical harmonic functions acting on homogeneous spaces G/H .

1.5.12 Generalised spherical harmonics

A function on a homogeneous space G/H is a function in $\mathcal{L}^2(G)$ that is constant on the left cosets i.e $f(gh) = f(g)$ for all $g \in G$ and $h \in H$. Similarly there is a left-regular group action of G on $\mathcal{L}^2(G/H, \mathbb{C})$, the space of square integrable complex-valued functions on the homogeneous space:

$$g \cdot f(g'H) = f(g^{-1}g'H) \quad (1.53)$$

for all $g, g' \in G$. It follows from Peter-Weyl's theorem that $\mathcal{L}^2(G/H)$ decomposes into irrep with multiplicity *at most* equal to the dimension of the respective irrep.

However there are situations – as we have previously seen for the rotation group – in which the decomposition $\mathcal{L}^2(G/H)$ is multiplicity-free. In such cases, the notion of spherical harmonics can be immediately generalised.

In particular it follows that H must satisfy a particular property: For any λ -irrep there exists a unique vector $|\mathbf{n}\rangle$ in the λ -irrep carrier space such that $v^\lambda(h)|\mathbf{n}\rangle = |\mathbf{n}\rangle$ for all $h \in H$. A subgroup

H of G that satisfies this is called a *massive subgroup*. In particular for any massive subgroup the decomposition of $\mathcal{L}^2(G/H)$ into irreducible component is multiplicity free. Moreover for each λ -irrep in $\mathcal{L}^2(G/H)$ we can define the *associated spherical harmonics* $Y_m^\lambda \in \mathcal{L}^2(G/H)$:

$$Y_m^\lambda(gH) := \langle \mathbf{n} | v^\lambda(g^{-1}) | e_m \rangle \quad (1.54)$$

with respect to an orthonormal basis $\{|e_m\rangle\}$ for the λ -irrep carrier space.

Therefore one can check the above are indeed well-defined (i.e invariant on left cosets) and form an orthonormal basis spanning the λ -irrep component of $\mathcal{L}^2(G/H)$:

$$\mathcal{L}^2(G/H) = \bigoplus_{\lambda \in \hat{G}} \text{span}_{\mathbb{C}} \{Y_m^\lambda : m = 1, 2, \dots, \dim(\lambda)\}. \quad (1.55)$$

As with the usual spherical harmonics there are many deep connections with partial differential operators on a manifold that are invariant under a "motion" group, Lie algebraic formulations. However we will be mostly concerned with the transformation property of these generalised spherical harmonics under the action of the group. This is directly related to the above decomposition and it is easy to check that indeed:

$$g \cdot Y_m^l(g_0H) = Y_m^\lambda(g^{-1}g_0H) = \langle \mathbf{n} | v^\lambda(g_0^{-1})v^\lambda(g) | e_m \rangle \quad (1.56)$$

$$= \sum_{m'} \langle \mathbf{n} | v^\lambda(g_0^{-1}) | e'_m \rangle \langle e'_m | v^\lambda(g) | e_m \rangle \quad (1.57)$$

$$= \sum_{m'} v_{m'm}^\lambda(g) Y_{m'}^l(g_0H) \quad (1.58)$$

for all $g \in G$ and cosets $g_0H \in G/H$, where the matrix coefficients are with respect to the fixed orthonormal basis $\{|e_m\rangle\}_{m=1}^{\dim(\lambda)}$.

A more detailed and rigorous account of associated spherical harmonics on homogeneous spaces can be found in the monograph on special functions by Vilenkin and Klimyk [59].

Chapter 2

A framework for quantum processes under symmetry principles

Much like a musical composition contains pure notes of different frequencies with varying time-weights, in the same way a quantum channel decomposes into components that respond to the underlying symmetry principle in a particular, quantifiable manner. This structure appears naturally as we view the space of superoperators on a given system to be the representation space of the symmetry group that decomposes into irreducible isotypical components. The formalism presented in this section goes beyond this, in that it aims to take into account the causal structure of quantum channels that distinguishes the input from the output systems. Therefore, the present framework gives a more fine-grained structure than the decomposition into isotopic components and allow for an intuitive diagrammatic representation.

An immediate advantage of this approach arises when dealing with implementations of a given target quantum channel via a globally symmetric interaction between system and environment. This type of quantum processing task appears for instance frequently in the context of quantum reference frames, quantum thermodynamics or general resource theories. In this setting, an important question is, What are the resources required to implement the (symmetry-breaking) target channel? Process modes allow to identify these minimal resource requirements without committing to a particular environment or interaction.

Globally symmetric channels on bipartite systems, may arise trivially as two independent symmetric channels on each of the subsystems but generally, ignoring (tracing out) one subsystem

results in a symmetry breaking channel. However, channels break the symmetry in different ways, so that relative to this internal symmetry-breaking degree of freedom globally symmetric channels have typically distinct features that lead to operational differences. These structural aspects are best represented in terms of the process mode formalism, resulting in a distinguished basis decomposition that can be connected at an abstract level with Feynman diagrams.

The framework we develop over the following sections represents the mathematical backbone on which the applications discussed in chapters 4 and 3 rest.

Previous work on modes of asymmetry by Marvian and Spekkens in [35,36] mainly focused on decomposition of states into symmetry breaking terms, and their use in an environment for simulation of channels under symmetry constraints. In here, we focus on quantum channels instead and explore the rich structures emergent in bipartite systems along with geometric interpretations for decomposition of symmetry-breaking channels. We emphasize the relation and differences between process modes and asymmetry modes at the end of section 2.1.1.

For the remainder of the chapter, unless specified otherwise, we will consider quantum channels $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$, where the input system \mathcal{H} and output system \mathcal{K} carry unitary representations U and U' of a compact group G . As detailed in Section 1.5.6, under the group action the channel transforms according to the representation \mathfrak{U} of G on $\mathcal{S}(\mathcal{H}, \mathcal{K})$ such that for any $g \in G$ it maps $\mathfrak{U}(g)[\mathcal{E}] = \mathcal{U}'_g \circ \mathcal{E} \circ \mathcal{U}_g^\dagger$, where $\mathcal{U}_g(\cdot) := U(g)(\cdot)U(g)^\dagger$.

2.1 Process modes – a diagrammatic representation

2.1.1 Irreducible tensor superoperators

The space of superoperators $\mathcal{S}(\mathcal{H}, \mathcal{K})$ with the representation \mathfrak{U} of G has a unique decomposition in terms of irreducible isotypical components. Moreover, the isomorphism between $\mathcal{S}(\mathcal{H}, \mathcal{K})$ and $\mathcal{H} \otimes \mathcal{H}^* \otimes \mathcal{K} \otimes \mathcal{K}^*$ implies that the irreps that appear in the decomposition of the space of superoperators are exactly the same as those in the unitary representation $U \otimes U^* \otimes U' \otimes (U')^*$. Since it shares the algebraic properties of bounded operator spaces, the notion of irreducible tensor operators described in 1.5.4 can be immediately generalised.

Definition 2.1.1. *For every irrep $\lambda \in \hat{G}$ the irreducible tensor superoperators are defined as a set of*

$\dim(\lambda)$ (dimension of λ) linear maps $\{\Phi_k^\lambda\}_{k=1}^{\dim(\lambda)}$ in $\mathcal{S}(\mathcal{H}, \mathcal{K})$ that transform as:

$$\mathfrak{U}(g)[\Phi_k^\lambda] = \sum_{k'} v_{k'k}^\lambda(g) \Phi_{k'}^\lambda, \quad (2.1)$$

for all $g \in G$, where $v_{kk'}^\lambda$ are matrix coefficients of the λ -irrep in the decomposition of $U \otimes U^* \otimes U' \otimes U'^*$.

The above definition warrants some further clarifications. Generally the irreps in the decomposition of $U \otimes U^* \otimes U' \otimes U'^*$ may have multiplicity greater than one and this fact is not explicitly taken into account in the definition of an irreducible tensor superoperators – what matters is only how they transform under the group action.

We will denote by $\text{Irrep}(\mathcal{H}, \mathcal{K})$ the non-equivalent irreps that appear in the decomposition of $U \otimes U^* \otimes U' \otimes U'^*$ (or equivalently in $\mathcal{S}(\mathcal{H}, \mathcal{K})$) into irreducible components. Moreover for each $\lambda \in \text{Irrep}(\mathcal{H}, \mathcal{K})$ we label the associated multiplicity generically by m_λ so that the unitary representation \mathfrak{U} has a unique decomposition into:

$$\mathfrak{U} \cong \bigoplus_{\lambda} V^\lambda \otimes \mathbb{C}^{|m_\lambda|} \quad (2.2)$$

where V^λ denotes the irreducible representation of G labelled by highest weight λ .

The irreducible tensor superoperators give a natural symmetry-adapted basis for the space of superoperators, which therefore decomposes into:

$$\mathcal{S}(\mathcal{H}, \mathcal{K}) \cong \bigoplus_{\lambda, m_\lambda} \text{span}\{\Phi_k^{\lambda, m_\lambda} : 1 \leq k \leq \dim(\lambda)\}. \quad (2.3)$$

As written, the decomposition in equation (2.3) assumes a particular choice of irreducible tensor superoperators for each λ and each multiplicity m_λ . We emphasize that there is unique coarse-grained structure given by the isotypical decomposition $\mathcal{S}(\mathcal{H}, \mathcal{K}) \cong \bigoplus_{\lambda} \mathcal{S}^\lambda$ with $\mathcal{S}^\lambda := \bigoplus_{m_\lambda} \text{span}\{\Phi_k^{\lambda, m_\lambda} : 1 \leq k \leq \dim(\lambda)\}$. In particular, for any map $\mathcal{E} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ its orthogonal projection onto the isotypical component \mathcal{S}^λ will be given by:

$$\mathcal{E}^\lambda := \dim(\lambda) \int \text{Tr}(v^\lambda(g)) \mathfrak{U}(g)[\mathcal{E}] dg \quad (2.4)$$

where $\text{Tr}(v^\lambda(g))$ is the character of the λ -irrep and integration is with respect to the Haar measure

on the compact group G . Note that projection above is independent of any choice of basis and \mathcal{E}^λ is uniquely defined for any given map \mathcal{E} .

However, the finer-grained structure given by further decomposing S^λ into a direct sum of λ -irreps presupposes a particular choice of basis. In order to construct sets of irreducible tensor superoperators in $\mathcal{S}(\mathcal{H}, \mathcal{K})$ defined by their transformation property it is important to emphasize that two choices of basis are needed: i) A basis of each irreducible representation of G , which gives the matrix coefficients $(v^\lambda(g))_{kk'}$ and ii) A basis for the underlying Hilbert spaces \mathcal{H} and \mathcal{K} respectively.

Given a full set of irreducible tensor superoperators $\{\Phi_k^{\lambda, m_\lambda}\}_{k=1}^{\dim(\lambda)} \subset \mathcal{S}(\mathcal{H}, \mathcal{K})$ for all $\lambda \in \text{Irrep}(\mathcal{H}, \mathcal{K})$ with corresponding multiplicity label m_λ , this forms a complete orthogonal basis for $\mathcal{S}(\mathcal{H}, \mathcal{K})$ in the following sense. For every linear map $\mathcal{E} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ there is a unique Choi operator $J[\mathcal{E}]$. Since the operator space has a Hilbert-Schmidt inner product, then the superoperator space can be given the inner product structure defined by $\langle \mathcal{E}, \mathcal{F} \rangle := \text{Tr}(J[\mathcal{E}]^\dagger J[\mathcal{F}])$ for any $\mathcal{E}, \mathcal{F} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$. It is with respect to this choice of inner product space that the irreducible tensor superoperators form an orthonormal basis such that $\langle \Phi_k^{\lambda, m_\lambda}, \Phi_{k'}^{\mu, m_\mu} \rangle \propto \delta_{\lambda, \mu} \delta_{k, k'} \delta_{m_\lambda, m_\mu}$, a direct consequence of Lemma 2.1.2 below.

Lemma 2.1.2. *Suppose that $\{\Phi_k^\lambda\}_{k=1}^{\dim(\lambda)} \subset \mathcal{S}(\mathcal{H}, \mathcal{K})$ are irreducible tensor superoperators. Then the set of operators $J[\Phi_k^\lambda]$ in $\mathcal{B}(\mathcal{K} \otimes \mathcal{H})$ form a set of irreducible tensor operators under the tensor product representation $\mathcal{U}' \otimes \mathcal{U}^*$, and are orthogonal with respect to the Hilbert-Schmidt inner product.¹*

Proof. From the Definition 2.1.1 and linearity of the Choi map it follows that $J[\mathfrak{U}(g)[\Phi_k^\lambda]] = \sum_{k'} v_{k'k}^\lambda(g) J[\Phi_{k'}^\lambda]$. However, for any linear map \mathcal{E} , $J[\mathfrak{U}(g)[\mathcal{E}]] = \mathcal{U}'_g \otimes \mathcal{U}^*_g(J[\mathcal{E}])$ and therefore $\mathcal{U}'_g \otimes \mathcal{U}^*_g(J[\Phi_k^\lambda]) = \sum_{k'} v_{k'k}^\lambda(g) J[\Phi_{k'}^\lambda]$ transform as λ irreducible tensor operators and in particular they are orthogonal. \square

For practical purposes, we will assume without loss of generality that the irreducible tensor superoperator basis are normalised such that $\langle \Phi_k^\lambda, \Phi_k^\lambda \rangle = 1$ for all irreps λ and vector component k . Therefore, any superoperator $\mathcal{E} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ acting on systems with an underlying symmetry can be written as a linear combination of irreducible tensor superoperators:

$$\mathcal{E} = \sum_{\lambda, m_\lambda, k} \alpha_{\lambda, m_\lambda, k} \Phi_k^{\lambda, m_\lambda} \quad (2.5)$$

¹A similar result holds for the Liouville representation of a superoperator

for some complex coefficients $\alpha_{\lambda, m_\lambda, k} = \langle \mathcal{E}, \Phi_k^{\lambda, m_\lambda} \rangle = \text{Tr}(J[\mathcal{E}]^\dagger J[\Phi_k^{\lambda, m_\lambda}])$. We will refer to the above as a *process mode decomposition*. To emphasize the physical aspects of our framework we refer to the irreducible tensor superoperators interchangeably as *process modes*.

Lemma 2.1.2 allows a constructive way of building irreducible tensor operators. We will deal with the question of how to concretely construct these basis in the following sections. For now, we address a type of decomposition of linear maps $\mathcal{E} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ that has been introduced as *modes of asymmetry* of quantum channels in [36], with the scope of differentiate between the approach taken herein. The (λ, k) asymmetry mode was defined as $\mathcal{E}_k^\lambda := \dim(\lambda) \int (v_{kk}^\lambda)^*(g) \mathfrak{U}(g)[\mathcal{E}] dg$, which do transform as irreducible tensor superoperators. We note that these depend on a choice of basis for the irreducible representations V^λ of G , but carry no assumption on the basis for the underlying input and output spaces. At an abstract level, it falls between an isotypical decomposition and a full, basis dependent, decomposition into irreducible subspaces. It only identifies different sectors in an isotypical component, without fully splitting it into specified λ -irreducible subspaces. In particular, if \mathcal{E} decomposes as in equation 2.5 in terms of irreducible tensor superoperators then the asymmetry modes are $\mathcal{E}_k^\lambda = \sum_{m_\lambda} \alpha_{\lambda, m_\lambda, k} \Phi_k^{\lambda, m_\lambda}$, where we sum over all multiplicities of the λ -irrep. Similarly, we also have the projection onto the isotypic component $\mathcal{E}^\lambda = \sum_k \mathcal{E}_k^\lambda$, where in this case we sum over all vector components. Therefore the decompositions into asymmetry modes, process modes or isotypical components represent different levels of coarse-graining a quantum channel (and generally linear maps) according to the underlying symmetry principles governing the physical systems involved. Different applications are more suitable to one or another type of such decomposition giving rise to different perspectives on the role symmetry plays in physics and particularly quantum information.

In summary, there are two different aspects we have tackled in the above discussion, which at its core they are fundamentally the same. On one hand we have the canonical isotypical decomposition of the superoperator space, and on the other hand a symmetry adapted orthogonal basis for $\mathcal{S}(\mathcal{H}, \mathcal{K})$ in the form of irreducible tensor superoperators. It is clear that the latter gives a particular decomposition into irreducible subspaces. Conversely, *any* orthogonal basis for each irreducible subspace in the decomposition of $\mathcal{S}(\mathcal{H}, \mathcal{K})$ will transform accordingly as irreducible tensor superoperators.

2.1.2 Process modes as a natural symmetry adapted basis for $\mathcal{S}(\mathcal{H}, \mathcal{K})$

In this section we delve into how process modes can be constructed generally and introduce a set of *canonical process modes* that have an intuitive diagrammatic representation that takes into account the causal structure of quantum channels.

We can view the space of superoperators as carrying a tensor product structure via the isomorphism of representations $\mathcal{S}(\mathcal{H}, \mathcal{K}) \cong \mathcal{B}(\mathcal{H})^* \otimes \mathcal{B}(\mathcal{K})$. This means that there is an intertwiner which respects this in the sense that any irrep $\lambda \in \text{Irrep}(\mathcal{H}, \mathcal{K})$ arises in some tensor product coupling between an irrep a in $\mathcal{B}(\mathcal{H})^*$ and \tilde{a} in $\mathcal{B}(\mathcal{K})$ (or equivalently in $U \otimes U^*$ and $U'^* \otimes U'$ respectively). Tracking all possible such couplings gives a useful way to label the multiplicities of irreps $\lambda \in \text{Irrep}(\mathcal{H}, \mathcal{K})$ in a constructive way that takes into account how superoperators act on the input and output spaces. We package this data into a compact notation $\lambda = (\lambda, m_\lambda)$ that includes both the irrep label and the multiplicity label. Moreover, under such a bottom up construction of the decomposition into irreducible components of $\mathcal{S}(\mathcal{H}, \mathcal{K})$ we can say that each multiplicity label $m_\lambda = (a, \tilde{a})$ is associated with a coupling of irreps in the input and output systems (which themselves may carry a multiplicity label).

Definition 2.1.3. *Let G be a compact group, U and U' unitary representations of \mathcal{H} and \mathcal{K} respectively. We define canonical process modes for the superoperator space $\mathcal{S}(\mathcal{H}, \mathcal{K})$ to be a complete set of irreducible tensor superoperators $\{\{\Phi_k^\lambda\}_{k=1}^{\dim(\lambda)}\}_\lambda$ of $\mathcal{S}(\mathcal{H}, \mathcal{K})$ that have as non-trivial domain and co-domain irreducible subspaces of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ respectively.*

A technical remark related to the above definition; for any general group G , there is a one to one correspondence between irreps a in the decomposition of $\mathcal{B}(\mathcal{H})$ with the adjoint action of U and those in $\mathcal{B}(\mathcal{H})^*$, space that carries the adjoint action of U^* . The following lemma gives a construction for the canonical process modes. A priori there is no assumption of a particular basis choice – except for the choice of tensor product. The choice of basis is reflected in the set of irreducible tensor operators for the input and output spaces and in the Clebsch-Gordan coefficients, but the definition itself is independent of these; there is still a freedom to choose particular basis for \mathcal{H} and \mathcal{K} , but we only impose an algebraic structure.

Lemma 2.1.4. *Let $\{T^a\}_a$ and $\{S^{\tilde{a}}\}_{\tilde{a}}$ denote a complete set of irreducible tensor operators for $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ respectively. The set of linear maps $\Phi_k^\lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ given by:*

$$\Phi_k^\lambda(\rho) = \sum_{m,n} \langle \tilde{a}, m; a, n | \lambda, k \rangle S_m^{\tilde{a}} \text{Tr}(T_n^a \rho) \quad (2.6)$$

for all $\rho \in \mathcal{B}(\mathcal{H})$ form a set of canonical process modes for all $\lambda = (\lambda, m_\lambda)$ where $\lambda \in \text{Irrep}(\mathcal{H}, \mathcal{K})$ with the multiplicity $m_\lambda = (\mathbf{a}, \tilde{\mathbf{a}})$.

Proof. We just need to check that under the group action \mathfrak{U}_g the set of superoperators $\{\Phi_k^\lambda\}_{k=1}^{\dim(\lambda)}$ defined above transform irreducibly. Since $\{S_m^{\tilde{\mathbf{a}}}\}$ in $\mathcal{B}(\mathcal{H})$ and $\{T_n^{\mathbf{a}}\}$ in $\mathcal{B}(\mathcal{K})$ are irreducible tensor operators then they will transform as:

$$\begin{aligned}\mathfrak{U}_g(T_n^{\mathbf{a}}) &= \sum_{nn'} v_{n'n}^{\mathbf{a}}(g) T_n^{\mathbf{a}} \\ \mathfrak{U}'_g(S_m^{\tilde{\mathbf{a}}}) &= \sum_{m'} v_{m'm}^{\tilde{\mathbf{a}}}(g) S_{m'}^{\tilde{\mathbf{a}}}\end{aligned}\tag{2.7}$$

and therefore the set of superoperators defined in the Equation 2.6 will satisfy:

$$\mathfrak{U}_g[\Phi_k^\lambda(\rho)] = \sum_{m,n,m',n'} \langle \tilde{\mathbf{a}}, m; a, n | \lambda, k \rangle v_{m'm}^{\tilde{\mathbf{a}}}(g) v_{n'n}^{\mathbf{a}}(g) S_{m'}^{\tilde{\mathbf{a}}} \text{Tr}(T_n^{\mathbf{a}} \rho).\tag{2.8}$$

Taking into account the definition of Clebsch-Gordan coefficients and the fact that they represent a unitary change of basis then they must satisfy the following relation between matrix coefficients of irreducible representations: $\sum_{k'} \langle \tilde{\mathbf{a}}, m'; a, n' | \lambda, k' \rangle v_{k'k}^\lambda = \sum_{m,n} \langle \tilde{\mathbf{a}}, m; a, n | \lambda, k \rangle v_{m'm}^{\tilde{\mathbf{a}}} v_{n'n}^{\mathbf{a}}$. Finally by substituting this into the previous equation we obtain that:

$$\begin{aligned}\mathfrak{U}_g[\Phi_k^\lambda(\rho)] &= \sum_{k',m',n'} \langle \tilde{\mathbf{a}}, m'; a, n' | \lambda, k' \rangle v_{k'k}^\lambda(g) S_{m'}^{\tilde{\mathbf{a}}} \text{Tr}(T_{n'}^{\mathbf{a}} \rho) \\ &= \sum_{k'} v_{k'k}^\lambda(g) \Phi_{k'}^\lambda.\end{aligned}\tag{2.9}$$

□

At a more abstract level, the canonical process modes give an explicit intertwiner map $\iota_C : \mathcal{B}(\mathcal{H})^* \otimes \mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{S}(\mathcal{H}, \mathcal{K})$, whereas the canonical process modes are the image of irreducible tensor operators with a choice of tensor product $\mathcal{B}(\mathcal{H})^* \otimes \mathcal{B}(\mathcal{K})$. The space of all intertwiners is the commutant of the group action \mathfrak{U} i.e $Comm := \{\iota : \mathcal{S}(\mathcal{H}, \mathcal{K}) \longrightarrow \mathcal{S}(\mathcal{H}, \mathcal{K}) : \forall g \in G \mathfrak{U}(g) \circ \iota = \iota \circ \mathfrak{U}(g)\}$. Therefore any set of irreducible tensor superoperators can be written as the image of the canonical process modes $\{\iota(\Phi_k^\lambda)\}_{\lambda,k}$ under some intertwiner $\iota \in Comm$.

Moreover we can view the process modes in a more compact notation as a column vector of

superoperators that transform irreducibly:

$$\Phi^\lambda := \left[\Phi_1^\lambda \quad \dots \quad \Phi_k^\lambda \quad \dots \quad \Phi_{\dim(\lambda)}^\lambda \right]^T. \quad (2.10)$$

As described, each such vector Φ^λ arises from couplings of irreps (a, \tilde{a}) and thus we can associate a diagram as in figure 2.1. The diagrammatic notation reflects a bottom-up approach in constructing irreducible tensor operators. It captures the fact that any λ -irrep subspace in the space of operators $\mathcal{S}(\mathcal{H}, \mathcal{K})$ arises from coupling particular a -irrep in $\mathcal{B}(\mathcal{H})$ and \tilde{a} -irrep in $\mathcal{B}(\mathcal{K})$. The edges are associated with irrep labels, while the vertex can be viewed as an intertwiner between $a \otimes \tilde{a}$ and λ (or equivalently between the trivial representation and $a \otimes \tilde{a} \otimes \lambda^*$) as given by the Clebsch-Gordan coefficients. The diagram contains all the information needed to write down an expression for the corresponding irreducible tensor superoperators. This scheme gives a tractable way of labelling multiplicities for the λ -irrep.

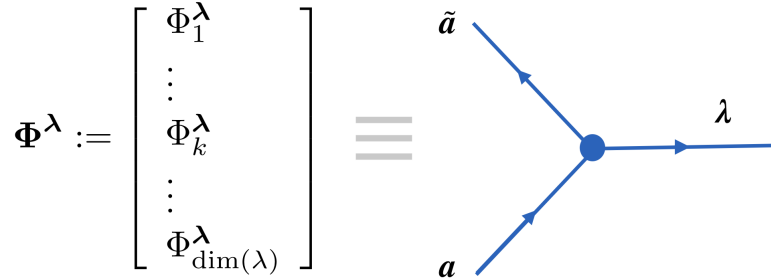


Figure 2.1: Process Mode Diagram

Diagrammatic representation of the $\dim(\lambda)$ canonical process modes components $\{\Phi_k^\lambda\}_{k=1}^{\dim(\lambda)}$ arising from the input-output coupling $(a, \tilde{a}) \xrightarrow{\lambda}$. The edges are labelled by irreps and the vertex corresponds to an intertwiner.

The diagram in figure 2.1.2 intuitively gives a pictorial representation of components in the harmonic analysis decomposition of any quantum channel. It shows how the asymmetry in the initial state is transformed into the asymmetry of the output state by means of interaction with the λ -irrep process mode. Time runs up the page, to reflect the ordering of events: the a -irrep state mode of the initial state ρ is selected by the edge labelled by a with an incoming arrow into the vertex and mapped under the process mode Φ^λ solely into the \tilde{a} -irrep state mode of the output state, described by the edge labelled by \tilde{a} with outgoing arrow. If a state ρ has no a -irrep

component then $\Phi_k^\lambda(\rho) = 0$. The vertical edge is labelled by λ , and captures the asymmetry provided by the environment to map incoming \mathbf{a} -state modes into outgoing $\tilde{\mathbf{a}}$ -irrep state modes. The diagram in figure 2.1.2 can also be viewed as a short-hand notation for the mathematical expression introducing canonical process modes in equation 2.6. Typically, graphical calculus for tensors will include all the labels (degrees of freedom) involved, and one may very well extend the process mode diagrams by adding vector component labels. Extending the process mode diagram notation to a well defined graphical calculus is beyond the aims of this thesis, and here we use them only for their descriptive power rather than as a full calculating tool. One can formalise the idea of combining different diagrams, and in some sense the diagrams we give in section 2.4.2 for symmetric channels on bipartite systems will describe one way of doing such a procedure.

At this stage a comment is warranted on the similarities between the diagrammatic representation of canonical process modes and Feynman diagrams. The latter come up in perturbative analysis of quantum field theory and are usually thought of as processes representing interactions between particles, with associated complex amplitude giving the probability that a particular process occurs. There is however a deeper perspective on Feynman diagrams [66,67] in which the edges label irreps (these correspond to elementary particles according to Wigner's classification) while vertices label intertwiners between the trivial representation and tensor product of irreps connected at said vertex (in physical terms, these are interactions between said particles).

Example 2.1.5. *SU(2) acting irreducibly on one qubit system*

We consider operations on single qubit system $\mathcal{H} \cong \mathcal{K} \cong \mathbb{C}^2$ with the 1/2-irrep of SU(2). As a representation space $\mathcal{B}(\mathcal{H}) \cong 1/2 \otimes 1/2^*$ it splits into a direct sum of $\mathcal{B}(\mathcal{H}) \cong 0 \oplus 1$. With respect to the spin angular momentum basis for \mathcal{H} given by $|0\rangle := |1/2, -1/2\rangle, |1\rangle := |1/2, 1/2\rangle$, the orthonormal irreducible tensor operators² are $\{T^0 := \mathbb{I}/\sqrt{2}\}$ for the 0-irrep and for the 1-irrep $\{T_{-1}^1 := -\sigma_+/ \sqrt{2}, T_0^1 := \sigma_Z / \sqrt{2}, T_1^1 := \sigma_- / \sqrt{2}\}$, where $\sigma_X, \sigma_Y, \sigma_Z$ are the Pauli matrices and $\sigma_\pm = \sigma_X \pm i\sigma_Y$.

The space of one qubit superoperators will decompose into a 2-irrep, a 1-irrep with multiplicity three, a 0-irrep with multiplicity 0 such that $\mathcal{S}(\mathcal{H}, \mathcal{H}) \cong 0 \oplus 0 \oplus 1 \oplus 1 \oplus 1 \oplus 2$. Each arises in a particular coupling of irreps in $\mathcal{B}(\mathcal{H})^*$ and $\mathcal{B}(\mathcal{H})$ corresponding to the input and output respectively. However for

²In this example we consider the standard basis for the irreducible representations of SU(2) – this fixes the matrix coefficients and the Clebsch-Gordan coefficients (for which we choose the Condon-Shortley phase convention). These aspects are detailed in section 1.5.7 of Chapter 1.

$SU(2)$, every irreducible representation is isomorphic to its dual and thus $\mathcal{S}(\mathcal{H}, \mathcal{H}) \cong (0 \oplus 1) \otimes (0 \oplus 1)$.

An integer spin value j labels each process mode, and it only takes on values $j = 0, 1$ and 2 . The $j = 0$ process mode components by definition transform trivially under the $SU(2)$ action and the most general form is

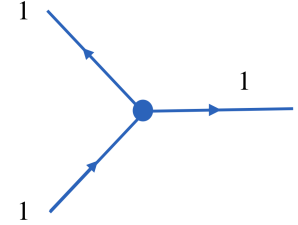
$$\Phi^0(\rho) = a\rho + b(\sigma_X\rho\sigma_X + \sigma_Y\rho\sigma_Y + \sigma_Z\rho\sigma_Z), \quad (2.11)$$

which in the case of quantum channels gives a directionless depolarization process on the qubit. These are the only $SU(2)$ -symmetric processes on a qubit.

The $j = 1$ process mode terms transform under the symmetry as a vector $(\Phi_{\mp 1}^1, \Phi_0^1, \Phi_{-1}^1)$ of process components. There are precisely three such process modes; one arises from coupling 1-irreps for both output and input

$$\Phi_{\mp 1}^{1,1} = -\frac{1}{2\sqrt{2}}(\sigma_{\pm}\text{Tr}(\sigma_Z\rho) - \sigma_Z\text{Tr}(\sigma_{\pm}\rho))$$

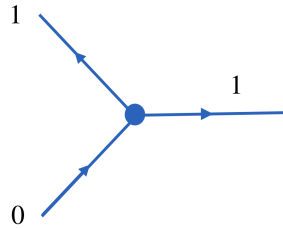
$$\Phi_0^{1,1} = \frac{i}{2\sqrt{2}}(\sigma_X\text{Tr}(\sigma_Y\rho) - \sigma_Y\text{Tr}(\sigma_X\rho))$$



and the other two arise from coupling a 1-irrep with a 0 irrep in either the output or input space according to:

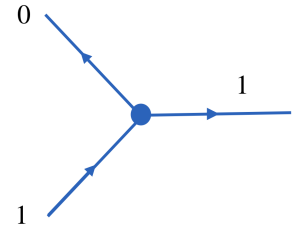
$$\Phi_{\mp 1}^{1,(0,1)}(\rho) = \mp \frac{1}{2}\sigma_{\pm}$$

$$\Phi_0^{1,(0,1)} = \frac{1}{2}\sigma_Z$$



$$\Phi_{\mp 1}^{1,(1,0)}(\rho) = \mp \frac{\mathbb{I}}{2}\text{Tr}(\sigma_{\pm}\rho)$$

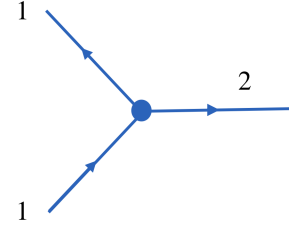
$$\Phi_0^{1,(1,0)} = \frac{\mathbb{I}}{2}\text{Tr}(\sigma_Z\rho).$$



Finally, for the $j = 2$ case, there is a single process mode term, which transforms as a 5-dimensional vector $(\Phi_2^2, \Phi_1^2, \Phi_0^2, \Phi_{-1}^2, \Phi_{-2}^2)$ of process components. These are given by coupling 1-irreps in both the

output and input space:

$$\begin{aligned}\Phi_{\mp 2}^2(\rho) &= \frac{1}{2}\sigma_{\pm}\text{Tr}(\sigma_{\pm}\rho) \\ \Phi_{\mp 1}^2(\rho) &= \mp\frac{1}{2\sqrt{2}}(\sigma_{\pm}\text{Tr}(\sigma_Z\rho) + \sigma_Z\text{Tr}(\sigma_{\pm}\rho)) \\ \Phi_0^2(\rho) &= -\frac{1}{2\sqrt{6}}(\sigma_X\text{Tr}(\sigma_X\rho) + \sigma_Y\text{Tr}(\sigma_Y\rho) - 2\sigma_Z\text{Tr}(\sigma_Z\rho)).\end{aligned}$$



It turns out that when dealing with quantum channels, the canonical process mode formalism gives rise to several particular features. For one, regardless of the symmetry group G , any operator space $\mathcal{B}(\mathcal{H})$ (and similarly $\mathcal{B}(\mathcal{K})$) contains the identity operator \mathbb{I} which transforms trivially under the adjoint representation. For the following result, we distinguish a trivial mode in the basis of irreducible tensor operators - and denote it by $\mathbf{0}$ with the corresponding irreducible tensor operator proportional to \mathbb{I} (i.e $\mathbb{I}/\sqrt{\dim(\mathcal{H})}$ for $\mathcal{B}(\mathcal{H})$ and $\mathbb{I}/\sqrt{\dim(\mathcal{K})}$ for $\mathcal{B}(\mathcal{K})$).

No quantum channel $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ contains the process modes corresponding to coupling $(\mathbf{a}, \mathbf{0})$ for all irreps \mathbf{a} in $U \otimes U^*$, except in the case $\mathbf{a} = \mathbf{0}$. Note that \mathbf{a} may be a trivial irrep too, but it has to represent a different multiplicity such that it is orthogonal to $\mathbf{0}$.

The proof is a direct consequence of the trace preserving condition for \mathcal{E} and orthonormality of irreducible tensor operators. Moreover the coefficient associated with the process mode $\Phi^{\mathbf{0}}(\rho) := \frac{\mathbb{I}}{\sqrt{\dim(\mathcal{K})}}\text{Tr}(\rho\frac{\mathbb{I}}{\sqrt{\dim(\mathcal{H})}})$ labelled by the coupling $(\mathbf{0}, \mathbf{0})$ in the decomposition of \mathcal{E} into process modes is $\sqrt{\dim(\mathcal{H})/\dim(\mathcal{K})}$.

Example 2.1.6. *Vibrational modes of molecules are characterised by an underlying symmetry principle in that each mode corresponds to a particular irreducible representation. Coupling of different vibrational modes or a vibrational mode with an external field is akin mathematically speaking with the couplings in the description of canonical process modes.*

The process mode formalism developed herein, apart from providing the mathematical toolkit with which to analyse quantum channels in the presence of a symmetry principle carries an inherent directionality given by the arrow of time. This arises from a particular choice of tensor product that reflects how symmetry breaking degrees of freedom³ in the input and output

³The notion of "symmetry breaking degrees of freedom" or more generally just "degrees of freedom" is at times loosely used in physics and can refer to several distinct ideas. In order to avoid using a vacuous description and hide complex ideas behind familiar terminology, further clarifications on the matter are needed.

Typically, degree of freedom refers to the dimensionality of the physical space related to a particular unchanged

systems interact.

Given a quantum channel \mathcal{E} that contains the process mode vector Φ^λ corresponding to coupling $(\mathbf{a}, \tilde{\mathbf{a}}) \xrightarrow{\lambda}$, it acts non-trivially only on components of an input state ρ corresponding to the irrep \mathbf{a} , which are transferred into components of the output state $\mathcal{E}(\rho)$ corresponding to the irrep $\tilde{\mathbf{a}}$.

We recall that a state $\rho \in \mathcal{B}(\mathcal{H})$ decomposes as:

$$\rho = \sum_{\mathbf{a}} \sum_{k=1}^{\dim(\mathbf{a})} \rho_{\mathbf{a},k} T_k^{\mathbf{a}} \quad (2.12)$$

in terms of a complete orthogonal basis of irreducible tensor operators for $\mathcal{B}(\mathcal{H})$, where the summation ranges over all irreps, multiplicities (which are compactly packaged into $\mathbf{a} = (a, m_a)$ with a the irrep label and m_a the multiplicity label) and vector components labelled by k . The complex coefficients in the above decomposition are uniquely given by $\rho_{\mathbf{a},k} = \text{Tr}(\rho(T_k^{\mathbf{a}})^\dagger)$. We define the state modes of ρ for any a -irreps and associated multiplicity as $\rho^{\mathbf{a}} := \sum_k \rho_{\mathbf{a},k} T_k^{\mathbf{a}}$. In previous work [36] the asymmetry modes of state ρ were defined as $\rho_k^{\mathbf{a}} = \sum_{m_a} \rho_{\mathbf{a},k} T_k^{\mathbf{a}}$ where the summation is over all multiplicities. Both the asymmetry modes and state modes transform irreducibly under the group action and while related, they represent distinct quantities. The present work will focus on state modes as they are more relevant within the context of process modes and quantum channels more generally. The process modes have the role of transferring *state modes* between the input and output system, in a way consistent with their transformation property under the underlying symmetry principles. Equivalently, we can say the process modes $\lambda = (\mathbf{a}, \tilde{\mathbf{a}})$ (and their diagrammatic representation) are objects labelled by an incoming state mode \mathbf{a} that evolves into an outgoing state mode $\tilde{\mathbf{a}}$ by way of an interaction with an external degree of freedom λ .

Of note in the definition of process modes is the temporal ordering of events in the sense that it respects the notion that there is an input and output space and that the incoming modes \mathbf{a} are transformed into outgoing modes $\tilde{\mathbf{a}}$. In figure 2.1.2 we illustrate this directionality of events with an example whereas a channel that transforms a polarised spin state into a maximally

property of a physical system. For instance a train confined to a track has a single degree of freedom when it comes to displacements in 3D space. In here, symmetry breaking degrees of freedom can be thought of as the "dimensionality of the non-trivial irreps", concept that can be associated to every "leg" of a process mode in its diagrammatic representation. For example, a process mode $(\mathbf{a}, \tilde{\mathbf{a}}) \xrightarrow{\lambda}$ acting on input state without an a -irrep mode produces no output (returns 0). In more physical terms, symmetry-breaking degrees of freedom are related to the coherence or more generally asymmetry properties of a state (or a system). An energy eigenstate coupled to a Hamiltonian driving transitions between different energy eigenstates will result in output state which exhibits coherence, that is dictated entirely by couplings of different energy eigenstates present in the driving Hamiltonian.

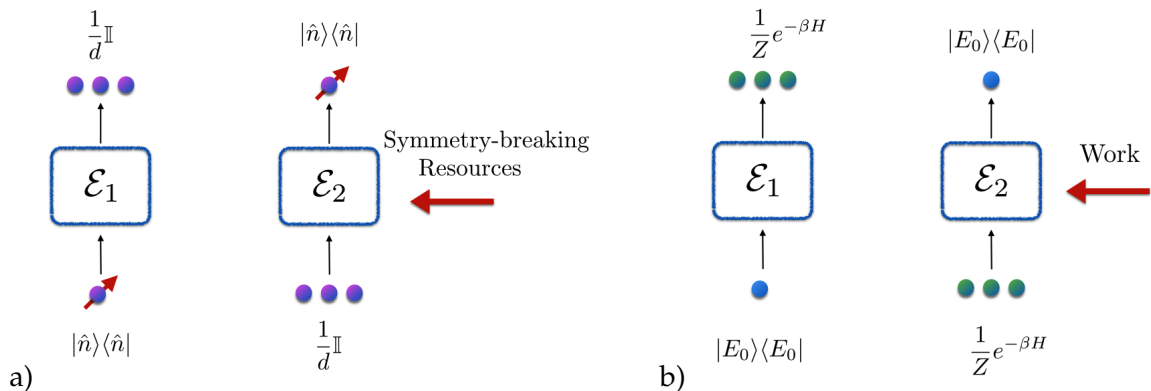


Figure 2.2: Directionality of process modes

- a) A quantum channel \mathcal{E}_2 that maps the maximally mixed state into spin state polarised along some direction \hat{n} requires at least one non-trivial process mode, while the reverse transformation may be achieved by a quantum channel \mathcal{E}_1 consisting solely of trivial process modes ($\lambda = 0$).
- b) The ground state $|E_0\rangle\langle E_0|$ of some system thermalises into the Gibbs state at temperature $T = 1/\beta$ without external interaction, but the reverse transformation requires a non-zero amount of ordered energy – mechanical work.

mixed state will have a fundamentally different process mode decomposition than a channel performing the inverse state transformation. It is in this sense that we say the process mode decomposition respects the causal structure of the quantum systems involved.

2.2 Simulation of open quantum systems and the process orbit of a channel

Simulation of quantum systems promises to be one of the key applications for quantum computers and can address a wide range of problems from quantum chemistry, material science, fundamental physics. Interactions involving many particles exhibiting a high degree of entanglement are considered to be generally difficult to simulate on classical computers – either analytically or numerically. There are only a handful of systems for which efficient classical simulation has been proven such as interaction-free fermionic modes [68]. On the other hand Hamiltonian simulation lies in BQP-complete, the class of decision problems which can be solved by a quantum computer in polynomial time. While much of the focus has been on closed-systems, there are many important problems that motivate studying the simulation of open quantum systems such as dissipative phenomena (engineering quantum state [69], dissipative quantum phase transitions [70]), thermalisation [71], modelling quantum effects in

biology [72,73], non-unitary quantum computation [74].

In broad terms, efficient simulation of the unitary dynamics $U(t)$ of a d -dimensional system involves engineering a sequence of unitary gates U_1, U_2, \dots, U_n that approximate U within some precision ϵ such that the number of gates n is polynomial in time t , the size of the system d and in $1/\epsilon$. Analogously, simulating the evolution of an open quantum system A described by the quantum channel \mathcal{E} involves exhibiting both the coherent evolution dynamics as well as dissipative dynamics given by an interaction with an environment system B . We can formalise this into the following definition.

Definition 2.2.1. Simulation of open-system dynamics

Given a target channel $\mathcal{E} : \mathcal{B}(\mathcal{H}_A) \longrightarrow \mathcal{B}(\mathcal{H}_A)$ on system A , a simulation of \mathcal{E} is a triplet $(E, \sigma_E, \mathcal{V})$ formed of the environment quantum system E with Hilbert space \mathcal{H}_E with an initial state $\sigma_E \in \mathcal{B}(\mathcal{H}_E)$ and a unitary $\mathcal{V} : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_E) \longrightarrow \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_E)$ such that for all $\rho \in \mathcal{B}(\mathcal{H}_A)$:

$$\mathcal{E}(\rho) = \text{Tr}_E(\mathcal{V}(\rho \otimes \sigma_E)). \tag{2.13}$$

Stinespring’s theorem [75] ensures that for any quantum channel (i.e CPTP map) there exists a simulation in the sense of the above definition. In this context an efficient simulation of \mathcal{E} must be given by i) an environment state σ_E that can be efficiently prepared and ii) an efficient unitary \mathcal{V} . These aspects are however beyond the reach of the present exposition. We will however be interested in a resource theoretic perspective. Suppose that \mathcal{V} in the above is restricted to the unitary free operations within a resource theory. Such a setting has been analysed in a number of different contexts such as quantum thermodynamics and quantum reference frames.

2.2.1 Process orbit and relative alignments

Under a symmetry principle, the simulation of open-system dynamics that arises from a symmetry-constrained unitary interaction with an ambient environment leads to a set of local “coordinates” $\{x_i\}$ in the environment with respect to which the target channel \mathcal{E} is induced.

Definition 2.2.2. *Let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$ be a superoperator with \mathcal{H} and \mathcal{K} carrying representations U*

respectively U' of a group G . Then we denote the process orbit of \mathcal{E} under the group action by:

$$\mathcal{M}(G, \mathcal{E}) := \{\mathcal{U}'_g \circ \mathcal{E} \circ \mathcal{U}_g^\dagger : \forall g \in G\}. \quad (2.14)$$

Any two quantum channels in the same process orbit are related by a group element in G and will contain the same decomposition into process modes (with different corresponding coefficients or weights). Moreover the space of all superoperators splits into a union of disjoint orbits i.e $\cup_{\mathcal{E}} \mathcal{M}(G, \mathcal{E}) = \mathcal{S}(\mathcal{H}, \mathcal{K})$. Therefore any quantum channel can be fully specified by providing its orbit and *alignment* with respect to some fixed representative element of each orbit. Let $\mathcal{E} \in \mathcal{M}(G, \mathcal{E}_0)$ then this alignment corresponds to the group element $g \in G$ for which $\mathcal{E} = \mathcal{U}'_g \circ \mathcal{E}_0 \circ \mathcal{U}_g^\dagger$.

Suppose that $(E, \mathcal{V}, \sigma_E)$ is a covariant-simulation of channel \mathcal{E} . Then, under a misalignment of the reference frame at system E with some group element $g \in G$ such that $\sigma_E \rightarrow \mathcal{U}_g^E(\sigma_E)$, then the simulation yields a different quantum channel $\mathcal{E}'(\rho) = \text{Tr}_E(\mathcal{V}(\rho \otimes \mathcal{U}_g^E(\sigma_E)))$. It follows that $\mathcal{E}' = \mathcal{U}_g \circ \mathcal{E} \circ \mathcal{U}_g^\dagger$ since:

$$\mathcal{E}'(\rho) = \mathcal{U}_g(\text{Tr}_E(U(g)^\dagger \otimes U^E(g)^\dagger(\mathcal{V}(\rho \otimes \mathcal{U}_g^E(\sigma_E)))U(g) \otimes U^E(g))) \quad (2.15)$$

$$= \mathcal{U}_g(\text{Tr}_E \mathcal{V}((U(g)^\dagger \otimes U^E(g)^\dagger(\rho \otimes \mathcal{U}_g^E(\sigma_E)))U(g) \otimes U^E(g))) \quad (2.16)$$

$$= \mathcal{U}_g(\text{Tr}_E(\mathcal{V}(U(g)^\dagger \rho U(g) \otimes \sigma_E))) \quad (2.17)$$

$$= \mathcal{U}_g \circ \mathcal{E} \circ \mathcal{U}_g^\dagger \quad (2.18)$$

where the first equality follows from cyclicity of trace and the second from the covariance property of the unitary channel \mathcal{V} . This means that the process orbit of \mathcal{E} remains invariant under misalignments in the environment system.

Equivalently the above calculation says that any channel \mathcal{E} within some fixed orbit $\mathcal{M}(G, \mathcal{E})$ admits a Stinespring dilation with the same covariant unitary \mathcal{V} . In some sense, simulating channels within the same orbit requires the same type of resources. Distinct orbits will most likely require different simulations $(E', \mathcal{V}', \sigma'_E)$. The symmetry breaking properties of the initial state in the environment are transferred into the simulated channel.

Lemma 2.2.3. *For any \mathcal{E} and group G the process orbit $\mathcal{M}(G, \mathcal{E})$ is a homogeneous space.*

Proof. A homogeneous space is a manifold with a transitive group action. In this case there is a

canonical group action $G \times \mathcal{M}(G, \mathcal{E}) \rightarrow \mathcal{M}(G, \mathcal{E})$ that maps $(g, \mathcal{F}) \rightarrow \mathfrak{U}_g[\mathcal{F}]$. This is clearly transitive by the definition of an orbit. \square

Under the group action given by the representation \mathfrak{U} on superoperators, the stabilizer (or isotropy) group $Stab(\mathcal{E}) = \{g \in G : \mathfrak{U}(g)[\mathcal{E}] = \mathcal{E}\}$ is a closed subgroup of G and the coset⁴ space $G/Stab(\mathcal{E})$ is homeomorphic (i.e topologically isomorphic) to the process orbit $\mathcal{M}(G, \mathcal{E})$.

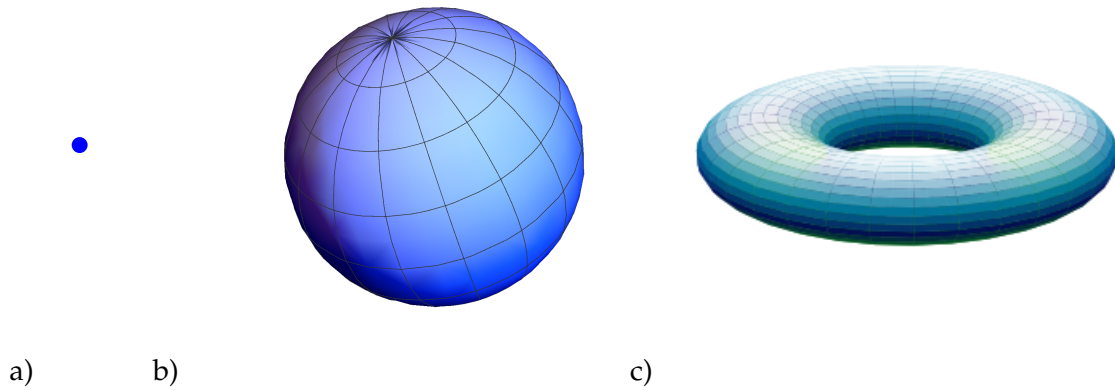


Figure 2.3: Examples of process orbits.

a) The process orbit of any symmetric quantum channel is a point b) The process orbit of axial quantum channels (that have a fixed direction) under rotational symmetry is a sphere c) General quantum channels will have more complex process orbits given by homogeneous spaces.

It should be remarked that the problem of determining all the orbits under a representation of a general finite or compact Lie group is a difficult task. Some general results are however known, but only for particular classes of representations (e.g multiplicity-free, representations on pre-homogeneous vector spaces).

Example 2.2.4. a) If \mathcal{E} is a covariant (symmetric) channel then $\mathfrak{U}_g[\mathcal{E}] = \mathcal{E}$ and then $\mathcal{M}(G, \mathcal{E}) \cong \{e\}$, where $\{e\}$ is the identity element in the group G . b) Consider a single qubit system with the defining spin-1/2 representation of $SU(2)$ and suppose \mathcal{E} is given by unitary rotations by angle θ around a fixed axis \hat{n} . Then $\mathcal{M}(SU(2), \mathcal{E}) \cong SU(2)/U(1) \cong S^2$.

One may consider the existence of additional geometrical structure on $\mathcal{M}(G, \mathcal{E})$ – this fits into the wider scope of the Erlangen⁵ program. For instance if H is a closed compact subgroup of a

⁴Given a subgroup H of a group G a coset space is given by $gH = \{gh, h \in H\}$, which describes an equivalence class with respect to the relation $g \approx g'$ if and only if there is $h \in H$ such that $g' = gh$

⁵The Erlangen program was initiated by Felix Klein in the 19th century with the aim to fully characterise different geometries using group theory.

connected Lie group G then there is at least one Riemannian metric on G/H that is invariant under the group action of G . Moreover the set of G -invariant Riemannian metrics on G/H are given exactly by the set of scalar products (invariant under the H action) on $\mathfrak{g}/\mathfrak{h}$ with the adjoint representation of G . For particular cases e.g $G = SU(2)$ with the maximal torus $H = U(1)$ one recovers the (unique) Fubini-Study metric. Unfortunately these details are beyond the scope of the current work. However, of relevance to the discussion herein is that any geometric parametrisation of the process orbit would give rise to a set of coordinates on the process orbit. Such coordinates will depend on the topology and geometry of the orbit, and the origin corresponds to a fixed representative channel. Any other quantum channel in the same orbit can be fully identified by a coordinate value.

Example 2.2.5. Consider a single spin-1/2 particle with Hamiltonian given by $H = \frac{\omega}{2}Z$ where Z is the Pauli matrix and ω is a precession frequency. The time evolution of the system is given by $U(t) = e^{-iHt}$. Consider a rotation about the x axis by an angle θ described by a quantum process $\mathcal{E}(\rho) = R_X(\theta)\rho R_X(\theta)^\dagger = e^{-i\theta X}\rho e^{i\theta X}$. The quantum process \mathcal{E} is not symmetric under time-translations because it matters if one first rotates and then evolves or evolve and then rotate. Mathematically this simply means that

$$U(t)R_X(\theta) \neq R_X(\theta)U(t), \quad (2.19)$$

for general θ, t . Framed another way, this means that

$$\mathcal{U}_t \circ \mathcal{E} - \mathcal{E} \circ \mathcal{U}_t \neq 0, \quad (2.20)$$

and which means that the quantum process does not commute with the group action. To study the degree to which this occurs we can define the family of quantum processes

$$\mathcal{E}_t := \mathcal{U}_t \circ \mathcal{E} \circ \mathcal{U}_{-t} \text{ for } t \in \mathbb{R} \quad (2.21)$$

that describe the action of the group on \mathcal{E} . We find that $\mathcal{E}_t(\rho) = V(t)\rho V(t)^\dagger$, where $V(t)$ is a unitary given by

$$V(t) = \exp[-i\theta(X \cos(\omega t) + Y \sin(\omega t))]. \quad (2.22)$$

It is clear that for $t = 0$ we recover the central process \mathcal{E} , but for $t > 0$ we have a whole family of unitary rotations around different axes. We see that $V(t)$ is periodic in t with period $\frac{2\pi}{\omega}$ and so the set of matrices $\{V(t)\}$ form a ring in the space of all matrices. Therefore we actually have a $G = U(1)$ symmetry group action. In exactly the same way the set of quantum processes $\{\mathcal{E}_t\}$ also form a ring of points in the set of

all quantum processes, describing how the symmetry group action affects \mathcal{E} . It is important to note that these set of processes depend both on the particular initial process \mathcal{E} and the group that act on it, and so for this set we write

$$\mathcal{M}(U(1), \mathcal{E}) := \{\mathcal{E}_t : \mathcal{E}_t(\rho) = V(t)\rho V(t)^\dagger\}, \quad (2.23)$$

which is the process orbit of \mathcal{E} under the phase group $U(1)$. Each point on the orbit is uniquely determined by the parameter $t \in [0, 2\pi/\omega]$, and so t provides a coordinate system on the orbit, with origin centered at \mathcal{E} . The basic shape of $\mathcal{M}(U(1), \mathcal{E})$ describes the way in which \mathcal{E} breaks the symmetry constraint. For example, note that if we send $\theta \rightarrow 0$ then the radius of the process orbit shrinks to zero, and it becomes a trivial point. This corresponds to \mathcal{E} being the identity unitary, which is symmetric under time translations. This is also true for any other symmetric process, and so we have the following simple geometric characterization: a process \mathcal{E} is symmetric if and only if its orbit is a single point.

We summarise the core physical motivations for introducing the process orbit of a quantum channel:

- i) Provides the minimal resources needed for covariant simulation
- ii) The choice of origin in the process orbit corresponds to a gauge degree of freedom that shows how the physics on the main system is related to the physics in its environment
- iii) Any parametrisation of the process orbit gives a set of (classical) coordinates that depend on the underlying geometric features.

These features will become relevant in the context of applications of the current framework that we develop in Chapters 4 and 3.

2.3 A polar-decomposition for quantum channels

In this section we describe connections between the process orbit and the process mode decomposition of a quantum channel. Recall that given a quantum channel $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ where \mathcal{H} and \mathcal{K} have representations U respectively U' of a compact group G then it decomposes in terms of process modes as:

$$\mathcal{E} = \sum_{\lambda} \alpha_{\lambda} \cdot \Phi^{\lambda} \quad (2.24)$$

where α_λ is a vector of complex coefficients.

We start by restricting to the $SU(2)$ group in the first instance, and then we give a generalisation to arbitrary finite and compact Lie groups in the subsequent subsection.

2.3.1 Case I: $SU(2)$ symmetry

For now, we restrict to the $SU(2)$ group and consider operations $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ where \mathcal{H} and \mathcal{K} have the unitary representations U and U' of $SU(2)$. If \mathcal{E} is a symmetric channel then the stabilizer subgroup is the whole $SU(2)$ group, and at the other extreme are channels for which the stabilizer contains only the identity. In between these two situations we find channels which are not symmetric, but still have some residual symmetry. Take for example a qubit system, the rotations, measurements or depolarizing along a fixed axis of the Bloch sphere corresponds to quantum channels which break the full rotational symmetry, but have a residual $U(1)$ symmetry, as they remain invariant under rotations along the fixed axis.

Definition 2.3.1. Axial quantum channels

Under an $SU(2)$ symmetry principle, \mathcal{E} is called an axial quantum channel if the stabilizer subgroup $Stab(\mathcal{E}) := \{h \in SU(2) : \mathcal{U}_h[\mathcal{E}] = \mathcal{E}\}$ is isomorphic to $U(1)$.

By the above definition it is clear that the process orbit of any axial quantum channel is a sphere since $S^2 \cong SU(2)/U(1)$. In the previous section we have explained how any quantum channel is determined by a label for which orbit it belongs to and an alignment with respect to the origin within that orbit. The fact that axial channels have a fixed (up to homeomorphism) process orbit, determined by a sphere in the case of $SU(2)$, implies that each quantum channel \mathcal{E} in that orbit will have associated a unique coordinate point $\hat{\mathbf{n}} = (\theta, \phi)$, a unit vector corresponding to a point on the sphere. Fix an axial channel \mathcal{E}_0 for instance, as representative of the orbit $\mathcal{M}(SU(2), \mathcal{E}_0)$ then the homeomorphism $F : \mathcal{M}(SU(2), \mathcal{E}_0) \rightarrow SU(2)/U(1) \rightarrow S^2$ bijectively maps any $\mathcal{E} \in \mathcal{M}(SU(2), \mathcal{E}_0)$ to points on the sphere S^2 given by unit vectors. Moreover for the channels in this orbit, the coefficients in the process mode decompositions are complex-valued functions on the sphere. If $F(\mathcal{E}) = \hat{\mathbf{n}}$ then $\mathcal{E} = \sum_{\lambda,k} \alpha_{\lambda,k}(\hat{\mathbf{n}}) \Phi_k^\lambda$, where each coefficient is a function $\alpha_{\lambda,k} \in \mathcal{L}^2(S^2)$. This follows easily from the following argument: i) the coefficients are uniquely given by $\alpha_{\lambda,k} = \langle \Phi_k^\lambda, \mathcal{E} \rangle$ and since F is bijective then $\alpha_{\lambda,k} : S^2 \rightarrow \mathbb{C}$ such that $\hat{\mathbf{n}} \rightarrow \langle \Phi_k^\lambda, F^{-1}[\hat{\mathbf{n}}] \rangle$, ii) looking at the decomposition in the Choi representation $J[\mathcal{E}] = \sum_{\lambda,k} \alpha_{\lambda,k}(\hat{\mathbf{n}}) J[\Phi_k^\lambda]$ and using

orthogonality of irreducible tensor superoperators $\text{Tr}[J[\mathcal{E}]^2] = \sum_{\lambda,k} |\alpha_{\lambda,k}(\hat{\mathbf{n}})|^2$ and therefore $\|\alpha_{\lambda,k}\|_{L^2}^2 = \int_{S^2} |\alpha_{\lambda,k}(\hat{\mathbf{n}})|^2 d\hat{\mathbf{n}} \leq \int_{S^2} \text{Tr}(J[F^{-1}(\hat{\mathbf{n}})]^2) d\hat{\mathbf{n}}$. This upper bound is the average purity of the Jamiolkowski state in the process orbit $\mathcal{M}(SU(2), \mathcal{E}_0)$ so it will be finite (since from Cauchy Schwartz, for any channel the purity of the Jamiolkowski state is less than or equal to $\dim(\mathcal{H})^2$) and hence $\alpha_{\lambda,k} \in \mathcal{L}^2(S^2)$.

The following result in Theorem 2.3.3 shows how the residual symmetry for axial channels reduces the number of free parameters in the description of \mathcal{E} in terms of process modes. These parameters split into intrinsic data and a gauge degree of freedom, which corresponds directly to the choice of origin in the orbit. We illustrate these aspects with an example, then proceed to give the general results.

Example 2.3.2. Axial unitary on a qubit

Let $V = e^{i\frac{\gamma}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}$ be a unitary rotation by angle γ around the $\hat{\mathbf{n}}$ axis of the Bloch sphere, where $\boldsymbol{\sigma} = (\sigma_X, \sigma_Y, \sigma_Z)$ is a vector of Pauli matrices. Under the 1/2 fundamental irrep of $SU(2)$, the quantum channel $\mathcal{V}(\rho) = V\rho V^\dagger$ decomposes into process modes $\mathcal{V} = \sum_{j,k} \alpha_{jk} \Phi_k^j$ with the irrep label j taking values of 0, 1 and 2 with multiplicities two, three and one respectively. From orthonormality of process modes the complex coefficients are given by $\alpha_{jk} = \text{Tr}(J[\Phi_k^j]^\dagger J[\mathcal{V}])$ where $J[\mathcal{V}] = |\text{vec}(V)\rangle\langle\text{vec}(V)|$ is the Choi operator of the unitary channel \mathcal{V} and $J[\Phi_k^j]$ is the Choi operator corresponding to the process mode Φ_k^j . Note that $|\text{vec}(A)\rangle$ corresponds to the vectorisation of an operator A with respect to the standard computational basis.

Therefore the coefficients are given by:

$$\alpha_{jk}(\hat{\mathbf{n}}) = \langle \text{vec}(e^{i\frac{\gamma}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}) | J[\Phi_k^j]^\dagger | \text{vec}(e^{-i\frac{\gamma}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}) \rangle \quad (2.25)$$

where we emphasize the explicit dependence of α_{jk} on the axis $\hat{\mathbf{n}}$. On the space of function on the sphere, there is a natural group action whereas for any $g \in SU(2)$ and $\alpha \in \mathcal{L}^2(S^2)$:

$$g \cdot \alpha(\hat{\mathbf{n}}) = \alpha(g^{-1} \cdot \hat{\mathbf{n}}) \quad (2.26)$$

where the group action of $SU(2)$ on the sphere S^2 is defined by $(g \cdot \hat{\mathbf{n}}) \cdot \boldsymbol{\sigma} = g(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})g^{-1}$. Under this group action we have that the coefficients in the process mode decomposition transform irreducibly, just

like spherical harmonic functions. This follows from:

$$\alpha_{jk}(g^{-1} \cdot \hat{\mathbf{n}}) = \langle \text{vec}(e^{i\frac{\gamma}{2}} g^\dagger \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} g) | J[\Phi_k^j]^\dagger | \text{vec}(e^{-i\frac{\gamma}{2}} g^\dagger \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} g) \rangle \quad (2.27)$$

$$= \langle \text{vec}(g^\dagger e^{i\frac{\gamma}{2}} \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} g) | J[\Phi_k^j]^\dagger | \text{vec}(g^\dagger e^{-i\frac{\gamma}{2}} \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} g) \rangle \quad (2.28)$$

$$= \langle \text{vec}(e^{i\frac{\gamma}{2}} \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) | (g \otimes g^* J[\Phi_k^j] g^\dagger \otimes g^T)^\dagger | \text{vec}(e^{-i\frac{\gamma}{2}} \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) \rangle \quad (2.29)$$

$$= \sum_{k'} (v_{k'k}^j(g))^* \alpha_{jk}(\hat{\mathbf{n}}) \quad (2.30)$$

where the third equality comes from the properties of vectorisation and the last equality from the transformation property of irreducible tensor superoperators and the group action on the Choi representation. The equation (2.30) implies that the vector $\alpha_j(\hat{\mathbf{n}})$ transforms within the j^* irrep in the decomposition of $\mathcal{L}^2(S^2)$. This component is spanned by the spherical harmonics $\{Y_k^{j^*}\}_k$.

With respect to the canonical process modes in Example 2.1.5, denote the coefficients by α_0 and α'_0 for the 0-irreps arising from couplings (0, 0) and (1, 1), α_1 and α'_1 for couplings (1, 1) $\xrightarrow{1}$ respectively (0, 1) $\xrightarrow{1}$ and α_2 for the coupling (1, 1) $\xrightarrow{2}$. For any qubit quantum channel we have that $\alpha_0 = 1$ and the rest of the coefficients for the unitary channel \mathcal{V} are given by:

$$\alpha_0 = -\frac{1}{\sqrt{3}}(1 + 2 \cos \gamma) \quad (2.31)$$

$$\alpha_1(\hat{\mathbf{n}}) = -i\sqrt{2} \sin 2\gamma \left(-\frac{x - iy}{\sqrt{2}}, z, \frac{x + iy}{\sqrt{2}} \right) \quad (2.32)$$

$$\alpha'_1(\hat{\mathbf{n}}) = \mathbf{0} \quad (2.33)$$

$$\alpha_2(\hat{\mathbf{n}}) = \sin^2 \gamma \left((x - iy)^2, -2(x - iy)z, -\sqrt{\frac{2}{3}}(1 - 3z^2), 2(x + iy)z, (x + iy)^2 \right) \quad (2.34)$$

where $\hat{\mathbf{n}} = (x, y, z)$ are the cartesian coordinates of the unit vector $\hat{\mathbf{n}}$. The above reveal that indeed the coefficients are proportional to the dual spherical harmonics given by $Y_k^{1^*} = (-1)^{1-k} Y_k^1$ and $Y_2^{2^*} = (-1)^k Y_{-k}^2$ with the vector component k ranging from $-j, \dots, j$ and where the form of the spherical harmonics $\mathbf{Y}^\lambda(\hat{\mathbf{n}})$ corresponds to the standard choice of basis for irreps of $SU(2)$.

The core result of this section describes how the coefficients in the process mode expansion split into an invariant term and a particular choice of gauge for the process orbit. We will refer to this result (and its subsequent generalisation in theorem 2.3.5) as a polar decomposition for quantum channels. The use of such a name can be motivated by the following example: A complex number $z \in \mathbb{C}$ can be decomposed into the magnitude $r = |z|$ and complex exponential of its argument $\phi := \arg(z)$ such that $z = re^{i\phi}$. The magnitude r is invariant under rotations

of the complex plane around the origin $z = 0$, which can be viewed as the action of the group $U(1)$ on \mathbb{C} . The orbits of this group action are circles centred around the origin and the spherical harmonics for $U(1)$ correspond to complex exponentials. The polar decomposition of channels gives a harmonic analysis of the quantum channel under a general group G . Traditionally such techniques have proven useful in a range of applications e.g signal processing, abelian/non-abelian (quantum) Fourier transform and therefore it is natural to extend such type of analyses to quantum channels.

Theorem 2.3.3. (*Polar decomposition of channels under $SU(2)$*) Let \mathcal{E} be an axial quantum channel, and suppose that \mathcal{E} corresponds to point $\hat{\mathbf{n}}$ on S^2 , the process orbit $\mathcal{M}(SU(2), \mathcal{E})$. Then there is a basis of process modes such that in the decomposition of $\mathcal{E} = \sum_{\lambda} \alpha_{\lambda} \cdot \Phi^{\lambda}$, the vector coefficients α are proportional to spherical harmonics:

$$\alpha_{\lambda} = a_{\lambda} \mathbf{Y}^{\lambda*}(\hat{\mathbf{n}}) \quad (2.35)$$

where the complex coefficient a_{λ} does not depend on the vector component.

Proof. Recall that the homeomorphism $F : \mathcal{M}(SU(2), \mathcal{E}) \rightarrow SU(2)/U(1) \rightarrow S^2$ bijectively maps \mathcal{E} to the unit vector $\hat{\mathbf{n}} \in S^2$ such that $F[\mathcal{E}] = \hat{\mathbf{n}}$ with inverse $\mathcal{E} = F^{-1}(\hat{\mathbf{n}})$.

$SU(2)$ acts transitively on the orbit $\mathcal{M}(SU(2), \mathcal{E})$ via $g \rightarrow \mathfrak{U}_g$. Under the homeomorphisms above the axial operation $\mathfrak{U}_g[\mathcal{E}] \in \mathcal{M}(SU(2), \mathcal{E})$ has a corresponding unique vector in $\hat{\mathbf{n}}' \in S^2$ such that $\mathbf{n}' = F[\mathfrak{U}_g[\mathcal{E}]]$ where \mathbf{n}' depends on the fixed $g \in SU(2)$. The transitive group action on the orbit induces a transitive group action on S^2 given by $g \cdot \hat{\mathbf{n}} := \hat{\mathbf{n}}' = F[\mathfrak{U}_g(F^{-1}(\hat{\mathbf{n}}))]$. In turn since the coefficients $\alpha_{\lambda,k} \in \mathcal{L}^2(S^2)$ we have an induced group action on the function space given by $g \cdot \alpha_{\lambda,k}(\hat{\mathbf{n}}) = \alpha_{\lambda,k}(g^{-1} \cdot \hat{\mathbf{n}})$. Moreover this group action will be equivalent to the left regular representation since we can identify the orbit with the coset space.

As \mathcal{E} transforms under the group action it generates only operations in $\mathcal{M}(SU(2), \mathcal{E})$ so we can explicitly show how the coefficients transform under the induced group action.

Claim 1 *The alpha-coefficients will transform under the induced group action as:*

$$g \cdot \alpha_{\lambda,k}(\hat{\mathbf{n}}) = \sum_{k'} v_{k'k}^{\lambda*}(g) \alpha_{\lambda,k'}(\hat{\mathbf{n}}) \quad (2.36)$$

for any $g \in SU(2)$ and any irrep $\lambda \in \text{Irrep}(A, A')$ with $v_{k'k}^{\lambda*}$ matrix coefficients of the dual irrep j^* .

Proof of claim:

We have that:

$$\mathfrak{U}_g[\mathcal{E}] = \sum_{\lambda,k} \alpha_{\lambda,k}(\hat{\mathbf{n}}') \Phi_k^\lambda. \quad (2.37)$$

We want to express $\alpha_{\lambda,k}(\hat{\mathbf{n}}')$ in terms of $\alpha_{\lambda,k}(\hat{\mathbf{n}})$ to explicitly show how the alpha coefficients transform under the group action. Since Φ_k^λ transform as irreducible tensor superoperators we have that:

$$\begin{aligned} \mathfrak{U}_g[\mathcal{E}] &= \sum_{\lambda,k} \alpha_{\lambda,k}(\hat{\mathbf{n}}) \mathfrak{U}_g[\Phi_k^\lambda] \\ &= \sum_{\lambda,k} v_{k'k}^\lambda(g) \alpha_{\lambda,k}(\hat{\mathbf{n}}) \Phi_{k'}^\lambda. \end{aligned} \quad (2.38)$$

Then equating the two different expressions for $\mathfrak{U}_g[\mathcal{E}]$ and using orthogonality of irreducible tensor superoperators implies that:

$$\alpha_{\lambda,k}(\hat{\mathbf{n}}') = \sum_{\lambda,k'} v_{kk'}^\lambda(g) \alpha_{\lambda,k'}(\hat{\mathbf{n}}). \quad (2.39)$$

Since $\hat{\mathbf{n}}' = g \cdot \hat{\mathbf{n}}$ we can re-write the above as $g^{-1} \cdot \alpha_{\lambda,k}(\hat{\mathbf{n}}) = \sum_{\lambda,k'} v_{kk'}^\lambda(g) \alpha_{\lambda,k'}(\hat{\mathbf{n}})$. However there is nothing special in our choice of the particular element g and corresponding point $\hat{\mathbf{n}}'$ so it turns out that this occurs for all elements $g \in SU(2)$. Because we can use $(v_{kk'}^\lambda(g))^* = v_{k'k}^\lambda(g^{-1})$ and $(v_{kk'}^\lambda(g))^* = v_{kk'}^{\lambda^*}(g)$ where λ^* is the dual irrep this leads to the desired transformation

$$g \cdot \alpha_{\lambda,k}(\hat{\mathbf{n}}) = \alpha_{\lambda,k}(g^{-1} \cdot \hat{\mathbf{n}}) = \sum_{\lambda,k'} v_{kk'}^{\lambda^*}(g) \alpha_{\lambda,k'}(\hat{\mathbf{n}}). \quad (2.40)$$

This ends our proof of Claim 1.

We have previously shown that $\alpha_{\lambda,k} \in \mathcal{L}^2(S^2, \mathbb{C})$. However square integrable functions on the sphere decompose $\mathcal{L}^2(S^2, \mathbb{C})$ into a complete set of spherical harmonics. It is important to mention at this point that the irrep-decomposition of $\mathcal{L}^2(S^2, \mathbb{C})$ is multiplicity-free and each λ -irrep component is spanned by a complete basis of orthonormal functions $Y_{\lambda,k}$ for $k = -\lambda, \dots, \lambda$ which correspond to the spherical harmonics. The fact that the isotypical decomposition is multiplicity free is exactly what allows us to define spherical harmonics for this homogeneous space in the first place. Since $\alpha_{\lambda,k}$ are functions on the manifold $\mathcal{M}(SU(2), \mathcal{E})$ and transform according to Equation (2.40) then they lie in the λ^* -irrep component of $\mathcal{L}^2(S^2, \mathbb{C})$ that is spanned by spherical harmonics Y_{λ^*k} . Therefore we can write them in terms of a spherical harmonics

basis such that:

$$\alpha_{\lambda,k}(\hat{\mathbf{n}}) = \sum_{k'} a_{\lambda k'}^{(k)} Y_{\lambda^* k'}(\hat{\mathbf{n}}) \quad (2.41)$$

for some complex coefficients $a_{\lambda k'}^k$ depending on some fixed λ and k . We show in the following that $a_{\lambda k'}^k = \delta_{kk'} a_{\lambda}$ for some complex number a_{λ} that is independent on the vector component k . To do so note that the above equation holds for all $\hat{\mathbf{n}} \in S^2$. So we have:

$$\begin{aligned} \alpha_{\lambda k}(g^{-1} \cdot \hat{\mathbf{n}}) &= \sum_{k'} a_{\lambda k'}^{(k)} (Y_{\lambda k'}((g^{-1} \cdot \hat{\mathbf{n}}))) \\ &= \sum_{k',m} a_{\lambda k'}^{(k)} v_{k'm}^{\lambda^*}(g) Y_{\lambda^* m}(\hat{\mathbf{n}}) \end{aligned} \quad (2.42)$$

and similarly directly from Claim 1 we have that the coefficients transform as:

$$\begin{aligned} \alpha_{\lambda k}(g^{-1} \cdot \hat{\mathbf{n}}) &= \sum_{k'} v_{kk'}^{\lambda^*}(g) \alpha_{\lambda k'}(\hat{\mathbf{n}}) \\ &= \sum_{m,k'} v_{kk'}^{\lambda^*}(g) a_{\lambda m}^{(k')} Y_{\lambda^* m}(\hat{\mathbf{n}}). \end{aligned} \quad (2.43)$$

Using orthonormality of spherical harmonics we can equate the two different forms for the transformed alpha-coefficients to obtain that for all m the following holds:

$$\sum_{k'} v_{kk'}^{\lambda^*}(g) a_{\lambda m}^{(k')} = \sum_{k'} a_{\lambda k'}^{(k)} v_{k'm}^{\lambda^*}(g). \quad (2.44)$$

Now we can use orthonormality of matrix coefficients for the λ^* irrep to multiply both sides by $(v_{ls}^{\lambda^*})^*(g)$ and integrate over all group elements to get

$$\sum_{k'} \delta_{kl} \delta_{k's} a_{\lambda m}^{(k')} = \sum_{k'} \delta_{k'l} \delta_{ms} a_{\lambda k'}^{(k)} \quad (2.45)$$

and therefore reduce to $a_{\lambda m}^{(s)} \delta_{kl} = \delta_{ms} a_{\lambda l}^{(k)}$ for all m . This implies that $a_{\lambda k'}^{(k)}$ is non-zero if and only if $k = k'$ and moreover it is independent on the vector component k . Therefore we denote it simply by $a_{\lambda} := a_{\lambda k}^{(k)}$. It follows immediately that the coefficients are then given by $\alpha_{\lambda,k} = a_{\lambda} Y_{\lambda^*,k}$ or equivalently using the vector notation, $\boldsymbol{\alpha}_{\lambda} = a_{\lambda} \mathbf{Y}^{\lambda^*}$. Furthermore since for $SU(2)$ any irrep is isomorphic to its dual, under the choice of Condon-Shortly phase the spherical harmonics for the dual representation are given by $Y_{\lambda^*,k} = (-1)^k Y_{\lambda,-k}$. Therefore under some basis choice for

Quantum channel	(a_0, a'_1, a_1, a_2)
Dephasing channel: $\mathcal{E}(\rho) = p\rho + (1-p)\sum_k \text{Tr}(\Pi_k\rho)\Pi_k$	$\left(\frac{2p-1}{\sqrt{3}}, 0, 1-p, 0\right)$
Projective measurements: $\mathcal{E}(\rho) = \sum_k \Pi_k \text{Tr}(\Pi_k\rho)$	$\left(-\frac{1}{\sqrt{3}}, 0, 0, 1, 0\right)$
Rotation about an axis: $\mathcal{E}(\rho) = e^{i\frac{\gamma}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}\rho e^{-i\frac{\gamma}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}$	$\left(-\frac{1}{\sqrt{3}}(1+2\cos\gamma), 0, -i\sqrt{2}\sin 2\gamma, 2\sin^2\gamma\right)$
State preparation $\mathcal{E}(\rho) = \frac{1}{2}(\mathbb{I} + p\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})$	$(0, p, 0, 0)$
Depolarising channel: $\mathcal{E}(\rho) = p\rho + (1-p)\frac{1}{2}\mathbb{I}$	$\left(\frac{1-4p}{\sqrt{3}}, 0, 0, 0\right)$

Figure 2.4: Axial channels and resource demands.

Any single qubit axial quantum channel with fixed axis $\hat{\mathbf{n}}$ decomposes into canonical process modes as $\mathcal{E} = \mathbb{I}/2 + a_0\Phi^0 + a'_1\mathbf{Y}^{1*}(\hat{\mathbf{n}}) \cdot \boldsymbol{\Phi}^{1,(0,1)} + a_1\mathbf{Y}^{1*}(\hat{\mathbf{n}}) \cdot \boldsymbol{\Phi}^1 + a_2\mathbf{Y}^{2*}(\hat{\mathbf{n}}) \cdot \boldsymbol{\Phi}^2$. The un-normalised dual spherical harmonics are $\mathbf{Y}^{1*} = \left(-\frac{x-iy}{\sqrt{2}}, z, \frac{x+iy}{\sqrt{2}}\right)$ and $\mathbf{Y}^{2*} = \frac{1}{2}\left((x-iy)^2, -2(x-iy)z, -\sqrt{\frac{2}{3}}(1-3z^2), 2(x+iy)z, (x+iy)^2\right)$. The table shows the intrinsic resource demands (a_0, a'_1, a_1, a_2) for various examples of channels. These are constant for any channel within the same process orbit. Here $\Pi_0 = |\hat{\mathbf{n}}\rangle\langle\hat{\mathbf{n}}|$ denotes the projection onto the direction $\hat{\mathbf{n}}$ and $\Pi_1 = \mathbb{I} - \Pi_0$.

the irreducible tensor operators and spherical harmonics the coefficients will take the form:

$$\alpha_{\lambda,k}(\hat{\mathbf{n}}) = a_{\lambda}(-1)^k Y_{\lambda,-k}(\hat{\mathbf{n}}) \quad (2.46)$$

for any $\hat{\mathbf{n}} \in S^2$ and some a_{λ} independent on the vector component labelled by k . \square

A general quantum channel on a d -dimensional system requires $d^4 - d^2$ real independent parameters [76]. For a qubit system, there are 12 independent real parameters. Axial channels are a particular class for which the orbit under the action of $SU(2)$ is isomorphic to a sphere. These are fully characterised by 7 real independent parameters for qubit systems with the fundamental representation of $SU(2)$. Three parameters describe the ‘‘position’’ of the channel within its orbit, while the rest of the free parameters are associated with different orbits and remain constant within every fixed orbit. The same classification into ‘‘intrinsic data’’ and ‘‘gauge degree of freedom’’ holds for higher dimensional systems under the action of $SU(2)$.

In Figure 2.4 we give the intrinsic data for several examples of one qubit axial channels and of the depolarising channel. The decomposition is computed with respect to a basis of canonical process modes as given in Example 2.1.5. The depolarising channel is fully symmetric under

rotations, there is no residual symmetry and so its orbit consists of a single point. For the dephasing channel, projective measurement, state preparation and rotation along a fixed axis defined by the unit vector $\hat{\mathbf{n}}$ the stabilizer subgroup is given by: $\{e^{i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}} : \theta \in [-\pi, \pi]\} \cong U(1)$. This allows to associate for every such channel a unit vector $\hat{\mathbf{n}}$ on an *distinct* S^2 process orbit. Each of these classes of channels belongs to a different orbit, while the stabilizer subgroup is the same. The association between channels and points on S^2 is certainly non-unique, and the intrinsic data does indeed depend on such a choice of origin for the orbit. In particular, if \mathcal{E} corresponds to $\hat{\mathbf{n}} \in S^2$ then for any $\mathcal{E}' \in \mathcal{M}(SU(2), \mathcal{E})$ in the same orbit there is a group element $g \in SU(2)$ such that $\mathfrak{U}_g[\mathcal{E}] = \mathcal{E}'$ and therefore \mathcal{E}' will correspond to a unit vector $g \cdot \hat{\mathbf{n}}$, which is uniquely determined by $(g \cdot \hat{\mathbf{n}}) \cdot \boldsymbol{\sigma} = g\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}g^{-1}$. This explicitly gives the isomorphism $\mathfrak{U}_g[\mathcal{E}] \longleftrightarrow g \cdot \hat{\mathbf{n}}$ between S^2 and the process orbit $\mathcal{M}(SU(2), \mathcal{E})$.

This prompts an important question: given an arbitrary axial quantum channel \mathcal{E} , how is the choice of origin of its orbit related to the choice of basis for the process mode decomposition?

It is clear that if one fixes a basis for the process modes, then the coefficients in the decomposition of a given \mathcal{E} are uniquely determined. According to Theorem 2.3.3 then this gives a corresponding unit vector $\hat{\mathbf{n}} \in S^2$ for \mathcal{E} .

Conversely, given \mathcal{E} since its stabilizer is an abelian group, all elements are commuting so there is an orthogonal basis $\{|\hat{\mathbf{n}}\rangle, |\hat{\mathbf{n}}^\perp\rangle\}$ that simultaneously diagonalises each element in the stabilizer subgroup. This determines a unit vector $\hat{\mathbf{n}} \in S^2$ such that $Stab(\mathcal{E}) = \{e^{i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}} : \theta \in [-\pi, \pi]\}$. Therefore we can associate \mathcal{E} with a unit vector $\hat{\mathbf{n}}$.

2.3.2 General case of compact Lie group symmetry

In this section we generalise some of the previous results to general compact Lie group G . We consider quantum channels $\mathcal{E} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$, with \mathcal{H} and \mathcal{K} carrying the unitary representations U and U' . Then G acts on the space of superoperator $\mathcal{S}(\mathcal{H}, \mathcal{K})$ via the (continuous) group action $(g, \mathcal{E}) \longrightarrow \mathfrak{U}_g[\mathcal{E}]$. Under this group action, the space $\mathcal{S}(\mathcal{H}, \mathcal{K})$ splits into a disjoint union of orbits $\mathcal{S}(\mathcal{H}, \mathcal{K}) = \cup_{\mathcal{E}} \mathcal{M}(G, \mathcal{E})$. Note that if \mathcal{E} is a CPTP map then so is every linear map in its orbit, because it is a composition of CPTP maps. Lemma 2.3.4 below ensures each orbit is a homogeneous space. Moreover for any channels \mathcal{E} and \mathcal{F} if their corresponding stabilizer subgroups are conjugate in G , then the quotients $G/Stab(\mathcal{E}) \cong G/Stab(\mathcal{F})$ and therefore their

orbits have the same type. This gives an equivalence relation on the space of orbits and classifies orbits according to their type.

Lemma 2.3.4. *Let G be a compact Lie group. For any channel \mathcal{E} , there is a homeomorphism between the process orbit $\mathcal{M}(G, \mathcal{E})$ and the homogeneous space $G/Stab(\mathcal{E})$:*

$$\mathcal{M}(G, \mathcal{E}) \cong G/Stab(\mathcal{E}). \quad (2.47)$$

Proof. We show that the mapping $f : G/Stab(\mathcal{E}) \rightarrow \mathcal{M}(G, \mathcal{E})$ taking every left coset $gStab(\mathcal{E}) \in G/Stab(\mathcal{E})$ to a channel $\mathcal{F} \in \mathcal{M}(G, \mathcal{E})$ such that $\mathcal{F} = \mathfrak{U}_g[\mathcal{E}]$ defines a continuous bijection (which for a compact group G becomes a homeomorphism).

We have the series of equivalences $f(g_1Stab(\mathcal{E})) = f(g_2Stab(\mathcal{E}))$ iff $\mathfrak{U}_{g_1}[\mathcal{E}] = \mathfrak{U}_{g_2}[\mathcal{E}]$ iff $\mathcal{E} = \mathfrak{U}_{g_1^{-1}} \circ \mathfrak{U}_{g_2}[\mathcal{E}] = \mathfrak{U}_{g_1^{-1}g_2}[\mathcal{E}]$ iff $g_1^{-1}g_2 \in Stab(\mathcal{E})$ iff $g_1Stab(\mathcal{E}) = g_2Stab(\mathcal{E})$. One way establishes that f is well defined the other that it is injective. Surjectivity is clear since for any $\mathcal{F} \in \mathcal{M}(G, \mathcal{E})$ there is $g \in G$ such that $\mathcal{F} = \mathfrak{U}_g[\mathcal{E}] = f(gStab(\mathcal{E}))$ (as G acts transitively on $\mathcal{M}(G, \mathcal{E})$).

Moreover the group action $g \rightarrow \mathfrak{U}_g[\mathcal{E}]$ is continuous so the induced map f will be continuous with respect to the quotient topology. f is also an open map because $G/Stab(\mathcal{E})$ is compact (as the cosets of a compact group are compact), therefore the inverse of f is continuous and thus f is a homeomorphism. Note that for Lie groups the homeomorphism becomes a diffeomorphism since in this case the orbit and coset space are smooth manifolds.

□

Let us fix a generic quantum channel \mathcal{E} and denote by $H := Stab(\mathcal{E})$ the stabilizer subgroup of G . This also fixes the orbit type of \mathcal{E} to be the homogeneous space G/H with transitive group action of G given left multiplication. Within an orbit the coefficients of any channel in the process mode decomposition are complex functions on this homogeneous space. Thus for each process mode in a complete orthonormal basis of irreducible tensor superoperators $\{\Phi_k^\lambda\}_{\lambda,k}$ for $\mathcal{S}(\mathcal{H}, \mathcal{K})$ we denote:

$$\alpha_{\lambda,k}(gH) := \langle \Phi_k^\lambda, \mathfrak{U}_g[\mathcal{E}] \rangle. \quad (2.48)$$

These are well defined functions $\alpha_{\lambda,k} : G/H \rightarrow \mathbb{C}$ because if $gH = g'H$ then $g^{-1}g' \in H$ and thus $\mathfrak{U}_g[\mathcal{E}] = \mathfrak{U}_{g'}[\mathcal{E}]$ so $\alpha_{\lambda,k}(gH) = \alpha_{\lambda,k}(g'H)$ for all ireps λ with vector component

k . Moreover we have that $\alpha_{\lambda,k} \in \mathcal{L}^2(G/H)$. This follows from the following argument $\sum_{\lambda,k} \|\alpha_{\lambda,k}\|_{L^2}^2 = \sum_{\lambda,k} \int_{G/H} \alpha_{\lambda,k}(gH) \langle \Phi_k^\lambda, \mathfrak{U}_g[\mathcal{E}] \rangle^* d gH = \int_{G/H} \langle \mathfrak{U}_g[\mathcal{E}], \sum_{\lambda,k} \alpha_{\lambda,k}(gH) \Phi_k^\lambda \rangle d gH$ and therefore $\sum_{\lambda,k} \|\alpha_{\lambda,k}\|_{L^2}^2 = \int_{G/H} \langle \mathfrak{U}_g[\mathcal{E}], \mathfrak{U}_g[\mathcal{E}] \rangle d \mu(gH) = \int_{G/H} \text{Tr}(J[\mathfrak{U}_g[\mathcal{E}]]) d \mu(gH)$ so that the RHS is $\leq d^2 \int_{G/H} d \mu(gH)$, where in the last inequality we use the fact that $\text{Tr}(J[\mathcal{E}]) = d$ for all \mathcal{E} and d the dimension of the input system \mathcal{H} . Thus $\sum_{\lambda,k} \|\alpha_{\lambda,k}\|_{L^2}^2 \leq d^2$ and so $\alpha_{\lambda,k}$ are square integrable complex functions on G/H . The integration is with respect to the induced Haar measure on the homogeneous space G/H , for more details see Chapter 1.

Theorem 2.3.5. *Let G be a compact group with unitary representations U and U' on the Hilbert spaces \mathcal{H} and \mathcal{K} . Consider the quantum channel \mathcal{E} in $\mathcal{S}(\mathcal{H}, \mathcal{K})$ with stabilizer H , a closed subgroup of G . Then the following hold:*

- i) *The functions $\alpha_{\lambda,k} \in \mathcal{L}^2(G/H)$ transform as a λ^* -irreducible representation under the action of G .*
- ii) *If H is a massive⁶ subgroup, then there is a basis of irreducible tensor superoperators such that:*

$$\mathfrak{U}_g[\mathcal{E}] = \sum_{\lambda,k} a_\lambda \mathbf{Y}^{\lambda^*}(gH) \cdot \Phi^\lambda \quad (2.49)$$

where $\mathbf{Y}^{\lambda^*}(\cdot)$ is the vector of associated spherical harmonics $Y_k^{\lambda^*} \in \mathcal{L}^2(G/H)$ for the λ^* -irrep and a_λ are complex coefficients independent on the vector component label k .

Proof. i) We have shown that the coefficients $\alpha_{\lambda,k}$ are square integrable functions on the homogeneous space G/H , where $\alpha_{\lambda,k}(gH) = \langle \Phi_k^\lambda, \mathfrak{U}_g[\mathcal{E}] \rangle$ for $\{\Phi_k^\lambda\}$ an orthonormal basis of irreducible tensor superoperators. There is a transitive left action of G on $\mathcal{L}^2(G/H)$ such that for any $f \in \mathcal{L}^2(G/H)$ and any $g \in G$:

$$g \cdot f(\mathbf{x}) = f(g^{-1} \cdot \mathbf{x}) \quad (2.50)$$

where by \mathbf{x} we denote a point on the homogeneous space G/H (\mathbf{x} can be thought to represent a particular coset in G/H). We show that under this group action, the coefficients transform irreducibly.

Under the left regular action on $\mathcal{L}^2(G/H)$, for any $g, g_0 \in G$ we have $g \cdot \alpha_{\lambda,k}(g_0H) = \alpha_{\lambda,k}(g^{-1}g_0H)$.

⁶This is a technical assumption – see 1.5.12 for more details on massive subgroups and their relation to principal orbits. It allows for spherical harmonics to be defined and they will span the multiplicity free irreducible subspaces in the decomposition of $\mathcal{L}^2(G/H)$. In particular one can think of H for instance as a maximal torus in G since it will always satisfy the requirement. In principle it holds for more general subgroups H as long as the homogenous space G/H is a symmetric space.

However, directly from the definition of such coefficients the RHS is:

$$\alpha_{\lambda,k}(g^{-1}g_0H) = \langle \Phi_k^\lambda, \mathfrak{U}_{g^{-1}g_0}[\mathcal{E}] \rangle = \langle \mathfrak{U}_g[\Phi_k^\lambda], \mathfrak{U}_{g_0}[\mathcal{E}] \rangle. \quad (2.51)$$

The irreducible tensor superoperators transform as $\mathfrak{U}_g[\Phi_k^\lambda] = \sum_{k'} v_{k'k}^\lambda(g) \Phi_{k'}^\lambda$ and therefore substituting into the above equations we obtain that:

$$g \cdot \alpha_{\lambda,k}(g_0H) = \langle \sum_{k'} v_{k'k}^\lambda(g) \Phi_{k'}^\lambda, \mathfrak{U}_{g_0}[\mathcal{E}] \rangle \quad (2.52)$$

$$= \sum_{k'} (v_{k'k}^\lambda(g))^* \langle \Phi_{k'}^\lambda, \mathfrak{U}_{g_0}[\mathcal{E}] \rangle. \quad (2.53)$$

However for any λ -irrep $(v_{k'k}^\lambda(g))^* = v_{k'k}^{\lambda^*}(g)$. Therefore we have that the coefficients in the process mode decomposition transform as λ^* irreducible representations:

$$g \cdot \alpha_{\lambda,k}(g_0H) = \sum_{\lambda,k'} v_{k'k}^{\lambda^*}(g) \alpha_{\lambda,k}(g_0H). \quad (2.54)$$

ii) Because H is a massive group, the space $\mathcal{L}^2(G/H)$ decomposes under the left regular representation into irreducible subspaces all appearing with multiplicity one. Unlike the decomposition of $\mathcal{L}^2(G)$ not all irreps of G appear in the decomposition of $\mathcal{L}^2(G/H)$. However each such λ irreducible subspace has a basis of associated spherical harmonic functions $Y_{\lambda,k} \in \mathcal{L}^2(G/H)$ with the vector label component ranging from $k = 1, \dots, \dim(\lambda)$. The vector of spherical harmonics \mathbf{Y}^λ with components $Y_{\lambda,k}$ transforms like the λ -irrep and therefore up to a change of basis $\alpha_\lambda = a_\lambda \mathbf{Y}^{\lambda^*}$ for some complex coefficient a_λ that does not depend on the vector component. \square

2.4 Symmetric quantum channels on bipartite systems

2.4.1 Structure theorem for symmetric channels

In this section we consider quantum channels on bipartite systems AB , $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ where $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{K} = \mathcal{K}_A \otimes \mathcal{K}_B$ carry the tensor product representation $U_A \otimes U_B$ respectively $U'_A \otimes U'_B$ of a finite or compact group G . We say that \mathcal{E} is *globally symmetric* if it is invariant under the group action $\mathcal{E} = \mathfrak{U}_g[\mathcal{E}] = (U'_A(g) \otimes U'_B(g)) \circ \mathcal{E} \circ (U_A(g)^\dagger \otimes U_B(g)^\dagger)$ for all $g \in G$. The following theorem reveals the structure of such an operator with respect to a

bipartite splitting of the input and output system into A and B . The local process modes for system A and B must be coupled in particular ways to give rise to a globally symmetric channel.

Theorem 2.4.1. *Every symmetric quantum channel $\mathcal{E} : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \longrightarrow \mathcal{B}(\mathcal{K}_A \otimes \mathcal{K}_B)$ decomposes into symmetric superoperators $\chi^{\lambda_{AB}} := \sum_{k=1}^{\dim(\lambda)} \Phi_k^{\lambda_A} \otimes \Phi_k^{\lambda_B^*}$ such that:*

$$\mathcal{E} = \sum_{\lambda_{AB}} c_{\lambda_{AB}} \chi^{\lambda_{AB}} \quad (2.55)$$

where $c_{\lambda_{AB}} \in \mathbb{C}$ and $\lambda_A = (\lambda, m_A)$ and $\lambda_B^* = (\lambda^*, m_B)$, for any choice of multiplicity labels m_A, m_B , and where $\{\Phi_k^{\lambda_A}\}$ (respectively $\{\Phi_k^{\lambda_B^*}\}$) is any complete set of irreducible tensor superoperators for $\mathcal{S}(\mathcal{H}_A, \mathcal{K}_{A'})$ (respectively $\mathcal{S}(\mathcal{H}_B, \mathcal{K}_{B'})$). The summation ranges over all irreps $\lambda \in \text{Irrep}(A, A')$ for which there is $\lambda^* \in \text{Irrep}(B, B')$ and their associated multiplicities m_A and m_B are labelled collectively by $\lambda_{AB} = (\lambda, m_A, m_B)$.

Proof. First step, we show that for any set of local irreducible tensor superoperators $\{\Phi_k^{\lambda_A}\}_{k, \lambda_A} \in \mathcal{S}(\mathcal{H}_A, \mathcal{K}_A)$ on system A and $\{\Phi_k^{\lambda_B}\}_{k, \lambda_B} \in \mathcal{S}(\mathcal{H}_B, \mathcal{K}_B)$ on system B the superoperators $\chi^{\lambda_{AB}} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ are symmetric. Let us denote for any $\mathcal{E}_A \in \mathcal{S}(\mathcal{H}_A, \mathcal{K}_A)$ the group action $\mathfrak{U}_g^A(\mathcal{E}_A) := \mathcal{U}'_A(g) \circ \mathcal{E}_A \circ \mathcal{U}_{A'}^\dagger$, and similarly for system B . Then:

$$\mathfrak{U}_g[\chi^{\lambda_{AB}}] = \sum_{k=1}^{\dim(\lambda)} \mathfrak{U}_g^A[\Phi_k^{\lambda_A}] \otimes \mathfrak{U}_g^B[\Phi_k^{\lambda_B^*}] \quad (2.56)$$

$$= \sum_{k, m, n} v_{mk}^\lambda(g) v_{nk}^{\lambda^*}(g) \Phi_m^{\lambda_A} \otimes \Phi_n^{\lambda_B} \quad (2.57)$$

where we have used that the Φ^{λ_A} on system A and $\Phi^{\lambda_B^*}$ on system B transform as λ and λ^* -irreps respectively. However $\sum_k v_{mk}^\lambda(g) v_{nk}^{\lambda^*}(g) = \sum_k v_{mk}^\lambda(g) (v_{nk}^\lambda)^*(g) = \sum_k v_{mk}^\lambda(g) v_{kn}^\lambda(g^{-1}) = (v^\lambda(e))_{m,n} = \delta_{nm}$, in which we have used that $v^{\lambda^*}(g) = v^\lambda(g^{-1})^T$ and thus it follows for all $g \in G$ that:

$$\mathfrak{U}_g[\chi^{\lambda_{AB}}] = \chi^{\lambda_{AB}} \quad (2.58)$$

is indeed a symmetric superoperator in $\mathcal{S}(\mathcal{H}, \mathcal{K})$.

Second, we must show that the invariant subspace of $\mathcal{S}(\mathcal{H}, \mathcal{K})$ is fully spanned by $\chi^{\lambda_{AB}}$ as we range over all labels λ_{AB} . We use the following core result in representation theory: Let V_A and V_B be a λ_A -irrep respectively λ_B -irrep, then there is at most one trivial 0-irrep in the decomposition of the tensor product $V_A \otimes V_B$ which appears if and only if $V_A \cong V_B^*$ as

representations. The proof of this result is immediate by looking at the characters.

Moreover as representation spaces we have that $\mathcal{S}(\mathcal{H}, \mathcal{K}) \cong \mathcal{S}(\mathcal{H}_A, \mathcal{K}_A) \otimes \mathcal{S}(\mathcal{H}_B, \mathcal{K}_B)$. The irreps in the decomposition of $\mathcal{S}(\mathcal{H}, \mathcal{K})$ are those that appear in $U \otimes U^* \otimes U'^* \otimes U'$ where $U = U_A \otimes U_B$ and $U' = U'_A \otimes U'_B$. We write $\mathcal{S}(\mathcal{H}_A, \mathcal{K}_A) \cong \bigoplus_{\lambda_A} V_{\lambda_A}$ and $\mathcal{S}(\mathcal{H}_B, \mathcal{K}_B) \cong \bigoplus_{\mu_B} V_{\mu_B}$ for the decomposition into irreducible subspaces of superoperators on A and B respectively, where $\lambda_A = (\lambda, m_A)$ and $\mu_B = (\mu, m_B)$ label both the irrep and its corresponding multiplicity. We can write:

$$\mathcal{S}(\mathcal{H}, \mathcal{K}) \cong \bigoplus_{\lambda_A, \mu_B} V_{\lambda_A} \otimes V_{\mu_B}. \quad (2.59)$$

Any symmetric \mathcal{E} lies fully within the trivial subspace of $\mathcal{S}(\mathcal{H}, \mathcal{K})$. Each subspace $V_{\lambda_A} \otimes V_{\mu_B}$ in the above orthogonal decomposition contains at most one trivial subspace if and only if $\mu \cong \lambda^*$. Then each such trivial subspace is uniquely characterised by a label $\lambda_{AB} = (\lambda_A, \lambda_B^*)$ with $\mu_B = \lambda_B^* = (\lambda^*, m_B)$ corresponding to the coupling $V_{\lambda_A} \otimes V_{\lambda_B^*}$ that has a unique 0-irrep spanned by $\chi^{\lambda_{AB}}$. Moreover orthogonality of the local irreducible tensor superoperators implies that $\langle \chi^{\lambda_{AB}}, \chi^{\tilde{\lambda}_{AB}} \rangle \propto \delta_{\lambda, \tilde{\lambda}} \delta_{m_A, \tilde{m}_B} \delta_{m_B, \tilde{m}_A}$ so each symmetric superoperator lies in orthogonal subspaces. Therefore $\text{span}\{\chi^{\lambda_{AB}} : \forall \lambda_{AB}\} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ is exactly the invariant subspace of the superoperator space. \square

What are the assumptions that come into play in the above structure theorem for symmetric channels on bipartite systems? Once the local set of irreducible tensor superoperators on system A and B are fixed, Theorem 2.4.1 gives a complete orthogonal basis for all symmetric superoperators in $\mathcal{S}(\mathcal{H}, \mathcal{K})$. However, a priori there is no assumption that requires to choose a *specific* set of local ITS. The structure theorem we give only *assumes that there is a tensor product structure* arising from a bipartition of the input and output Hilbert spaces \mathcal{H} and \mathcal{K} into system AB . There is freedom in both the choice of bipartite split and the local basis of ITS for A and B .

2.4.2 Diagrammatic representation of symmetric channels on bipartite systems

We have seen in Section 2.1.2 how to construct process modes from coupling irreducible tensor operators for the input and output Hilbert spaces and how to represent them diagrammatically. The structure of symmetric channels on bipartite systems allows to build a similar diagrammatic representation for the basis of symmetric superoperators $\chi^{\lambda_{AB}}$ in Theorem 2.4.1. This relies on a particular choice of tensor product structure in the space of superoperators $\mathcal{S}(\mathcal{H}, \mathcal{K})$ on system

AB with $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{K} = \mathcal{K}_A \otimes \mathcal{K}_B$ having representations $U = U_A \otimes U_B$ respectively $U' = U'_A \otimes U'_B$. As representation spaces we have the series of isomorphisms:

$$\mathcal{S}(\mathcal{H}, \mathcal{K}) \cong \mathcal{S}(\mathcal{H}_A, \mathcal{K}_A) \otimes \mathcal{S}(\mathcal{H}_B, \mathcal{K}_B) \cong \mathcal{B}(\mathcal{K}_A) \otimes \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{K}_B) \otimes \mathcal{B}(\mathcal{H}_B) \quad (2.60)$$

This allows to classify a basis for the symmetric superoperators in $\mathcal{S}(\mathcal{H}, \mathcal{K})$ in terms of the coupling of irreps between the input and output across the bipartite split into system A and B . Each element in this basis is characterised by a diagram θ as in Figure 2.5 that corresponds to coupling the process modes corresponding to the diagram $\lambda_A = (\mathbf{a}, \tilde{\mathbf{a}}) \xrightarrow{\lambda}$ at A and process modes given by $\lambda_B^* = (\mathbf{b}, \tilde{\mathbf{b}}) \xrightarrow{\lambda^*}$ on system B . This data is packaged together in the θ label, which is descriptively given by $\theta = (\mathbf{a}, \tilde{\mathbf{a}}) \xrightarrow{\lambda} (\mathbf{b}, \tilde{\mathbf{b}})$.

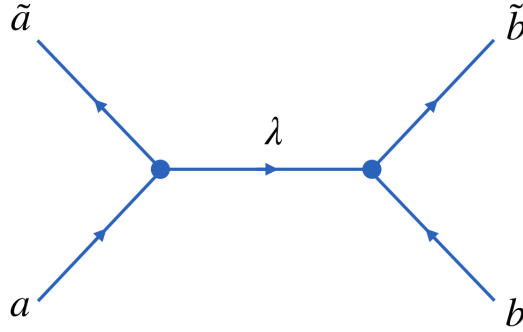


Figure 2.5: Diagrammatic representation of symmetric superoperators.

Each such diagram denoted generically by $\theta := (\mathbf{a}, \tilde{\mathbf{a}}) \xrightarrow{\lambda} (\mathbf{b}, \tilde{\mathbf{b}})$ corresponds to a symmetric superoperator $\Phi_\theta = \sum_\lambda \Phi_k^{\lambda_A} \otimes \Phi_k^{\lambda_B^*}$. The directionality on the horizontal leg is chosen such that the dual representation acts at the incoming vertex.

The symmetric superoperator Φ_θ corresponding to diagram θ is:

$$\Phi_\theta(\rho) = \sum_{k,m,m',n,n'} \langle \tilde{\mathbf{a}}, m; \mathbf{a}, n | \lambda, k \rangle \langle \tilde{\mathbf{b}}, m'; \mathbf{b}, n | \lambda^*, k \rangle \tilde{T}_m^{\tilde{\mathbf{a}}} \otimes \tilde{S}_{m'}^{\tilde{\mathbf{b}}} \text{Tr}(T_n^{\mathbf{a}} \otimes S_{n'}^{\mathbf{b}} \rho) \quad (2.61)$$

where $\{T_m^{\mathbf{a}}\}_{\mathbf{a},m}$ and $\{\tilde{T}_{m'}^{\tilde{\mathbf{a}}}\}_{\tilde{\mathbf{a}},m'}$ are complete sets of irreducible tensor operators for the input space $\mathcal{B}(\mathcal{H}_A)$ and output space $\mathcal{B}(\mathcal{K}_A)$ on system A and similarly $\{S_m^{\mathbf{a}}\}_{\mathbf{a},m}$ and $\{\tilde{S}_{m'}^{\tilde{\mathbf{a}}}\}_{\tilde{\mathbf{a}},m'}$ for $\mathcal{B}(\mathcal{H}_B)$ respectively $\mathcal{B}(\mathcal{K}_B)$ on system B . Every set of ITOs will contain at least one operator invariant under the group action. Therefore there is always a choice of basis which distinguishes the identity $\frac{\mathbb{I}}{\sqrt{d}}$. To refer to this element in a set of orthonormal ITOs we denote the corresponding

label $\mathbf{a} = 0$ and reserve the notation $\mathbf{a} \cong 0$ for the operators that transform trivially under the group action, but are orthogonal to the identity (i.e they are trace free symmetric operators).

One can classify the set of all possible diagrams for a given physical system and symmetry in three broad classes:

- Local diagrams for which $\lambda \cong 0$
- Asymmetry injection diagrams for which one of the output modes $\tilde{\mathbf{a}} = 0$ or $\tilde{\mathbf{b}} = 0$
- Relational diagrams.

Local diagrams – these correspond to symmetric superoperators where the local process modes are themselves symmetric. They take the general form $\Phi^{0,A} \otimes \Phi^{0,B}$ where for all $g \in G$ $\mathcal{U}_g[\Phi^{0,A}] = \Phi^{0,A}$ and $\mathcal{U}_g[\Phi^{0,B}] = \Phi^{0,B}$. There is no asymmetry flowing between subsystems A and B .

Asymmetry injection diagrams that have the form $\theta_{\text{inject}} = (\mathbf{a}, 0) \xrightarrow{\lambda} (\mathbf{b}, \tilde{\mathbf{b}})$ correspond exactly to symmetric superoperators $\Phi_{\theta_{\text{inject}}}$ for which tracing out system A gives a non-trivial action on system B . Similarly if $\tilde{\mathbf{b}} = 0$ then such diagrams will give non-zero local operation at A upon tracing out the system B . More generally, only quantum channels that contain at least an injection diagram in their decomposition can give a non-trivial local simulation on one of the subsystems.

Relational diagrams will give no contribution to local simulation, but still allow flow of asymmetry between the subsystems A and B as these diagrams will have $\lambda \not\cong 0$.

2.4.3 Example: symmetric channels on two qubits under $SU(2)$ symmetry

In this section we characterise the symmetric channels on two qubit systems under $SU(2)$ symmetry where each qubit system carries the fundamental representation of highest weight $1/2$. With a slight abuse of notation, whereas the irrep label denotes the subspace transforming under the said irrep then $\mathcal{H}_A \cong \mathcal{K}_A \cong \mathcal{H}_B \cong \mathcal{K}_B \cong 1/2$ and:

$$\mathcal{S}(\mathcal{H}_B, \mathcal{K}_B) \cong \mathcal{S}(\mathcal{H}_A, \mathcal{K}_A) \cong 0 \oplus 0 \oplus 1 \oplus 1 \oplus 1 \oplus 2 \quad (2.62)$$

For $SU(2)$ every irrep is isomorphic to its dual, and therefore the number of symmetric superoperators will be the sum of the squares of the multiplicity of each irrep appearing in the

above decomposition. In total we have a basis consisting of $2^2 + 3^2 + 1 = 14$ for the space of symmetric superoperators on $\mathcal{S}(1/2 \otimes 1/2, 1/2 \otimes 1/2)$. For quantum channels however, the trace preserving condition implies the diagram $(1, 0) \xrightarrow{1} (1, 0)$ cannot appear. Therefore in terms of the diagrammatic decomposition, any quantum channel can be decomposed into at most 13 different types of symmetric superoperators Φ_θ corresponding to the diagrams in Figure 2.6. These are classified in terms of the directionality of symmetry breaking degrees of freedom exchanged between A and B .

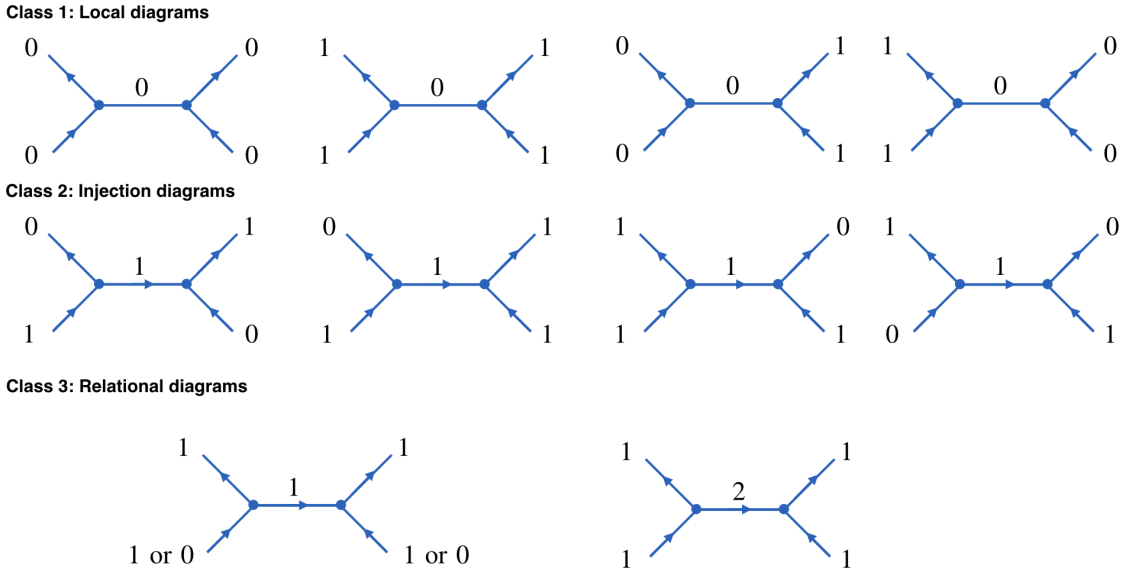


Figure 2.6: The set of 2-qubit $SU(2)$ -diagrams.

All possible diagrammatic terms allowed for a 2-qubit symmetric quantum channel. The space of valid channels is 13-dimensional.

Local channels on two qubits

Channels \mathcal{E} that contain only diagrams for which $\lambda \cong 0$. \mathcal{E} are also constructed out of locally symmetric terms. This leads to a two-parameter family of channels given by the product of partial depolarising channels $\mathcal{E} = \mathcal{D}_{p_A} \otimes \mathcal{D}_{p_B}$, where $p_A, p_B \in [-\frac{1}{3}, 1]$ and

$$\mathcal{D}_p(\rho) = p\rho + \frac{1}{4}(1-p)\mathbb{I}. \quad (2.63)$$

These channels involve only two process modes locally, $id(\rho) = \rho$ and $\Phi^0(\rho) = \sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z$.

Asymmetry injection channels on AB and simulation of channels at A .

In this section we take a closer look at quantum channels on qubits that are built only from diagrams in the first and second class. These are the only terms that contribute to any protocol which aims to simulate or induce a target quantum channel local at A (or B) using asymmetry resources at in system B (or A).

The general structure of these channels is:

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_{in,A} + \mathcal{E}_{in,B}, \quad (2.64)$$

where $\mathcal{E}_0(\rho) = \frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}$ and $\mathcal{E}_{in,A}$, $\mathcal{E}_{in,B}$ contain class-1 and class-2 diagrams such that for any ρ we have $\text{Tr}_A[\mathcal{E}_{in,A}(\rho)] = \text{Tr}_B[\mathcal{E}_{in,B}(\rho)] = 0$, and $\text{Tr}_A[\mathcal{E}_{in,B}(\rho)] \neq 0$, $\text{Tr}_B[\mathcal{E}_{in,A}(\rho)] \neq 0$ so these components describe the injection of asymmetry into A and B respectively. In this we also include diagrams that maintain the local Bloch vectors (which are class-1 diagrams) since we are interested in inducing local channels.

For a qubit under $SU(2)$ symmetry, each of these two terms form a 3-parameter family of maps given by

$$\begin{aligned} \mathcal{E}_{in,A} &= x\Phi_{\theta_1} + y\Phi_{\theta_2} + z\Phi_{\theta_3} \\ \mathcal{E}_{in,B} &= x'\Phi_{\theta'_1} + y'\Phi_{\theta'_2} + z'\Phi_{\theta'_3} \end{aligned} \quad (2.65)$$

for coefficients x, y, z, x', y', z' chosen such that \mathcal{E} is a valid quantum channel.

The diagrams involved are given by

$$\begin{aligned} \theta_1 &= [(1, 1) \xrightarrow{0} (0, 0)] \\ \theta_2 &= [(0, 1) \xrightarrow{1} (1, 0)] \\ \theta_3 &= [(1, 1) \xrightarrow{1} (1, 0)] \end{aligned} \quad (2.66)$$

for simulation at A , and by

$$\begin{aligned}\theta'_1 &= [(0, 0) \xrightarrow{0} (1, 1)] \\ \theta'_2 &= [(1, 0) \xrightarrow{1} (0, 1)] \\ \theta'_3 &= [(1, 0) \xrightarrow{1} (1, 1)]\end{aligned}\tag{2.67}$$

for simulation at B .

More generally these channels can be described by their action on a generic two qubit state. Any two qubit density matrix ρ_{AB} takes the canonical form:

$$\rho_{AB} = \frac{1}{4} \left(\mathbb{I} \otimes \mathbb{I} + \mathbf{a} \cdot \boldsymbol{\sigma} \otimes \mathbb{I} + \mathbb{I} \otimes \mathbf{b} \cdot \boldsymbol{\sigma} + \sum_{i,j} T_{ij} \sigma_i \otimes \sigma_j \right)\tag{2.68}$$

where local Bloch vectors \mathbf{a} and \mathbf{b} together with the correlation matrix T_{ij} are chosen such that ρ_{AB} is a positive matrix with trace 1. Given the above parameterisation of the general channel \mathcal{E} , we have that

$$\mathcal{E}(\rho_{AB}) = \frac{1}{4} \left(\mathbb{I} \otimes \mathbb{I} + \tilde{\mathbf{a}} \cdot \boldsymbol{\sigma} \otimes \mathbb{I} + \mathbb{I} \otimes \tilde{\mathbf{b}} \cdot \boldsymbol{\sigma} \right),\tag{2.69}$$

where the local Bloch vectors of the output states are given by

$$\tilde{\mathbf{a}} = - \left(\frac{x\mathbf{a}}{\sqrt{3}} + y\mathbf{b} + \frac{z\mathbf{T}}{\sqrt{2}} \right)\tag{2.70}$$

$$\tilde{\mathbf{b}} = - \left(\frac{x'\mathbf{b}}{\sqrt{3}} + y'\mathbf{a} + \frac{z'\mathbf{T}}{\sqrt{2}} \right).\tag{2.71}$$

Here \mathbf{T} denotes the vector with components $\mathbf{T}_k := \sum_{i,j} \epsilon_{kij} T_{ij}$. The geometric significance of this can be seen for the case of initial product states $\rho_{AB} = \rho_A \otimes \rho_B$ with local Bloch vectors \mathbf{a} and \mathbf{b} . In such case the correlation matrix takes the form of $T_{ij} = a_i b_j$ and therefore since $\epsilon_{kij} a_i b_j = (\mathbf{a} \times \mathbf{b})_k$ and so the vector $\mathbf{T} = \mathbf{a} \times \mathbf{b}$ is cross-product between the input Bloch vectors at A and B . More generally \mathbf{T} is a vector component that describes the joint asymmetry of A and B , in contrast to \mathbf{a} and \mathbf{b} , which are purely local terms.

We now restrict to the set of channels that yield a non-trivial simulation at A . Recall that this consists of the class-1 and class 2 diagrams:

$$\mathcal{E} = \Phi_0 + x\Phi_{\theta_1} + y\Phi_{\theta_2} + z\Phi_{\theta_3}\tag{2.72}$$

or equivalently:

$$\mathcal{E} = \begin{array}{c} 0 \\ \diagdown \\ \bullet \\ \diagup \\ 0 \end{array} \begin{array}{c} 0 \\ \diagdown \\ \bullet \\ \diagup \\ 0 \end{array} + x \begin{array}{c} 1 \\ \diagdown \\ \bullet \\ \diagup \\ 1 \end{array} \begin{array}{c} 0 \\ \diagdown \\ \bullet \\ \diagup \\ 0 \end{array} + y \begin{array}{c} 1 \\ \diagdown \\ \bullet \\ \diagup \\ 0 \end{array} \begin{array}{c} 0 \\ \diagdown \\ \bullet \\ \diagup \\ 1 \end{array} + z \begin{array}{c} 1 \\ \diagdown \\ \bullet \\ \diagup \\ 1 \end{array} \begin{array}{c} 0 \\ \diagdown \\ \bullet \\ \diagup \\ 1 \end{array} .$$

This is a 3-parameter family of quantum channels characterised by (x, y, z) and so we can plot the allowed region in 3-D. Imposing CPTP conditions for \mathcal{E} leads to a convex region of allowed values for (x, y, z) in \mathbb{R}^3 . The set of all such valid quantum channels is given by the convex set bounded by the paraboloid

$$x^2 + x \left(\frac{2 - 6y}{\sqrt{3}} \right) + 2y + 3y^2 + 6z^2 = 1 \quad (2.73)$$

and the plane $\sqrt{3}(1 + y) + x = 0$.

We can make a change of coordinates for which the quartic boundary (paraboloid) reduces to one of the 17 standard forms. Let

$$\begin{aligned} X &= (1 + \sqrt{3}x - 3y)/2 \\ Y &= 1 - 3y \\ Z &= 3z/\sqrt{2}. \end{aligned} \quad (2.74)$$

Then the region of parameters (X, Y, Z) is given by the three dimensional convex set bounded by an elliptic paraboloid described by the equation,

$$X^2 + Z^2 = Y, \quad (2.75)$$

and the plane $2 + X - Y = 0$. In Figure 2.7 we show this parameter region while highlighting the points corresponding to distinguished extremal channels. In particular, we find that the vertex of the paraboloid at $(X, Y, Z) = (0, 0, 0)$ corresponds to the quantum channel

$$\mathcal{E}_{\text{U-NOT}}(\rho) = \frac{1}{4} \left(\mathbb{I} - \frac{1}{3} \mathbf{b} \cdot \boldsymbol{\sigma} \otimes \mathbb{I} \right) \quad (2.76)$$

which is the result of discarding the input at A , performing an approximate Universal NOT gate on system B (mapping local Bloch vectors $\mathbf{b} \rightarrow -\frac{1}{3}\mathbf{b}$), and then injecting this into the

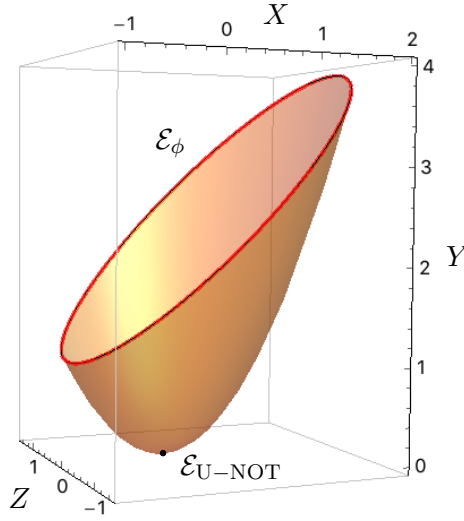


Figure 2.7: The set of induced channels on A via a globally symmetric channel on AB . Shown is the allowed parameter region for the rotated coefficients corresponding to each diagram appearing in a general quantum channel that injects asymmetry. The boundary is described by the intersection of a plane and an elliptic paraboloid.

output system at A . This approximate U-NOT is the optimal “spin inversion” that is allowed by quantum mechanics [77].

The intersection of the two boundary regions is an ellipse that we parametrise by a single coordinate ϕ . The coordinates for this parametrisation are $Z(\phi) = \frac{3}{2} \cos \phi$, $Y(\phi) = \frac{3}{2} \sin \phi + \frac{5}{2}$ and $X(\phi) = \frac{3}{2} \sin \phi + \frac{1}{2}$ resulting in the corresponding one-parameter family of channels with:

$$\mathcal{E}_{\phi}(\rho) = \frac{1}{4} (\mathbb{I} \otimes \mathbb{I} + \tilde{\mathbf{a}} \cdot \boldsymbol{\sigma} \otimes \mathbb{I}), \quad (2.77)$$

where

$$\tilde{\mathbf{a}} = \frac{1}{2} ((\mathbf{a} + \mathbf{b}) + (\mathbf{b} - \mathbf{a}) \sin \phi + \mathbf{T} \cos \phi). \quad (2.78)$$

If the input state ρ_{AB} is a tensor product state with local Bloch vectors \mathbf{a} and \mathbf{b} at A respectively B , then under \mathcal{E}_{ϕ} we get output state $\rho'_A(\phi) \otimes \frac{\mathbb{I}}{2}$ where ρ'_A has Bloch vector $\tilde{\mathbf{a}} = \frac{1}{2} ((\mathbf{a} + \mathbf{b}) + (\mathbf{b} - \mathbf{a}) \sin \phi + \mathbf{a} \times \mathbf{b} \cos \phi)$. This corresponds geometrically to an ellipse parametrised by ϕ with orientation and size determined by $\frac{\mathbf{b}-\mathbf{a}}{2}$ and $\mathbf{a} \times \mathbf{b}$ that is displaced by vector $\frac{\mathbf{a}+\mathbf{b}}{2}$.

Note that for $\phi = \frac{\pi}{2}$ we have that the output on A is the input state on B . Therefore the line joining $\mathcal{E}_{\text{UNOT}}$ and $\mathcal{E}_{\phi=\frac{\pi}{2}}$ is the set of general depolarization channels on B where the output is sent to the output on system A .

Finally, to a good approximation, the set of all channels using a qubit at B to induce a non-symmetric channel on A is given by the convex hull of the optimal U-NOT gate and the set of channels \mathcal{E}_ϕ with $0 \leq \phi \leq 2\pi$.

Purely relational channels

In this section we consider quantum channels which contain only class-3 diagrams (and the trivial diagram $(0, 0) \xrightarrow{0} (0, 0)$ which outputs the identity operator and is needed for satisfying the completely positive and trace preserving condition). For a qubit with SU(2) symmetry there are 5 diagrams in total within class-3 and therefore the most general quantum channels that involve only these type of diagrams take the form of:

$$\mathcal{E} = \Phi_0 + x_4\Phi_{\theta_4} + x_5\Phi_{\theta_5} + x_6\Phi_{\theta_6} + x_7\Phi_{\theta_7} + x_8\Phi_{\theta_8} \quad (2.79)$$

where

$$\begin{aligned} \theta_4 &= [(0, 1) \xrightarrow{1} (0, 1)] & \theta_7 &= [(0, 1) \xrightarrow{1} (1, 1)] \\ \theta_5 &= [(1, 1) \xrightarrow{1} (1, 1)] & \theta_8 &= [(1, 1) \xrightarrow{2} (1, 1)]. \\ \theta_6 &= [(1, 1) \xrightarrow{1} (0, 1)] \end{aligned} \quad (2.80)$$

Any such quantum channel has the property that the output state $\mathcal{E}(\rho)$ always has maximally mixed marginals for all initial states. More precisely, for any $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ we have that $\text{Tr}_A(\mathcal{E}(\rho)) = \text{Tr}_B(\mathcal{E}(\rho)) = \frac{1}{2}\mathbb{I}$. This implies that we must have

$$\mathcal{E}(\rho_{AB}) = \frac{1}{4} \left(\mathbb{I} \otimes \mathbb{I} + \sum_{i,j} R_{ij} \sigma_i \otimes \sigma_j \right) \quad (2.81)$$

for some correlation matrix R_{ij} that depends on both ρ_{AB} and the particular relational channel. Given that the input state ρ_{AB} takes the general form $\rho_{AB} = \frac{1}{4}(\mathbb{I} \otimes \mathbb{I} + \mathbf{a} \cdot \boldsymbol{\sigma} \otimes \mathbb{I} + \mathbb{I} \otimes \mathbf{b} \cdot \boldsymbol{\sigma} + \sum_{i,j} T_{ij} \sigma_i \otimes \sigma_j)$ then we can precisely identify the contribution of each class-3 diagram to the

output state determined by the tensor R_{ij} . We have the following tensors:

$$\begin{aligned}
\theta_4 : R^{\theta_4} &= -\frac{\mathbb{I}}{4} \\
\theta_5 : R^{\theta_5} &= \frac{1}{8}(T^T + \text{Tr}(T)\mathbb{I}) \\
\theta_6 : R_{ij}^{\theta_6} &= i\frac{\sqrt{2}}{2}(\epsilon_{ijk}a_k) \\
\theta_7 : R_{ij}^{\theta_7} &= i\frac{\sqrt{2}}{2}(-\epsilon_{ijk}b_k) \\
\theta_8 : R^{\theta_8} &= \frac{1}{8}(T^T - \frac{2}{3}T + \text{Tr}(T)\mathbb{I}),
\end{aligned} \tag{2.82}$$

where we have denoted by R^{θ_m} the correlation matrix of the output under applying the super-operator Φ_{θ_m} to ρ_{AB} . In other words, $\Phi_{\theta_m}(\rho_{AB}) = \sum_{i,j} R_{ij}^{\theta_m} \sigma_i \otimes \sigma_j$.

To explore such channels, we restrict for simplicity to those diagrams that are invariant under swapping A and B . For this, the most general form we analyse in the following is:

$$\mathcal{E} = \Phi_0 + x\Phi_{\theta_4} + y\Phi_{\theta_5} + z\Phi_{\theta_8} \tag{2.83}$$

or equivalently

$$\mathcal{E} = \text{Diagram 1} + x \text{Diagram 2} + y \text{Diagram 3} + z \text{Diagram 4}$$

for real parameters x, y, z . Imposing that \mathcal{E} is a valid quantum channel restricts to a convex region of allowed parameters $(x, y, z) \in \mathbb{R}^3$. This region is characterised via its boundary surfaces that are given by the following quartics:

$$\begin{aligned}
(9x + 3y + 5z - 3)^2 &= (5z + 21y - 12)^2 - 108(1 - 2y)^2 \\
(6y + 3x)^2 &= 6x + 3 + 20z \\
y^2 &= \left(\frac{1-x}{2}\right)^2 \text{ for } 0 < x \leq 1 \\
y^2 &= \frac{(x + 5/3)^2}{4} - \frac{4}{9} \text{ for } -1/3 \leq x \leq 0.
\end{aligned}$$

In other words the boundary is the intersection of an elliptic cone, a parabolic cylinder, two intersecting planes and a hyperbolic cylinder respectively.

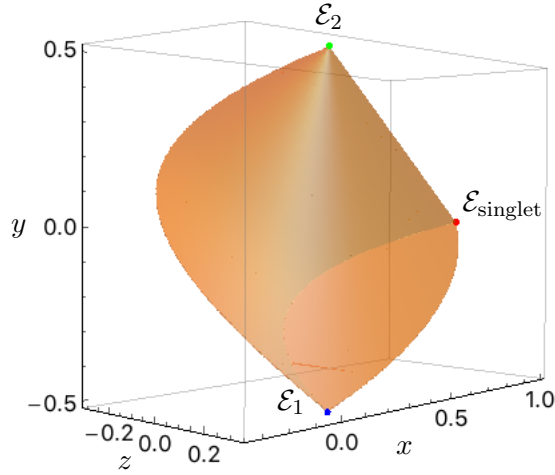


Figure 2.8: $SU(2)$ -symmetric, relational channels on 2-qubits.

The allowed parameter region for the relational channels including only class-3 diagrams invariant under swapping A and B . The right-most red point is the singlet channel $\mathcal{E}_{\text{singlet}}$, blue point is \mathcal{E}_1 and green point is \mathcal{E}_2

There are distinguished simple channels that correspond to points on the surface boundary. For instance the point $(1, 0, 0)$ is the unique intersection of the two intersecting planes, the parabolic cylinder, the elliptic cone. It corresponds to the singlet preparation channel $\mathcal{E}_{\text{singlet}}(\rho) = |\psi^-\rangle\langle\psi^-|$ for any 2-qubit state ρ .

In addition there are two points that lie at the intersection of the elliptic cone the parabolic cylinder and the hyperbolic cylinder and each of them are in one of the two planes and correspond to imposing that $x = 0$. These channels are unital, so they do not displace the maximally mixed state. More concretely, these two distinguished channels take the form:

$$\mathcal{E}_1 = \Phi_0 - \frac{1}{2}\Phi_{\theta_5} + \frac{3}{10}\Phi_{\theta_8} \quad (2.84)$$

$$\mathcal{E}_2 = \Phi_0 + \frac{1}{2}\Phi_{\theta_5} + \frac{3}{10}\Phi_{\theta_8} \quad (2.85)$$

where \mathcal{E}_1 corresponds to the point $(0, -\frac{1}{2}, \frac{3}{10})$ and \mathcal{E}_2 to the point $(0, \frac{1}{2}, \frac{3}{10})$.

Both are only sensitive to the T_{ij} components of the input state ρ . Therefore without loss of generality, we look at how they act on input states that have no local Bloch vectors: $\rho_{AB} = \frac{1}{4}(\mathbb{I} + \sum_{i,j} T_{ij}\sigma_i \otimes \sigma_j)$. Moreover, up to local unitaries any such state can be brought to a canonical form $\rho_{AB} = \frac{1}{4}(\mathbb{I} + \sum_i t_i\sigma_i \otimes \sigma_i)$ specified by a single vector (t_1, t_2, t_3) and often referred to as T -states. It is well-known that the range of these parameters lie in a tetrahedron

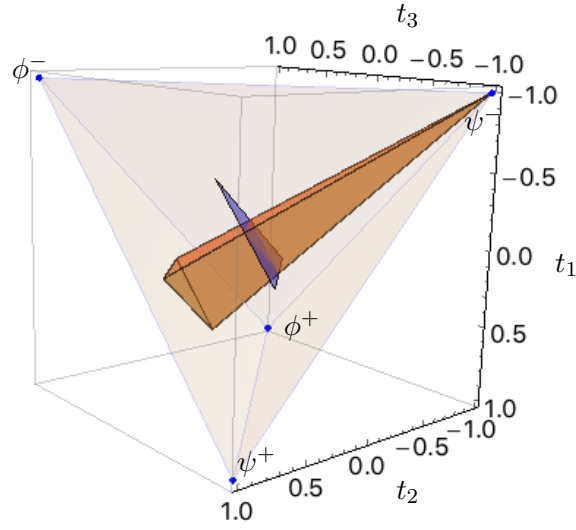


Figure 2.9: T-state transformations.

The set of 2-qubit states with maximally mixed marginals modulo local choice of bases (or “T-states”) have a tetrahedral state space with the four Bell states at the extremal points. Under the extremal channels \mathcal{E}_1 and \mathcal{E}_2 the set of T-states is mapped into the (blue) triangle and inner (brown) tetrahedron respectively.

whose vertices correspond to the Bell states [76]. The extremal channels \mathcal{E}_1 and \mathcal{E}_2 act on the Bell states ϕ^\pm, ψ^\pm in the following way:

$$\begin{aligned}
 \mathcal{E}_1(\phi^\pm) &= \frac{3}{20}\mathbb{I} + \frac{3}{10}\phi^\pm + \frac{1}{10}\psi^- \\
 \mathcal{E}_1(\psi^+) &= \frac{3}{20}\mathbb{I} + \frac{3}{10}\psi^+ + \frac{1}{10}\psi^- \\
 \mathcal{E}_1(\psi^-) &= \frac{1}{4}\mathbb{I}
 \end{aligned} \tag{2.86}$$

and

$$\begin{aligned}
 \mathcal{E}_2(\phi^\pm) &= \frac{2}{5}\mathbb{I} - \frac{1}{5}\phi^\pm - \frac{2}{5}\psi^- \\
 \mathcal{E}_2(\psi^+) &= \frac{2}{5}\mathbb{I} - \frac{1}{5}\psi^+ - \frac{2}{5}\psi^- \\
 \mathcal{E}_2(\psi^-) &= \psi^-
 \end{aligned} \tag{2.87}$$

The convex hull of these images of the set of Bell states under \mathcal{E}_1 and \mathcal{E}_2 will give the action of these channels in the more general cases. \mathcal{E}_1 and \mathcal{E}_2 will map T -states to T -states and the image of the tetrahedral state space is graphically displayed in Figure 2.9.

The preceding analysis can be used on more general bipartite quantum systems, where it allows a compact book-keeping for simplifying the analysis of quantum channels .

Symmetric unitary channels on two qubits

The space of invariant Hermitian matrices on two qubits under the $1/2 \otimes 1/2$ representation of $SU(2)$ is spanned by \mathbb{I} and $\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$, and therefore the set of unitaries invariant under the group action is a two-parameter family given by:

$$\exp[i(s\mathbb{I} + t(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z))]. \quad (2.88)$$

The first term is a phase term and so $V(t) = e^{it(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)}$ is the only non-trivial unitary interaction present. The quantum channel $\mathcal{E}(\rho) = V(t)\rho V(t)^\dagger$ has a process mode decomposition into diagrammatic form as in Figure 2.10.

Note that because V is symmetric under swapping A and B , we have this additional symmetry reflected in the diagram contributions. The aim of this examples stands to illustrate how quantum channels that have a simple description in terms of their Kraus decomposition (unitary channel have a single Kraus operator) can hide non-trivial structure in terms of the exchange of asymmetry between the bipartite subsystems.

$$\begin{aligned}
\mathcal{E} = & \begin{array}{c} 0 \\ \diagdown \\ \text{---} \\ \diagup \\ 0 \end{array} \begin{array}{c} 0 \\ \diagdown \\ \text{---} \\ \diagup \\ 0 \end{array} - \sqrt{3} \cos^2 2t \left(\begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} 0 \\ \diagdown \\ \text{---} \\ \diagup \\ 0 \end{array} + \begin{array}{c} 0 \\ \diagdown \\ \text{---} \\ \diagup \\ 0 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \right) + (2 + \cos 4t) \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \\
& - 3 \sin^2 2t \left(\begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 0 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} + \begin{array}{c} 0 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 0 \end{array} \right) + 5 \sin^2 2t \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} 2 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \\
& + \frac{i \sin 4t}{\sqrt{2}} \left(\begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} 0 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} + \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 0 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \right. \\
& \left. + \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} + \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 0 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array} \right)
\end{aligned}$$

Figure 2.10: Diagrammatic decomposition of $SU(2)$ symmetric unitary channel
The decomposition of the channel given by the symmetric unitary $V(t) = e^{it(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)}$ on two qubits.

Chapter 3

Irreversibility in the symmetry breaking degrees of freedom

One of the most fundamental ways to mark the difference between classical and quantum mechanics appears in the form of no-go theorems such as “no-cloning” or “no information without disturbance”. These can all be related under the notion of incompatibility of devices: given two physical devices these are compatible if and only if they can be viewed as subcomponents of an allowed physical device. Whenever we consider physical devices subjected to symmetry constraints there are limitations as to what types of transformations can be achieved. For example the Wigner-Araki-Yanase theorem states that under a symmetry corresponding to an additive conserved charge, measurements described by observables that do not commute with said charge are not allowed [78, 79]. Typically these limitations can be lifted by using an external system, initialised in a symmetry-breaking state. Once used, these symmetry-breaking resources are subject to irreversibility. A clock functions by breaking time-translation symmetry. However when used as a quantum mechanical system to implement timed operations its ability to be used subsequently as a clock is affected. More specifically, under a globally symmetric process $\rho_A \otimes \sigma_B \rightarrow \mathcal{V}(\rho_A \otimes \sigma_B)$, the external system B suffers a back-action $\sigma_B \rightarrow \sigma'_B = \text{Tr}_A(\mathcal{V}(\rho_A \otimes \sigma_B))$ that affects the ability of σ_B to act as a reference state to break the symmetry. Generally σ'_B breaks the symmetry in a much weaker form than the original state σ_B , and thus we say that this gives rise to an irreversibility under the symmetry constraint. There are situations where with respect to using system B in a specific way via some fixed protocol, such a type of irreversibility can be avoided.

We start by establishing a formal set-up that probes for repeatable use of resources from a reservoir to perform a particular processing task, with an emphasis on studying irreversibility in the environment involved in a simulation of a target quantum channel. The work of Johan Åberg [80] on catalytic coherence highlights particular models in which coherence from a reservoir can be used repeatedly in order to perform the same task without degradation relative to the outcome of interest. This provides motivation for our first application of the framework developed in Chapter 2. Using the process mode formalism we characterise a class of protocols that allows for repeatable use of coherence resources from a reservoir and establish links with incompatibility and broadcasting. We provide a different perspective between the interplay of quantum coherence and $U(1)$ symmetry that arises from energy conservation constraints and identify the core features necessary for re-use of coherence.

3.1 Irreversibility with respect to a fixed processing task

We aim to formalise the previous discussion. Suppose that performing a particular quantum processing task denoted by \mathcal{P} makes use of a system B . We write $\mathcal{P}(\sigma_B)$ for the outcomes of implementing the processing task with a use of system B initialised in state σ_B . The reduced state in system B becomes $\sigma_B \rightarrow \sigma'_B$. If $\mathcal{P}(\sigma_B) \neq \mathcal{P}(\sigma'_B)$ then we say that system B suffers an irreversible process with respect to the processing task \mathcal{P} . More interestingly is the situation when $\mathcal{P}(\sigma_B) = \mathcal{P}(\sigma'_B)$ so that the reduced state in system B changes, but from the point of view of implementing \mathcal{P} it remains unchanged. This is a situation which allows to investigate the subtle boundary between reversibility and irreversibility in quantum mechanics.

There are many examples that can be analysed in such terms ranging from entanglement catalysis, cryptographic protocols, quantum thermodynamics processes, use of reference frames under a symmetry principle. For example, it is well-known that not all bipartite states can be interconverted into one another via local operations and classical communication alone. For pure states, Nielsen's criterion provides a necessary and sufficient condition in terms of majorization. However there are examples where $|\phi\rangle_{AA'} \xrightarrow{LOCC} |\psi\rangle_{AA'}$ but there exists a *catalyst* state χ which makes the transformation $|\phi\rangle_{AA'} \otimes |\chi\rangle_B \xrightarrow{LOCC} |\psi\rangle_{AA'} \otimes |\chi\rangle_B$ possible. With respect to the processing task of transforming $|\phi\rangle_{AA'}$ into $|\psi\rangle_{AA'}$ via local operations, the system B aids this process without suffering any irreversibility $\sigma_B = \sigma_{B'} = |\chi\rangle_B \langle \chi|$ and thus it can be re-used to perform the same task again. A more non-trivial example in which irreversibility crops in comes from

the use of embezzling states. Suppose that the processing task of interest \mathcal{P} is to implement an LO protocol converting any two pure arbitrary bipartite states on AA' within δ -error by making use of system B . The embezzling protocol [81] provides such an example by using embezzling states $|\chi(d)\rangle := \frac{1}{C(d)} \sum_{j=1}^d \frac{1}{\sqrt{j}} |j\rangle |j\rangle$ to implement the transformation $|\chi(d)\rangle \xrightarrow{LOCC} |\mu(d)\rangle \otimes |\psi_{AA'}\rangle$ with fidelity $\langle \chi(d) | \mu(d) \rangle = 1 - \epsilon$, where the error depends on the dimension of the auxiliary system B such that $\epsilon \rightarrow 0$ as $d \rightarrow \infty$. Initially $\sigma_B = |\chi(d)\rangle \langle \chi(d)|$ and after one use of system B for the embezzling protocol the state will change into $\sigma'_B = |\mu(d)\rangle \langle \mu(d)|$. However as long as the dimension d is large enough such that $(1 - \epsilon)^2 \geq (1 - \delta)$, the system B can be re-used to convert two different states within δ -error via LO. While the state in system B changes, its ability to perform the processing task of interest does not.

The purpose of the examples above is to illustrate that there are many useful scenarios in quantum information theory which operate at this interface between reversibility and irreversibility. The present work contributes to this aim, by investigating this boundary when the quantum processes involved operate under a (global) symmetry constraint.

3.1.1 Repeatable use of quantum resources for simulation protocols

The focus of our analysis consists of studying irreversibility in the use of an environment system B with respect to the processing task of simulating local dynamics at A by way of interaction with B . In subsequent sections we restrict to interactions that are subjected to symmetry constraints, but for now we make no such assumption and treat the problem in the general terms of a resource theory. Suppose that \mathfrak{F} denotes the free operations of the resource theory and \mathfrak{R} the resource states.

More specifically, given a fixed initial state σ_B then this is used in order to implement a simulation $\mathcal{E} : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}'_A)$ on system A given by

$$\mathcal{E}(\rho) = \text{Tr}_{B'} V(\rho_A \otimes \sigma_B) V^\dagger,$$

where $V : \mathcal{H}_A \otimes \mathcal{H}_B \rightarrow \mathcal{H}'_A \otimes \mathcal{H}'_B$ is an isometry. Denote by \mathcal{P} the protocol that probes the use of a resource state in system B to induce a channel on system A . The protocol specifies a *free isometry* $\mathcal{V} : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}'_A \otimes \mathcal{H}'_B)$ with $V \in \mathfrak{F}$ and takes as input a particular state $\sigma_B \in \mathcal{B}(\mathcal{H}_B)$ to output channel \mathcal{E} as above; we can write $\mathcal{P}(\mathcal{V}, \sigma_B) = \mathcal{E}$.

Typically one is interested to simulate a particular target operation \mathcal{E}_{target} on system A using resource states in system B . However, given access to only system B as an environment there might not always exist an *exact* simulation. A particular protocol \mathcal{P} as described above will provide an approximate simulation $\mathcal{P}(\mathcal{V}, \sigma_B) = \mathcal{E} \approx \mathcal{E}_{target}$ for *some* resource state σ_B . The performance of the protocol \mathcal{P} is generally determined by the distance between the induced channel \mathcal{E} and the target \mathcal{E}_{target} (as measured for instance by the diamond norm $\|\mathcal{E} - \mathcal{E}_{target}\|_{\diamond}$).

For simplicity, to emphasize the conceptual ideas involved we restrict to $\mathcal{H}'_B \cong \mathcal{H}_B$ and protocols that specify a single free isometry \mathcal{V} (as opposed to a set of isometries designed to provide approximate simulations of a set of target channels).

We write $\mathcal{P} \circ \mathcal{P}(\mathcal{V}, \sigma_B)$ to denote two subsequent uses of the system B under the protocol \mathcal{P} . Explicitly it involves the following steps:

- System B initialised in state σ_B
- Apply the free isometry \mathcal{V} on system B and A_1
- Apply the free isometry \mathcal{V} on system B and a *different* system A_2 , isomorphic to A_1
- Output the channel \mathcal{E}_2 induced on system A_2 , which will generally depend on both σ_B *and* the initial state of system A_1 .

More specifically the induced channel $\mathcal{E}_2 : \mathcal{B}(\mathcal{H}_{A_2}) \rightarrow \mathcal{B}(\mathcal{H}'_{A_2})$ on the second system A_2 takes the form $\mathcal{E}_2(\rho_2) = \text{Tr}_{A'_1 B}(\mathcal{V}_2 \circ \mathcal{V}_1(\rho_1 \otimes \rho_2 \otimes \sigma_B))$ where $V_1, V_2 : \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_B \rightarrow \mathcal{H}'_{A_1} \otimes \mathcal{H}'_{A_2} \otimes \mathcal{H}_B$ are isometries such that $V_1 = \mathbb{I}_{A_2} \otimes V$ acts trivially on system A_2 and with the free isometry V on A_1 and B and similarly $V_2 = \mathbb{I}_{A_1} \otimes V$ acts trivially on system A_1 and with V on A_2 and B . Suppose that we denote the back-action channel on system B after a single application of the protocol by $\mathcal{F}_\rho(\sigma_B) = \text{Tr}_{A'} \mathcal{V}(\rho \otimes \sigma_B)$ for a fixed initial state $\rho \in \mathcal{B}(\mathcal{H}_A)$. Then the channel \mathcal{E}_2 induced upon a second application of the protocol that uses the same environment system B can be equivalently thought of as $\mathcal{E}_2(\rho_2) = \text{Tr}_B \mathcal{V}(\rho_2 \otimes \mathcal{F}_{\rho_1}(\sigma_B))$. This is the channel induced by a single application of the protocol to the state $\sigma'_B = \mathcal{F}_{\rho_1}(\sigma_B)$.

We will shortly see that a protocol \mathcal{P} is twice repeatable if $\mathcal{P} \circ \mathcal{P}(\mathcal{V}, \sigma_B) = \mathcal{P}(\mathcal{V}, \sigma_B)$, which means that the channel induced on system A by using σ_B is the same after either applying the protocol twice or a single time $\mathcal{E}_2 = \mathcal{E}$. Note that these statements are independent on the initial states in the systems A_1 or A_2 . Indeed the focus is not on single shot state transformations but,

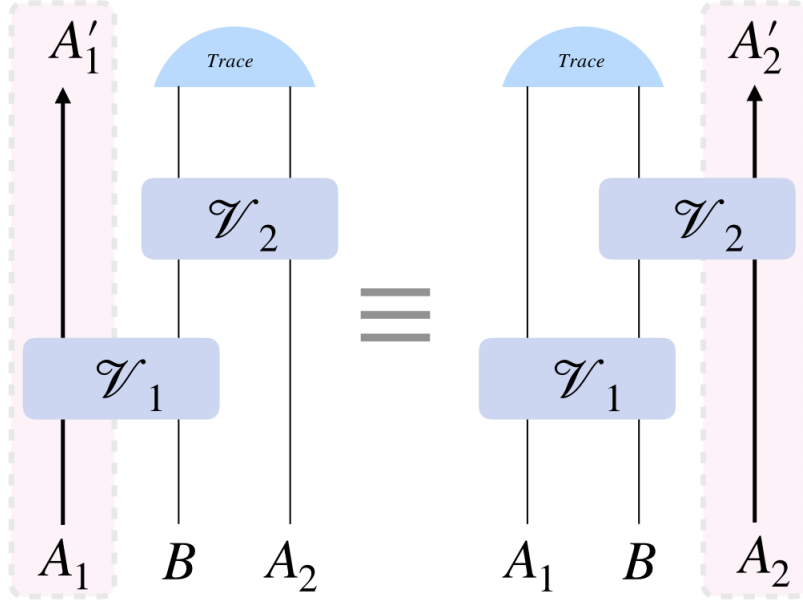


Figure 3.1: Illustration of a 2-repeatable protocol on system \mathcal{H}_B .

more generally we aim to investigate the capability of a reservoir to reliably implement the same local channel subsequently.

An elementary aspect of the repeatable use of some resource state σ_B that induces a map \mathcal{E} on system A is that a subsequent uses of system B will also result in exactly the same quantum process.

In general we may allow for different isometries to be applied at each step, and also we could allow for the output system of the environment B to change. Thus we give the definition of n -repeatability.

Definition 3.1.1. Let A_1, \dots, A_n be n isomorphic systems $A_i \cong A$ and $\mathcal{E}_{\text{target}} : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_{A'})$ a target quantum channel. We say that the protocol \mathcal{P} for $\mathcal{E}_{\text{target}}$ using system B is n -repeatable if it specifies a circuit of (symmetric) operations $\mathcal{W} = \mathcal{V}_n \circ \mathcal{V}_{n-1} \circ \dots \circ \mathcal{V}_1$ with $\mathcal{V}_i : \mathcal{B}(\mathcal{H}_{A_i} \otimes \mathcal{H}_{B_{i-1}}) \rightarrow \mathcal{B}(\mathcal{H}_{A'_i} \otimes \mathcal{H}_{B_i})$ for all i initially acting on $B_0 = B$ such that for any $\sigma_B \in \mathcal{H}_B$ and any k :

$$\text{Tr}_{\setminus k, B_n}(\mathcal{W}(\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n \otimes \sigma_B)) = \mathcal{E}(\rho_k) \quad (3.1)$$

the induced process \mathcal{E} is the same on all subsystems A_k and is an approximation of $\mathcal{E}_{\text{target}}$ using σ . In particular \mathcal{E} will depend on σ but not on k .

In particular, in the case of the previous protocol \mathcal{P} , the circuit of operations consists of applying

the same isometry \mathcal{V} to the different systems A_i and B , where the system $B = B_i$ remains unchanged. In such a particular case it follows from the above definition that \mathcal{P} is a n -repeatable protocol on system \mathcal{H}_B if for all $\sigma_B \in \mathcal{B}(\mathcal{H}_B)$ and all $i \leq n$:

$$\underbrace{\mathcal{P} \circ \dots \circ \mathcal{P}}_{i \text{ times}}(\mathcal{V}, \sigma_B) = \mathcal{P}(\mathcal{V}, \sigma_B),$$

where $\mathcal{E} = \mathcal{P}(\mathcal{V}, \sigma_B)$ is the channel induced by a single application of the protocol to the initial state σ_B .

3.1.2 Repeatable use of resources and connections with quantum-classical channel from the reference system into the multiple simulations

Suppose a protocol is n -repeatable for any arbitrary finite n then we say that the protocol is *arbitrarily repeatable*. For example: If the interaction between system A and B is given by a unitary acting only locally $U_A \otimes U_B$ then the protocol that simulates dynamics at A using $\mathcal{V} = \mathcal{U}_A \otimes \mathcal{U}_B$ is arbitrarily repeatable in a trivial way. We will look at more complex examples in the following sections.

The fact that the induced channel on system k depends only on the state σ_B in the reservoir B and not on systems A_1, \dots, A_n is essential. This implies that applying the simulation protocol repeatedly for n times as in definition 3.1.1 can be described for a fixed $\rho \in \mathcal{B}(\mathcal{H}_A)$ by a channel $\Lambda_\rho : \mathcal{B}(\mathcal{H}_B) \longrightarrow \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \dots \mathcal{H}_{A_n})$ given by $\Lambda_\rho(\sigma_B) = \text{Tr}_{B_n} \mathcal{W}(\rho^{\otimes n} \otimes \sigma_B)$.

The n -repeatability ensures that for every fixed $\rho \in \mathcal{B}(\mathcal{H}_A)$ and any $\sigma \in \mathcal{B}(\mathcal{H}_B)$ the state $\Lambda_\rho(\sigma_B)$ has same marginals on each subsystem A_i so that for all k , $\text{Tr}_{\setminus k}(\Lambda_\rho(\sigma_B)) = \mathcal{E}(\rho)$ where we trace over all but the k th system, gives a channel independent of k . This property reveals a connection with the notion of n -extendible quantum channels [82], which we recall in the following definition. ¹

Definition 3.1.2. A quantum channel $\mathcal{C} : \mathcal{B}(\mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}_A)$ is said to be n -extendible if there exists a quantum process $\Lambda : \mathcal{B}(\mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \dots \otimes \mathcal{H}_{A_n})$ with $\mathcal{H}_{A_i} \cong \mathcal{H}_A$ for all i , such that for all $X \in \mathcal{B}(\mathcal{H}_B)$ we have that $\text{Tr}_{\setminus i} \Lambda(X) = \mathcal{C}(X)$, where all but the i -th system has been traced out.

¹The definition below that we use for k -extendibility of channels is a (possibly) stronger version of the usually encountered k -extendibility of channels which is typically directly related to the k -extendibility of the Choi representation

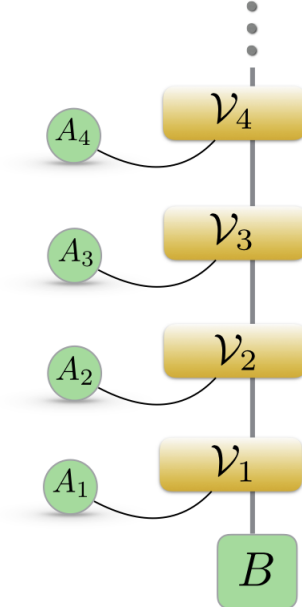


Figure 3.2: Circuit of symmetric operations \mathcal{W} that induce identical local simulations. This gives a protocol that implements arbitrary repeatable use of symmetry-breaking operations.

We also recall the definition of a k -extendible state [83]. A bipartite state $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ is k -extendible with respect to A if there exists a state $\rho_{A_1 A_2 \dots A_k B} \in \mathcal{H}_A^{\otimes k} \otimes \mathcal{H}_B$ that is invariant under permutations of the A system and have same marginals $\rho_{AB} = \text{Tr}_{\bar{i}}(\rho_{A_1 A_2 \dots A_k B})$, where we trace all but the i -th A system for any i . The following result establishes the connections between k -extendibility of states and k -extendibility of channels as is definition 3.1.2 via the Choi operator.

Lemma 3.1.3. *Suppose $\mathcal{C} : \mathcal{B}(\mathcal{H}_B) \longrightarrow \mathcal{B}(\mathcal{H}_A)$ is an n -extendible quantum channel, then the corresponding Choi operator $J[\mathcal{C}]_{AB} = id \otimes \mathcal{C}(|\Phi^+\rangle\langle\Phi^+|)$ is n -extendible state, where $|\Phi^+\rangle = \sum_i |ii\rangle$ is the maximally mixed state for $\{|i\rangle\}_{i=1}^{\dim(\mathcal{H}_B)}$, an orthonormal basis for the input system \mathcal{H}_B .*

Proof. From the definition it follows that there is a channel $\Lambda : \mathcal{B}(\mathcal{H}_B) \longrightarrow \mathcal{B}(\mathcal{H}_{A_1} \otimes \dots \otimes \mathcal{H}_{A_n})$ such that $\text{Tr}_{\setminus i}(\Lambda)(X) = \mathcal{C}(X)$ and denote by $J[\Lambda]_{BA_1, \dots, A_n} = id \otimes \Lambda(|\Phi^+\rangle\langle\Phi^+|)$ its corresponding Choi operator. It follows that $\text{Tr}_{\setminus i}(J[\Lambda]_{BA_1, \dots, A_n}) = J[\mathcal{C}]_{BA}$. In addition $J[\Lambda]_{BA_1, \dots, A_n}$ can be chosen to be invariant under permutations of A_1, \dots, A_n . Denote this action $V_\pi \in \mathcal{B}(\mathcal{H}_{A_1}, \dots, \mathcal{H}_{A_n})$ for every $\pi \in S_n$ and construct the channel $\tilde{\Lambda} : \mathcal{B}(\mathcal{H}_B) \longrightarrow \mathcal{B}(\mathcal{H}_{A_1} \otimes \dots \otimes \mathcal{H}_{A_n})$ given by:

$$\tilde{\Lambda} = \frac{1}{\dim(S_n)} \sum_{\pi \in S_n} V_\pi \Lambda V_\pi^\dagger \quad (3.2)$$

This is invariant under permutations by construction and it is an n -extension of \mathcal{C} because

$$\mathrm{Tr}_{\setminus i}(V_\pi \Lambda V_\pi^\dagger) = \mathrm{Tr}_{\setminus \pi^{-1}(i)} \Lambda = \mathcal{C} \quad (3.3)$$

for all $\pi \in S_n$. Therefore the Choi operator corresponding to $\tilde{\Lambda}$ will be a k -extendible state with a symmetric extension. \square

Therefore it follows that under an n -repeatable protocol given by the circuit of symmetric operations \mathcal{W} then for any *fixed* $\rho \in \mathcal{B}(\mathcal{H}_A)$, the channel $\mathrm{Tr}_{\setminus k}(\Lambda_\rho(\sigma_B)) : \mathcal{B}(\mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}_{A_k})$ is n -extendible where we recall that $\Lambda_\rho(\sigma_B) = \mathrm{Tr}_{B_n}(\mathcal{W}(\rho^{\otimes n} \otimes \sigma_B))$. The following lemma will help characterise what channels can be implemented locally at A under an arbitrarily repeatable protocol. In [83] it was shown that if a state ρ_{AB} is k -extendible for all finite k then it is a separable state. One direction is straightforward to check, whilst the other follows from the quantum de Finetti theorem. The idea is to use each k -extension to generate states in $\mathcal{H}_A^{\otimes k}$, which are separable as a consequence of the de Finetti theorem, and then argue that ρ_{AB} itself must be separable.

Lemma 3.1.4. *Suppose that a quantum channel $\mathcal{C} : \mathcal{B}(\mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}_A)$ is n -extendible for every finite n . Then it is an entanglement-breaking channel so it takes the form of a measure and prepare channel:*

$$\mathcal{C}(\sigma) = \sum_a \mathrm{Tr}(N_a \sigma) \rho_a$$

where $\{N_a\}$ is a POVM on $\mathcal{B}(\mathcal{H}_B)$ and $\rho_a \in \mathcal{B}(\mathcal{H}_A)$ fixed states.

Proof. By the corollary 3.1.3 it follows that the Choi operator for \mathcal{C} given by J_{BA} is n -extendible for all finite n so that there is a symmetric state J_{BA_1, \dots, A_n} which is invariant under any permutation $\pi \in S_n$ of the systems A_1, \dots, A_n . Therefore it follows that J_{BA} is separable (from [83]). Finally, separable Choi operators correspond to entanglement-breaking channels. \square

We emphasise that the proof techniques for the above lemmas are the same even in the case where \mathcal{H}_B is an infinite dimensional system, but \mathcal{H}_A is finite dimensional. Two further clarifications need to be made in this direction to be fully mathematically precise. First, we can extend the definition of a Choi operator to those channels for which the input system is infinite dimensional. This has previously been done already by Holevo in [84], where it is also shown that one does not need to employ a limiting approximation to extend the maximally entangled state $|\Phi^+\rangle$ to

infinite dimensions. In addition it is also shown that even in this infinite dimensional setting separable Choi operators correspond to entanglement-breaking channels. Second, the core result in establishing the lemma 3.1.4 is that a bipartite state is k -extendible for all finite k if and only if it is separable. If the extension is carried with respect to A , then the other system B can be infinite dimensional and the same result holds. For instance, [85] gives finite upper bounds on how close a k -extendible state can be to a separable state. The bound is proportional to the dimensionality of the extension space, that is of A and $1/k$, but does not depend on the dimension of B .

This allows to show the following theorem, that characterises what types of channels can be induced by using arbitrarily repeatable protocols. Note however that the result does not assume the circuit of isometries/unitaries specifying \mathcal{P} to be symmetric.

Theorem 3.1.5. *Let B be a quantum system with Hilbert space \mathcal{H}_B . Suppose that the quantum channel $\mathcal{E} : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}'_A)$ on system A is simulated by a state $\sigma \in \mathcal{B}(\mathcal{H}_B)$ via an arbitrarily repeatable protocol \mathcal{P} . Then there exists a POVM $\{M_a\}$ on \mathcal{H}_B and completely positive maps $\Phi_a : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}'_A)$ such that:*

$$\mathcal{E}(\rho) = \sum_a \text{Tr}(M_a \sigma) \Phi_a(\rho). \quad (3.4)$$

Proof. For every finite n the protocol \mathcal{P} gives a circuit of operations \mathcal{W} as in definition 3.1.1 such that for fixed state ρ the induced channel is $\mathcal{E}(\rho) = \text{Tr}_{\setminus k, B} \text{Tr}(\mathcal{W}(\rho^{\otimes n} \otimes \sigma))$. As \mathcal{E} explicitly depends on σ but not on k (or n) we denote for every fixed ρ the channel $\mathcal{C}_\rho(\sigma) : \mathcal{B}(\mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}'_A)$ defined by $\mathcal{C}_\rho(\sigma) = \mathcal{E}(\rho)$. Repeatability ensures that \mathcal{C}_ρ is n -extendible for every fixed ρ . Since \mathcal{P} is arbitrarily repeatable then it follows from Lemma 3.1.4 that \mathcal{C}_ρ takes the form of a measure and prepare channel.

Therefore for each fixed $\rho \in \mathcal{B}(\mathcal{H}_A)$ there exists a POVM $\{N_i\}$ on \mathcal{H}_B and quantum states $\rho_i \in \mathcal{B}(\mathcal{H}'_A)$ such that:

$$\mathcal{C}_\rho(\sigma) = \sum_i \text{Tr}(N_i \sigma) \rho_i. \quad (3.5)$$

Note that there could be dependence on ρ in either ρ_i or N_i , however this can be simplified by noting that one can decompose any POVM into a convex combination of extremal POVMs. We can write

$$N_i = \sum_k p_k M_{k,i} \quad (3.6)$$

where $\mathcal{M}_k = (M_{k,i})$ is an extremal POVM for each k , and p_k is a probability distribution. This implies that

$$\mathcal{C}_\rho(\sigma) = \text{Tr}_{\setminus k, B_n}(\mathcal{W}(\rho^{\otimes n} \otimes \sigma)) = \sum_{i,k} p_k \text{Tr}(M_{k,i}\sigma)\rho_i \quad (3.7)$$

$$= \sum_{i,k} \text{Tr}(M_{k,i}\sigma)\Phi_{k,i}(\rho) \quad (3.8)$$

where $\Phi_{k,i}(\rho) := p_k \rho_i$ is a completely-positive linear map on ρ (since it maps density matrices to fixed positive operators), and which implies that the POVM acting on B can be chosen to be independent of the input state ρ on A . Introducing the single index $a = (k, i)$ completes the proof. \square

3.2 Demystifying Åberg's catalytic coherence

Typically resources such as coherence, entanglement or asymmetry degrade upon use [86,87]. The work of Johan Åberg [80] aims to investigate repeatable use of the coherence reservoir in implementing the same target operation that generates coherence on several qubit systems. The catalytic coherence protocol gives a particular non-trivial example of an arbitrary repeatable protocol as previously described in this chapter. In what follows we give a brief overview of the core results in [80] by emphasizing the connection with process modes.

The particular set-up consists of a coherence reservoir described by \mathcal{H}_E an infinite-dimensional (ladder) system with equally spaced energy eigenstates $\{|j\rangle\}_{j \in \mathbb{Z}}$ and the main system given by \mathcal{H}_A with orthonormal basis $\{|\phi_m\rangle\}_{m=1}^d$. The reservoir interacts with system A under a $U(1)$ symmetry constraint, a consequence of imposing conservation of total energy. The group $U(1)$ acts on system A and the ladder system via the tensor product representation $U_E \otimes U_A$ where $U_E(\theta)|n\rangle = e^{in\theta}|n\rangle$ and $U_A(\theta)|\phi_m\rangle = e^{im\theta}|\phi_m\rangle$.

A state $\sigma \in \mathcal{B}(\mathcal{H}_E)$ simulates quantum operations $\mathcal{E}_\sigma(\cdot) = \text{Tr}_E(V_U(\cdot) \otimes \sigma V_U^\dagger)$ via a set of globally symmetric unitary interactions $V_U \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_E)$:

$$V_U = \sum_{mn} U_{mn} |\phi_m\rangle \langle \phi_n| \otimes \Delta^{n-m} \quad (3.9)$$

where $\Delta^n := \sum_j |j+n\rangle \langle j|$ and U_{mn} are matrix coefficients for some arbitrary unitary $U \in \mathcal{B}(\mathcal{H})$

with respect to the orthonormal basis $\{|\phi_m\rangle\}_m$. The unitary V_U commutes with this group action and thus $[V_U, (U_A \otimes U_E)(\theta)] = 0$ for all $\theta \in U(1) \cong S^1$.

The induced operations on system A take the general form:

$$\mathcal{E}_\sigma(\rho) = \sum_{n,n',m,m'} K_{mn}\rho K_{m'n'}^\dagger \text{Tr}(\Delta^{n-m+m'-n'}\sigma) \quad (3.10)$$

where $K_{mn} := C_{mn}|\phi_m\rangle\langle\phi_n|$. In more compact notation, writing $\Phi^\lambda(\rho) := \sum_{\substack{n,n',m,m' \\ \lambda=n-m+m'-n'}} K_{mn}\rho K_{m'n'}^\dagger$

we get that:

$$\mathcal{E}_\sigma(\rho) = \sum_\lambda \Phi^\lambda(\rho) \text{Tr}(\Delta^\lambda\sigma) \quad (3.11)$$

and we observe that Φ^λ is an irreducible tensor superoperator transforming as a λ -irrep of $U(1)$ under the group action on system A , $\mathcal{U}_A(\theta)[\Phi^\lambda] = e^{i\lambda\theta}\Phi^\lambda$. For every interaction between systems A and E as above, the reservoir suffers a *back-action* depending on the initial state in the main system given by $\mathcal{F}(\sigma) = \text{Tr}_S(V_U\rho \otimes \sigma V_U^\dagger)$.

The catalytic coherence property. The coherence reservoir can be used arbitrarily many times to perform the same task – induce a target operation on system A . The operations induced by states σ and $\mathcal{F}(\sigma)$ in a simulation involving the set of $U(1)$ -symmetric unitaries V_U are the same:

$$\mathcal{E}_\sigma = \mathcal{E}_{\mathcal{F}(\sigma)}. \quad (3.12)$$

While the reduced state in the coherence reservoir changes $\sigma \rightarrow \mathcal{F}(\sigma)$, the expectation values of the operators $\langle\Delta^\lambda\rangle$ remain unchanged. For the protocol specified, these coefficients are the only quantities involved in the simulation \mathcal{E}_σ . States in the coherence reservoir are subject to irreversibility, but their coherence properties relevant for this set-up do not degrade.

Different aspects of Åberg's paper have been debated and challenged² in [88], while others have

²A particular point of discontent raised by [88] was whether catalytic coherence protocol gives rise to a state that has an unbounded amount of coherence. On physical grounds one cannot increase the amount of coherence under time-translation symmetric operations (i.e under unitary evolutions that commute with the Hamiltonian). The problem raised was the following: Start with the initial state $\rho = |0\rangle\langle 0|$ which is transformed into $\mathcal{E}_\sigma(\rho) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1| + (1 - \frac{1}{L})(|0\rangle\langle 1| + |1\rangle\langle 0|))$ by use of a particular unitary U (which would implement $|0\rangle \rightarrow \frac{|0\rangle+|1\rangle}{\sqrt{2}}$) and some reservoir state in a coherent superposition of L energy eigenstates $\sigma = \frac{1}{\sqrt{L}} \sum_{j=1}^L |j\rangle$ in the reservoir with initial asymmetry measure $A(\rho \otimes \sigma) = \frac{1}{L}$ (the asymmetry measure quantifies the amount of asymmetry(coherence) present and is defined as $A(\sigma) = S(\mathcal{G}(\sigma)) - S(\sigma)$ where $S(\cdot)$ is the Von Neumann entropy and $\mathcal{G}(X) = \int U(\theta)XU^\dagger(\theta)d\theta$ the average state under all group actions). Then it is argued that the asymmetry in the final state diverges in the number n of repeated experiments since $A(\rho_S^{\otimes n}) \approx \frac{1}{2} \ln(n\pi e/2)$, which contradicts the idea that the amount of asymmetry (or coherence) does not increase under symmetric operations. However, the resolution is that subsequent uses of Åberg's protocol will not result in implementing a product state $\rho_S^{\otimes n}$ but rather a

taken the ideas further to exploit repeatable work-extraction [89], the role of clocks in quantum thermodynamics [90,91] or autonomous quantum machines [92,93]. Perhaps indeed the use of the word "catalytic" is unfortunate leading to a misrepresentation of the subtle conceptual ideas behind the catalytic coherence protocol (particularly since the state in the reservoir is subject to irreversible change). Nevertheless we highlight the more important aspects:

- Catalytic coherence is about investigating properties of a coherence reservoir, a system with a particular structure undergoing a $U(1)$ symmetry principle. The emphasis is *not* on the implementation of a target operation, but rather on how the resources in the coherence reservoir change upon multiple subsequent uses. This is in contrast with a large body of works in the context of reference frames whereas the accent falls on a fixed target operation and investigates what minimal or optimal external resources can be used to implement such operations. These are two different sides of a coin, and while related are inherently distinct features.
- The coherence reservoir can be re-used arbitrarily many times to produce the same *local* channels on many subsystems. All of these subsystems are necessarily correlated via their interaction with the common coherence reservoir.
- The structure of the particular model, that is the form of the unitaries V_U is essential for the catalytic coherence property to hold. We show in the following section that it is essentially the unique interaction model with a system formed of an infinite ladder of energies that allows for repeatable use of coherence.

Let us expand on the second point above, by describing the catalytic coherence protocol in slightly different, more operational terms. Suppose that Alice is in charge of system \mathcal{H}_E , the coherence reservoir and there are countably many Bobs, each having a qubit system labelled by A_1, A_2, \dots, A_n . Alice prepares her system in a state σ , picks a unitary U and subsequently interacts it with systems A_1, A_2, \dots, A_n (in some arbitrary order) using the $U(1)$ -symmetric unitary V_U . Therefore Alice prepares:

$$\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n \otimes \sigma \longrightarrow (V_U^1 \otimes \dots \otimes V_U^n)(\rho_1 \otimes \dots \otimes \rho_n \otimes \sigma)((V_U^1)^\dagger \otimes \dots \otimes (V_U^n)^\dagger),$$

where V_U^i is a unitary interaction of the form 3.9 that acts only on system E and the i 'th qubit. There's no ambiguity in this notation because all such unitaries commute through on the system multipartite system with correlations such that the marginals at each site are the same.

E. The qubit systems are then returned to the Bobs (which are otherwise isolated from one another). Globally, such a multipartite interaction builds correlations between the qubit systems A_1, \dots, A_n . However, the essence of Aberg's work is that, each Bob will locally have implemented the *same* operation, so that for the purpose of local experiments in the i th Bob's lab their system will be prepared in state $\mathcal{E}_\sigma(\rho_i)$. The reduced state on all of Bob's system $\rho'_{1,2,\dots,n}$ will not be a product state of the form $\mathcal{E}_\sigma(\rho_1) \otimes \dots \otimes \mathcal{E}_\sigma(\rho_n)$ but will have marginals $\text{Tr}_i(\rho_{1,2,\dots,n}) = \mathcal{E}_\sigma(\rho_i)$, where we trace over all but the i 'th system.

As we have emphasized, the induced operations under this set-up depend only on particular degrees of freedom in the resource state of the coherence reservoir. These are given by the expectations values $\langle \Delta^\lambda \rangle$. Therefore with respect to Aberg's protocol, the states in the reservoir can be partitioned into equivalence classes $\mathcal{C}_{\mathbf{k}} = \{\sigma \in \mathcal{B}(\mathcal{H}_E) : \langle \Delta^n \rangle_\sigma = k_n\}$ for any vector of $\mathbf{k} = (k_{-d}, \dots, k_0, \dots, k_d)$ with $k_n^* = k_{-n}$ and the dimension of d is related to the number of modes required to simulate operations on A , so it will be related to the dimension of A . Under $U(1)$ symmetry \mathcal{H}_A decomposes into $\dim(\mathcal{H}_A)$ irreducible representations of various weights, labelled by integer numbers n . Aberg's protocol is insensitive to which state in a particular class has been used in the reservoir.

It is this information alone that is shared with all Bobs. In an intuitive sense, Alice's state acts as a reference frame for lifting the $U(1)$ symmetry principle and involves broadcasting of particular degrees of freedom related to the ladder operators Δ^λ . In more detail given $\rho \in \mathcal{B}(\mathcal{H})$ then say that ρ can be broadcasted if there is a channel $\Lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ such that $\text{Tr}_1 \Lambda(\rho) = \text{Tr}_2 \Lambda(\rho) = \rho$. The no-broadcasting theorem [94] says that a set of states $\{\rho_1, \dots, \rho_k, \dots\}$ can be simultaneously broadcasted (i.e there is a channel Λ such that for all k it satisfies $\text{Tr}_1 \Lambda(\rho_k) = \text{Tr}_2 \Lambda(\rho_k) = \rho_k$) if and only if they commute $[\rho_i, \rho_j] = 0$ for all i, j . Analogously, the ladder operators commute $[\Delta^n, \Delta^m] = 0$ for all integers $m, n \in \mathbb{Z}$ therefore all representative states $\sigma_B(\mathbf{k}) = \sum k_{-n} \Delta^n$ for every equivalence class $\mathcal{C}(\mathbf{k})$ will commute with each other. In this sense the information flow from the coherence reservoir to each of the different simulations on systems A_i involves broadcasting of compatible degrees of freedom.

The above analysis of the catalytic coherence protocol illustrates that it is a particular example of an arbitrary repeatable protocol under a $U(1)$ global symmetry constraint as in Definition 3.1.1. In particular it follows from Theorem 3.1.5 that the local channels induced by this model of interaction take the form of measure and prepare channels from the coherence reservoir into

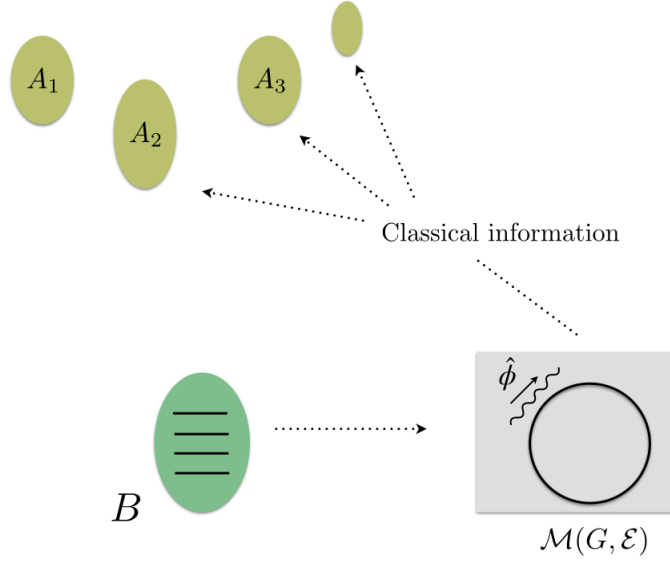


Figure 3.3: Aberg's protocol as broadcasting of reference frame data

the simulation. For example in the case of a qubit $d = 2$ we have that:

$$M_a = x_a \mathbb{I} + y_a \cos \hat{\phi} + z_a \sin \hat{\phi}, \quad (3.13)$$

where we have $\Delta^{\pm 1} = \cos \hat{\phi} \pm i \sin \hat{\phi}$ and $\hat{\phi}$ is an observable (self-adjoint operator) that is conjugate with generator of translations [95] which is given by the ladder operators Δ (see section 3.3.1 for more details). Therefore the operations implemented on a qubit system by a catalytic coherence protocol must take the form:

$$\mathcal{E}_\sigma(\rho) = \Phi_1(\rho) \text{Tr}(M_1 \sigma) + \Phi_2(\rho) \text{Tr}(M_2 \sigma) + \Phi_3(\rho) \text{Tr}(M_3 \sigma) \quad (3.14)$$

where $M_1 + M_2 + M_3 = \mathbb{I}$. For finite d -dimensional subsystems A , we also see that the POVM required only involves modes no larger than d . In such a case the set $\{M_a\}$ will be composed of additional terms $\cos \lambda \hat{\phi}$ and $\sin \lambda \hat{\phi}$ with λ an integer that labels the $U(1)$ irreps in $\mathcal{B}(\mathcal{H}_A)$.

Definition 3.2.1. We call a catalytic coherence protocol any simulation protocol that implements target operations of the form $\mathcal{E}(\rho) : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$:

$$\mathcal{E}(\rho) = \sum_{\lambda} \text{Tr}(\Delta^\lambda \sigma) \Phi^\lambda(\rho) \quad (3.15)$$

for some $\sigma \in \mathcal{B}(\mathcal{H}_E)$, where Φ^λ are irreducible tensor superoperators in $\mathcal{S}(\mathcal{H}, \mathcal{H})$ under $U(1)$ symmetry.

3.3 Repeated use of symmetry breaking resources: a process mode analysis

3.3.1 Asymptotic reference frames

A quantum system of unbounded size can play the same role as a classical reference frame. The space $\mathcal{L}^2(G)$ of complex valued functions on a compact group G with the left-regular representation can act as a perfect classical reference frame that lifts the symmetry constraint. For every group element $g \in G$ there exists a wavefunction on the group, $|g\rangle$ which can be thought of as a state in the separable Hilbert space $\mathcal{L}^2(G)$ such that $\{|g\rangle\}_{g \in G}$ forms an orthogonal basis $\langle g|h\rangle = \delta(gh^{-1})$. Therefore there is a perfect encoding of group elements $g \rightarrow |g\rangle$ into states that can be perfectly discriminated. For an infinite group, such an encoding is not possible if the size of the reference frame is bounded. It is in this sense that we refer to an unbounded quantum system described by $\mathcal{L}^2(G)$ as a classical reference (or the classical limit of a QRF). We will also refer to the set of orthogonal states $\{|g\rangle\}_{g \in G}$ as the *asymptotic reference frames* of $\mathcal{L}^2(G)$. This in turn allows that for every \mathcal{E}_{target} operation on some system A , there will be a covariant simulation using $\mathcal{L}^2(G)$ as an environment initialised in a sufficiently symmetry-breaking state.

U(1) symmetry

There is a physical realisation of $\mathcal{L}^2(U(1))$ using a system \mathcal{H}_E consisting of an equally spaced "ladder" of energies $\{|n\rangle\}_{n \in \mathbb{Z}}$ corresponding to Hamiltonian $H = c \sum_{n \in \mathbb{Z}} n |n\rangle \langle n|$ for some scaling constant factor c , which for simplicity we take to be $c = 1$. There is a $U(1)$ group action on \mathcal{H}_{ladder} , induced by time-translations $t \rightarrow e^{iHt}$ which acts on \mathcal{H}_{ladder} by displacing states $|\psi(0)\rangle \rightarrow |\psi(t)\rangle := |e^{iHt}\psi(0)\rangle$. However the structure of the Hamiltonian is such that any state displaced at times equal to integer multiples of 2π returns to the initial state, $|\psi(2\pi)\rangle = |\psi(0)\rangle$. Therefore the action of the non-compact translation group for this system reduces to a representation for the compact $U(1)$ group:

$$U^B : \theta \rightarrow e^{iH\theta} \quad \theta \in [0, 2\pi].$$

On \mathcal{H}_E we can define an orthonormal basis $\{|\theta\rangle\}_{\theta \in S^1}$ that encodes every group element $\theta \rightarrow |\theta\rangle$.

In terms of the energy eigenstates these are defined by:

$$|\theta\rangle = (2\pi)^{-1/2} \sum_m e^{-i\theta m} |m\rangle \quad (3.16)$$

and much as functions in $\mathcal{L}^2(S^1)$ they should be viewed in a distributional sense where $\langle\theta|\theta'\rangle = \delta(\theta - \theta')$. We will refer to the set $\{|\theta\rangle\}_{\theta \in S^1}$ as *asymptotic reference frames* under $U(1)$. In [95] it was shown that the associated phase angle observable $\hat{\phi} := \int \theta |\theta\rangle \langle\theta| d\theta$ is canonically conjugate to angular momentum along the z -axis. Hence $\hat{\phi}$ generates the (additive) unitary group of shift operators $\Delta^n := \sum_m |m+n\rangle \langle m|$. These act on the ladder system by shifting the energy eigenstates and are equivalently given by:

$$\Delta^n = e^{in\hat{\phi}} \quad (3.17)$$

for all integers n .

3.3.2 Arbitrary repeatable use of coherence resources

Quantum coherence underpins a wide range of quantum mechanical phenomena exhibiting non-classical behaviour from the wave particle duality and interference of particles in superposition, entanglement to its crucial role in quantum computation. Over the last decade, significant efforts and advancements have been made in controlling and manipulating quantum mechanical systems that exhibit a high degree of coherence, also prompting foundational questions on how to define and quantify coherence in rigorous terms [25]. These have found answers in resource theoretic formulations, which aim to describe coherence in an operational way from a set of simple principles. With respect to a fixed orthonormal basis, states that exhibit coherence are those which are not diagonal with respect to the distinguished basis – these are *resource states*. The free states in such a theory of coherence are incoherent states; these are diagonal in the said basis and form a classical probabilistic mixture of basis states. Typically, for all practical purposes there will be a distinguished basis, such as the eigenstates of an observable or of a Hamiltonian driving the system of interest. In such situations superselection rules or conservation laws hold. For example, when the total energy is conserved the only way to transform a system in an energy eigenstate into a coherent state is by introducing an external source of coherence. Such a setting necessarily introduces a certain degree of irreversibility.

In the following we show that any arbitrarily repeatable protocol for simulation under global energy conservation which has a classical asymptotic limit must necessarily be a catalytic coherence protocol as defined in 3.2.1.

Theorem 3.3.1. *If a protocol \mathcal{P} to induce a local process $\mathcal{E}_{\text{target}}$ on A using a ladder system B satisfies:*

i Global $U(1)$ symmetry.

ii Arbitrary repeatability.

iii Asymptotic reference frames on B are not disturbed.

iv Asymptotic reference frames on B yield perfect simulations of $\mathcal{E}_{\text{target}}$.

then \mathcal{P} is a catalytic coherence protocol.

This provides a clear physical interpretation of the repeatable use of quantum coherence in simple physical terms. Note it does *not* imply that the system B is in some perfectly coherent state, or that the state of B stays the same – the repeatability holds irrespective of the state on B . The proof of this result is straightforward using process modes, and is given as follows.

Proof. From (ii) we have that since the protocol \mathcal{P} for target map $\mathcal{E}_{\text{target}} \in \mathcal{S}(A, A')$ is arbitrarily repeatable on system B then from Theorem 3.1.5 it follows that there is a POVM $\{M_a\}$ such that the protocol \mathcal{P} induces the channel $\mathcal{E}_\sigma(\rho) = \sum_a \text{Tr}(M_a \sigma) \Phi_a(\rho)$. We introduce the extra label \mathcal{E}_σ to emphasize that different reference states σ will induce different channels.

Hypothesis (i) assumes that there is a $U(1)$ tensor product representation on the bipartite system AB . Explicitly denote by U and U^B the representations acting on \mathcal{H}_A and \mathcal{H}_B with the adjoint representations \mathcal{U} respectively \mathcal{U}^B . Therefore channels $\Phi_a \in \mathcal{S}(A, A')$ will have a process mode decomposition $\Phi_a = \sum_\lambda c_{\lambda,a} \Phi^\lambda$ for complex coefficients $c_{\lambda,a}$, where λ includes both a highest weight irrep label and a multiplicity label. Note that as $U(1)$ is abelian all its irreps are one-dimensional so the irreducible tensor superoperator basis $\{\Phi^\lambda\}$ carries a single irrep label and transforms as $\mathcal{U}_\theta[\Phi^\lambda] = \mathcal{U}_\theta \circ \Phi^\lambda \circ \mathcal{U}_\theta^\dagger = e^{i\lambda\theta} \Phi^\lambda$. It follows that the induced target maps have the form:

$$\mathcal{E}_\sigma(\rho) = \sum_\lambda \text{Tr} \left(\sum_a (c_{\lambda,a} M_a) \sigma \right) \Phi^\lambda(\rho). \quad (3.18)$$

We simplify the above equation using the notation $X^\lambda := \sum_a c_{\lambda,a} M_a$ to get the compact form for the induced map:

$$\mathcal{E}_\sigma(\rho) = \sum_\lambda \text{Tr}(X^\lambda \sigma) \Phi^\lambda. \quad (3.19)$$

As a direct consequence of the global $U(1)$ symmetry the action of the symmetry group on σ will generate the orbit of \mathcal{E}_σ . More concretely for any $\sigma \in \mathcal{B}(\mathcal{H}_B)$:

$$\mathcal{E}_{\mathcal{U}_\theta^B(\sigma)} = \mathcal{U}_\theta \circ \mathcal{E}_\sigma \circ \mathcal{U}_\theta^\dagger. \quad (3.20)$$

Now we substitute equation (3.19) into (3.20) to get that:

$$\sum_\lambda \text{Tr}(X^\lambda \mathcal{U}_\theta^B(\sigma)) \Phi^\lambda = \sum_\lambda \text{Tr}(X^\lambda \sigma) \mathcal{U}_\theta \circ \Phi^\lambda \circ \mathcal{U}_\theta^\dagger. \quad (3.21)$$

The irreducible tensor operators Φ^λ are orthonormal and therefore the associated coefficients must be equal in the above. Since $\mathcal{U}_\theta \circ \Phi^\lambda \circ \mathcal{U}_\theta^\dagger = e^{i\lambda\theta} \Phi^\lambda$ it follows that for all λ -irreps and all $\theta \in U(1)$:

$$\text{Tr}(X^\lambda \mathcal{U}_\theta(\sigma)) = \text{Tr}(X^\lambda \sigma) e^{i\lambda\theta}. \quad (3.22)$$

Using cyclicity of the trace in the left-hand side of the above we move the group action \mathcal{U}_θ on to the POVM element. The equation (3.22) holds for all $\sigma \in \mathcal{B}(\mathcal{H}_B)$ so:

$$\mathcal{U}_\theta^\dagger(X^\lambda) = e^{i\lambda\theta} X^\lambda. \quad (3.23)$$

Assumption iii is equivalent to the statement that the POVM effects $\{M_a\}$ must all commute with the self adjoint operator $\hat{\Phi}$ associated with the asymptotic reference frames $\{|\theta\rangle\}_{\theta \in U(1)}$, given by $\hat{\Phi} := \int_0^{2\pi} \theta |\theta\rangle \langle \theta| d\theta$. In particular, $[X^\lambda, \hat{\Phi}] = 0$, and therefore M_a (and each X^λ) will be diagonal in the asymptotic reference frame basis. Finally, we can write this as $X^\lambda = \int \langle \theta | X^\lambda | \theta \rangle |\theta\rangle \langle \theta| d\theta$.

However, the operators X^λ transform under the group action according to equation 3.23 and the $U(1)$ asymptotic reference frames satisfy $|\theta\rangle = U(\theta)^\dagger |0\rangle$. Thus

$$\langle \theta | X^\lambda | \theta \rangle = \langle 0 | \mathcal{U}_\theta(X^\lambda) | 0 \rangle = e^{-i\lambda\theta} \langle 0 | X^\lambda | 0 \rangle.$$

Altogether,

$$X^\lambda = \alpha_\lambda(\mathcal{E}_0) \int e^{-i\lambda\theta} |\theta\rangle \langle \theta| d\theta \quad (3.24)$$

where $\alpha_\lambda(\mathcal{E}_0)$ is a constant that depends only on \mathcal{E}_0 the origin in the process orbit of \mathcal{E} , corresponding to the reference frame $\sigma = |0\rangle\langle 0|$.

However (see [96]) the displacement operators can be written as $\Delta^\lambda = e^{i\lambda\hat{\Phi}}$. This implies that $X^\lambda \propto \Delta^{-\lambda}$ and the maps induced by the protocol must take the form of:

$$\mathcal{E}_\sigma(\rho) = \sum_\lambda \alpha_\lambda(\mathcal{E}_0) \text{Tr}(\Delta^{-\lambda}\sigma)\Phi^\lambda(\rho). \quad (3.25)$$

Therefore the protocol under assumptions i)-iv) must necessarily be a catalytic coherence protocol as defined in 3.2.1 □

3.3.3 Discussion: Are non-commutative symmetry breaking resources reusable?

Given the analysis we have provided for quantum coherence, we might wonder if a similar construction applies for more general groups. For such cases, there is one simple way in which an environment B can be used in a repeatable way – namely we can embed the system's Hilbert space into the space of wavefunctions on G and perform the measurement that estimates groups elements $\{|g\rangle\langle g|\}$ on this infinite dimensional space. Since this extracts all the reference data from B into a classical form it can be copied and repeatedly used. However this assumes a very particular interaction, and that B can physically be embedded in the required infinite-dimensional system (which is a non-trivial assumption).

We can therefore ask if repeatability can occur for a general group G and a finite dimensional system B ? For simplicity we can restrict to $G = SU(2)$ and consider just the set of all axial channels as our target quantum channels. As already described, the process orbit $\mathcal{M}(G, \mathcal{E})$ for these quantum channels is the 2-sphere S^2 , with coordinates (θ, ϕ) . Now if arbitrary repeatability is present then we have by the same analysis that $\mathcal{E} = \sum_a \text{Tr}(M_a\sigma)\mathcal{E}_a$ where the POVM elements on B must supply the coordinates on \mathcal{M} via the condition

$$\sum_a c_{j,m,a} \text{Tr}(M_a\sigma_B) = a_j(\mathcal{E}) Y_{jm}(\theta, \phi), \quad (3.26)$$

where $\{a_j(\mathcal{E})\}$ are the invariant data for the channel orbit, and $c_{j,a}$ are the coefficients of \mathcal{E}_a in the $SU(2)$ process mode decomposition.

However, now an important distinction is made with the $U(1)$ coherence case. The POVM

that extracts the reference data from B must estimate a point on a sphere. In the classical limit one can have perfect resolution of any point $(\theta, \phi) \in S^2$, however for finite dimensional B it is impossible to provide a perfect encoding of the point. Moreover, we know that quantum mechanics on S^2 is a phase space and so in the case that B is finite dimensional there will be a non-trivial uncertainty relation present. If, for example, B is a d -dimensional spin, then one has operators $\hat{X}_i := \frac{r}{d^2-1} J_i$ for B that constitute non-commuting coordinates such that $\hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 = r^2$. This defines the so-called “fuzzy sphere” [97] in non-commutative geometry where one has a discrete representation of spherical geometry. In the $d \rightarrow \infty$ limit this coincides with classical geometry, however for finite d has a fundamental lower bound on resolution and complementarity in measurements.

Therefore if one is using the system B within some globally symmetric channel to represent $\mathcal{M}(G, \mathcal{E}) \cong S^2$, then the complementarity in measurements on this phase space will imply incompatibility in the use of symmetry-breaking resources. This incompatibility is not present for coherence, since essentially only $\hat{\phi}$ is needed to supply the reference data.

More generally, the channel orbit perspective suggests a form of quantum-mechanical irreversibility in the use of symmetry breaking resources that depends on whether the geometry that can be induced on $\mathcal{M}(G, \mathcal{E})$ by B is non-commuting or not. This is consistent with the asymptotic limit of classical reference frames in quantum theory for an arbitrary group G , and also with the case of quantum coherence, however we must leave any further analysis to later work.

Chapter 4

How to gauge general quantum channels?

Gauge theories have played a deeply important role in the development of modern physics [4] and underpin our understanding of the fundamental forces of nature. Traditionally, gauge symmetries are viewed as mathematical redundancies in the description of a physical system and introduce interaction mediators – gauge fields.

Recently, quantum information theory tools have paved novel paths in other areas of theoretical physics. In particular, the AdS/CFT dual correspondence has been linked to error correcting codes [98,99], and tensor networks approaches to holography revealed entanglement features [100]. Moreover quantum simulation of lattice gauge theories [101, 102] and tensor network approaches to group renormalisation have been widely investigated. In this context of bridging quantum information with high energy physics concepts, gauge symmetries have received relatively little consideration not least because dualities act on quantities that are gauge invariant. However in [103] it was argued that gauge invariance explains the correspondence between a bulk operator and multiple distinct boundary operators unveiling a connection to the emergence of spacetime and therefore provides a motivation for introducing gauge symmetries in the context of quantum information.

We aim to address this gap and use the formalism of process modes to give an information theoretic perspective on gauge symmetries. Therefore the approach we take is operational. The core results describe a procedure to gauge global symmetries of quantum processes to local

symmetry acting on additional degrees of freedom. The main advantage of the result is that it does not rely on a Lagrangian formulation – the usual formalism to describe gauge theories. While our prescription recovers both the usual results on gauging lattice systems undergoing a unitary evolution [104–106] and on gauging quantum states [107], it goes beyond these examples and allows for analysis of general symmetric dynamics in which irreversibility is present as described by the language of quantum operations.

The present work is also motivated by the study of entanglement in systems with gauge symmetries, which poses significant challenges, particularly since local symmetry constraints (in the form of Gauss’s law) on the physical states prevent introducing a tensor product structure between different regions. Thus, there may be conceptual or technical advantages to express gauge symmetries in a manner that does not involve a Lagrangian formulation but uses the more suitable notion of quantum channels. One should expect that applications involving novel statements about entanglement in quantum field theories would require additional conceptual leaps beyond the aims of the present work. A more tangible motivation and future application of the framework concerns the measurability of gauge-invariant observables. It is known that Wilson loops in gauge theories are not measurable in relativistic physics if one has a non-abelian symmetry. This non-trivial result raises the question as to what POVMs can be measured in such non-abelian gauge theories. The present analysis seems suitable to address this, at least in a toy-model scenario, however we leave it for future work. To summarise, the increasing body of work that connects quantum information and high energy physics concepts motivates building a broader framework to provide a toolkit to analyse gauge theories in the more general setting of quantum channels. Thus it gives a unifying formalism that can deal with both traditional Lagrangian formulations and newer approaches using tensor network states.

4.1 From global to local gauge symmetries

4.1.1 What does gauging mean?

A *gauge* typically refers to a choice of a particular coordinate system that depends on an underlying parameter space such as a location in space-time. One may change these coordinates associated to every point in the parameter space and such mappings between different gauges are called *gauge transformations*.

For example the purification of a mixed state is non-unique, there is a unitary freedom on the purifying space that gives the same density matrix. Picking a particular purification represents a choice of gauge, while the set of (local) unitaries correspond to gauge transformations. The set of all purifications of ρ can be thought of as a fibre associated to state ρ .

Gauge theories are constructed out of mathematical or physical objects that remain invariant under the local gauge transformations, which form particular representations of a gauge symmetry group G .

Gauging refers to mapping a global symmetry to a local symmetry in such a way as to typically satisfy some "minimal coupling" condition. Such a procedure is not unique, and in fact even the minimal coupling principle is ambiguous to define precisely [108]. Generally it amounts to replacing derivatives ∂_x that do not transform under the local symmetry with covariant derivatives depending on a gauge potential operator $\partial_x + iA_x$ to construct a gauge-invariant Lagrangian.

4.1.2 Global vs local symmetry for many body systems

Consider a multipartite system consisting of subsystems A_1, A_2, \dots, A_n with the Hilbert space $\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \dots \otimes \mathcal{H}_{A_n}$. The compact (Lie) group G acts on each of the subsystems via a unitary representation U_i that maps every group element to an operator $U_i(g) \in \mathcal{B}(\mathcal{H}_{A_i})$. On the full system there is a *global* symmetry action given by the tensor product representation $U_1(g) \otimes \dots \otimes U_n(g)$, where the same group element $g \in G$ is applied to each subsystem. A *local* symmetry action acts on the system with $U_1(g_1) \otimes \dots \otimes U_n(g_n)$ where *independent* group elements act on different subsystems. This can be viewed as a representation U on $\mathcal{H}_{A_1} \otimes \dots \otimes \mathcal{H}_{A_n}$ for the local symmetry group $G \times G \times \dots \times G =: G^{\times n}$. For this, we denote generically by $\mathbf{g} := (g_1, g_2, \dots, g_n)$ group elements in $G^{\times n}$ and write $U(\mathbf{g}) = U(g_1) \otimes \dots \otimes U(g_n)$. Using similar notation as in the previous chapters, the corresponding action on the space of operators in $\mathcal{B}(\mathcal{H}_{A_1} \otimes \dots \otimes \mathcal{H}_{A_n})$ is $\mathcal{U}_{\mathbf{g}}$. This allows to formalise what it means for a quantum process $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ acting on this multipartite system to be globally or locally symmetric.

Definition 4.1.1. Globally and locally symmetric processes

(i) The quantum process \mathcal{E} is globally symmetric if $\mathfrak{U}_{\mathbf{g}} \circ \mathcal{E} \circ \mathfrak{U}_{\mathbf{g}}^\dagger = \mathcal{E}$ for all $\mathbf{g} = (g, g, \dots, g)$ with g a group element in G .

(ii) The quantum process \mathcal{E} is locally symmetric if $\mathfrak{U}_{\mathbf{g}} \circ \mathcal{E} \circ \mathfrak{U}_{\mathbf{g}}^\dagger = \mathcal{E}$ where $\mathbf{g} = (g_1, g_2, \dots, g_n)$ holds for all group elements $g_i \in G$.

Remark: The global symmetry will be considered to be a tensor product representation of the group G , while the local symmetry is a representation of the local symmetry group $G^{\times n}$. The irreducible representations of $G^{\times n}$ are tensor products of irreducible representations of G . For example, if U_1, U_2, \dots, U_n are irreducible representations of G then $U(\mathbf{g}) = U_1(g_1) \otimes \dots \otimes U_n(g_n)$ is an irreducible representation of $G^{\times n}$.

4.2 Gauging beyond Lagrangian formulations

Typically, gauging a physical theory assumes that the dynamics involved admits a Lagrangian description. We make no such assumption. We describe gauging of globally symmetric quantum processes, as described in quantum information theory in the language of completely positive trace preserving operations. While this trivially includes dynamics given by a Hamiltonian or Lagrangian, it goes much further in that the formalism of quantum operations can also describe interactions with an environment. We will restrict to discrete systems, with the additional remark that continuous systems can generally be expected to have discretised approximations as lattice formulations.

Given a globally symmetric quantum process \mathcal{E} on subsystems A_1, A_2, \dots, A_n such that it is not symmetric under the local group action, the goal is to extend it to a locally symmetric process $\tilde{\mathcal{E}}$ acting on the subsystem and additional ("gauge fields") systems.

The procedure for gauging globally symmetric quantum processes informally follows the algorithm:

- *Introduce "static" background systems.* Define a collection of systems acting as quantum reference frames that encode relational data.
- *Specify dynamics for the background systems.* Define a globally symmetric quantum process describing dynamics of the additional systems and how it transforms under the local group action.
- *Gauging the global symmetry.* Perform a uniform average over the local symmetry group to discard the relational data between subsystems.

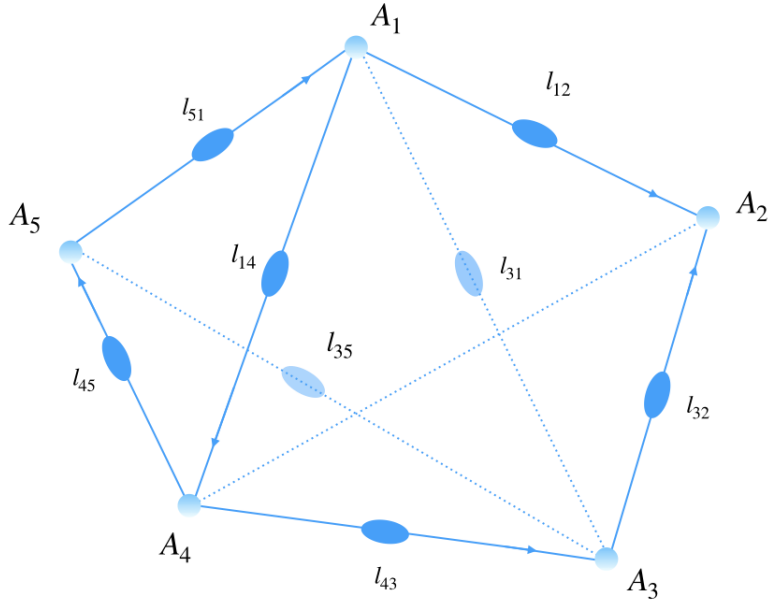


Figure 4.1: Lattice associated with multipartite systems undergoing a globally symmetric dynamics.

Before delving into the technicalities of the gauging procedure, we assume for simplicity that the multipartite system has an associated graph $\Gamma = (V, E)$ whereas each subsystem corresponds to a vertex $x \in V$ and E denotes the set of edges connecting them with each link between vertices x and y given by $l = [x, y] \in E$. Moreover each edge will have a direction, which may be chosen arbitrarily but must remain fixed throughout the analysis. This is to ensure that relative alignments between adjacent subsystems are viewed in a consistent way.

We introduce such a graph structure associated with the many body system in order to facilitate the gauging procedure. This comes by analogy with lattice gauge systems, where the matter systems live on the vertices of a graph and the additional degrees of freedom, the gauge fields, are linked with the edges of the graph. The state of the gauge-matter system becomes invariant under the local symmetry.

The formalism in Chapter 2, particularly the structure theorem for globally symmetric channels on bipartite systems forms an important starting point for gauging of more general globally symmetric dynamics on multipartite systems.

For that, we identify a particular class of such processes, that decomposes into 2-local terms

which are invariant under the global symmetry.

Definition 4.2.1. A globally symmetric quantum process $\mathcal{E} : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}'_A)$ on multipartite system $\mathcal{H}_A = \mathcal{H}_{A_1} \otimes \dots \otimes \mathcal{H}_{A_n}$ is called 2-symmetric if it can be written as:

$$\mathcal{E} = \sum_{\{(l_k, \theta_k)\}} c_{\{(l_k, \theta_k)\}} \chi^{(l_1, \theta_1)} \otimes \chi^{(l_2, \theta_2)} \otimes \dots \otimes \chi^{(l_r, \theta_r)} \quad (4.1)$$

with $c_{\{(l_k, \theta_k)\}} \in \mathbb{C}$, where we range over all ordered links $l = [xy] \in E$, between A_x and A_y , and where $\chi^{(l, \theta)}$ is a θ -diagram term on A_x and A_y .

Recall from Theorem 2.4.1 each of the terms $\chi^{(l, \theta)}$ take the form $\chi^{(l, \theta)} = \sum_k \Phi_k^{\lambda_x} \otimes \Phi_k^{\lambda_y^*}$, where $\{\Phi_k^{\lambda_x}\}_k$ and $\{\Phi_k^{\lambda_y^*}\}_k$ transform under the λ respectively λ^* irreps of G . These operators have associated a θ -diagram labelling the incoming and outgoing modes (and consequently the multiplicities of λ_x respectively λ_y^*). The present analysis is however independent on the particular choice of these local irreducible tensor superoperators so that a θ diagram representation is not essential¹. For clarity we emphasize that the decomposition in equation 4.1 is unambiguous for the terms where the edges l_1, \dots, l_r have no common vertices. In contrast, if two links share a vertex the tensor product should be understood as a composition of linear maps on the common vertex. For example if $l_1 = [x \rightarrow y]$ and $l_2 = [y \rightarrow z]$ then the term $\chi^{l_1, \theta_1} \otimes \chi^{l_2, \theta_2} = \sum_{k, m} \Phi_k^{\lambda_x} \otimes \Phi_k^{\lambda_y^*} \circ \Phi_m^{\mu_y} \otimes \Phi_m^{\mu_z^*}$.

One can extend the definition of 2-symmetric processes to generators of the unitary evolution.

Definition 4.2.2. Let H be a Hamiltonian describing the dynamics of the multipartite system $\mathcal{H}_A = \mathcal{H}_{A_1} \otimes \dots \otimes \mathcal{H}_{A_n}$. Then we say that its corresponding Liouville operator $\mathcal{L}(\rho) := [\rho, H]$ is 2-symmetric if:

$$H = \sum_{\{l, \theta_i\}} H^{l, \theta_1} \otimes \dots \otimes H^{l_r, \theta_r}$$

where each H^{l_i, θ_i} acts on the system $A_x \otimes A_y$ corresponding to the link l_i and is an invariant operator (labelled by θ_i) under the global symmetry, i.e commutes with the group action $[U_{A_x}(g) \otimes U_{A_y}(g), H^{l_i, \theta_i}] = 0$

A particular example of a 2-symmetric process comes from the spin lattice model with pairwise Heisenberg interactions $H = \sum_l H_l$ where H_l acts non-trivially only on subsystems A_x and A_y linked by l . Each 2-local Hamiltonian term is $H_l = c_l(X_x \otimes X_y - Y_x \otimes Y_y + Z_x \otimes Z_y)$ for

¹For the present application only the λ -irrep symmetry breaking degree of freedom between the subsystems is important.

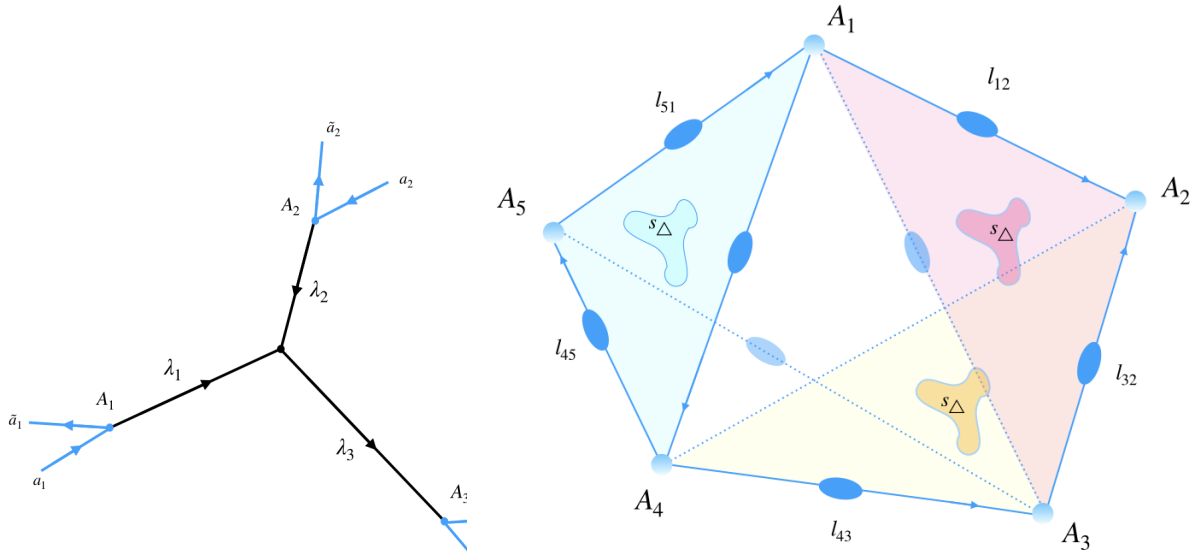


Figure 4.2: 3-symmetric channels.

Gauging k -symmetric processes that have higher order symmetry breaking interactions between sites and requires additional quantum reference frames associated with k simplices. A structure theorem describing globally symmetric processes on multipartite system consisting of three or more subsystems leads to generalisations of θ -diagrams which correspond to intertwiners between trivial representation and tensor products of three (or more) irreps $\lambda_1 \otimes \lambda_2 \otimes \lambda_3$.

constant c_l and Pauli operators X, Y, Z with subscript x or y if applied at the respective sites. Each H_l is invariant under $U \otimes U$ for any unitary $U \in SU(2)$ acting on the $l = [xy]$. Under this dynamics, any state ρ will evolve according to Liouville's equation $\frac{d\rho}{dt} = \mathcal{L}(\rho) = i[\rho, H]$. In this example the generator of the dynamics will be 2-symmetric as the Hamiltonian is constructed out of globally invariant 2-local terms. In general however, the unitary evolution given by $\rho \rightarrow e^{iHt} \rho e^{-iHt}$ will not give rise to a 2-symmetric process itself. However, it will be up to a first order approximation in t , which is dominated then by the 2-symmetric generator since $e^{iHt} \rho e^{-iHt} = \mathbb{I} + it[H, \rho] + \mathcal{O}(t^2)$.

Typically, a globally symmetric quantum process may not be 2-symmetric. In such cases the decomposition into local terms might involve interactions involving three or more sites. One can generalise the Definition 4.2.1 to k -symmetric processes and beyond. In order to describe such terms that also involve three or more sites in a non-trivial way, one needs to take into account not only the links between two sites but also polygons between multiple sites. To gauge such processes will require additional reference frame systems associated with simplices between three or more sites. Moreover a similar structure theorem for tripartite systems, will have more

complicated features and additional multiplicities to account for. We analyse in the following the gauging procedure for the restricted class of 2-symmetric processes. The general processes could be treated in a similar manner, with the caveat that they will require additional quantum reference frame systems.

4.2.1 I. Inclusion of background reference frame systems

We introduce an array of systems associated with every link $l \in E$ and given by Hilbert spaces \mathcal{H}_l . Their role is to act as quantum reference frames encoding the *relative alignment* between subsystems placed at the vertices connected by a given link.

Every Hilbert space \mathcal{H}_l associated with the directed link $l = [x \rightarrow y]$ carries a representation $W_l : G \times G \rightarrow \mathcal{B}(\mathcal{H}_l)$ with the corresponding adjoint representation \mathcal{W}_l acting on operators on \mathcal{H}_l such that $W_l(g_x, g_y) = L(g_x)R(g_y)$ where R, L are two distinct (commuting) representations of G . These give the "right" or "left" group action such that R and L take on the group element associated with the outgoing and incoming vertex respectively.

This gives a representation $W : G^{\times n} \rightarrow \mathcal{B}(\bigotimes_{l \in E} \mathcal{H}_l)$ of the local symmetry group acting on all the quantum reference frame systems \mathcal{H}_l , where $W = \bigotimes_{l \in E} W_l$ with W_l acting trivially on all but the link l .

There will always be a choice of systems \mathcal{H}_l such that it encodes every group element $g \in G$ perfectly in the sense that states associated to different group elements can be perfectly discriminated. For infinite groups such a perfect encoding requires the systems \mathcal{H}_l to be infinite-dimensional, and it is realised by $\mathcal{L}^2(G)$ the space of square integrable functions on the group G . This encodes every group element $h \in G$ to a state $|h\rangle$, which can be thought of in a distributional sense as a wavefunction $\psi_h(g) = \delta(g - h)$ localised at the point h of the group manifold. The set $\{|h\rangle\}_{h \in G}$ forms a set of perfectly distinguishable states. The right group action maps $R(g_y)|h\rangle = |hg_y^{-1}\rangle$ and the left group action $L(g_x)|h\rangle = |g_x h\rangle$.

Therefore, the quantum reference frames on the links are viewed as $\mathcal{H}_l \cong \mathcal{L}^2(G)$ with the group action:

$$\mathcal{W}_l(g_x, g_y)(|h\rangle\langle h|) = |g_x h g_y^{-1}\rangle\langle g_x h g_y^{-1}| \quad (4.2)$$

for all $h \in G$ and any $(g_x, g_y) \in G \times G$.

The collection of systems \mathcal{H}_l on the links provide the quantum reference frames for the group G , however it is not necessary to make assumptions on the structure of \mathcal{H}_l . So far we assumed that $\mathcal{H}_l = L^2(G)$, and indeed much of the discussions in future sections will rely on this, however it is worth discussing here what happens when we relax such assumption. We may for instance require that \mathcal{H}_l are finite dimensional quantum reference frames for an infinite group G . In such a case, we can no longer associate group elements $g \in G$ with states $|g\rangle$ that can be perfectly discriminated. The finite dimensionality will put constraints on how well one can encode group elements and typically different group elements will correspond to states in $\mathcal{B}(\mathcal{H}_l)$ that have a non-zero overlap. However, the core feature that allows to gauge a globally symmetric channel to one with local symmetry is how the quantum processes that describe the initial evolution of systems \mathcal{H}_l transform under the action of group elements applied to the vertices of the link l . As such, one can devise a gauging procedure, even in those cases where the quantum reference frame do not permit a perfect encoding of group elements.

Suppose that every group element $h \in G$ will be encoded in some state $\sigma_h \in \mathcal{B}(\mathcal{H}_l)$ (which may have non-zero overlap with one another). Under the local symmetry group transformation with $\mathbf{g} = (g_1, g_2, \dots, g_n)$ the states of the reference will be mapped to:

$$\sigma_h \longrightarrow \sigma_{g_x h g_y^{-1}} \quad (4.3)$$

for all $h \in G$ and any g_x and g_y group elements applied locally on systems situated at vertices x and y .

The above transformation property ensures that the state in \mathcal{H}_l will carry information on the relative alignment between systems at the vertices x and y . To illustrate how this occurs, we restrict to the two systems at x and y who are initially in an arbitrary state ρ_{xy} . The initial state in the system \mathcal{H}_l is σ_e , state that encodes the identity element. Then under a *local* symmetry transformation $(\rho_{xy} \otimes \sigma_e) \xrightarrow{\mathbf{g}} \mathcal{U}_{\mathbf{g}}(\rho_{xy}) \otimes \sigma_{g_x g_y^{-1}}$. However the local group action on the system A_x and A_y can be re-written as $U(g_x) \otimes U(g_y) = (U(g_x g_y^{-1}) \otimes \mathbb{I})(U(g_y) \otimes U(g_y))$. Then $g_x g_y^{-1}$ gives the relative alignment between system A_x and A_y after a local symmetry transformation and this group element is encoded into the reference system \mathcal{H}_l that changes into the state $\sigma_{g_x g_y^{-1}}$.

4.2.2 II. Dynamics on the reference frame system

The quantum reference frame systems \mathcal{H}_l on the links become dynamical objects themselves. As the states of the multipartite system A evolve under the quantum process \mathcal{E} , similarly the quantum reference frames will evolve according to a process that must be made explicit to account for the additional degrees of freedom. This needs to satisfy two initial requirements:

- the processes on the quantum reference frames are invariant under the global symmetry
- the processes on each \mathcal{H}_l must transform under the local symmetry in a particular way as to encode the relative alignments between group elements applied to the systems at the two vertices of the link l .

This suggests to define process gauge couplings that act on the links and respond to the local symmetry in a particular way, analogous to the process modes that we introduced previously.

First, we make an additional remark on the notation used throughout the remainder of the chapter. The total "gauged" Hilbert space consists of systems and quantum reference frames on the links $\mathcal{H}_{\text{tot}} = \mathcal{H}_{A_1} \otimes \dots \otimes \mathcal{H}_{A_n} \otimes_{l \in E} \mathcal{H}_l$. The local group action on the system and links is given by $U_{\text{tot}}(\mathbf{g}) = U_{A_1}(g_1) \otimes \dots \otimes U_{A_n}(g_n) \otimes_{l=[x \rightarrow y] \in E} W_l(g_x, g_y)$ where group element g_x is applied to the system A_x and associated with the vertex x . We will denote $\mathfrak{U}_{\mathbf{g}}$ this group action lifted on the space of superoperators $\mathcal{S}(\mathcal{H}_{\text{tot}}, \mathcal{H}_{\text{tot}})$. For simplicity we will keep the same notation $\mathfrak{U}_{\mathbf{g}}$ for the group action on the space of superoperators on the links alone which carry the representation $\otimes_{l=[x \rightarrow y] \in E} W_l(g_x, g_y)$.

Definition 4.2.3. A process gauge coupling, $\{\mathcal{A}_{jk}^{(l,\lambda)}\}$ for a quantum reference frame on a link l is a set of superoperators $\mathcal{A}_{jk}^{(l,\lambda)} : \mathcal{B}(\mathcal{H}_l) \rightarrow \mathcal{B}(\mathcal{H}_l)$ such that under the local symmetry action with $\mathbf{g} = (g_1, g_2, \dots, g_n)$:

$$\mathfrak{U}_{\mathbf{g}}[\mathcal{A}_{jk}^{(l,\lambda)}] = \sum_{m,n} v^\lambda (g_x^{-1})_{jm} v^\lambda (g_y)_{nk} \mathcal{A}_{mn}^{(l,\lambda)}, \quad (4.4)$$

with v^λ the λ -irrep of G and where x and y are the endpoints of the directed link $l = [x \rightarrow y]$.

A process gauge coupling $\{\mathcal{A}_{jk}^{(l,\lambda)}\}$ can be packaged into a compact notation $\mathcal{A}^{(l,\lambda)}$ comprising of

a matrix of superoperators on \mathcal{H}_l :

$$\mathcal{A}^{(l,\lambda)} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1d} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \cdots & \mathcal{A}_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{A}_{d1} & \mathcal{A}_{d2} & \cdots & \mathcal{A}_{dd} \end{bmatrix} \quad (4.5)$$

where d is the dimension of the λ -irrep of the compact group G . A local symmetry transformation on \mathcal{H}_l that applies group element g_x on subsystem A_x and g_y on subsystem A_y acts on the matrix object $\mathcal{A}^{(l,\lambda)}$ via right multiplication by $v^\lambda(g_y)$ and left multiplication by $v^\lambda(g_x^{-1})$, which are $d \times d$ matrices. In compact form:

$$\mathfrak{U}_g[\mathcal{A}^{(l,\lambda)}] = v^\lambda(g_x^{-1})\mathcal{A}^{(l,\lambda)}v^\lambda(g_y) \quad (4.6)$$

Since we have restricted the present analysis to 2-symmetric processes, then it is sufficient to explain how to introduce the dynamical degrees of freedom described by \mathcal{H}_l for a general bipartite (globally symmetric) superoperator term $\chi^{(l,\theta)} = \sum_k \Phi_k^{\lambda_x} \otimes \Phi_k^{\lambda_y^*}$ acting non-trivially on subsystems A_x and A_y connected by link l .

Under the local symmetry, $\chi^{(l,\theta)}$ transforms non-trivially as:

$$\chi^{(l,\theta)} \longrightarrow \sum_{k,m,n} v_{mk}^\lambda(g_x)v^{\lambda^*}(g_y)_{nk}\Phi_m^{\lambda_x} \otimes \Phi_n^{\lambda_y^*} = \sum_{m,n} v_{mn}^\lambda(g_xg_y^{-1})\Phi_m^{\lambda_x} \otimes \Phi_n^{\lambda_y^*}. \quad (4.7)$$

To gauge the diagram terms $\chi^{(l,\theta)}$ the process on the additional degrees of freedom must transform in such a way as to counteract the matrix coefficients terms that depend on the local group elements in the above.

First we make explicit the background process involving the additional quantum reference frame system \mathcal{H}_l :

$$\chi^{(l,\theta)} \longrightarrow \chi'^{(l,\theta)} := \chi^{(l,\theta)} \otimes \mathcal{A} = \left(\sum_k \Phi_k^{\lambda_x} \otimes \Phi_k^{\lambda_y^*} \right) \otimes \mathcal{A}^l \quad (4.8)$$

where $\mathcal{A}^l := \sum_\lambda d_\lambda \sum_k \mathcal{A}_{kk}^{(l,\lambda)}$ and the summation is over all irreps λ of G with dimension d_λ and over all diagonal components of the matrix of process gauge couplings $\mathcal{A}^{(l,\lambda)}$. \mathcal{A} can be thought of as a background scalar, that is invariant under the global symmetry and makes the quantum reference frame on the link l into dynamical objects. Directly from Definition

4.2.3 it follows that \mathcal{A}^l is globally symmetric $\mathfrak{U}_{\mathbf{g}}[\mathcal{A}^l] = \sum_{\lambda} d_{\lambda} \sum_{k,m,n} v^{\lambda}(g^{-1})_{km} v^{\lambda}(g)_{nk} \mathcal{A}_{mn}^{(l,\lambda)} = \sum_{\lambda} d_{\lambda} \sum_{mn} \delta_{mn} \mathcal{A}_{mn}^{(l,\lambda)} = \mathcal{A}^l$ where we used that the irreducible representation v^{λ} is unitary so $\sum_k v^{\lambda}(g^{-1})_{mk} v^{\lambda}(g)_{kn} = (v^{\lambda}(g^{-1})v^{\lambda}(g))_{mn} = \delta_{mn}$.

Therefore $\chi'^{(l,\theta)}$ is symmetric under the global group action but transforms non-trivially under the local symmetry.

4.2.3 III. Gauging globally symmetric processes to local symmetry

The final step involves promoting the global symmetry to a local symmetry. Once the background quantum reference frames and process mode couplings are explicitly introduced then we discard the relative alignments by averaging over the local group action. Explicitly this mapping is achieved by:

$$\chi'^{(l,\theta)} \xrightarrow{\mathfrak{G}} \tilde{\chi}^{(l,\theta)} := \mathcal{G}[\chi'^{(l,\theta)}] \quad (4.9)$$

where \mathcal{G} denotes the G -twirling over the local group $G^{\times n}$ and is given by:

$$\mathcal{G}[\chi'^{(l,\theta)}] = \int_{G^{\times n}} \mathfrak{U}_{\mathbf{g}}[\chi'^{(l,\theta)}] d\mathbf{g}. \quad (4.10)$$

The G -twirling operation projects onto the symmetric subspace, herein the locally symmetric superoperators then the gauged superoperator $\mathfrak{G}[\chi'^{(l,\theta)}]$ is locally symmetric since for all $\mathbf{h} \in G^{\times n}$ we have $\mathfrak{U}_{\mathbf{h}}[\mathfrak{G}[\chi'^{(l,\theta)}]] = \mathfrak{U}_{\mathbf{h}}[\mathcal{G}[\chi'^{(l,\theta)}]] = \int_{G^{\times n}} \mathfrak{U}_{\mathbf{h}} \mathfrak{U}_{\mathbf{g}}[\chi'^{(l,\theta)}] d\mathbf{g} = \int_{G^{\times n}} \mathfrak{U}_{\mathbf{g}}[\chi'^{(l,\theta)}] d\mathbf{g}$, where the last equality follows from the invariance of Haar measure under multiplication by group elements.

In terms of the process gauge couplings, the gauging map takes a simple form:

$$\mathfrak{G}[\chi'^{(l,\theta)}] = (\mathbf{\Phi}^{\lambda_x})^T \cdot \mathcal{A}^{(l,\lambda)} \cdot \mathbf{\Phi}^{\lambda_y} = \sum_{n,m} \Phi_m^{\lambda_x} \otimes \Phi_n^{\lambda_y} \otimes \mathcal{A}_{mn}^{(l,\lambda)}.$$

where $\mathbf{\Phi}^{\lambda_x}$ denotes the vector of process modes $\{\Phi_k^{\lambda_x}\}_{k=1}^{dim(\lambda)}$ and similarly on system at y .

A quick check shows that indeed the gauging map takes the above form:

$$\begin{aligned}
\mathcal{G}[\chi^{(l,\theta)}] &= \sum_{\mu} d_{\mu} \int_{G^{\times n}} \mathfrak{U}_{\mathbf{g}} \left[\sum_{k,j} \Phi_k^{\lambda_x} \otimes \Phi_k^{\lambda_y^*} \otimes \mathcal{A}_{jj}^{(l,\mu)} \right] d\mathbf{g} \quad (4.11) \\
&= \sum_{\substack{\mu,k,m,n \\ j,m',n'}} d_{\mu} \int_{G^{\times n}} v^{\lambda}(g_x)_{mk} v^{\lambda^*}(g_y)_{nk} \Phi_m^{\lambda_x} \otimes \Phi_n^{\lambda_y^*} \otimes (v^{\mu}(g_x^{-1})_{jm'} v^{\mu}(g_y)_{n'j}) \mathcal{A}_{m'n'}^{(l,\mu)} d\mathbf{g} \\
&= \sum_{\substack{k,m,n \\ j,m',n'}} d_{\mu} \left(\int_{G^{\times 2}} \left(v^{\lambda}(g_x)_{mk} v^{\mu}(g_x^{-1})_{jm'} v^{\mu}(g_y)_{n'j} v^{\lambda^*}(g_y)_{nk} \right) d g_x d g_y \right) \Phi_m^{\lambda_x} \otimes \Phi_n^{\lambda_y^*} \otimes \mathcal{A}_{m'n'}^{(l,\mu)}
\end{aligned}$$

where the last line follows from the fact we act with the local group averaging operation on a term that acts non-trivially only on sites x and y (and the joining link between them). The integration in the brackets can be done independently as each of the terms are dependent on either g_x or g_y but not both. Therefore we have that:

$$\int_G v^{\lambda}(g_x)_{mk} v^{\mu}(g_x^{-1})_{jm'} d g_x = \int_G v^{\lambda}(g_x)_{mk} v^{\mu}(g_x)_{m'j}^* d g_x = d_{\lambda}^{-1} \delta_{\mu,\lambda} \delta_{kj} \delta_{mm'} \quad (4.12)$$

where the last equality follows from Schur orthogonality of the matrix coefficients. Similarly we also have:

$$\int_G v^{\mu}(g_y)_{n'j} v^{\lambda^*}(g_y)_{nk} d g_y = \int_G v^{\mu}(g_y)_{n'j} v^{\lambda}(g_y)_{nk}^* d g_y d g_x = d_{\lambda}^{-1} \delta_{\mu,\lambda} \delta_{kj} \delta_{nn'} \quad (4.13)$$

By substituting back into the equation 4.12 above we obtain the gauged version of the superoperator term $\chi^{(l,\theta)}$. We have that:

$$\begin{aligned}
\mathfrak{G}[\chi^{(l,\theta)}] &= \sum_{\substack{k,m,n \\ j,m'n'}} d_{\lambda}^{-1} \delta_{kj} \delta_{mm'} \delta_{nn'} \Phi_m^{\lambda_x} \otimes \Phi_n^{\lambda_y^*} \otimes \mathcal{A}_{m'n'}^{(l,\lambda)} \quad (4.14) \\
&= \sum_{n,m} \Phi_m^{\lambda_x} \otimes \Phi_n^{\lambda_y^*} \otimes \mathcal{A}_{mn}^{(l,\lambda)}
\end{aligned}$$

Finally, linearity ensures that any 2-symmetric quantum process \mathcal{E} can be gauged into $\mathfrak{G}(\mathcal{E})$, which is invariant under the local symmetry acting on the subsystems and additional quantum reference frames. The gauging procedure described in this section can then be summarised by the following theorem.

Theorem 4.2.4. *Given a 2-symmetric quantum operation $\mathcal{E} : \mathcal{B}(\mathcal{H}_A) \longrightarrow \mathcal{B}(\mathcal{H}_{A'})$ on a multipartite*

system then it can be gauged into a quantum operation $\mathfrak{G}(\mathcal{E})$ given by:

$$\mathfrak{G}[\mathcal{E}] = \sum_{\{(l_k, \theta_k)\}} c_{\{(l_k, \theta_k)\}} \tilde{\chi}^{(l_1, \theta_1)} \otimes \tilde{\chi}^{(l_2, \theta_2)} \otimes \dots \otimes \tilde{\chi}^{(l_r, \theta_r)}. \quad (4.15)$$

where for each i , $\tilde{\chi}^{(l_i, \theta_i)} = \mathfrak{G}[\chi^{(l_i, \theta_i)}] = \sum_{m,n} \Phi_m^{\lambda_x} \otimes \mathcal{A}_{mn}^\mu \otimes \Phi_n^{\lambda_y^*}$ is the corresponding gauged superoperator acting only on the subsystems joined by the link l_i .

Example 4.2.5. Suppose that the Hilbert space \mathcal{H}_l has a set of perfectly distinguishable basis states $\{|g\rangle\}$ encoding group elements. Let the background scalar specifying the process on the quantum reference frames be given by $\mathcal{A}^l(\sigma) = |e\rangle\langle e|$ for any $\sigma \in \mathcal{B}(\mathcal{H}_l)$ and every link l and irrep λ , where $|e\rangle$ is the encoding of the identity element of G . Then under the local symmetry $\mathfrak{U}_g[\mathcal{A}^l] = |g_x g_y^{-1}\rangle\langle g_x g_y^{-1}|$. Moreover, the local symmetry group average of the superoperator $\chi^{(l, \lambda)} \otimes \mathcal{A}^l$ will be given by:

$$\begin{aligned} \chi^{(l, \lambda)} &\longrightarrow \chi^{(l, \lambda)} \otimes \mathcal{A} \\ &\xrightarrow{\mathfrak{G}} \int_{G \times 2} \sum_k \mathfrak{U}_{g_x}[\Phi_k^{\lambda_x}] \otimes \mathfrak{U}_{g_y}[\Phi_k^{\lambda_y^*}] \otimes |g_x g_y^{-1}\rangle\langle g_x g_y^{-1}| d g_x d g_y \\ &= \sum_{m,n} \Phi_m^{\lambda_x} \otimes \Phi_n^{\lambda_y^*} \otimes L_{mn} \end{aligned} \quad (4.16)$$

where the operator $L_{mn} := \int_g v_{mn}^\lambda(g) |g\rangle\langle g| d g \in \mathcal{B}(\mathcal{H}_l)$ arises from the invariance of the Haar measure under group multiplication and a change of variables in the above integration. In this particular example the gauge couplings take the form $\mathcal{A}_{mn}^{(l, \lambda)}(\sigma_l) = L_{mn}$ for any $\sigma_l \in \mathcal{B}(\mathcal{H}_l)$.

4.3 Gauge fixing

In the context of gauge theory one can also consider the reverse process of gauging, namely fixing a gauge to map a local gauge symmetry to a global one. A gauged theory allows for many different physical configurations, all of which are related by gauge transformation and describe the same physics. Choosing a particular gauge removes the redundancy in the physical description that comes from the local gauge symmetry and is a convenient procedure to simplify calculations. For example in classical electromagnetism gauge fixing amounts to a particular choice of representative from all equivalent potentials. The choice itself often takes into account the practical physical problem of interest. In quantum field theory additional problems appear in that it becomes necessary to fix a gauge to avoid dealing with sums of integrals over all field

configuration that leads to undefined objects.

We discuss in the following how gauge fixing can be introduced in the formalism. We will assume that the quantum reference frame can perfectly encode group elements in the basis $\{|g\rangle\}_{g \in G}$. When dealing with gauged locally symmetric quantum operations, the procedure of fixing a particular gauge can be viewed as a pre- and post-selecting the reference frames onto particular group elements. As with the usual gauge theory formulations, the construction presented here is not necessarily unique.

In the same spirit as in the previous section, we illustrate gauge fixing for a two lattice site with subsystems A_x and A_y linked together by l , which contains the reference frame \mathcal{H}_l . Suppose that the quantum process $\tilde{\mathcal{E}} \in \mathcal{S}(\mathcal{H}_{A_x} \otimes \mathcal{H}_{A_y} \otimes \mathcal{H}_l)$ is invariant under the local group action such that $\mathfrak{U}_{(g_x, g_y)}[\tilde{\mathcal{E}}] = \tilde{\mathcal{E}}$.

For any group elements $h_1, h_2 \in G$ we pre- and post-select on the system \mathcal{H}_l before and after the quantum process \mathcal{E} such that we map:

$$\tilde{\mathcal{E}} \longrightarrow \tilde{\mathcal{E}}_{h_1, h_2} = (id \otimes \Pi_{h_2}) \circ \tilde{\mathcal{E}} \circ (id \otimes \Pi_{h_1}) \quad (4.17)$$

where $\Pi_h(\sigma) = |h\rangle\langle h|\sigma|h\rangle\langle h|$ is the projection onto the state $|h\rangle$ for any $\sigma \in \mathcal{B}(\mathcal{H}_l)$. This achieves the role of breaking the $G^{\times 2}$ symmetry action to a global symmetry only.

Under the local group action:

$$\mathfrak{U}_{(g_x, g_y)}[\tilde{\mathcal{E}}_{h_1, h_2}] = \tilde{\mathcal{E}}_{(g_x h_1 g_y^{-1}, g_x h_2 g_y^{-1})} \quad (4.18)$$

for any $g_x, g_y \in G$. This follows immediately from local invariance of \mathcal{E} . Explicitly for any $\tau \in \mathcal{B}(\mathcal{H}_{A_x} \otimes \mathcal{H}_{A_y} \otimes \mathcal{H}_l)$ we get:

$$\begin{aligned} \mathfrak{U}_{(g_x, g_y)}[\tilde{\mathcal{E}}_{h_1, h_2}(\tau)] &= \mathcal{U}_{(g_x, g_y)} \circ (id \otimes \Pi_{h_2}) \circ \tilde{\mathcal{E}} \circ (id \otimes \Pi_{h_1}) \circ \mathcal{U}_{(g_x, g_y)}^\dagger \\ &= U(g_x) \otimes U(g_y) \langle h_2 | \tilde{\mathcal{E}} \left(\langle h_1 | \mathcal{U}_{(g_x, g_y)}^\dagger(\tau) | h_1 \rangle \otimes | h_1 \rangle \langle h_1 | \right) | h_2 \rangle (U^\dagger(g_x) \otimes U^\dagger(g_y)) \otimes | g_x h_2 g_y^{-1} \rangle \langle g_x h_2 g_y^{-1} |. \end{aligned} \quad (4.19)$$

However as $\tilde{\mathcal{E}}$ is invariant under $\mathfrak{U}_{(g_x, g_y)}$ it follows that $\langle h_2 | \tilde{\mathcal{E}}(\langle h_1 | \mathcal{U}_{(g_x, g_y)}^\dagger(\tau) | h_1 \rangle \otimes | h_1 \rangle \langle h_1 |) | h_2 \rangle = \langle h_2 | \mathcal{U}_{(g_x, g_y)}^\dagger(\tilde{\mathcal{E}}(\mathcal{U}_{(g_x, g_y)}(\langle h_1 | \mathcal{U}_{(g_x, g_y)}^\dagger(\tau) | h_1 \rangle \otimes | h_1 \rangle \langle h_1 |))) | h_2 \rangle$. Therefore the local group actions on

A_x and A_y will cancel out in the above equation in order to give:

$$\mathfrak{U}_{(g_x, g_y)}[\tilde{\mathcal{E}}_{h_1, h_2}] = \langle g_x h_2 g_y^{-1} | \tilde{\mathcal{E}}(\langle g_x h_1 g_y^{-1} | \tau | g_x h_1 g_y^{-1} \rangle \otimes |g_x h_1 g_y^{-1}\rangle \langle g_x h_1 g_y^{-1}|) | g_x h_2 g_y^{-1} \rangle \otimes |g_x h_2 g_y^{-1}\rangle \langle g_x h_2 g_y^{-1}|$$

which is exactly the expansion of equation 4.18. Moreover, the operation $\tilde{\mathcal{E}}_{h_1, h_2}$ obtained after pre and post-selecting with group elements h_1 and respectively h_2 is invariant under the *local* symmetry if and only if the following conditions hold:

$$g_x h_1 g_y^{-1} = h_1 \quad \text{and} \quad g_x h_2 g_y^{-1} = h_2 \quad (4.20)$$

Therefore this implies that $\mathfrak{U}_{(h_1 g_y h_1^{-1}, g_y)}[\tilde{\mathcal{E}}_{h_1, h_2}] = \tilde{\mathcal{E}}_{h_1, h_2}$ where $g_y h_2 h_1^{-1} = h_2 h_1^{-1} g_y$. For the case where $h_1 = h_2 = h$ the constraint is trivially satisfied for all $g_y \in G$. The symmetry has been broken from a local symmetry to a global symmetry, up to a re-alignment of the local subsystems corresponding to $h \in G$. Equivalently,

$$\mathfrak{U}_{(h g h^{-1}, g)}[\tilde{\mathcal{E}}_{h, h}] = \tilde{\mathcal{E}}_{h, h} \quad (4.21)$$

and moreover since $\mathfrak{U}_{g_x, g_y}[\tilde{\mathcal{E}}_{h, h}] = \tilde{\mathcal{E}}_{h, h}$ holds if and only if $g_x = h g_y h^{-1}$ it follows that indeed the largest isotropy group of $\tilde{\mathcal{E}}_{h, h}$ is G (global symmetry) and *not* $G \times G$ (local symmetry) as was the case for the initial process $\tilde{\mathcal{E}}$.

Typically pre and post selection is a trace decreasing operation so fixing the gauge in the manner described above will result in a trace decreasing quantum channel that is invariant under the global symmetry. This does not pose a significant issue as we will illustrate by revisiting the example 4.2.5, which describes the gauging of a bipartite superoperator diagram term $\chi^{l, \lambda}$ with process gauge coupling superoperators given by state preparation. The gauged term acting on both systems and quantum reference frame took the form:

$$\mathfrak{G}(\chi^{l, \lambda}) = \sum_{m, n} \Phi_m^{\lambda_x} \otimes \Phi_n^{\lambda_y^*} \otimes \int v_{mn}^{\lambda}(g) |g\rangle \langle g| dg \quad (4.22)$$

Suppose we pre and post select the gauged term by projecting the quantum reference frame

onto the *fixed* state $|h\rangle$. Then we will obtain:

$$(id \otimes \Pi_h) \circ \mathfrak{G}(\chi^{l,\lambda}) \circ (id \otimes \Pi_h) = \sum_{m,n} \Phi_m^{\lambda_x} \otimes \Phi_n^{\lambda_y^*} \otimes \int \langle h|v_{mn}^\lambda(g)|g\rangle \langle g|h\rangle dg |h\rangle \langle h| \quad (4.23)$$

$$= \sum_{m,n} \Phi_m^{\lambda_x} \otimes \Phi_n^{\lambda_y^*} \otimes v_{mn}^\lambda(h) |h\rangle \langle h| \quad (4.24)$$

This is invariant under the (global) group action $\mathfrak{U}_{(g_x, g_y)}$ where $g_x = gh^{-1}$ and $g_y = h^{-1}g$ for all $g \in G$. If $h = e$, the identity element then we find that $id \otimes \Pi_e \circ \mathfrak{G}\chi^{l,\lambda} \circ id \otimes \Pi_e = \chi^{l,\lambda} \otimes |e\rangle \langle e|$, a particular case in which gauge fixing is trace preserving since $v_{mn}^\lambda(e) = \delta_{mn}$ for all λ . But even in general, while the resulting quantum channel after gauge fixing will be trace decreasing, the relevant features remain, particularly the local process modes involved are not affected and up to an overall realignment according to group element h they couple just as in the initial gauged term $\mathfrak{G}(\chi^{l,\lambda})$.

4.4 Examples of gauge theories revisited: a quantum information theory perspective

This purpose of this section is to illustrate the wide applicability of the gauging method that we have presented. In particular cases, it recovers well-known methods for gauging lattice gauge theories [109] or gauging quantum states [107]. These correspond to quantum channels that are unitary or state preparation respectively. The analysis presented goes much further than unitary dynamics and states, in that it allows to gauge essentially *any* globally symmetric quantum process allowed.

4.4.1 U(1) symmetry

Consider the abelian group $G = U(1)$, whose irreducible representations are one-dimensional and labelled by $\lambda = n$ for all integers $n \in \mathbb{Z}$.

The building blocks for processes on bipartite systems on A_x and A_y that are invariant under the global U(1) symmetry are of the form $\chi^{(l,n)} = \Phi^{\mathbf{n}_x} \otimes \Phi^{-\mathbf{n}_y}$ where the local process modes transform under the U(1) symmetry principle as $\mathfrak{U}_\phi[\Phi^{\mathbf{n}_x}] = e^{in\phi} \Phi^{\mathbf{n}_x}$ and $\mathfrak{U}_\phi[\Phi^{-\mathbf{n}_y}] = e^{-in\phi} \Phi^{-\mathbf{n}_y}$ for any angle $\phi \in S^1 \cong U(1)$. (These superoperators give rise to any 2-symmetric quantum

process. More general globally symmetric operations will contain terms acting non-trivially on 3 or more sites. For $U(1)$ these take a simpler form $\chi^\Delta = \Phi^{n_x} \otimes \Phi^{m_y} \otimes \Phi^{-(n+m)_z}$ for any triangle $\Delta = (x, y, z)$ with similar generalisations for more interactions. For simplicity the following discussion will also involve only 2-site interactions constructed out of superoperators of the form $\chi^{(l,n)}$.)

I. Introduce quantum reference frame \mathcal{H}_l on link l and its local group transformation

For a directed link $l = [x \rightarrow y]$ joining subsystems A_x and A_y we associate quantum reference frames \mathcal{H}_l spanned by orthogonal basis states $\{|m\rangle\}_{m \in \mathbb{Z}}$. Every group element $\phi \in S^1$ can be encoded in a coherent state on the circle:

$$|\phi\rangle = (2\pi)^{-1} \sum_{m \in \mathbb{Z}} e^{-im\phi} |m\rangle. \quad (4.25)$$

These form an orthonormal set of eigenvectors for the self-adjoint operator $\hat{\phi} = \int_{S^1} \phi |\phi\rangle \langle \phi| d\phi$, which is canonically conjugate to the angular momentum in the z -direction. In a distributional sense $\langle \phi | \theta \rangle = \delta(\phi - \theta)$ the coherent states perfectly encode all group elements of $U(1)$. Under the de-synchronised local group action they transform according to:

$$\mathcal{W}_{(g_x, g_y)}^l(|\phi\rangle \langle \phi|) = |g_x + \phi - g_y\rangle \langle g_x + \phi - g_y| \quad (4.26)$$

for all $g_x, g_y \in U(1)$.

II. Add a quantum process acting on \mathcal{H}_l that is invariant under the global symmetry

The simplest process on the quantum reference frames is the depolarizing channel $\mathcal{A}(\sigma_l) = |0\rangle \langle 0|$ where every state $\sigma_l \in \mathcal{B}(\mathcal{H}_l)$ is mapped into the coherent state for $\phi = 0$. This process is invariant under the global symmetry but non-trivial under the local symmetry.

III. Gauging the global U(1) symmetry to a local symmetry

Gauging of the globally symmetric superoperator $\chi^{(l,n)} = \Phi^{\mathbf{n}_x} \otimes \Phi^{-\mathbf{n}_y}$ involves a local group averaging to de-synchronize the relative alignment of system x and y .

$$\chi^{(l,n)} \longrightarrow \mathfrak{G}[\chi^{(l,n)}] = \int_{U(1) \times U(1)} e^{in(g_x - g_y)} \Phi^{\mathbf{n}_x} \otimes \Phi^{-\mathbf{n}_y} \otimes |g_x - g_y\rangle \langle g_x - g_y| dg_x dg_y. \quad (4.27)$$

However (translation) invariance of the Haar measure for U(1) leads to:

$$\mathfrak{G}(\chi^{(l,n)}) = \Phi^{\mathbf{n}_x} \otimes \Phi^{-\mathbf{n}_y} \otimes \int_{S^1} e^{in\phi} |\phi\rangle \langle \phi| d\phi = \Phi^{\mathbf{n}_x} \otimes \Phi^{-\mathbf{n}_y} \otimes \Delta^n \quad (4.28)$$

where $\Delta^n = \sum_{m \in \mathbb{Z}} |m+n\rangle \langle m|$.

Discussion

What if instead of $\mathcal{A}(\sigma_l) = |0\rangle \langle 0|$, we would choose the process on the reference frames to be $\mathcal{A}(\sigma_l) = |\phi_0\rangle \langle \phi_0|$ – this is a valid process that is also invariant under the global symmetry? In such a case the gauging procedure gives an additional phase $\mathfrak{G}[\chi^{(l,n)}] = \Phi^{\mathbf{n}_x} \otimes \Phi^{-\mathbf{n}_y} \otimes e^{in\phi_0} \Delta^n$.

What about gauge fixing a locally symmetric process of the form in equation 4.28 by pre and post-selecting on a fixed group element ϕ_0 ? This will map $\chi^{(l,n)}$ to a globally symmetric term $\tilde{\chi}_{\phi_0} = \Phi^{\mathbf{n}_x} \otimes \Phi^{-\mathbf{n}_y} \otimes e^{in\phi_0} |\phi_0\rangle \langle \phi_0|$. One way of interpreting this is that by fixing the gauge to be ϕ_0 we break the local symmetry to a global symmetry *and* an additional phase factor depending on the choice of gauge emerges.

In some ways this is reminiscent of the Bohm-Aharonov effect whereas a choice of non-zero gauge gives rise to an observable effect.

4.4.2 Unitary dynamics for a lattice gauge theory

We consider the unitary dynamics for a lattice gauge theory described in terms of the Kogut-Susskind Hamiltonian approach. The aim is to illustrate how this well-known formalism connects with the gauging procedure we developed. A full treatment on lattice gauge theories or their Hamiltonian formulation is beyond the goals of this thesis, and can be found in many excellent accounts [4].

Consider a 2D square lattice $\Gamma = (V, E)$ with a Hamiltonian describing nearest-neighbour interactions given by $H = \sum_x N(x) + \sum_{x, \epsilon \sim x} K(x, \epsilon)$, where $\epsilon \sim x$ denotes summation over vertices linked to x situated at an ϵ distance apart. The term $N(x) := \sum_k a_k^\dagger(x) a_k(x)$ corresponds to the local particle density observable at site x and the kinetic term corresponds to a hopping interaction between neighbouring sites and is given by the hermitian operator:

$$K(\mathbf{x}, \epsilon) := \sum_k a_k^\dagger(\mathbf{x} + \epsilon) a_k(\mathbf{x}) + a_k^\dagger(\mathbf{x}) a_k(\mathbf{x} + \epsilon). \quad (4.29)$$

The operators $a_k^\dagger(\mathbf{x})$ and $a_k(\mathbf{x})$ correspond to the creation or annihilation of particle in mode k at site \mathbf{x} . At every site in the lattice there is an associated d -dimensional Hilbert space $\mathcal{H}_\mathbf{x} = \text{span}_{\mathbb{C}}\{|k\rangle_\mathbf{x}\}_k$. We assume that there is an irreducible representation U of a compact group G acting on each $\mathcal{H}_\mathbf{x}$ with matrix coefficients $u_{ij}(g)$ for $i, j = 1, \dots, d$ and $g \in G$. The full Hilbert space \mathcal{H} is the Fock space, generated by the creation operators $\{a_k^\dagger(\mathbf{x})\}_{\mathbf{x} \in V, k}$ acting on a fiducial state $|\Omega\rangle$. Under the group action, the creation operators transform irreducibly as $a_k^\dagger(\mathbf{x}) \xrightarrow{U(g)} \sum_j u_{kj}(g) a_j^\dagger(\mathbf{x})$. Under the local group action, the transformation $U(g_\mathbf{x})$ is applied at site \mathbf{x} , and collectively we denote this by $\mathcal{U}_\mathbf{g}$ acting on operators where $\mathbf{g} = (\dots, g_\mathbf{x}, \dots)$.

The Hamiltonian is invariant under the global action where we apply the same group element $g_\mathbf{x} = g$ at every site $\forall \mathbf{x} \in V$. The unitary evolution generated by the Hamiltonian $\mathcal{E}(\rho) = e^{-iHt} \rho e^{iHt}$, for any $\rho \in \mathcal{B}(\mathcal{H})$ is therefore globally symmetric so that $\mathcal{U}_\mathbf{g}[\mathcal{E}] = \mathcal{E}$ for $\mathbf{g} = (g, g, \dots, g, \dots)$. The particle number operator $N = \sum_\mathbf{x} N(\mathbf{x})$ is also invariant under the local symmetry, however the hopping potential is not.

As previously done, to gauge dynamics to a local symmetry we introduce Hilbert spaces \mathcal{H}_l associated with the edges of the lattice. Each \mathcal{H}_l is spanned by a set of perfectly distinguishable states encoding group elements $\{|g\rangle_l\}_{g \in G}$ which transform under the local group action according to $\mathcal{U}_{(g_x, g_y)}(|g\rangle_l \langle g|_l) = |g_x g_y^{-1}\rangle_l \langle g_x g_y^{-1}|_l$ on the link l joining sites x and y .

For simplicity, we gauge the generator $\mathcal{L}(\rho) = -i[H, \rho]$ of the unitary dynamics. The first term of the Hamiltonian N is locally symmetric and the kinetic term is mapped under the gauging procedure to:

$$K(\mathbf{x}, \epsilon) \longrightarrow \tilde{K}(\mathbf{x}, \epsilon) = \sum_{mn} a_m^\dagger(\mathbf{x}) \otimes L_{mn}(l) \otimes a_n(\mathbf{x} + \epsilon) + h.c \quad (4.30)$$

where the link operator $L_{mn}(l) = \int_G u_{mn}(h) |h\rangle_l \langle h|_l dh$ acts on the space \mathcal{H}_l that connects sites \mathbf{x}

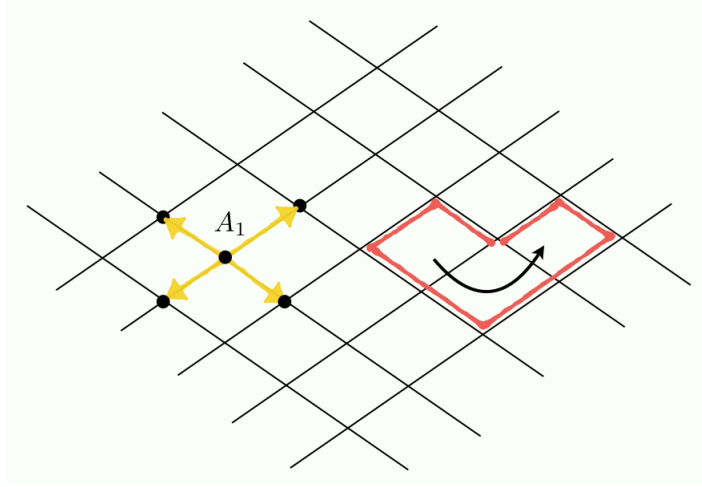


Figure 4.3: Lattice gauge theory

Subsystems are located at the vertices of the lattice, while quantum reference frames on the links. The generators $\{J_a\}$ of the local group action at a subsystem A_1 act non-trivially on the vertex as well as the 4 directed links surrounding it (yellow radial arrows). Gauss' law is satisfied if the state of the system is a symmetric state, $\rho = \mathcal{G}(\rho)$, under the full local symmetry. Within a fully symmetric scenario, the only observables that can be measured are those that are invariant under the symmetry group. Wilson loops (e.g. the red-loop shown) are defined purely on the reference frames, and are examples of such measurable observables.

and $\mathbf{x} + \epsilon$. Under the local gauge transformation the link operators are mapped to $\mathcal{U}_{\mathbf{g}}(L_{mn}) = \sum_{j,k} u_{mj}(g_{\mathbf{x}+\epsilon})u_{kn}(g_{\mathbf{x}}^{-1})L_{jk}$.

Finally the gauged process is generated by $\tilde{\mathcal{L}} = \mathcal{L}_0 + i\mathcal{L}_K - i\mathcal{L}_{K^*}$, where $\mathcal{L}(\rho) = -i[\tilde{N}, \rho]$ with $\tilde{N} = N \otimes id$ acting trivially on the gauge degree of freedom on the links and kinetic terms $\mathcal{L}_K(\rho) = i \sum_{\mathbf{x}=\epsilon} \tilde{K}(\mathbf{x} + \epsilon)\rho$ and $\mathcal{L}_{K^*}(\rho) = i \sum_{\mathbf{x}=\epsilon} \rho \tilde{K}(\mathbf{x} + \epsilon)$ for $\rho \in \mathcal{B}(\mathcal{H} \otimes_{l \in E} \mathcal{H}_l)$ state acting on the lattice sites and the links.

The process $\tilde{\mathcal{E}}$ generated by $\tilde{\mathcal{L}}$ is locally symmetric and describes the gauged dynamics at every lattice site.

4.4.3 Gauging quantum states

In this subsection we restrict the formalism for gauging quantum channels to quantum states, and therefore illustrate how previous work on gauging (pure) quantum states in [107] fits into

the general framework developed here. The work in [107] was motivated by tensor network approaches applied to quantum field theories and quantum chemistry. Their aim was to study tensor network states in the context of systems with gauge symmetry and apply gauging of quantum states to projected entangled pair states (PEPS) thus obtaining a toy model to study the phase diagrams of a \mathbb{Z}_2 gauge theory. For a multipartite pure state $|\psi\rangle \in \mathcal{H}_{A_1} \otimes \dots \mathcal{H}_{A_n}$ which is invariant under the global group action $U_1(g) \otimes U_2(g) \otimes \dots \otimes U_n(g)|\psi\rangle = |\psi\rangle$ for all $g \in G$ the gauging prescription in [107] maps $|\psi\rangle$ to a locally invariant state $P(|\psi\rangle \otimes \bigotimes_{l \in E} |e\rangle_l)$ acting on the vertices and the links, where P is the projector onto the physical subspace (i.e the trivial representation of the local group) with $P = \int U_{A_1}(g_1) \otimes \dots \otimes U_{A_n}(g_n) \bigotimes_{l=[x \rightarrow y] \in E} W_l(g_x, g_y) d\mathbf{g}$. Recall that we have denoted by $W_l(g_x, g_y) = L(g_x)R(g_y)$ the left and right action of G acting on Hilbert space $\mathcal{L}^2(G)$ associated with the links so if e denotes the identity group element then $W_l(g_x, g_y)|e\rangle_l = |g_x g_y^{-1}\rangle_l$. Therefore the gauging map given in [107] is:

$$|\psi\rangle \longrightarrow \int U_{\mathbf{g}}|\psi\rangle \bigotimes_{l=[x \rightarrow y]} |g_x g_y^{-1}\rangle d\mathbf{g} \quad (4.31)$$

Now we apply the gauging prescription described in Section 4.2 to the case where the quantum process \mathcal{E} is state preparation i.e $\mathcal{E} : 1 \longrightarrow |\psi\rangle\langle\psi|$. Thus formally the gauging map is of the form:

$$\mathfrak{G}(|\psi\rangle\langle\psi|) = \int U_{\mathbf{g}}|\psi\rangle\langle\psi|U_{\mathbf{g}}^\dagger \bigotimes_{l=[x \rightarrow y]} |g_x g_y^{-1}\rangle\langle g_x g_y^{-1}| d\mathbf{g} \quad (4.32)$$

where $\mathbf{g} = (g_1, \dots, g_n)$ and integration is over the group $G \times \dots \times G$. At this stage, the above holds any pure state $|\psi\rangle$ there is no extra assumption whether $|\psi\rangle$ is 2-symmetric or not. However to make the connection between the formalism here and the gauging procedure for quantum states in [107] we do not need to go further and specify what the gauge coupling processes are for this particular channel. The invariance property of Haar measure under group translations is enough to ensure that equation 4.31 and 4.32 are equivalent and give the same gauging map.

For completeness, we mention that the 2-symmetric assumption would allow to evaluate equation 4.32. What does a 2-symmetric state look like? For any three vertices i, j, k let ρ_{ijk} be the marginal state on these three subsystems A_i, A_j and A_k , that is $\rho_{ijk} = \text{Tr}_{i, \bar{j}, k}(|\psi\rangle\langle\psi|)$ where we trace all but the subsystems i, j, k . Then ψ is 2-symmetric if equivalently for any triplet of vertices the marginal state ρ_{ijk} acts trivially on at least one subsystem. In other words ρ_{ijk} must be either of $\rho_{ij} \otimes \mathbb{I}/d_{A_k}, \rho_{jk} \otimes \mathbb{I}/d_{A_i}$ or $\rho_{ik} \otimes \mathbb{I}/d_{A_j}$.

4.5 Connections between resource theories and Gauss's law

For gauge theories, the physically allowed states are those invariant under the local symmetry (i.e the gauge-invariant states). Gauging a global symmetry, as we have seen in the previous sections, involves introducing additional degrees of freedom described by quantum reference frames, together with their associated dynamical process and transformation under the local group symmetry. The total Hilbert space for the systems and QRFs is:

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_{A_1} \otimes \dots \otimes \mathcal{H}_{A_n} \bigotimes_{l=[x \rightarrow y]} \mathcal{H}_l. \quad (4.33)$$

Only a small subset of all states in \mathcal{H}_{tot} are physical since gauging is meant to introduce mathematical redundancies in the description, with the intent to map the global constraints to more tractable local ones. The physical quantities are those that remain unchanged under the gauge transformations. In our set-up, $|\psi\rangle \in \mathcal{H}_{\text{phys}} \subset \mathcal{H}_{\text{tot}}$ belongs to the physical Hilbert space if

$$U_{\text{tot}}(\mathbf{g})|\psi\rangle = |\psi\rangle \quad (4.34)$$

for all $\mathbf{g} \in G^{\times n}$, where the local symmetry $U_{\text{tot}} = U \otimes W$ acts on both the subsystems A_x (the "matter" systems) and the quantum reference frames \mathcal{H}_l (the "gauge fields").

An alternative description involves the generators of the symmetry. For a compact connected Lie group, the exponential map $\exp : \mathfrak{g} \rightarrow G$ is surjective and therefore representations of the group can be given in terms of representations of the Lie algebra. Therefore we can say:

$$U_x(g_x) = e^{i \sum_c Q_c(x) \theta_c(g_x)} \quad (4.35)$$

$$W_{l=[x \rightarrow y]}(g_x, g_y) = L(g_x)R(g_y) = e^{i \sum_c E_c^- \theta_c(g_x) + E_c^+ \theta_c(g_y)} \quad (4.36)$$

where $\theta_c(g) \in \mathbb{R}$ and $Q_c(x) \in \mathcal{B}(\mathcal{H}_{A_x})$, $E_c^-, E_c^+ \in \mathcal{B}(\mathcal{H}_l)$ are Hermitian operators that satisfy the commutation relations for generators of the Lie algebra representations: $[E_a^-, E_b^+] = 0$, $[E_a^\pm, E_b^\pm] = i \sum_c f_{abc} E_c^\pm$ and $[Q_a, Q_b] = i \sum_c f_{abc} Q_c$, where f_{abc} are the structure constants of \mathfrak{g} . The number of generators is the dimension of the Lie group.

Therefore it follows that the generator at each site x of the local symmetry is given by:

$$G(x) = \sum_c \left(Q_c(x) + \sum_{\substack{l \in E \\ \forall y: l=[y \rightarrow x]}} E_c^-(l) + \sum_{\substack{l \in E \\ \forall y: l=[x \rightarrow y]}} E_c^+(l) \right)$$

The operator $G(x)$ acts non-trivially only on the Hilbert space \mathcal{H}_{A_x} associated with vertex x and the links that connect x , that is on the space $\bigotimes_{y:l=[xy]} \mathcal{H}_l$. Therefore $G := \sum_x G(x)$ is the generator of the local symmetry U_{tot} acting on \mathcal{H}_{tot} .

Therefore the constraint in equation 4.34 that $|\psi\rangle$ belongs to the physical subspace can be equivalently cast in terms of the generators so $|\psi\rangle \in \mathcal{H}_{\text{phys}}$ then

$$G|\psi\rangle = |\psi\rangle \quad (4.37)$$

which implies that the physical states satisfy $G(x)|\psi\rangle = |\psi\rangle$ for all vertices x associated to subsystem A_x . This is a generalised form of Gauss's law.

In the context of classical electrodynamics, which is a classical U(1) gauge theory, Gauss's law relates the electric charge distribution with the electric field.

Example 4.5.1. For an abelian group $G = U(1)$ any representation has a single generator. Moreover the generators for the left and right representations are the same up to an overall minus sign $E^-(l) = -E^+(l)$ for any link l . Therefore on a square lattice of spacing 2ϵ horizontally and $2\epsilon'$ vertically the physical states must satisfy:

$$J(x)|\psi\rangle = 0 \quad (4.38)$$

where $J(x) = E(x + \epsilon) - E(x - \epsilon) + E(x + \epsilon') - E(x - \epsilon') - Q(x)$. In the continuous limit $\epsilon, \epsilon' \rightarrow 0$ we get the Gauss law in the usual form $(\nabla E(x) - Q(x))|\Psi\rangle = 0$ relating the divergence of the electric field E at point x with the local charge density $Q(x)$.

From the resource-theoretic perspective, Gauss's law coincides with the statement that one can only freely prepare states which are invariant under the local symmetry constraints.

In the resource theory of asymmetry (and similarly quantum reference frames) the 'free states' of the theory are those that are invariant under the symmetry constraints. An information-theoretic result states that under the local group symmetry, any state ρ cannot be distinguished from its

G-twirled state $\mathcal{G}[\rho]$ i.e from its projection on the invariant subspace:

$$\mathcal{G}[\rho] = \int_{G^{\times n}} \mathcal{U}_{\mathbf{g}}[\rho] \mathbf{g}. \quad (4.39)$$

In the resource theory of asymmetry for a local gauge symmetry group G , the free states of the theory coincide with the set of all convex mixtures of pure quantum states $|\Psi\rangle\langle\Psi|$ that obey the generalized Gauss's Law. The set of free operations within the resource theory coincide with the set of all locally gauge-invariant processes.

4.6 Further discussion

In this chapter we gave a procedure to gauge any general quantum process with global symmetry to a local symmetry at the expense of introducing additional degrees of freedom. To do so we leveraged on the structure of globally symmetric quantum channels on bipartite systems, which was developed in Chapter 2. The gauging analysis assumed a particular property that the processes involved must be 2-symmetric with the caveat that generalisations dealing with channels built out of local terms involving three or more subsystems can be treated in a similar way.

Typically in quantum field theories the Lagrangian describing the physical systems are truncated to a reasonable approximation that deals with the energy regimes of interest. This is related to the so-called UV cut-off in which higher order terms in the Lagrangian are suppressed.

Our restriction to the 2-symmetric processes should be viewed in an analogous way, only that in this case it is the higher order symmetry breaking interactions between three or more sites that are suppressed.

The construction we gave is certainly not unique. However it follows the same spirit as the minimal coupling principle in that no local symmetry breaking degrees of freedom of the globally symmetric processes are ignored and the gauged process contains all the information about the initial dynamics. Setting the "gauge field" to zero amounts to setting the gauge coupling processes $\mathcal{A}_{mn}^\lambda = \delta_{mn}\mathcal{A}$ for all m, n and irrep λ . This recovers exactly the initial globally symmetric process \mathcal{E} . By contrast, suppose that one did not introduce the additional background degrees of freedom and projected the global symmetry to a local symmetry by directly averaging

\mathcal{E} over the local symmetry group. The resulting process would also be invariant under the local symmetry, however all the initial symmetry breaking degrees of freedom would be lost and the only remaining information refers to the diagrams $\theta = (a, \tilde{a}) \xrightarrow{0} (b, \tilde{b})$ with $\lambda = 0$ contained in \mathcal{E} .

The analysis presented does not explicitly rely on the existence of an asymptotic classical regime, and the quantum reference frames need not be infinite dimensional so as to perfectly encode elements of a compact Lie group. However it is still the case that some choice of group action must be made, even when \mathcal{H}_l are finite-dimensional. It will be the case, that gauging particular quantum processes will necessarily result in a loss of information, as not all modes can be supported by a finite-sized reference. A trivial example is that gauging channels that are also locally symmetric does not require an infinite dimensional quantum reference frame. Therefore there will be limitations in how well the gauged dynamics reproduces the initial globally symmetric dynamics when some choice of gauge is made. Many other interesting features relating to use of finite dimensional reference frames on the links may also emerge.

A fundamental question in a gauge field theory concerns which measurements are allowed by the physics? Wilson loops correspond to gauge-invariant observables defined on the quantum reference frames, associated with links that form a closed path. However it was shown in [110] that relativistic causality places constraints on the measurements of Wilson loops.

Therefore it would be of interest to extend the formalism presented herein to allow for causal structures in order to connect with the wide area of tools on quantum measurements in an information theoretic setting. Such a long term goal may well give insights into delineating the physical content from the mathematical redundancies in gauge theories.

Chapter 5

Beyond Noether's Theorem

Conservation laws play an essential role in all areas of modern physics. In classical physics Noether's theorem fully quantifies how conserved currents emerge from *global* continuous symmetries of the dynamics. In quantum mechanics, Schrodinger's equation implies directly that the expectation value of an observable commuting with the Hamiltonian remains constant in time. Therefore the dynamics of closed quantum system incorporates Noether's theorem at a fundamental level. However, for general quantum processes there is a breakdown between conservation laws and symmetries. In [34], Marvian and Spekkens show that under an open system dynamics the consequences of symmetry are no longer fully captured by conservation laws in the sense that in order to characterise state transitions $\rho \rightarrow \sigma$ under a (non-unitary) symmetric dynamics \mathcal{E} then the necessary and sufficient conditions cannot be described only by the expectation value of the generators of the symmetry. In such cases novel measures of asymmetry need to be introduced, and in [14] it is shown that to fully characterise state transitions under symmetric dynamics a set of infinite entropic conditions are required. Whenever dissipative phenomena comes into play, both in the classical and quantum world there is a departure from Noether's theorem as these types of systems cannot be described via a stationary action integral. Baez gives in [111] a generalisation for classical Markov processes showing that an observable commutes with the Hamiltonian if and only if both its expectation value and standard deviation are constant in time. Dynamics of quantum systems are generally described by completely positive trace preserving maps – which may be viewed as arising from a unitary interaction with an environment that is discarded afterwards. In such a scenario, Noether's charges no longer capture all the consequences of the symmetric dynamics and describing state transformation

requires introducing a set of information-theoretic measures [34]. Moreover, the general necessary and sufficient conditions for state transformations under symmetry constraints involve an infinite set of entropic conditions [14].

This prompts the main question we address in this chapter: *How robust are conservation laws under a general symmetric quantum process?* We aim to explore the divide between conservation laws and symmetry. Specifically we quantify the trade-off relations between loss of purity (decoherence) and violations of conservation laws for systems that undergo a symmetric general quantum process. This is a difficult task to tackle in general terms because there is no a priori control over the degrees of freedom that are lost by decoherence nor is it easily determined which ones are relevant for the symmetry in question. One can show that small perturbations from a symmetric unitary dynamics lead to small deviations from conservation laws, while the converse is not always true and involves additional subtleties. However, for particular types of symmetry (that involves no multiplicities, for example a spin system under the defining representation of the rotation group) the conservation laws are robust under a symmetric open system dynamics.

Moreover there are fundamental limits to how much a conservation law is violated under a quantum channel subject to a continuous symmetry constraint. In particular for spin systems, we give the maximal deviations from conservation laws achievable under any symmetric dynamics under $SU(2)$.

5.1 Noether's charges and symmetric dynamics

Let \mathcal{H} be a d -dimensional quantum system carrying a representation $U : G \rightarrow \mathcal{B}(\mathcal{H})$ of the compact Lie group G . We will consider general quantum channels that are invariant under this symmetry principle. These are CPTP operations $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ with $\mathcal{U}_g \circ \mathcal{E} \circ \mathcal{U}_g^\dagger = \mathcal{E}$ where $\mathcal{U}_g(\cdot) = U(g)(\cdot)U(g)^\dagger$ for any $g \in G$. The connection between Lie groups and Lie algebras is established via the exponential map $exp : \mathfrak{g} \rightarrow G$. This is a diffeomorphism between a neighbourhood of $0 \in \mathfrak{g}$ and a neighbourhood of the identity element in G ; for matrix Lie groups it is given directly in terms of exponentiation of matrices.

To every finite-dimensional representation U of a Lie group there is a unique corresponding Lie algebra representation $R_U : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathcal{H})$ where $\mathfrak{gl}(\mathcal{H})$ denotes the space of linear maps on \mathcal{H}

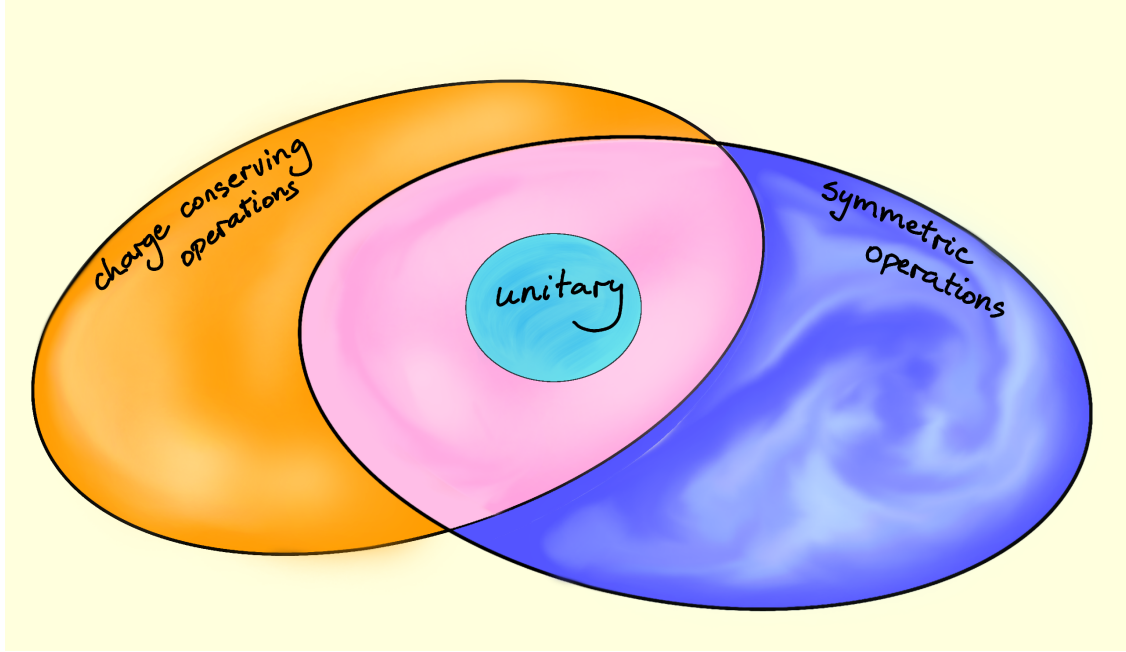


Figure 5.1: Symmetric operations vs charge conserving operations.

endowed with a Lie bracket structure given by the commutator. The Lie algebra representation ρ has n generators which we denote by J_1, J_2, \dots, J_n , where n is the dimension of the Lie group G . These can be thought of as the image of the generators of the Lie algebra \mathfrak{g} under ρ and satisfy the same commutation relations:

$$[J_i, J_j] = \sum_k f_{ijk} J_k \quad (5.1)$$

where f_{ijk} are the structure constants for \mathfrak{g} . As per the physics convention we will assume that the generators are Hermitian, so that the exponentiation carries an extra factor of i .

It is worth mentioning at this point that conversely, a Lie algebra representation only gives rise to a representation for the component of the group G that is connected to the identity. The exponentiation map gives a one-to-one correspondence between the representations of the Lie algebra and Lie groups whenever the latter are simply connected. For example the group of rotations $SO(3)$ and $SU(2)$ have isomorphic Lie algebras $\mathfrak{so}(3) = \mathfrak{su}(2)$ however any even dimensional irreducible representation of the Lie algebra does not lift to a representation of the full group $SO(3)$ with two distinct connected components, which only has odd dimensional irreducible representations.

The statement that a quantum operation \mathcal{E} on \mathcal{H} is invariant under the symmetry principle has an equivalent formulation in the algebraic setting. \mathcal{E} is symmetric if and only if for all generators

J_k with k ranging from 1 to n it holds that:

$$[J_k, \mathcal{E}(X)] = \mathcal{E}([J_k, X]) \quad (5.2)$$

for any operator $X \in \mathcal{B}(\mathcal{H})$.

The major feat of Noether's theorem has been to associate generators of the symmetry with conserved charges. A quantum process \mathcal{E} for which the generators $\{J_k\}_k$ are conserved is one for which the expectation value of any J_k with respect to the initial state ρ and final state $\mathcal{E}(\rho)$ remains unchanged for all states ρ :

$$\langle J_k \rangle_\rho = \langle J_k \rangle_{\mathcal{E}(\rho)} \quad \text{or equivalently} \quad \text{Tr}(J_k \rho) = \text{Tr}(J_k \mathcal{E}(\rho)). \quad (5.3)$$

There are many subtleties that arise with Noether's theorem as applied to quantum field theory or even classical field theory. For instance, Noether's charges need not necessarily form a Lie algebra, leading to anomalies. This is however beyond the aims of this chapter.

5.2 Quantifying deviations from conservation laws

Expectation values for the generators are typically not necessarily constant under a symmetric quantum process that is not unitary. In the following we consider a quantum operation \mathcal{E} on system \mathcal{H} and a set of observables (charges) $\{J_k\}_{k=1}^n$. For now, we make no assumption that the observables form a Lie algebra or that there is an underlying symmetry principle.

We introduce several measures to quantify how much the expectation values of these observables change under the dynamics specified by \mathcal{E} , and in effect probing quantitatively the departure from a conservation law under a general quantum process.

The *directional deviation* for the expectation value of the observable J_k with respect to the state $\rho \in \mathcal{B}(\mathcal{H})$ is given by

$$\Delta_{J_k}(\rho, \mathcal{E}) := \text{Tr}((\mathcal{E}(\rho) - \rho)J_k). \quad (5.4)$$

For multiple charges, one may be equally interested in the deviation from a conservation law for all of them. Therefore, we also define the *global deviation from conservation law* as the l_2 norm

of the directional deviations for all generators:

$$\Delta_{l_2}(\rho, \mathcal{E}) := \sum_{k=1}^n |\Delta_{J_k}(\rho, \mathcal{E})|^2, \quad (5.5)$$

which is additionally motivated by the fact that l_2 norm is rotationally invariant. It holds that $\Delta_{l_2}(\rho, \mathcal{E}) = 0$ if and only if the expectation values of all generators are conserved.

Although both Δ_{J_k} and Δ_{l_2} are state-dependent, it is straightforward to use these quantities to define state-independent measures. First, we can quantify the *worst-case deviation* by considering the maximal violation of the conservation law over all possible states,

$$\Delta_{\max}(\mathcal{E}) := \sup_{\rho} \Delta_{l_2}(\rho, \mathcal{E}). \quad (5.6)$$

Another way to quantify the violation in a state-independent way is to use *average deviation*,

$$\Delta(\mathcal{E}) := \int \Delta_{l_2}(|\psi\rangle\langle\psi|, \mathcal{E}) d\psi \quad (5.7)$$

where we integrate with respect to the induced Haar measure on the space of pure states.

Recall that any quantum operation \mathcal{E} has a (non-unique) Kraus decomposition $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$ with $\sum_i A_i^\dagger A_i = \mathbb{I}$. Using this notation, the directional deviation $\Delta_{J_k}(\rho, \mathcal{E})$ takes the form of:

$$\begin{aligned} \Delta_{J_k}(\rho, \mathcal{E}) &= \text{Tr} \sum_i \rho \left(A_i^\dagger J_k A_i - J_k \right) \\ &= \text{Tr} \sum_i \rho \left(A_i^\dagger A_i J_k + A_i^\dagger [J_k, A_i] - J_k \right) \\ &= \text{Tr} \sum_i \rho A_i^\dagger [J_k, A_i]. \end{aligned} \quad (5.8)$$

We can thus view the directional deviation of a generator J_k as the expectation value of an observable Q_k :

$$\Delta_{J_k}(\rho, \mathcal{E}) = \text{Tr}(\rho Q_k), \quad Q_k := \sum_i A_i^\dagger [J_k, A_i], \quad (5.9)$$

which depends on the both the generator J_k and the Kraus operators describing a given quantum operation \mathcal{E} . It is straightforward to check that Q_k is indeed hermitian.

This leads to a compact formulation for the average deviation:

$$\Delta(\mathcal{E}) = \frac{1}{d(d+1)} \left(\sum_k \text{Tr}(Q_k)^2 + \text{Tr}(Q_k^2) \right). \quad (5.10)$$

This follows from the following argument involving the second moment of the random variable $\text{Tr}(Q_k|\psi\rangle\langle\psi|)$. Using the definitions introduced in this section we have:

$$\Delta(\mathcal{E}) = \sum_k \int (\text{Tr}Q_k|\psi\rangle\langle\psi|)^2 d\psi. \quad (5.11)$$

However, since

$$(\text{Tr}Q_k|\psi\rangle\langle\psi|)^2 = \text{Tr}(Q_k \otimes Q_k)(|\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|), \quad (5.12)$$

we can use a general expression (see for instance [112]):

$$\int |\psi\rangle\langle\psi|^{\otimes n} d\psi = \frac{(d-1)!}{(d+n-1)!} \sum_{\pi \in S_n} P_\pi, \quad (5.13)$$

with P_π denoting representations of the symmetric group, i.e., permutations. For $n = 2$ the only permutations are identity and transposition (swap). Therefore, Eq. (5.11) becomes

$$\Delta(\mathcal{E}) = \sum_k \frac{(d-1)!}{(d+1)!} (\text{Tr}(Q_k^{\otimes 2}) + \text{Tr}(P_{SWAP}Q_k^{\otimes 2})), \quad (5.14)$$

which directly implies equation (5.10).

5.3 Quantifying openness of a quantum system

Every quantum process can be realised by a unitary evolution on an extended system including an environment followed by tracing out the extra degrees of freedom. We review some measures that quantify how far is the process from a unitary dynamics.

5.3.1 Definitions and properties

Worst-case scenario: The diamond norm

The diamond norm is often used in quantum information as a measure of distinguishability of quantum channels, not least because it carries an operational interpretation.

For a superoperator \mathcal{E} the *diamond norm* is defined by

$$\|\mathcal{E}\|_{\diamond} := \sup_{A \in \mathcal{B}(\mathcal{K} \otimes \mathcal{H})} \|(id \otimes \mathcal{E})(A)\|_1 \quad (5.15)$$

where id is the identity superoperator acting on the auxiliary Hilbert space \mathcal{K} and $\|A\|_1 := \text{Tr}(\sqrt{A^\dagger A})$ is the Schatten 1-norm. The diamond norm introduces a distance measure on the set of superoperators, and therefore on quantum channels.

It has the following operational meaning as the optimal probability of distinguishing between two given channels. More precisely, given a single use of a quantum channel chosen uniformly at random to be either \mathcal{E} or \mathcal{F} , the aim is to maximise the probability of guessing the correct choice $p = \frac{1}{2} + \frac{\|\mathcal{E} - \mathcal{F}\|_{\diamond}}{4}$.

Average-case scenario: Unitarity

Finally, we need to characterise how much the dynamics generated by a given quantum channel \mathcal{E} deviates from the closed unitary dynamics. To this end we will employ the notion of *unitarity*, originally introduced in [113] as a way to quantify how well a quantum channel preserves purity on average. Unitarity u of a quantum channel \mathcal{E} is defined by

$$u(\mathcal{E}) := \frac{d-1}{d} \int \text{Tr} \left(\mathcal{E} \left(\psi - \frac{\mathbb{I}}{d} \right)^2 \right) d\psi, \quad (5.16)$$

where the integral is over all pure states $\psi = |\psi\rangle\langle\psi|$ distributed according to Haar-measure. The unitarity represents the output purity over all input pure states with the identity component subtracted. It is defined in such a way, as opposed to just the average output purity of the channel, because we want to distinguish for instance between a map that sends all states to a fixed pure state $\mathcal{E}(\rho) = |0\rangle\langle 0| \text{Tr}(\rho)$ and a unitary dynamics. As defined, unitarity then satisfies $u(\mathcal{E}) \leq 1$ with equality if and only if the operation is a unitary, and can be efficiently estimated

using a process similar to randomized benchmarking [113]. A useful characterization of the unitarity of \mathcal{E} involves the unital part of $L(\mathcal{E})$. Namely, if one chooses the first element of the orthonormal basis defining the Liouville representation to be $T_1 = \mathbb{I}/\sqrt{d}$, then the unitarity can be expressed as:

$$u(\mathcal{E}) = \frac{1}{d^2 - 1} \text{Tr}(L_u(\mathcal{E})^\dagger L_u(\mathcal{E})), \quad (5.17)$$

where $L_u(\mathcal{E})$ is the Liouville matrix restricted to acting on the remaining $d^2 - 1$ dimensional space spanned by $\{T_i\}_{i=2}^{d^2}$.

Every quantum channel \mathcal{E} acting on \mathcal{H} has a Stinespring dilation given by an environment system \mathcal{H}_E and an isometry $V : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}_E$ such that:

$$\mathcal{E}(\rho) = \text{Tr}_E(V\rho V^\dagger). \quad (5.18)$$

This representation is unique only up to a local isometry acting on system \mathcal{H}_E . The complementary channel $\tilde{\mathcal{E}} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}_E)$ describes the information flow from the system into the environment and is defined by:

$$\tilde{\mathcal{E}}(\rho) = \text{Tr}_S(V\rho V^\dagger) \quad (5.19)$$

where the partial trace above is taken over the main system with Hilbert space \mathcal{H} .

The following lemma presents an alternative formula for the unitarity, in terms of the purities of both the channel \mathcal{E} and its complementary $\tilde{\mathcal{E}}$ acting on the identity. This does not appear in previous work on unitarity.

Lemma 5.3.1. *The unitarity of the quantum channel \mathcal{E} acting on a d -dimensional Hilbert space is given by:*

$$u(\mathcal{E}) = \frac{d}{d^2 - 1} \left(d \text{Tr}(\tilde{\mathcal{E}}(\mathbb{I}/d)^2) - \text{Tr}(\mathcal{E}(\mathbb{I}/d)^2) \right) \quad (5.20)$$

where $\tilde{\mathcal{E}}$ is the corresponding complementary channel.

Proof. The unitarity of a quantum channel \mathcal{E} can be written in terms of the purity of its Choi operator $J_{\mathcal{E}}$ and as shown in [113] it is given by $u(\mathcal{E}) = \frac{d}{d^2 - 1} (d \text{Tr}(J_{\mathcal{E}}^2) - \text{Tr}(\mathcal{E}(\mathbb{I}/d)^2))$. We just need to compute $\text{Tr}(J_{\tilde{\mathcal{E}}}^2)$ in terms of the complementary channel. The Choi operator is given by $J_{\mathcal{E}} = \frac{1}{d}(\mathcal{E} \otimes \mathbb{I}_S)(\sum_{i,j} |i\rangle\langle j| \otimes |i\rangle\langle j|)$ where $|i\rangle$ is an orthonormal basis for system \mathcal{H}_S . Therefore

in terms of the isometry $V : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}_E$ we have that:

$$\begin{aligned}
\text{Tr}(J_{\tilde{\mathcal{E}}}^2) &= \frac{1}{d^2} \sum_{i,j} \text{Tr}(\mathcal{E}(|i\rangle\langle j|)\mathcal{E}(|j\rangle\langle i|)) & (5.21) \\
&= \frac{1}{d^2} \sum_{i,j,m,n} \langle n|\mathcal{E}(|i\rangle\langle j|)|m\rangle\langle m|\mathcal{E}(|j\rangle\langle i|)|n\rangle \\
&= \frac{1}{d^2} \sum_{i,j,m,n} \langle j|V^\dagger|m\rangle\langle n|\otimes\mathbb{I}_E V|i\rangle\langle i|V^\dagger|n\rangle\langle m|\otimes\mathbb{I}_E V|j\rangle \\
&= \frac{1}{d^2} \sum_{j,m,n} \langle j|V^\dagger|m\rangle\langle n|\otimes\mathbb{I}_E VV^\dagger|n\rangle\langle m|\otimes\mathbb{I}_E V|j\rangle \\
&= \frac{1}{d^2} \sum_{j,m} \langle j|V^\dagger|m\rangle\langle m|\otimes\text{Tr}_S(VV^\dagger)V|j\rangle \\
&= \frac{1}{d^2} \sum_j \langle j|V^\dagger\mathbb{I}_S\otimes\text{Tr}_S(VV^\dagger)V|j\rangle \\
&= \frac{1}{d^2} \text{Tr}(V^\dagger\mathbb{I}_S\otimes\text{Tr}_S(VV^\dagger)V) = \frac{1}{d^2} \text{Tr}(\tilde{\mathcal{E}}^\dagger(\tilde{\mathcal{E}}(\mathbb{I}))).
\end{aligned}$$

The above allows to re-write the unitarity as $u(\mathcal{E}) = \frac{d}{d^2-1} \left(\text{Tr}(\tilde{\mathcal{E}}^\dagger(\tilde{\mathcal{E}}(\mathbb{I}/d)) - \text{Tr}(\mathcal{E}(\mathbb{I}/d)^2) \right)$. Suppose that $|e_m\rangle$ is an orthonormal basis for the environment \mathcal{H}_E , then we have:

$$\begin{aligned}
\text{Tr}(\tilde{\mathcal{E}}^\dagger\tilde{\mathcal{E}}(\mathbb{I}/2)) &= \sum_{m,n} \text{Tr}(\tilde{\mathcal{E}}^\dagger(|e_m\rangle\langle e_n|)\langle e_m|\tilde{\mathcal{E}}(\mathbb{I}/d)|e_n\rangle) & (5.22) \\
&= \sum_{m,n,j} \langle j|\tilde{\mathcal{E}}^\dagger(|e_m\rangle\langle e_n|)|j\rangle\langle e_m|\tilde{\mathcal{E}}(\mathbb{I}/d)|e_n\rangle \\
&= \sum_{m,n} \langle e_n|\tilde{\mathcal{E}}(\mathbb{I})|e_m\rangle\langle e_m|\tilde{\mathcal{E}}(\mathbb{I}/d)|e_n\rangle \\
&= d\text{Tr}(\tilde{\mathcal{E}}(\mathbb{I}/d)^2).
\end{aligned}$$

Substituting this into the unitarity formula we get the required result. \square

5.3.2 Symmetry constraints

Symmetry constraints impose extra structure that gives a simple operational formulation for the unitarity of a symmetric quantum process \mathcal{E} . For any such channels the Kraus operators transform as irreducible tensor operators and therefore we can label them by j, m, α with j labels the j -irrep in $U \otimes U^*$ of multiplicity α , and m a vector component that ranges from 1 to the $\dim(j)$. Therefore any symmetric channel under the adjoint action of U on superoperators

admits a Kraus decomposition of the form:

$$\mathcal{E}(\rho) = \sum_{j,m,\alpha} A_{j,m,\alpha} \rho A_{j,m,\alpha}^\dagger \quad (5.23)$$

where $\sum_{j,m,\alpha} A_{j,m,\alpha}^\dagger A_{j,m,\alpha} = \mathbb{I}$ and they transform as $U(g)A_{j,m,\alpha}U(g)^\dagger = \sum_{m'} v_{m'm}^j(g)A_{j,m',\alpha}$ for all $g \in G$.

We can similarly express the complementary channel $\tilde{\mathcal{E}}$ of \mathcal{E} in terms of the above Kraus operators. The environment system \mathcal{H}_E has an orthonormal basis that we label correspondingly by $\{|j, m, \alpha\rangle\}_{j,m,\alpha}$ which has dimension equal to the number of Kraus operators. Then \mathcal{H}_E with the isometry $V : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}_E$ given by $V = \sum_{j,m,\alpha} |j, m, \alpha\rangle \otimes A_{j,m,\alpha}$ is a Stinespring dilation with $\mathcal{E}(\rho) = \text{Tr}_E V \rho V^\dagger$. Therefore the complementary channel (which will be also symmetric) will take the form of:

$$\tilde{\mathcal{E}}(\rho) = \sum_{j,m,j',m'} |j, m, \alpha\rangle \langle j', m', \alpha' | \text{Tr}(\rho A_{j',m',\alpha'}^\dagger A_{j,m,\alpha}). \quad (5.24)$$

Since the Kraus operators form a system of irreducible tensors this implies that in particular they are orthogonal with respect to the Hilbert-Schmidt inner product, but not necessarily normalised. Equivalently this means that $\text{Tr}(A_{j',m',\alpha'}^\dagger A_{j,m,\alpha}) = \delta_{jj'} \delta_{mm'} \text{Tr}(A_{j,m,\alpha}^\dagger A_{j,m,\alpha})$. Therefore we get the following

$$\tilde{\mathcal{E}}(\mathbb{I}/d) = \sum_{j,m,\alpha} \frac{1}{d} |j, m, \alpha\rangle \langle j, m, \alpha | \text{Tr}(A_{j,m,\alpha}^\dagger A_{j,m,\alpha}) \quad (5.25)$$

with the corresponding purity given by:

$$\text{Tr}(\tilde{\mathcal{E}}(\mathbb{I}/d)^2) = \frac{1}{d^2} \sum_{j,m,\alpha} \text{Tr}(A_{j,m,\alpha}^\dagger A_{j,m,\alpha})^2. \quad (5.26)$$

We can denote by $E_{j,m,\alpha} := A_{j,m,\alpha}^\dagger A_{j,m,\alpha}$ which forms a POVM giving rise to the following probability distribution $p(j, m, \alpha) = \text{Tr}(A_{j,m,\alpha}^\dagger A_{j,m,\alpha})/d$. This allows a compact formulation for unitarity, that takes into account the symmetry constraints on \mathcal{E} as described in terms of Kraus operators. Therefore we obtain:

$$u(\mathcal{E}) := \frac{d^2}{d^2 - 1} \sum_{j,m,\alpha} p(j, m, \alpha)^2 - \frac{d \text{Tr}(\mathcal{E}(\mathbb{I}/d)^2)}{d^2 - 1}. \quad (5.27)$$

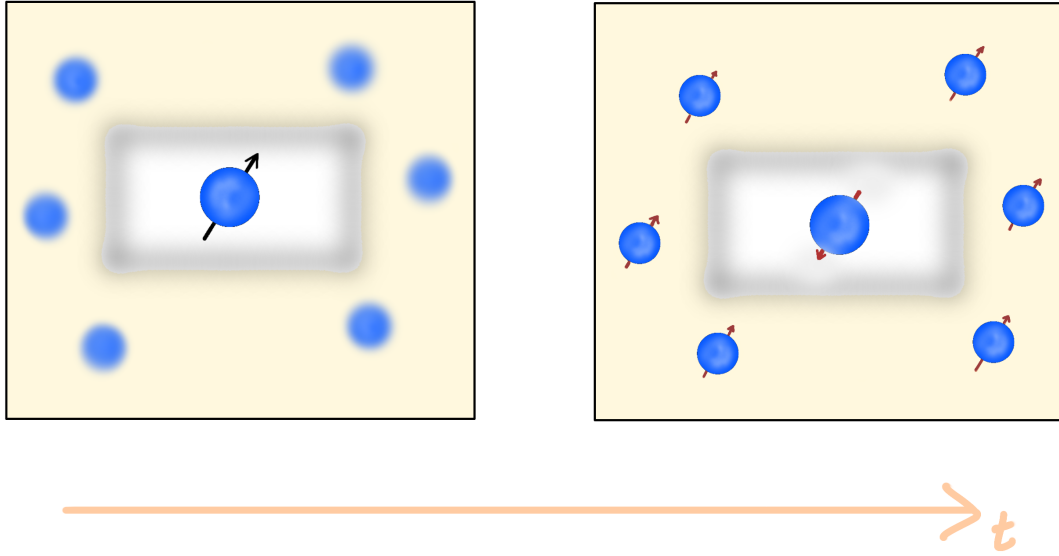


Figure 5.2: Spin polarization of an environment.

Under a symmetric open systems dynamics, the expectation values of the angular momentum can either decrease or increase. A rotationally symmetric channel that achieves maximal inversion of polarisation will have a complementary symmetric channel that maximally increases spin.

5.4 Maximal deviations from conservation laws for spin systems

Can angular momentum increase under symmetric dynamics? Suppose for example that the generators of the symmetry in question are the spin angular momenta $\mathbf{S} = (S_x, S_y, S_z)$ along Cartesian coordinates x, y, z acting on a qubit system. An initial state of the qubit system takes the form $\rho = \frac{1}{2}(\mathbb{I} + \mathbf{P} \cdot \mathbf{S})$ where \mathbf{P} is the spin polarization (or equivalently the Bloch vector for single qubit systems). Any symmetric quantum operation \mathcal{E} can only change the magnitude of the polarization vector and not the angles relative to the coordinate system such that $\mathbf{P} \xrightarrow{\mathcal{E}} \mathbf{P}' = f_1 \mathbf{P}$ holds for any \mathbf{P} . There are fundamental limitations that bounds the values of f_1 to the interval $f_1 \in [-\frac{1}{3}, 1]$. The extremal channel that gives a factor of $f_1 = -1/3$ corresponds to the optimal approximate universal NOT operation:

$$\mathcal{E}_{-\frac{1}{3}}(\rho) = \frac{\mathbb{I}}{4} + \frac{3}{4}(\sigma_X \rho \sigma_X + \sigma_Y \rho \sigma_Y + \sigma_Z \rho \sigma_Z). \quad (5.28)$$

However, every symmetric quantum operation can be realised by a symmetric unitary operation

acting on the initial system and an environment in a symmetric state. Additionally, the expectation values of the angular momenta are conserved under symmetric unitary dynamics. It follows that the environment that implements $\mathcal{E}_{-1/3}$ becomes heavily polarised. The complementary channel $\tilde{\mathcal{E}}_{-1/3}$ captures the flow of angular momenta from the qubit system into the environment. While $\tilde{\mathcal{E}}_{-1/3}$ is a symmetric channel the expectation value of the spin increases by $4/3$ so that for any ρ with $\langle \mathbf{S} \rangle_\rho = 1$ then $\langle \mathbf{S}_{\text{env}} \rangle_{\tilde{\mathcal{E}}_{-1/3}(\rho)} = \frac{4}{3}$. Therefore, while $\tilde{\mathcal{E}}_{-1/3}$ is symmetric, the expectation value of the angular momentum increases. Alternatively this example also illustrates that the angular momentum operators are not asymmetry monotones.

Quantifying how much can spin angular momentum be reversed under a symmetric dynamics is equivalent to determining the maximal deviation from a conservation law. The two concepts are different facets of the same coin and allow for a characterisation (at least for particular types of symmetries) of what is the maximal departure from Noether's theorem when we allow symmetric open system dynamics.

In this section, particularly subsections and we fully describe the set of extremal symmetric channels for spin systems. The analysis extends to symmetries for general (Lie) groups G , as long as the operator spaces have a multiplicity-free decomposition.

Further, such an analysis will be useful in both determining the maximal spin reversal, or equivalently the maximal deviation from a conservation law as well as a quantitative trade-offs between decoherence from unitary dynamics and deviations from conserved quantities.

5.4.1 Simplex representation of symmetric channels with multiplicity-free irreducible decomposition of the operator space

In here we show that the Liouville representation for particular types of symmetries takes on a simple characterisation where every symmetric channel corresponds to a vector in a complex vector space with the set of all symmetric channels forming a simplex.

If $\mathcal{E} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ is symmetric with respect to unitary representations U and U' of group G acting on \mathcal{H} and \mathcal{K} respectively, then equivalently its Liouville matrix $L(\mathcal{E})$ commutes with the group action:

$$(U' \otimes (U')^*)(g)L(\mathcal{E}) = L(\mathcal{E})(U \otimes U^*)(g) \quad (5.29)$$

for all $g \in G$. Because of the above relation, the Liouville matrix takes on a block structure with

each block corresponding to a irreducible representation (and its multiplicity space) that appears in the decomposition of both $U \otimes U^*$ and $U' \otimes (U')^*$. Without loss of generality these tensor representations decompose respectively as $\mathcal{H} \otimes \mathcal{H}^* \cong \bigoplus_{\lambda} \mathcal{H}_{\lambda} \otimes \mathbb{C}^{m_{\lambda}}$ and $\mathcal{K} \otimes \mathcal{K}^* \cong \bigoplus_{\lambda} \mathcal{H}_{\lambda} \otimes \mathbb{C}^{n_{\lambda}}$ where the group acts trivially on the multiplicity spaces, and \mathcal{H}_{λ} are carrier spaces for the λ -irreducible representations that appear in each decomposition. Then Schur's lemma implies that the Liouville operator acts non-trivially only on the multiplicity spaces:

$$L(\mathcal{E}) = \bigoplus_{\lambda} \mathbb{I}^{(\lambda)} \otimes L^{\lambda} \quad (5.30)$$

where λ ranges over all irreps that appear in both $U \otimes U^*$ and $U' \otimes (U')^*$ with L^{λ} a non-trivial $n_{\lambda} \times m_{\lambda}$ matrix with m_{λ} and n_{λ} the dimension of the multiplicity space of the λ irrep in $U \otimes U^*$ and $U' \otimes (U')^*$ respectively.

We focus on quantum channels $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ where the unitary representation U and U' of the group G act on \mathcal{H} and \mathcal{K} such that both $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ have multiplicity-free decompositions into irreducible components. Under such constraints, for any symmetric \mathcal{E} its Liouville matrix $L(\mathcal{E})$ is block diagonal and each block is proportional to the identity. For example this occurs for the group $SU(2)$, whenever \mathcal{H} and \mathcal{K} carry irreducible representations.

Theorem 5.4.1. *Let G be a group with U and U' unitary representation of the Hilbert spaces \mathcal{H} and \mathcal{K} of dimension $d_{\mathcal{H}}$ respectively $d_{\mathcal{K}}$. Suppose that $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ have multiplicity-free decompositions into irreps. Then if $\mathcal{E} \in \mathcal{S}(\mathcal{H}, \mathcal{H})$ is a symmetric operation the following hold:*

i) *The Liouville representation of \mathcal{E} has the form:*

$$L(\mathcal{E}) = \bigoplus_{\lambda} f_{\lambda} \mathbb{I}^{(\lambda)} \quad (5.31)$$

for some complex coefficients $f_{\lambda} \in \mathbb{C}$, where λ labels the irreducible representations in $U \otimes U^$ and $\mathbb{I}^{(\lambda)}$ is the identity acting on the subspace of dimension $\dim(\lambda)$.*

ii) *With respect to a complete basis of irreducible tensor operators¹ $\{T_k^{\lambda}\}_{\lambda,k}$ and $\{S_m^{\mu}\}_{\mu,m}$ for $\mathcal{B}(\mathcal{H})$ respectively $\mathcal{B}(\mathcal{K})$ then the symmetric operation \mathcal{E} acts on a state $\rho \in \mathcal{B}(\mathcal{H})$ as:*

$$\mathcal{E}(\rho) = \frac{\mathbb{I}}{d_{\mathcal{K}}} + \sum_{\lambda} f_{\lambda} \mathbf{r}^{\lambda} \cdot \mathbf{S}^{\lambda} \quad (5.32)$$

¹Note that the irreducible tensor operators must be chosen in such a way as to transform according to the same matrix coefficients. Otherwise the λ -block in $L(\mathcal{E})$ will not be the identity matrix but rather a unitary (corresponding to the isomorphism between λ -irrep subspaces of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$).

where the vector of coefficients \mathbf{r}^λ is defined by $r_k^\lambda := \text{Tr}(\rho(T_k^\lambda)^\dagger)$ such that $\rho = \frac{\mathbb{I}}{d_{\mathcal{H}}} + \sum_{\lambda} \mathbf{r}^\lambda \cdot \mathbf{T}^\lambda$.

Proof. i) Since $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ have a multiplicity free decomposition into irreps then in equation (5.30) the operators L^λ are one-dimensional as they act on the multiplicity space and therefore correspond to a single complex number f_λ . Hence the Liouville operator $L(\mathcal{E})$ is given by:

$$L(\mathcal{E}) = \bigoplus_{\lambda} f_{\lambda} \mathbb{I}^{\lambda} \quad (5.33)$$

where λ ranges over all irreps that appear in both $U \otimes U^*$ and $U' \otimes U'^*$. Note that $L(\mathcal{E})$ is a $d_{\mathcal{H}}^2 \times d_{\mathcal{K}}^2$ matrix which is zero when restricted to linear maps between \mathcal{H}^λ and \mathcal{H}^μ with non-isomorphic representations $\lambda \neq \mu$.

ii) From the definition of the Liouville representation $L(\mathcal{E})|A\rangle\rangle = |L(\mathcal{E}(A))\rangle\rangle$ and its block structure in equation (5.30) for symmetric operations \mathcal{E} it follows that in such a case it maps the irreducible tensor operator basis $\{T_k^\lambda\}_{\lambda,k}$ for $\mathcal{B}(\mathcal{H})$ and $\{S_k^\lambda\}_{\lambda,k}$ for $\mathcal{B}(\mathcal{K})$ accordingly such that $\mathcal{E}(T_k^\lambda) = f_\lambda S_k^\lambda$ for all λ and k . More generally it follows that any symmetric operation \mathcal{E} , where $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ have a multiplicity-free decomposition into irreps takes the form:

$$\mathcal{E}(\rho) = \sum_{\lambda} f_{\lambda} \left(\sum_k \text{Tr}((T_k^\lambda)^\dagger \rho) S_k^\lambda \right) \quad (5.34)$$

for any $\rho \in \mathcal{B}(\mathcal{H})$. □

The above result implies that to any symmetric operation $\mathcal{E} \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ one can associate a vector of complex coefficients $\mathbf{f} := (\dots, f_\lambda, \dots)$ with dimension equal to the number of irreps common for the decompositions of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$. We emphasize that the coefficients f_λ are associated with the whole λ -irrep subspace, so they are independent on the particular choice of the orthonormal operator basis. In this sense they are uniquely associated to the given symmetric channel \mathcal{E} . The identity operator in $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ are symmetric under the group action so they span the trivial subspaces. Since the decomposition is multiplicity free it follows that $\mathcal{E}(\mathbb{I}) = f_0 \mathbb{I}$. For trace preserving operations we get $f_0 = 1$.

For the group $G = SU(2)$ it is always the case that there is a choice of a complete set of hermitian irreducible tensor operators so that in this case $L(\mathcal{E})$ is hermitian and the coefficients f_λ are real. Moreover, imposing complete positivity on \mathcal{E} leads to a set of convex inequalities on the coefficients f_λ such that $\{\mathbf{f}(\mathcal{E}) : \mathcal{E} \text{ symmetric}\}$ forms a simplex.

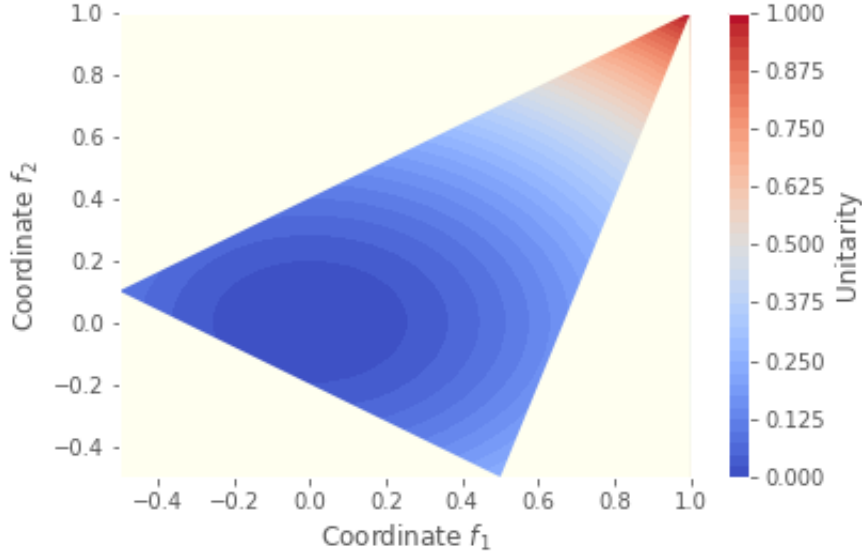


Figure 5.3: SU(2)-invariant qutrit channels.

For a qutrit \mathcal{H} with 1-irrep of SU(2) the operator space decomposes $\mathcal{B}(\mathcal{H}) \cong 0 \oplus 1 \oplus 2$. Any symmetric operation has a Liouville representation with three diagonal blocks and acts as a multiple of identity on the corresponding subspaces. The coordinates f_1 and f_2 correspond to the 1-irrep respectively 2-irrep blocks. The set of all channels is determined by the allowed regions for f_1 and f_2 as imposed by the CPTP conditions and is the convex hull of three extremal channels. Contour plot shows unitarity levels for the corresponding channels.

Example 5.4.2. For a qutrit $\mathcal{H} \cong \mathbb{C}^3$ carrying the 1-irrep of SU(2) there are three extremal channels Φ_0, Φ_1 and Φ_2 which are respectively characterised by the parameters $f^0 = (1, 1), f^1 = (\frac{1}{2}, -\frac{1}{2})$ and $f^2 = (-\frac{1}{2}, \frac{1}{10})$. Note that f^0 corresponds to the identity channel – the only symmetric unitary operation.

5.4.2 Liouville representation of extremal SU(2)-symmetric channels

We consider SU(2) irreducibly symmetric channels $\mathcal{E} : \mathcal{B}(\mathcal{H}_{j_1}) \longrightarrow \mathcal{B}(\mathcal{H}_{j_2})$ with highest weights (half integers) j_1 and j_2 labelling the respective irreducible representations. Denote by $C(j_1, j_2)$ the set of all such symmetric channels. Then there is a full characterisation of the extremal points of $C(j_1, j_2)$ – these have been previously analysed in [114]. In the last section 5.6.1 of this chapter we give an overview of that work by setting it in an information-theoretic language and give simplified proofs. We summarise those results in the following.

Theorem 5.4.3. Let $\Phi : \mathcal{B}(\mathcal{H}_{j_1}) \longrightarrow \mathcal{B}(\mathcal{H}_{j_2})$, where \mathcal{H}_{j_1} and \mathcal{H}_{j_2} carry irreducible representations V^{j_1} and V^{j_2} of SU(2) with dimensions $2j_1 + 1$ respectively $2j_2 + 1$. The set of all such symmetric quantum operations $C(j_1, j_2)$ has $2 \max(j_1, j_2) + 1$ extremal points $\Phi_l \in \mathcal{S}(\mathcal{H}_{j_1}, \mathcal{H}_{j_2})$, each labelled by

$l \in \{|j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2\}$ such that:

$$\Phi_l(|j_1, n\rangle\langle j_1, m|) = \sum_{k=-l}^l \langle j_2, m - k; l, k | j_1, m\rangle \langle j_2, n - k; l, k | j_1, n\rangle |j_2, n - k\rangle\langle j_2, m - k| \quad (5.35)$$

where $\{|j_1, m\rangle\}_{m=-j_1}^{j_1}$ and $\{|j_2, m\rangle\}_{m=-j_2}^{j_2}$ are orthonormal basis for \mathcal{H}_{j_1} and \mathcal{H}_{j_2} respectively.

However since the input and output spaces carry irreducible representations j_1 and j_2 of $SU(2)$ this means that the decomposition of the operator spaces into irreducible components is multiplicity free and therefore the previous results on the structure of the corresponding Liouville representation holds.

For each extremal channel Φ_l there is a corresponding unique vector \mathbf{f}^l of real coefficients. Therefore each of the vectors \mathbf{f}^l represent the extremal points that form a simplex in \mathbb{R}^d , where $d = 2\min(j_1, j_2)$. Since we have a full characterisation of the channels Φ_l then we can give closed form formulas for the vectors \mathbf{f}^l in terms of Clebsch-Gordan coefficients. In doing so, we will make use of the Wigner-Eckart theorem.

Theorem 5.4.4. *Let $\{T_k^\lambda\}_{k,\lambda}$ and $\{S_k^\lambda\}_{\lambda,k}$ be complete sets of orthonormal irreducible tensor operators for $\mathcal{B}(\mathcal{H}_{j_1})$ and $\mathcal{B}(\mathcal{H}_{j_2})$. Then the extremal channels $\Phi_l \in \mathcal{S}(\mathcal{H}_{j_1}, \mathcal{H}_{j_2})$ are:*

$$\Phi_l(\rho) = \sum_{\lambda} f_{\lambda}^l \left(\sum_k \text{Tr}((T_k^\lambda)^\dagger \rho) S_k^\lambda \right) \quad (5.36)$$

where the coefficients f_{λ}^l of the vector \mathbf{f}^l corresponding to channel Φ_l take the form:

$$f_{\lambda}^l = \frac{\langle j_1 || T^\lambda || j_1 \rangle}{\langle j_2 || S^\lambda || j_2 \rangle} \sum_{s=-l}^l \frac{\langle j_1, j_2 + s; \lambda, 0 | j_1, j_2 + s \rangle}{\langle j_2, j_2; \lambda, 0 | j_2, j_2 \rangle} \langle j_2, j_2; l, s | j_1, j_2 + s \rangle^2. \quad (5.37)$$

To simplify the exposition, we prove this theorem at the end of the present chapter in section 5.6.2

5.4.3 Unitarity of $SU(2)$ -symmetric dynamics

For symmetric operations acting on an irreducible space under $SU(2)$, it follows that the unitarity admits a simple formula. Suppose that \mathcal{H}_J carries the J -irrep of dimension $2J + 1$. Since $SU(2)$

is simply connected, there is a one-to-one correspondence with representations of the Lie algebra $\mathfrak{su}(2)$ such that the symmetry on \mathcal{H}_J has the generators $\{J_x, J_y, J_z\}$. In physical terms these correspond to angular momentum operators on \mathcal{H}_J along the three Cartesian axis.

Any symmetric operation $\mathcal{E} : \mathcal{B}(\mathcal{H}_J) \rightarrow \mathcal{B}(\mathcal{H}_J)$ acting on such a system admits Kraus operators $\{A_{j,m}\}$ where j labels irreps in $\mathcal{B}(\mathcal{H}_J)$ with $j \in \{0, 1, \dots, 2J\}$ and the vector component label is an integer $m \in [-j, j]$. For $SU(2)$ the commutation relations with the generators take the form:

$$\begin{aligned} [J_z, A_{j,m}] &= mA_{j,m} \\ [J_{\pm}, A_{j,m}] &= \sqrt{(j \mp m)(j \pm m + 1)}A_{j,m\pm 1} \end{aligned} \quad (5.38)$$

where in the above we have defined the ladder operators $J_{\pm} := J_x \pm iJ_y$. Therefore the directional deviations are given by

$$\begin{aligned} \Delta_z(\rho, \mathcal{E}) &= \sum_{j,m} m \text{Tr}(\rho A_{j,m}^{\dagger} A_{j,m}) \\ \Delta_{\pm}(\rho, \mathcal{E}) &= \sum_{j,m} \sqrt{(j \mp m)(j \pm m + 1)} \text{Tr}(\rho A_{j,m}^{\dagger} A_{j,m\pm 1}) \end{aligned} \quad (5.39)$$

where $\Delta_x = (\Delta_+ + \Delta_-)/2$ and $\Delta_y = (\Delta_+ - \Delta_-)/2i$. More specifically we have that

$$\Delta_z(\rho, \mathcal{E}) = \sum_{j,m} p(j, m|\rho)m = \sum_m p(m|\rho)m = \langle Q_z \rangle_{\rho}, \quad (5.40)$$

where $p(m|\rho) = \sum_j p(j, m|\rho) = \sum_j \text{Tr}(E_{j,m}\rho)$ is the probability distribution for the measurement on state ρ and $E_{j,m} = A_{j,m}^{\dagger} A_{j,m}$.

In the case of irreducible $SU(2)$ symmetric operations we have that \mathcal{E} is always a unital channel, and the decomposition is multiplicity free so that the unitarity becomes simply:

$$u(\mathcal{E}) = \frac{1}{d^2 - 1} \left(d^2 \sum_{j,m} p(j, m)^2 - 1 \right) \quad (5.41)$$

where we recall that the probability is $p(j, m) = p(j, m|\mathbb{I}/d) = \text{Tr}(A_{j,m}^{\dagger} A_{j,m})/d$. Additionally, this can be interpreted as the measurement probability distribution of output state of the complementary channel acting on the maximally mixed state, that is $p(j, m) = \text{Tr}(\tilde{\mathcal{E}}(\mathbb{I}/d)|j, m\rangle\langle j, m|)$. We can think of this as describing the coherent flow of information into the environment.

Alternatively, $p(j, m)$ can be related to the coefficients for \mathcal{E} in the Liouville representation. Recall from section 5.4.2 that any symmetric channel \mathcal{E} has an associated vector of $2J$ coefficients $\mathbf{f}(\mathcal{E}) = (f_1, \dots, f_{2J})$. The Liouville representation is block diagonal on each irrep in $j \otimes j^*$ that decomposes $\mathcal{B}(\mathcal{H})$ into irreducible subspaces and moreover $f_j = \text{Tr}(A_{jm}^\dagger A_{jm})$ is independent of m . The unitarity for an $SU(2)$ - irreducibly symmetric channel can be recast as:

$$u(\mathcal{E}) = \frac{1}{d^2 - 1} \left(\sum_{j=1}^{2J} (2\lambda + 1) f_j^2 \right). \quad (5.42)$$

5.4.4 Maximal spin reversal under symmetric dynamics

The spin polarisation \mathbf{P} is a vector whose components are the expectation values of the spin operators along the three spatial components of Cartesian axis $\mathbf{S} = (S_x, S_y, S_z)$.

For a spin-1/2 particle the spin operators are related to the Pauli matrices such that $\mathbf{S} = \frac{\hbar}{2} \boldsymbol{\sigma}$, which transforms irreducibly under the action of $SU(2)$ according to the fundamental irreducible representation of $SO(3)$. The spin polarisation cannot be universally reversed such that $\mathbf{P} \rightarrow -\mathbf{P}$ for all states under a valid CPTP map \mathcal{E} . However, the optimal approximate UNOT operation maps $\mathbf{P} \rightarrow -\frac{1}{3}\mathbf{P}$ so that we can say that for spin-1/2 particle the maximal spin reversal can be achieved by a factor of $\frac{1}{3}$.

How much can higher dimensional spins be universally reversed under a symmetric dynamics? What is the maximal spin reversal for quantum operations between different spin systems? These questions, while of interest on their own, are connected with the maximal deviations from conservation laws under a symmetric dynamics. The connection lies in the fact that the symmetric channels that achieves maximal spin reversal will also give the maximal deviation from a conservation law and vice-versa. We address these questions using the previous results on extremal channels with $SU(2)$ irreducible symmetry.

For higher dimensional spin j system with Hilbert space \mathcal{H}_j , the spin operators in Cartesian coordinates S_x, S_y, S_z are observables in $\mathcal{B}(\mathcal{H}_j)$ that satisfy the same commutation relations as the Pauli matrices. These can also be viewed as generators in the $\mathfrak{su}(2)$ Lie algebra representation corresponding to the unitary J -irrep of $SU(2)$ on \mathcal{H}_j . The Pauli matrices transform as $g\boldsymbol{\sigma}g^\dagger = V^1(g)\boldsymbol{\sigma}$ for all $g \in SU(2)$ where V^1 is the 1-irrep of $SU(2)$. Therefore the spin operators $\mathbf{S}^{(j)} = (S_x, S_y, S_z)$ will similarly transform as $V^j(g)\mathbf{S}(V^j(g))^\dagger = V^1(g)\mathbf{S}$ for all $g \in SU(2)$ where

V^j is the unitary j -irrep on \mathcal{H}_j .

For any state $\rho \in \mathcal{B}(\mathcal{H}^j)$ we can define the spin polarisation vector $\mathbf{P}(\rho) = (P_x(\rho), P_y(\rho), P_z(\rho))$ to be the expectation values of the spin operator along the cartesian axis. That is $P_a(\rho) := \text{Tr}(S_a \rho)$ for $a \in \{x, y, z\}$.

Under a symmetric quantum operation $\mathcal{E} : \mathcal{B}(\mathcal{H}_{j_1}) \longrightarrow \mathcal{B}(\mathcal{H}_{j_2})$ we ask what is the maximal spin reversal factor f such that $\mathbf{P} \xrightarrow{\mathcal{E}} -f\mathbf{P}$? Equivalently for any $\rho \in \mathcal{B}(\mathcal{H}_{j_1})$

$$\text{Tr}(\mathcal{E}(\rho)\mathbf{S}^{(j_2)}) = -f\text{Tr}(\rho\mathbf{S}^{(j_1)}) \quad (5.43)$$

and the aim is to determine a symmetric channel \mathcal{E} for which f is maximised. The spin operators span the 1-irrep subspace in the decomposition of $\mathcal{B}(\mathcal{H}_{j_1})$ respectively $\mathcal{B}(\mathcal{H}_{j_2})$ into irreps. According to the results section 5.4.1 and theorem 5.4.1, the λ -irrep component is mapped under \mathcal{E} into the λ -irrep component in a rigid way with some scaling coefficient f_λ for each irrep λ . Therefore, we have with this notation that $-f = f_1$ where f_1 is the coefficient corresponding to the 1-irrep subspace.

Any symmetric \mathcal{E} can be written as a linear combination of extremal channels:

$$\mathcal{E}(\rho) = \sum_l p_l \Phi_l(\rho) \quad (5.44)$$

where $p_l \geq 0$ and $\sum_l p_l = 1$. Each extremal channel is associated with a vector of coefficients \mathbf{f}^l with f_1^l corresponding to the 1-irrep. Suppose that \mathbf{f} corresponds to \mathcal{E} , then its 1-irrep coefficient is $f_1 = \sum_l p_l f_1^l$. The aim is to minimise f_1 (equivalently maximise $f = -f_1$) over all probability distributions. In effect, we are optimizing over a convex region and therefore the minimum will be attained on the boundary. In this case it will be given by one of the extremal points so that:

$$f = \max(-f_1) = \min_{\{p_l\}} \sum_l p_l f_1^l = f_1^{l_{\max}} \quad (5.45)$$

for some l_{\max} . We show in the appendix that $l_{\max} = j_1 + j_2$ and summarise the result in the following theorem.

Theorem 5.4.5. *The maximal spin polarisation reversal is achieved by an $SU(2)$ - irreducibly extremal channel $\mathcal{E} = \mathcal{E}^{(j_1+j_2)} : \mathcal{B}(\mathcal{H}_{j_1}) \longrightarrow \mathcal{B}(\mathcal{H}_{j_2})$ and therefore for any state ρ the sign of the expectation value*

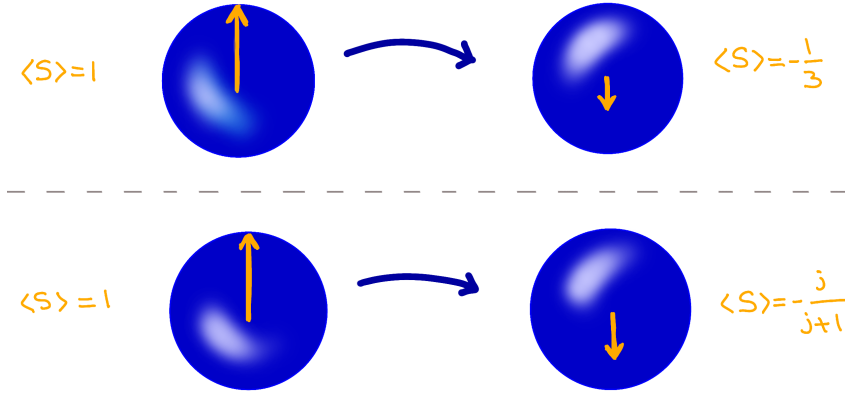


Figure 5.4: Maximal spin reversal under symmetric dynamics for 1/2-spin particle and j-spin particle.

of spin angular momentum is reversed by a factor of

$$\mathbf{P}(\mathcal{E}(\rho)) = -\frac{j_2}{j_1 + 1} \frac{(2j_2 + 1)}{(2j_1 + 1)} \mathbf{P}(\rho). \quad (5.46)$$

If the input and output systems have the same dimension $j_1 = j_2$ then then maximal spin reversal takes the simple form:

$$\mathbf{P}(\mathcal{E}(\rho)) = -\frac{j}{j + 1} \mathbf{P}(\rho). \quad (5.47)$$

It follows that the maximal spin reversal is achieved by the extremal channel that requires the largest environment. Indeed, for every extremal channel Φ_l , its minimal Stinespring dilation (and thus the number of Kraus operators) has dimension $2l + 1$. Consequently, this means that the larger the environment, the more spin can be reversed. Note that in the "classical" macroscopic limit $j_1 = j_2 = j \rightarrow \infty$ we get $f \rightarrow -1$ so that indeed spin can be perfectly reversed. While for finite dimensional systems perfect reversal of spin is not allowed by quantum theory, the above results give fundamental limits on maximal spin reversal under any valid symmetric quantum process.

Similarly, one can ask what is the maximal scaling of the spin polarisation vector for any fixed state and what covariant operation achieves the maximal spin amplification factor

$$A := \max_{p_i} \sum_l p_l f_1^l \quad (5.48)$$

As before, the maximum is attained on the boundary, and in the appendix we show that the extremal channel $\Phi_{|j_1-j_2|}$ achieves this and compute the factor $f_1^{|j_1-j_2|}$.

Theorem 5.4.6. *The maximal spin amplification under $\mathcal{E} : \mathcal{B}(\mathcal{H}^{j_1}) \longrightarrow \mathcal{B}(\mathcal{H}^{j_2})$ is achieved by an $SU(2)$ -irreducibly covariant extremal channel $\mathcal{E} = \Phi_{(|j_1-j_2|)}$ such that for any state ρ the expectation value the spin angular momentum scales by a factor of either:*

i) if $j_1 > j_2$

$$A = \frac{j_2}{j_1} \quad (5.49)$$

ii) if $j_2 < j_1$

$$A = \frac{j_2 + 1}{j_1 + 1} \quad (5.50)$$

iii) if $j_1 = j_2$ then $A = 1$ where

$$\mathbf{P}(\rho) \xrightarrow{\mathcal{E}} \mathbf{P}(\mathcal{E}(\rho)) = A\mathbf{P}(\rho). \quad (5.51)$$

Both of these results are derived in full detail in the last section of this chapter, respectively in subsections 5.6.3 and 5.6.4 and are a consequence of theorem 5.4.4.

5.4.5 Maximal deviation for spin angular momentum conservation

The results from the previous sections can be directly used to determine deviations from conservation laws of spin angular momentum. While we have defined quantifiers to probe this for operations with the same input and output system, the generalisation to different systems is straightforward. Let $\mathcal{E} : \mathcal{B}(\mathcal{H}_{j_1}) \longrightarrow \mathcal{B}(\mathcal{H}_{j_2})$ be a symmetric quantum operation and suppose that the generators of the j_1 irrep on \mathcal{H}_{j_1} and respectively j_2 irrep on \mathcal{H}_{j_2} . These generators correspond to spin angular momentum operators S_x, S_y, S_z for the input system respectively T_x, T_y, T_z for the output system. Then the deviation along direction $k \in \{x, y, z\}$ from a spin conservation law for the symmetric dynamics \mathcal{E} acting on initial state ρ is given by $\Delta_k(\rho, \mathcal{E}) = \text{Tr}(\mathcal{E}(\rho)T_k) - \text{Tr}(\rho S_k)$. Suppose that f corresponds to the diagonal factors in the Liouville representation of \mathcal{E} . Then since the spin operators transform as the 1-irrep of $SU(2)$ it follows that for all $\rho \in \mathcal{B}(\mathcal{H})$ we have $\mathcal{E}(S_k) = f_1 T_k$, and since this is hermitian $f_1 \in \mathbb{R}$ and therefore $\text{Tr}(\mathcal{E}(\rho)T_k) = f_1 \text{Tr}(\rho S_k)$ so that:

$$\Delta_k(\rho, \mathcal{E}) = (f_1 - 1)\text{Tr}(\rho S_k) \quad (5.52)$$

Which means that the average deviation will be:

$$\Delta(\mathcal{E}) = (f_1 - 1)^2 \sum_k \int \langle \psi | S_k | \psi \rangle^2 d\psi = (f_1 - 1)^2 \frac{3}{2j_1(2j_1 + 1)} \quad (5.53)$$

We have seen that $f_1 \in [f_1^{j_1-j_2}, f_1^{j_1+j_2}]$. Notice that the maximum deviation $\Delta(\mathcal{E})$ will be achieved by the minimal negative value possible for f_1 . It follows that:

$$\Delta(\mathcal{E}) = \left(1 + \frac{j_1}{j_2 + 1} \sqrt{\frac{j_2(j_2 + 1)(2j_1 + 1)}{j_1(j_1 + 1)(2j_2 + 1)}} \right)^2 \frac{3}{2j_1(2j_1 + 1)} \quad (5.54)$$

Equivalently, maximal deviation from a conservation law for spin system is achieved by a symmetric operation that maximally reverses spin.

In this case it is interesting to compare the average deviation from a conservation law with the unitarity given in equation (5.42) and we investigate in the following section the relationship between these two quantities.

5.5 Trade-off relations between unitarity and deviation from conservation laws

Building up on the results developed so far in this chapter we address the core question of interest, namely how to quantify the departure from conservation laws under symmetric dynamics subject to environment interactions. How robust to errors are conservation laws? We expect the deviations from conservation laws to be small under a symmetric dynamics that approximates unitary dynamics within a small error. Intuitively this is analogous to the classical case, where a small amount of dissipation has only a small effect on the the departure from conservation laws. Does the converse also hold in the quantum setting? In other words if we measure small errors in the conservation laws what does this say about the unitarity of the dynamics? A first step towards analysing this is to characterise those symmetric quantum operation for which conservation laws hold exactly but are not unitary.

In particular notice that $\Delta(\rho, \mathcal{E}) = 0$ if and only if all generators are conserved i.e $\Delta_k(\rho, \mathcal{E}) = 0$. Whenever \mathcal{E} is a symmetric unitary process \mathcal{U} , then Noether's theorem ensures that we have $\Delta(\rho, \mathcal{U}) = 0$ for all $\rho \in \mathcal{B}(\mathcal{H})$. In general, even if \mathcal{E} is a symmetric operation but not unitary, then

we expect $\Delta(\rho, \mathcal{E}) \neq 0$. Small perturbations from a unitary dynamics should result in small corrections to the conservation laws. Our aim is to characterise when each of the following two qualitative statements hold:

- If \mathcal{E} is close to a symmetric unitary then the deviation from a conserved law $\Delta(\rho, \mathcal{E})$ is small for all ρ .
- If the deviation $\Delta(\rho, \mathcal{E})$ is small then \mathcal{E} is close to a symmetric unitary.

5.5.1 Dynamics that approximates symmetric unitary have approximate conservation laws

We consider Hilbert systems \mathcal{H} that carry a representation of a Lie group that has generators J_1, \dots, J_n . Given *any* symmetric unitary U it commutes with the generators of the symmetry and $\Delta_k(\rho, \mathcal{U}) = 0$ so that we can write for all quantum operations $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$:

$$|\Delta_k(\rho, \mathcal{E})| = |\Delta_k(\rho, \mathcal{E}) - \Delta_k(\rho, \mathcal{U})| = |\text{Tr}((\mathcal{U} - \mathcal{E})(\rho)J_k)|.$$

For any two matrices A and B , Hölder's inequality holds for the Schatten p -norms such that $|\langle A, B \rangle| \leq \|A\|_1 \|B\|_\infty$. Therefore from the above we get the following upper bound for all k and ρ :

$$|\Delta_k(\rho, \mathcal{E})| \leq \|(\mathcal{U} - \mathcal{E})(\rho)\|_1 \|J_k\|_\infty \tag{5.55}$$

$$\leq \|\mathcal{U} - \mathcal{E}\|_\diamond \|J_k\|_\infty \tag{5.56}$$

where the last inequality is state-independent and follows directly from the definition of the diamond norm.

Recall that the deviation from a conservation law is $\Delta_{l_2}(\rho, \mathcal{E}) = \sum_k |\Delta_k(\rho, \mathcal{E})|^2$ and it follow from the above that it has a state-independent upper bound:

$$\Delta(\rho, \mathcal{E}) \leq \|\mathcal{U} - \mathcal{E}\|_\diamond^2 \sum \|J_k\|_\infty^2. \tag{5.57}$$

Since the bound does not depend on the state ρ then it holds equally well for the maximal and average deviations from a conservation law under the dynamics given by \mathcal{E} . We summarise

these in the following lemma.

Lemma 5.5.1. *Given an operation $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$, where the Hilbert space \mathcal{H} carries a Lie group representation with generators $\{J_1, \dots, J_n\}$. Then for any symmetric unitary U :*

$$\Delta(\mathcal{E}) \leq \Delta_{max}(\mathcal{E}) \leq \|U - \mathcal{E}\|_{\diamond}^2 \left(\sum_{k=1}^n \|J_k\|_{\infty}^2 \right) \quad (5.58)$$

The above analysis tells us that for any dynamics \mathcal{E} that is difficult to distinguish from a symmetric unitary then all the conservation laws hold within a small threshold error.

If \mathcal{E} is a channel that approximates a symmetric unitary dynamics such that $\|U - \mathcal{E}\|_{\diamond}^2 \leq \epsilon$ for some small $\epsilon > 0$, then lemma 5.5.1 above implies that the deviation from the conservation law is also small such as $\Delta(\mathcal{E}) \leq C\epsilon$, where C is a constant determine by the operator norm of the generators. In essence approximations of symmetric unitary dynamics lead to approximate version of Noether's theorem.

In the next subsection we determine under what conditions does the converse hold, meaning that we are determining when approximate conservation laws lead to small amount of decoherence, under a symmetric open system dynamics.

5.5.2 When do conservation laws hold (beyond unitary dynamics)?

Clearly it holds true that for symmetric unitary evolutions the generators of the symmetries are conserved, but what about non-unitary symmetric dynamics for which the conservation laws hold? In the following we explain how this situation may occur.

We use the following result from theorem 4.25 in [115]: *Let $\Phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a unital CPTP map such that it has Kraus decomposition $\Phi(X) = \sum_i A_i X A_i^\dagger$ for any $X \in \mathcal{B}(\mathcal{H})$. Then it holds that $\Phi(X) = X$ if and only if $[X, A_i] = 0$ for all i .*

Suppose \mathcal{E} is a symmetric channel; we want to characterise \mathcal{E} for which all the charges J_k are conserved and therefore satisfy $\Delta_{l_2}(\rho, \mathcal{E}) = 0$ for all states $\rho \in \mathcal{B}(\mathcal{H})$. Equivalently, $\text{Tr}(\rho J_k) = \text{Tr}(\mathcal{E}(\rho) J_k)$ for all k and all $\rho \in \mathcal{B}(\mathcal{H})$. Correspondingly, the adjoint channel \mathcal{E}^\dagger that represents \mathcal{E} in the Heisenberg picture satisfies $\mathcal{E}^\dagger(J_k) = J_k$ for all k . Note that if \mathcal{E} is a CPTP map then \mathcal{E}^\dagger is CP and unital.

Recall that any symmetric channel \mathcal{E} admits a Kraus decomposition $\{A_{j,m,\alpha}\}_{j,m,\alpha}$ where j labels irreps in $\mathcal{B}(\mathcal{H})$ of multiplicity α and vector component m . Then it follows that $\mathcal{E}^\dagger(X) = \sum_{j,m,\alpha} A_{j,m,\alpha}^\dagger X A_{j,m,\alpha}$. If in addition \mathcal{E}^\dagger is trace-preserving (or equivalently \mathcal{E} is unital) then the result stated above applies directly. Therefore $\mathcal{E}^\dagger(J_k) = J_k$ holds if and only if $[A_{j,m,\alpha}, J_k] = 0$ for all j, m, α and k .

Since from symmetry considerations the $A_{j,m,\alpha}$ form irreducible tensor operators the commutation relation implies that \mathcal{E} contains only those Kraus operators that transform trivially and therefore $j = 0$ (and consequently $m = 0$). In conclusion, for unital symmetric channels conservation laws hold if and only if \mathcal{E} takes the form:

$$\mathcal{E}_{\text{conserved}}(\rho) = \sum_{\alpha} A_{0,\alpha} \rho A_{0,\alpha}^\dagger \quad (5.59)$$

where each Kraus operator $A_{0,\alpha}$ commutes with the group action and satisfies $\sum_{\alpha} A_{0,\alpha}^\dagger A_{0,\alpha} = \sum_{\alpha} A_{0,\alpha} A_{0,\alpha}^\dagger = \mathbb{I}$.

The above consideration fully work out the case of when conservation laws hold for a unital symmetric dynamics \mathcal{E} . In general it may also be possible for conservation laws to hold for non-unital operations, but a full characterisation of the dynamics for which this happens remains open.

Whenever $\mathcal{B}(\mathcal{H})$ contains a single trivial subspace then there is no symmetric channel other than identity for which conservation laws hold. This is the case for instance when \mathcal{H} carries an irreducible representation of $SU(2)$.

The following provides an example of a *non-unitary* symmetric dynamics \mathcal{E} for which all conservation laws relevant for the symmetry hold. This illustrates that probing conservation laws for a physical realisation of a symmetric dynamics is not always sufficient to decide if there are decoherence effects present.

Example 5.5.2. Consider a two qubit system where the first qubit transforms under the 1/2-irrep of $SU(2)$ and the second transforms trivially. The conserved charges generating the symmetry are the spin operators on the first system. Let $\mathcal{E}_{AB}(\rho \otimes \sigma) := \rho \otimes \mathcal{E}_B(\sigma)$. This is covariant under the symmetry, the conservation laws hold for all states, however it is not a unitary operation as we are free to choose any CPTP \mathcal{E}_B on system B .

As we shall see in the next section, there are particular types of symmetries which guarantee a

certain robustness to conservation laws. In such cases, approximate conservation laws hold if and only if the dynamics is close to a unitary symmetric evolution.

5.5.3 Spin systems: approximate deviations from conservation laws lead to approximate symmetric unitary dynamics

In this subsection we will restrict to SU(2) symmetry given by the j -irrep of dimension $d = 2j + 1$ action on \mathcal{H} and we probe for conservation of spin along the directions given by a cartesian reference frame. Then, as described in Section 5.4.2 any symmetric channel \mathcal{E} will be described entirely by the coefficients f_λ for λ an irrep in $j \otimes j^*$. Therefore \mathcal{E} has an associated vector of $2j$ coefficients $\mathbf{f}(\mathcal{E}) = (f_1, \dots, f_{2j})$. Recall that the Liouville representation is block diagonal on each irrep in $j \otimes j^*$ that decomposes $\mathcal{B}(\mathcal{H})$ into irreducible subspaces and we have seen that the unitarity of such a channel is given by:

$$u(\mathcal{E}) = \frac{1}{d^2 - 1} \left(\sum_{\lambda=1}^{2j} (2\lambda + 1) f_\lambda^2 \right). \quad (5.60)$$

Similarly the average deviation was evaluated in section ?? to be:

$$\Delta(\mathcal{E}) = (f_1 - 1)^2 C(d) \quad (5.61)$$

for a constant factor $C(d) = \frac{3}{d(d-1)}$.

Theorem 5.5.3. *Suppose that \mathcal{E} is an SU(2)-irreducible covariant operation for which the average deviation from spin conservation is upper bounded by $\Delta(\mathcal{E}) \leq \epsilon^2 C(d)$ for some small $\epsilon > 0$. Then the unitarity of \mathcal{E} is:*

$$u(\mathcal{E}) = 1 - \mathcal{O}(\epsilon) \quad (5.62)$$

Proof. We aim to show that if $\Delta(\mathcal{E}) \leq \epsilon^2 C(d)$ for some small ϵ then unitarity of \mathcal{E} is close to 1. We have equivalently that $|f_1 - 1| \leq \epsilon$, and therefore $f_1 = 1 + \mathcal{O}(\epsilon)$. However the vertex $(1, 1, \dots, 1) \in \mathbb{R}^{d-1}$ forms an extremal point of the simplex of covariant operations and represents the identity channel. Moreover we can show from the exact formulas for the SU(2) extremal channels that $|f_k| \leq 1$ for all k and if $f_1 = 1$ then the only valid CPTP channel is the identity. Then (for any $\epsilon \leq \frac{1}{j(2j+1)}$) the hyperplane $f_1 = 1 - \epsilon$ separates the vertex $(1, 1, \dots, 1)$ from the rest of the $d - 2$ extremal points, which we will denote by the vectors $\mathbf{x}^1, \dots, \mathbf{x}^{d-2}$. Suppose that the

line that connects each extremal point \mathbf{x}^l with $(1, 1, \dots, 1)$ intersects the hyperplane at point \mathbf{y}^l . The latter satisfies $y_k^l = 1 - \epsilon \frac{1-x_k^l}{1-x_1^l}$. Any covariant channel with coordinates $\mathbf{f} = (f_1, f_2, \dots, f_{d-1})$ for which $|f_1 - 1| \leq \epsilon$ will lie in the simplex generated by $(1, 1, \dots, 1)$ and the points \mathbf{y}^l for l ranging from 1 to $d - 2$. Since the factors $\frac{1-x_k^l}{1-x_1^l}$ are constant and fixed by the dimension of the system, this implies that necessarily $\mathbf{f} = (1 - \mathcal{O}(\epsilon), \dots, 1 - \mathcal{O}(\epsilon))$. Hence the unitarity is close to one $u(\mathcal{E}) = 1 - \mathcal{O}(\epsilon)$. \square

The lemma above illustrates that some form of robustness of conservation laws may be expected even in the presence of decoherence for particular types of symmetries. In such cases small deviations from conservation laws imply that the symmetric dynamics involved is approximately unitary. Generally however there are significant limitations when the operator spaces contain trivial subspaces with dimension greater than one as we have shown in the preceding section.

5.6 Supplementary Results

This section contains some supplementary results and calculations that have been used in the present chapter.

5.6.1 Extremal irreducible $SU(2)$ -symmetric channels

We focus on the group $SU(2)$ acting irreducibly on the input and output Hilbert spaces and describe the extremal points of the set of symmetric channels under this group action. These have been examined in detail in previous literature under the name of EPOSIC channels [114]. The results in this subsection provide an overview of that work and we give alternative simpler proofs for some of the results, by casting it into a quantum-information friendly notation.

Consider a covariant quantum channel $\Phi : \mathcal{B}(\mathcal{H}_{j_1}) \rightarrow \mathcal{B}(\mathcal{H}_{j_2})$ whose input and output spaces carry irreducible representations of $SU(2)$ labelled by the highest weights j_1 and j_2 respectively. Recall that all irreducible representations of $SU(2)$ are indexed by highest weights that are half integer numbers i.e $0, 1/2, 1, 3/2, \dots$ and the carrier spaces correspond every possible dimension. The input space \mathcal{H}_{j_1} has dimension $2j_1 + 1$ and \mathcal{H}_{j_2} has dimension $2j_2 + 1$. If we denote by U_{j_1} and U_{j_2} the corresponding representations associating to every group element g an operator acting on \mathcal{H}_{j_1} and \mathcal{H}_{j_2} respectively then since Φ is $SU(2)$ -covariant it satisfies $U_{j_2}(g)\Phi(\rho)U_{j_2}(g)^\dagger =$

$\Phi(U_{j_1}(g)\rho U_{j_1}(g)^\dagger)$ for all $g \in SU(2)$. We will denote by $C(j_1, j_2)$ the convex set of all such $SU(2)$ symmetric quantum channels. The corresponding Choi matrix $J(\Phi) : \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \rightarrow \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$ defined via $J(\Phi) := Id \otimes \Phi(|\text{vec}(\mathbb{I})\rangle\langle\text{vec}(\mathbb{I})|)$ satisfies the covariance relation given by $[U_{j_1}(g) \otimes U_{j_2}^*(g), J(\Phi)] = 0$ for all $g \in SU(2)$. Moreover it satisfies the trace-preserving property: $\text{Tr}_2(J(\Phi)) = \mathbb{I}$. The problem of characterising the extremal points of $C(j_1, j_2)$ is equivalent to determining the extremal operators $J[\Phi]$ that satisfy the covariance property. For now we focus on the latter.

Recall the Clebsch-Gordan decomposition of a tensor product of $SU(2)$ irreps into irreducible components:

$$j_1 \otimes j_2^* \cong j_1 \otimes j_2 \cong \bigoplus_{l=|j_1-j_2|}^{j_1+j_2} l \quad (5.63)$$

where each irrep appears with multiplicity at most one. This will be a crucial point of the analysis. Any Choi operator will act on the tensor product space $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$, and we will show that for every l -irrep in the unique decomposition above there is an associated extremal point in the space of operators on $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$ that commute with the group action $j_1 \otimes j_2^*$.

The following lemma is a straightforward extension of Schur's lemma that is useful to characterise the structure of operators that commute with a given group action on the input and output spaces.

Lemma 5.6.1. *Suppose that V_1 and V_2 are representations of $SU(2)$ where each irreducible component in the decomposition of V_1 and V_2 has multiplicity one. Then any operator $J : V_1 \rightarrow V_2$ that commutes with the group action is a linear combination of projectors onto each irrep appearing in both V_1 and V_2 .*

Proof. By assumption, $V_1 = \bigoplus_{\mu} V_{\mu}$ and $V_2 = \bigoplus_{\lambda} V_{\lambda}$ decompose into irreps, each appearing at most once. Then if we restrict J to each irreducible subspace we get $J_{\mu,\lambda} : V_{\mu} \rightarrow V_{\lambda}$ with $J = \bigoplus_{\lambda,\mu} J_{\mu,\lambda}$. Because the decomposition is semisimple it means that each $J_{\mu,\lambda}$ commutes with the (restricted) group action. By Schur's lemma this implies that $J_{\mu,\lambda} \propto \delta_{\mu,\lambda} \mathbb{I}$. Therefore J is a linear combination of projectors onto each irrep that appears in both V_1 and V_2 . \square

Definition 5.6.2. *For every integer $|j_1 - j_2| \leq l \leq j_1 + j_2$ we define the orthogonal projection $\Pi_l : \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \rightarrow \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$ onto the l -irrep in the decomposition of $j_1 \otimes j_2^*$.*

Since each Π_l projects onto the l -irrep that has dimension $2l + 1$ then $\text{Tr}_2(\Pi_l) = \frac{2l+1}{2j_1+1} \mathbb{I}$. Therefore because any projector is a positive operator $\Pi_l \geq 0$, we can construct Choi operators $J_l :=$

$\frac{2j_1+1}{2l+1}\Pi_l$ that correspond to valid quantum channels (i.e CPTP) $\Phi_l : \mathcal{B}(\mathcal{H}_{j_1}) \longrightarrow \mathcal{B}(\mathcal{H}_j)$. In total there will be $2 \min(j_1, j_2) + 1$ different covariant quantum channels Φ_l and the following result shows that they fully characterise the extremal points.

Theorem 5.6.3. *Any Choi operator J that satisfies the covariance property $[U_{j_1}(g) \otimes U_{j_2}^*(g), J] = 0$ can be written as a unique convex combination of the Choi operators J_l for $|j_1 - j_2| \leq l \leq j_1 + j_2$. Equivalently $\{\Phi_l : l \in [|j_1 - j_2|, j_1 + j_2], l \in \mathbb{Z}\}$ are the extremal points of the set $C(j_1, j_2)$ of $SU(2)$ -covariant operations.*

Proof. We have that $J : \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \longrightarrow \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$ with the input and output carrying the $U_{j_1} \otimes U_{j_2}^*$ representation of $SU(2)$ that decomposes into irreducible components according to Eqn 5.63. Moreover J commutes with the group action so according to Lemma 5.6.1 we have that any Choi operator can be written as $J = \sum_l p_l J_l$. By the uniqueness of Choi representation, the same result holds for the corresponding quantum channels. \square

There is a natural symmetry-adapted orthonormal basis for \mathcal{H}_{j_1} and \mathcal{H}_{j_2} which is given by angular momentum states $\{|j_1, m\rangle, -j_1 \leq m \leq j_1\}$ and $\{|j_2, m\rangle, -j_2 \leq m \leq j_2\}$. Moreover the dual states correspond to $|j^*, m\rangle = (-1)^{j-m}|j, -m\rangle$ for any j and m . This gives the action of the intertwiner between the j -irrep and j^* -irrep of $SU(2)$ and allows a simple characterisation for the projection Π_l defined previously.

From coupling of angular momentum states we have that:

$$|l, k\rangle = \sum_{m, m'} \langle j_1, m; j_2, m' | l, k \rangle |j_1, m\rangle |j_2^*, m'\rangle \quad (5.64)$$

with $-l \leq k \leq l$ form an orthonormal basis for the l -irrep in the decomposition of $j_1 \otimes j_2^*$ where the coupling Clebsch-Gordan coefficients are real due to the choice of phase (i.e the standard Condon-Shortley) for the dual states. Then the projection Π_l onto the l -irrep subspace in the decomposition of $j_1 \otimes j_2^*$ takes the form of:

$$\Pi_l = \sum_{k=-l}^l |l, k\rangle \langle l, k| \quad (5.65)$$

Then, by taking into account the connection with the dual states and the Clebsch-Gordan expansion for states expressed in equation 5.64 we can re-write the projector Π_l in terms of the

standard angular momentum basis states in $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$.

$$\Pi_l = \sum_{m,n,m',n'} \sum_{k=-l}^l \langle j_1, m; j_2, m' | l, k \rangle \langle j_1, n; j_2, n' | l, k \rangle (-1)^{2j_2 - (m' + n')} |j_1, n\rangle |j_2, -n'\rangle \langle j_1, m | \langle j_2, -m' | \quad (5.66)$$

To simplify the expressions we can make use of the symmetries for the Clebsch-Gordan coefficients, namely that: $\langle j_1, m; j_2, m' | l, k \rangle = (-1)^{j_2 - m'} \sqrt{\frac{2l+1}{2j_2+1}} \langle j_2, -m'; l, k | j_1, m \rangle$ together with the selection rule stating this CGC is non-zero if and only if $m + m' = k$. Then above expression for the projection gives the extremal Choi operators J_l :

$$J_l = \sum_{m,n} \sum_{k=-l}^l \langle j_2, m-k; l, k | j_1, m \rangle \langle j_2, n-k; l, k | j_1, n \rangle |j_1, n\rangle \langle j_1, m | \otimes |j_2, n-k\rangle \langle j_2, m-k|. \quad (5.67)$$

Equivalently, the extremal CPTP channels take the following simple form in terms of their action on the symmetry-adapted basis:

$$\Phi_l(|j_1, n\rangle \langle j_1, m|) = \sum_{k=-l}^l \langle j_2, m-k; l, k | j_1, m \rangle \langle j_2, n-k; l, k | j_1, n \rangle |j_2, n-k\rangle \langle j_2, m-k| \quad (5.68)$$

where l ranges from $|j_1 - j_2|$ to $j_1 + j_2$ giving rise to $2\min(j_1, j_2) + 1$ different points corresponding to the full set of extremal covariant channels. Equivalently we can write down for each extremal channel Φ_l a set of $2l + 1$ Kraus operators $A_k^l := \sum_n \langle j_2, n-k; l, k | j_1, n \rangle |j_2, n-k\rangle \langle j_1, n|$ for $-l \leq k \leq l$.

5.6.2 Simplex representation of irreducibly symmetric extremal channels – A proof of Theorem 5.6.2

As before, let $\{T_k^\lambda\}_{k,\lambda}$ and $\{S_k^\mu\}_{k,\lambda}$ be irreducible tensor operator basis for $\mathcal{B}(\mathcal{H}_{j_1})$ and $\mathcal{B}(\mathcal{H}_{j_2})$ respectively. We have that $\Phi_l(T_k^\lambda) = f_\mu^l S_k^\mu \delta_{\mu,\lambda}$ for any l, λ, μ and k . The vector \mathbf{f}^l has entries f_λ^l with λ ranging from 1 to $\min(2j_1, 2j_2)$; for $\lambda = 0$ trace preserving condition implies that $f_0^l = \frac{1}{2j_2+1}$ is constant for all covariant channels so we will not include it further into the vector definition of \mathbf{f}^l .

With respect of the angular momentum states that form basis for \mathcal{H}_{j_1} and \mathcal{H}_{j_2} for any λ -irrep there exists m', n' and k labels such that $\langle j_2, n' | S_k^\lambda | j_2, m' \rangle \neq 0$. Therefore we can conveniently

re-write each coefficient as:

$$f_\lambda^l = \frac{\langle j_2, n' | \Phi_l(T_k^\lambda) | j_2, m' \rangle}{\langle j_2, n' | S_k^\lambda | j_2, m' \rangle} \quad (5.69)$$

where we re-iterate that at the core of our previous analysis is that the quantity above is independent of m', n' and k and this followed solely as a consequence of covariance of Φ_l . The numerator can be written in an equivalent form by a basis expansion $\langle j_2, n' | \Phi_l(T_k^\lambda) | j_2, m' \rangle = \sum_{m,n} \langle j_1, n | T_k^\lambda | j_1, m \rangle \langle j_2, n' | \Phi_l(|j_1, n\rangle \langle j_1, m|) | j_2, m' \rangle$. Therefore, by using the specific action of Φ_l on angular momentum states given in we obtain that:

$$\begin{aligned} f_\lambda^l &= \sum_{m,n} \frac{\langle j_1, n | T_k^\lambda | j_1, m \rangle \langle j_2, n' | \Phi_l(|j_1, n\rangle \langle j_1, m|) | j_2, m' \rangle}{\langle j_2, n' | S_k^\lambda | j_2, m' \rangle} \\ &= \sum_{n,m=-j_1}^{j_1} \sum_{s=-l}^l \frac{\langle j_1, n | T_k^\lambda | j_1, m \rangle}{\langle j_2, n' | S_k^\lambda | j_2, m' \rangle} \langle j_2, m-s; l, s | j_1, m \rangle \langle j_2, n-s; l, s | j_1, n \rangle \delta_{n',n-s} \delta_{m',m-s} \\ &= \sum_{s=-l}^l \frac{\langle j_1, n'+s | T_k^\lambda | j_1, m'+s \rangle}{\langle j_2, n' | S_k^\lambda | j_2, m' \rangle} \langle j_2, m'; l, s | j_1, m'+s \rangle \langle j_2, n'; l, s | j_1, n'+s \rangle \end{aligned} \quad (5.70)$$

To simplify the above expression further we can employ the Wigner-Eckart theorem which says that the matrix elements of an irreducible tensor operators depends on the vector component labels only through the Clebsch-Gordan coefficients. In particular:

$$\langle j_2, n' | S_k^\lambda | j_2, m' \rangle = \langle j_2, m'; \lambda, k | j_2, n' \rangle \langle j_2 || S^\lambda || j_2 \rangle \quad (5.71)$$

where $\langle j_2 || S^\lambda || j_2 \rangle$ is the reduced matrix element which is independent of n', m' or k . We can also write down Wigner-Eckart for the T_k^λ irreducible operator. This leads to the following form for the vector of coefficients for the extremal channel labelled by l :

$$f_\lambda^l = \frac{\langle j_1 || T^\lambda || j_1 \rangle}{\langle j_2 || S^\lambda || j_2 \rangle} \sum_{s=-l}^l \frac{\langle j_1, m'+s; \lambda, k | j_1, n'+s \rangle}{\langle j_2, m'; \lambda, k | j_2, n' \rangle} \langle j_2, m'; l, s | j_1, m'+s \rangle \langle j_2, n'; l, s | j_1, n'+s \rangle \quad (5.72)$$

In particular, since the above factor has no dependence on the labels m', n' and k then we can take without loss of generality $k = 0, m' = n' = j_2$ such that we obtain a much simpler expression:

$$f_\lambda^l = \frac{\langle j_1 || T^\lambda || j_1 \rangle}{\langle j_2 || S^\lambda || j_2 \rangle} \sum_{s=-l}^l \frac{\langle j_1, j_2+s; \lambda, 0 | j_1, j_2+s \rangle}{\langle j_2, j_2; \lambda, 0 | j_2, j_2 \rangle} \langle j_2, j_2; l, s | j_1, j_2+s \rangle^2 \quad (5.73)$$

5.6.3 Maximal reversal of spin polarization for spin j systems

We will consider a particular case when the input and output spaces have the same dimension and both carry the same irrep j of $SU(2)$. Therefore, by using the specific form that the extremal channels take for $j_1 = j_2 = j$ we obtain the following:

$$f_\lambda^l = \sum_{s=-l}^l \frac{\langle j, j+s; \lambda, 0 | j, j+s \rangle}{\langle j, j; \lambda, 0 | j, j \rangle} (\langle j, j; l, s | j, j+s \rangle)^2. \quad (5.74)$$

We characterise the range of values that the coefficient f_λ^l for $\lambda = 1$ takes while varying over all extremal channels l . These factors correspond to how much the spin polarization can scale (up or down) under a symmetric operation. We show that $-\frac{j}{j+1} \leq f_1^l \leq 1$ where the upper bound is attained for $l = 1$, the identity channel and the lower bound is attained for $l = 2j$ which corresponds to the extremal channel with the maximal number of Kraus operators. From the general formula for f_λ^l above we have that:

$$f_1^l = \sum_{s=-l}^l \frac{\langle j, j+s; 1, 0 | j, j+s \rangle}{\langle j, j; 1, 0 | j, j \rangle} (\langle j, j; l, s | j, j+s \rangle)^2. \quad (5.75)$$

We can evaluate the Clebsch-Gordan coefficients above since they are much simplified by our particular choice of irrep $\lambda = 1$. We have that $\frac{\langle j, j+s; 1, 0 | j, j+s \rangle}{\langle j, j; 1, 0 | j, j \rangle} = \frac{j+s}{j}$ and also that $(\langle j, j; l, s | j, j+s \rangle)^2 = \frac{(2j+1)!(2j+s)!(l-s)!}{(2j-l)!(l+2j+1)!(l+s)!(-s)!}$ is non-zero for $s \leq 0$ resulting in the following form for the coefficient associated with spin:

$$f_1^l = \frac{(2j+1)!}{(2j-l)!(l+2j+1)!} \sum_{s=0}^l \frac{(j-s)(2j-s)!(l+s)!}{j(l-s)!s!}. \quad (5.76)$$

It turns out that the above expression can easily be evaluated in terms of products of binomial coefficients so that:

$$f_1^l = \binom{l+2j+1}{l}^{-1} \left(\sum_{s=0}^l \binom{2j-s}{l-s} \binom{l+s}{s} - \frac{(l+1)}{j} \binom{2j-s}{l-s} \binom{l+s}{s-1} \right). \quad (5.77)$$

We can compute each of the two sums above separately by using combinatorial identities $\sum_{s=0}^l \binom{2j-s}{l-s} \binom{l+s}{s} = \binom{l+2j+1}{l}$ and $\sum_{s=0}^l \binom{2j-s}{l-s} \binom{l+s}{s-1} = \binom{l+2j+1}{l-1}$ to obtain a closed form formula

for the coefficients:

$$f_1^l = 1 - \frac{l+1}{j} \binom{l+2j+1}{l-1} \binom{l+2j+1}{l}^{-1} \quad (5.78)$$

$$= 1 - \frac{l(l+1)}{2j(j+1)} \quad (5.79)$$

Therefore, under any SU(2) symmetric operations the spin polarisation can either remain the same (whenever $l = 0$, a channel that corresponds to the identity), decrease for $0 \leq f_1^l \leq 1$ or get inverted for $f_1^l \leq 0$. However, in this particular scenario the spin polarization will never increase. The maximal spin polarisation reversal will therefore be given by:

$$f_1^{2j} = -\frac{j}{j+1}. \quad (5.80)$$

5.6.4 Maximal spin polarisation reversal under symmetric operations on spin systems $j_1 \longrightarrow j_2$

More generally, we consider symmetric operations $\mathcal{E} : \mathcal{B}(\mathcal{H}_{j_1}) \longrightarrow \mathcal{B}(\mathcal{H}_{j_2})$ where \mathcal{H}_{j_1} and \mathcal{H}_{j_2} carry the $2j_1 + 1$ and $2j_2 + 1$ dimensional irreducible representations of SU(2). The spin operators for \mathcal{H}_1 are denoted by \mathbf{S}^1 and for \mathcal{H}_2 by \mathbf{T}^1 . The components of these spin operators both transform as an irreducible tensor operators for the 1-irrep in $\mathcal{B}(\mathcal{H}_{j_1})$ respectively $\mathcal{B}(\mathcal{H}_{j_2})$. The aim is to determine the maximal factor f such that approximately reverses spin polarisation $\mathbf{P}_1 \xrightarrow{\mathcal{E}} \mathbf{P}_2 = -f\mathbf{P}_1$ for all input states $\rho \in \mathcal{B}(\mathcal{H}_{j_1})$ where $\mathbf{P}_1 = \text{Tr}(\mathbf{S}^1 \rho)$ and $\mathbf{P}_2 = \text{Tr}(\mathbf{T}^1 \mathcal{E}(\rho))$.

The symmetric operation \mathcal{E} , in terms of the extremal channels Φ^l for $l \in \{|j_1 - j_2|, \dots, j_1 + j_2\}$ decomposes as:

$$\mathcal{E} = \sum_{l=|j_1-j_2|}^{j_1+j_2} p_l \Phi^l. \quad (5.81)$$

As in the previous case, the minimisation of the coefficient $f_1 = \sum_l p_l f_1^l$ corresponding to the 1-irrep is attained for one of the extremal channels l_{\max} so that $-f = \min_{p_l} \sum_l p_l f_1^l = f_1^{l_{\max}}$.

In Section 5.4.2 we derived the exact form for the coefficients f_λ^l that characterise the extremal channel. For $\lambda = 1$ we have obtained previously that:

$$f_1^l = \frac{\langle j_2 || S^1 || j_2 \rangle}{\langle j_1 || T^1 || j_1 \rangle} \sum_s \frac{\langle j_1, j_2 + s; 1, 0 | j_1, j_2 + s \rangle}{\langle j_2, j_2; 1, 0 | j_2, j_2 \rangle} \langle j_2, j_2; l, s | j_1, j_2 + s \rangle^2. \quad (5.82)$$

The 1-irrep subspace in $\mathcal{B}(\mathcal{H}_{j_1})$ and $\mathcal{B}(\mathcal{H}_{j_2})$ will be spanned by the spin operator components which transform as irreducible tensor operators. The reduced matrix elements will be independent on the vector label component and since there is a single 1-irrep in the decomposition of the corresponding spaces, then the ratio $\frac{\langle j_2 || S^1 || j_2 \rangle}{\langle j_1 || T^1 || j_1 \rangle}$ does not depend on the particular operator basis. For this we use Wigner-Eckart together with the standard form for irreducible tensor operators S_0^1 and T_0^1 to evaluate a particular matrix element. Then we get that:

$$\langle j_2 || S^1 || j_2 \rangle = \frac{\langle j_2, m | S_0^1 | j_2, m \rangle}{\langle j_2, m; 1, 0 | j_2, m \rangle} = \frac{\sqrt{3}}{\sqrt{2j_2 + 1}} \quad (5.83)$$

so that the ratio of the reduced matrix element of \mathbf{S}^1 to \mathbf{T}^1 is $\sqrt{\frac{2j_1+1}{2j_2+1}}$.

Using this result together with binomial expansions for the Clebsch-Gordan coefficients involved we can arrive at a closed form formula for f_1^l . First notice that one of the terms in the above equation for f_1^l has a simple expression $\langle j_2, j_2; l, s | j_1, j_2 + s \rangle^2 = \frac{2j_1+1}{2j_2+1} \binom{1+j_1+j_2+l}{j_1-j_2+l}^{-1} \binom{j_1+j_2+s}{l+s} \binom{l-s}{j_1-j_2-s}$. Remark that for the coefficients to be non-zero we need that $-l \geq s \geq l$ and $j_1 - j_2 - s \geq 0$ where we recall that l takes one of the positive values in the set $\{|j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2\}$.

Therefore, using the binomial expression the coefficients f_1^l will become:

$$f_1^l = \sqrt{\frac{j_2(j_2+1)(2j_1+1)}{j_1(j_1+1)(2j_2+1)}} \binom{1+j_1+j_2+l}{j_1-j_2+l}^{-1} \sum_{s=-l}^{j_1-j_2} \frac{j_2+s}{j_2} \binom{j_1+j_2+s}{l+s} \binom{l-s}{j_1-j_2-s} \quad (5.84)$$

where in the summation only terms for which the two binomials exist contribute i.e $j_1 - j_2 - s \geq 0$. These correspond exactly to non-zero values of the relevant CGC in the previous summation. Changing the dummy summation variable from s to $w = s + l$ we have the alternative formulation:

$$f_1^l = \sqrt{\frac{j_2(j_2+1)(2j_1+1)}{j_1(j_1+1)(2j_2+1)}} \binom{1+j_1+j_2+l}{j_1-j_2+l}^{-1} \sum_{w=0}^{j_1-j_2+l} \frac{j_2-l+w}{j_2} \binom{j_1+j_2-l+w}{w} \binom{2l-w}{j_1-j_2+l-w}.$$

To compute the above we make use of the following combinatorial property: $\sum_{w=0}^a \binom{a+b+w}{w} \binom{c-w}{a-w} = \frac{(1+a+b+c)!}{a!(1+b+c)!} = \binom{1+a+b+c}{a}$ for $c \geq a$ and similarly $\sum_{w=0}^a w \binom{a+b+w}{w} \binom{c-w}{a-w} = \frac{(1+a+b+c)!(1+a+b)}{(a-1)!(2+b+c)!} = \binom{1+a+b+c}{a} \frac{a(1+a+b)}{2+b+c}$ for $a \neq 0$ and $c \geq a$; if $a = 0$ the latter sum clearly becomes zero. Now this means the coefficients are given by:

$$f_1^l = \sqrt{\frac{j_2(j_2+1)(2j_1+1)}{j_1(j_1+1)(2j_2+1)}} \left(\frac{j_2-l}{j_2} + \frac{(j_1-j_2+l)(1+j_1+j_2-l)}{2j_2(1+j_2)} \right) \quad (5.85)$$

Simplifying things a little more:

$$f_1^l = \sqrt{\frac{j_2(j_2+1)(2j_1+1)}{j_1(j_1+1)(2j_2+1)} \left(\frac{j_1(j_1+1) + j_2(j_2+1) - l(l+1)}{2j_2(1+j_2)} \right)} \quad (5.86)$$

For the different extremal channels with l from $|j_2 - j_1|$ to $j_1 + j_2$ the maximal value is attained for the closest valid value of l to $\frac{j_1 - j_2 + 1}{2}$. The minimal value in turn will always be attained by $l_{\max} = j_1 + j_2$ which gives:

$$-f = f_1^{j_1+j_2} = -\frac{j_1}{j_2+1} \sqrt{\frac{j_2(j_2+1)(2j_1+1)}{j_1(j_1+1)(2j_2+1)}}. \quad (5.87)$$

This corresponds to the extremal channel that requires the largest environment. Indeed, for every extremal channel Φ^l , its minimal Stinespring dilation (and thus the number of Kraus operators) has dimension $2l + 1$. Consequently, this means that the larger the environment, the more spin can be reversed. Note that in the "classical" macroscopic limit $j_1 = j_2 = j \rightarrow \infty$ we get $f = 1$ so that indeed spin can be perfectly reversed. While for finite dimensional systems perfect reversal of spin is not allowed by the theory, the above results give fundamental limits on maximal spin reversal under any valid symmetric quantum process.

Similarly the analysis can also give how much spin angular momentum can increase under a symmetric operation. This will also be achieved by one of the extremal channels, by similar convexity arguments as for spin reversal. Therefore the maximal value for f_1^l is attained for $l = |j_1 - j_2|$, which also corresponds to the extremal channel requiring the least number of Kraus operator and thus has minimal dimension of the environment. We have two cases: i) if $j_1 \geq j_2$ then

$$f_1^{|j_1-j_2|} = \sqrt{\frac{j_2(j_1+1)(2j_1+1)}{j_1(j_2+1)(2j_2+1)}} \quad (5.88)$$

and ii) if $j_1 \leq j_2$ then:

$$f_1^{|j_1-j_2|} = \sqrt{\frac{j_1(j_2+1)(2j_1+1)}{j_2(j_1+1)(2j_2+1)}} \quad (5.89)$$

Chapter 6

Conclusion

Symmetries provide structure to physical theories. In the context of quantum information, symmetry principles have found many applications to quantum speed limits, quantum metrology, quantum thermodynamics.

In this thesis we have explored these structures in order to provide a framework with which to analyse quantum operations under the presence of a symmetry principle. This has led to a diagrammatic decomposition of quantum operations in terms of process modes, superoperators that transform irreducibly under the group action. In Chapter 2 we develop this formalism by extending the resource theory of asymmetry techniques introduced by Marvian and Spekkens to quantum operations. Particularly, we give a model-independent analysis of the minimal symmetry-breaking resources required to implement arbitrary quantum operations and show how they are related to coordinates on a process orbit. The core result gives a structure theorem for bipartite quantum channels that are invariant under a global symmetry.

Techniques from information theory have recently been successfully applied to other areas such as condensed matter and high energy physics. One particular issue of interest is studying entanglement in the context of quantum field theory. There are significant challenges to overcome because the local structure of such theories makes it difficult to separate the physical state space as determined by Gauss's laws into a tensor product structure. The notion of entanglement is highly dependent on a relative choice of such a multipartite tensor product and therefore it suits a resource theoretic formulation. Hence there are increasing motivations to connect traditional Lagrangian methods with the formalism of information theory.

In Chapter 4 we focussed on addressing the question of how quantum operations with a global symmetry can be gauged to a local symmetry. Based on the decomposition of symmetric channels acting on bipartite system we provide an information-theoretic perspective on gauging general quantum processes. The work presented herein is a step towards bridging the traditional Lagrangian viewpoint with quantum information theory approaches. It contributes to this aim by showing how traditional Lagrangian symmetry techniques can be greatly generalised to arbitrary quantum processes and can be reformulated in terms of purely information-theoretic concepts such as quantum reference frames, completely positive trace preserving maps, pre- and post- selection.

As emphasized, gauging physical theories is not a unique procedure. Can we however formulate a minimal coupling principle for gauging quantum channels? Intuitively, our formulation keeps track of all the local symmetry breaking degrees of freedom existing between subsystems in such a way that the gauged channel can recover the initial globally symmetric dynamics. When the additional quantum reference frame systems are finite-dimensional this places constraints on the recoverability of the initial channel. A set of physically motivated conditions that single out a particular gauging map (or procedure) would play a similar role to a minimal coupling principle. For example one can show for a bipartite system, that the difference in mutual information between subsystems of the gauged and un-gauged dynamics given access to the internal quantum reference frames is related to the difference in the local asymmetry measure. It should be expected that the (physical) information content of the gauged and un-gauged theory are the same. However we leave these lines of investigation for future work.

The other application of the process mode formalism looks at optimal multiple use of coherence resources from a reservoir to implement target operations on a system under a global symmetric dynamics as generated by conservation of total energy. We show that under $U(1)$ symmetry constraints, the catalytic coherence protocol is essentially the unique protocol that allows for arbitrary repeatability in the use of the same coherence reservoir to implement a particular task.

There are several directions in which one may extend the framework we have presented in this thesis, particularly in connection with the application of gauging quantum channels. First one may further develop structure theorems for symmetric channels on multipartite systems analogous with the main result in section 2.4 on the decomposition of symmetric channels on bipartite systems. Such results will find application in gauging quantum channels beyond the restriction to 2-symmetric processes to include those that have non-trivial symmetry breaking

relations between three or more subsystems, whilst maintaining a global symmetry constraint. Second, one can extend the diagrammatic notation to a consistent graphical calculus with the aim of simplifying generalisations beyond the bipartite scenario.

Finally in Chapter 5 we discuss the trade-off between irreversibility and conservation laws under a symmetric dynamics. We show that if the dynamics is a small perturbation of a symmetric unitary evolution then conservation laws hold approximately within a small error. This rises the question whether detecting deviations from conserved quantities in a system undergoing a symmetric quantum process can give a quantitative estimate on the amount of decoherence present. For particular symmetries given by irreducible representations of $SU(2)$ we discover a certain robustness to conservation laws in that a small deviation from them implies that the unitarity (a measure of how much on average a process deviates from a unitary evolution) is close to one. An open direction to explore is if these results, particularly the fundamental bounds on deviations from conservation of spin angular momentum have any applications in mitigating the noise in quantum computations subject to symmetry constraints.

In contrast with the early literature on quantum reference frames, the applications in Chapter 3 and 4 involve the notion of quantum reference frames in a dynamical way. We are interested in not only using the quantum reference frames to implement a symmetry-breaking quantum operation and respectively to encode relational data, but more importantly in how these systems evolve in a way that is compatible with the task required. This is an essential aspect that can lead to misinterpretation of results that require the use of *dynamical* quantum reference frames rather than a quantum system in a particular token state.

Of course, in the case of gauging quantum processes the dynamical features of the quantum reference frames appear naturally. We have not discussed much the philosophical implications of gauging dynamics by introducing additional systems given by Hilbert spaces associated with each link. Should we view the background quantum reference frames as physical systems themselves? Are they simply convenient mathematical redundancies? When we restrict our formalism to unitary interaction governed by gauge symmetries acting on a lattice then the role of the interaction mediators – gauge fields – is played by the quantum reference frames. An open direction to explore is how to view the Aharonov-Bohm effect, both for $U(1)$ but also for the more subtle non-abelian symmetries, within the perspective developed here. The Aharonov-Bohm [116] effect illustrates a measurable physical effect in the form of relative phase shifts in the wavefunctions of two electrons moving in opposite directions in a region outside of a long

cylindrical solenoid. In such an experiment, there is no electric or magnetic potential outside of the cylinder, however there are non-zero potentials. This quantum mechanical phenomena suggests that the electrons in a region with no electric or magnetic fields can be affected by the electromagnetic potential. Different choices of gauges for the potential lead to distinct relative phase shifts, which marks a departure from classical electromagnetism where the gauge degrees of freedom correspond to redundancies in the physical description. The model describing the Aharonov-Bohm effect involves a Hamiltonian description, which could be mapped to the quantum-information type of formalism we developed for gauge theories. It would also be of interest to investigate connections between this effect and the types of measurements physically accessible in a gauge field theory, particularly from the point of view we developed here where the role of the gauge fields is taken by fully quantum mechanical system – the quantum reference frames. More specifically, is there an effect analogous to Aharonov-Bohm for irreversible quantum processes, and if so does it have additional features? While it is unlikely this will answer the ontological question regarding the physical content of gauge fields, it has the potential to open novel information theoretic questions on gauge theories in general.

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