Erratum: Higher Order Elicitability and Osband's Principle

Tobias Fissler* Johanna F. Ziegel†

January 28, 2019

Abstract. This note corrects conditions in Proposition 3.4 and Theorem 5.2(ii) and comments on imprecisions in Propositions 4.2 and 4.4 in Fissler and Ziegel (2016a).

Keywords: Consistency; Decision theory; Elicitability; Expected Shortfall; Point forecasts; Propriety; Scoring functions; Scoring rules; Spectral risk measures; Value at Risk

AMS 2010 Subject Classification: 62C99; 91B06

1 Proposition 3.4

As detailed Brehmer (2017) there are two technicalities that need to be resolved in Fissler and Ziegel (2016a, Proposition 3.4): Firstly, due to the particular choice of the integration path in the original version of Fissler and Ziegel (2016a, Proposition 3.4), the image of the integration path is not necessarily contained in $\mathrm{int}(\mathsf{A})$. Secondly, one needs to assume that the identification function V is locally bounded *jointly* in the two components. Proposition 1 gives a refined version of Fissler and Ziegel (2016a, Proposition 3.4).

Proposition 1. *Assume that* $\mathrm{int}(\mathsf{A}) \subseteq \mathbb{R}^k$ *is simply connected and let* $T\colon \mathcal{F} \to \mathsf{A}$ *be a surjective, elicitable and identifiable functional with a strict* $\mathcal{F}$*-identification function* $V\colon \mathsf{A} \times \mathsf{O} \to \mathbb{R}^k$ *and a strictly* $\mathcal{F}$*-consistent scoring function* $S\colon \mathsf{A} \times \mathsf{O} \to \mathbb{R}$*. Suppose that Assumption (V1), (V2), (S1) from Fissler and Ziegel (2016a) are satisfied. Let* h *be the matrix-valued function appearing at Fissler and Ziegel (2016a, Equation (3.2)).*

*Imperial College London, Department of Mathematics, Section Statistics, 180 Queen's Gate, London SW7 2AZ, United Kingdom, e-mail: t.fissler@imperial.ac.uk

†University of Bern, Department of Mathematics and Statistics, Institute of Mathematical Statistics and Actuarial Science, Alpeneggstrasse 22, 3012 Bern, Switzerland, e-mail: johanna.ziegel@stat.unibe.ch

For any $F \in \mathcal{F}$ and any points $x, z \in \mathrm{int}(\mathsf{A})$ such that $\gamma\colon [0,1] \to \mathrm{int}(\mathsf{A})$ is an integration path with $\gamma(0) = x$, $\gamma(1) = z$ the score difference is necessarily of the form

$$\bar{S}(x, F) - \bar{S}(z, F) = \int_\gamma \mathrm{d}\bar{S}(\cdot, F) = \int_0^1 h\big(\gamma(\lambda)\big)\bar{V}\big(\gamma(\lambda), F\big)\gamma'(\lambda)\,\mathrm{d}\lambda\,. \tag{1}$$

Moreover, if Assumptions (F1) and (VS1) from Fissler and Ziegel (2016a) are satisfied and V is locally bounded, then there is a Lebesgue null set $N \subseteq \mathsf{A} \times \mathsf{O}$ such that for all $(x, y) \in N^c$, $(z, y) \in N^c$ it necessarily holds that

$$S(x, y) - S(z, y) = \int_\gamma \mathrm{d}S(\cdot, y) = \int_0^1 h\big(\gamma(\lambda)\big)V\big(\gamma(\lambda), y\big)\gamma'(\lambda)\,\mathrm{d}\lambda\,, \tag{2}$$

where again $\gamma\colon [0,1] \to \mathrm{int}(\mathsf{A})$ is an integration path with $\gamma(0) = x$, $\gamma(1) = z$.

Proof. Equation (1) follows from Fissler and Ziegel (2016a, Theorem 3.2) and Königsberger (2004, Satz 2, p. 183). The proof of (2) follows the lines of the original proof in Fissler and Ziegel (2016b); cf. Brehmer (2017, Theorem 1.31) for details. □

2 Theorem 5.2(ii)

The complication in the proof of Theorem 5.2(ii) in Fissler and Ziegel (2016a) can be found on p. 1702 directly under the equation defining the term R_2. We write "Due to the assumptions, the term $G_r(y) + (p_r/q_r)G_k(w)y$ is increasing in $y \in [t_r, x_r]$." However, our assumptions do not ensure that the interval $[t_r, x_r]$ is necessarily contained in the set $\mathsf{A}'_{r,w} = \{x_r \colon \exists (z_1, \ldots, z_k) \in \mathsf{A},\ x_r = z_r,\ x_k = z_k\}$. Hence, the condition at Equation (5.3) cannot readily be applied.

In Subsection 2.1 we give a counterexample which demonstrates that the complication described above can indeed lead to a scoring function satisfying the conditions of Theorem 5.2(ii) in Fissler and Ziegel (2016a) which is not consistent. In Subsection 2.2 we introduce a condition on the action domain which, in combination with the conditions in Theorem 5.2(ii), ensures (strict) consistency of the scoring function. We end with some remarks as to when this additional condition is satisfied.

2.1 Counterexample

Using the same notation as in Fissler and Ziegel (2016a), we confine ourselves to presenting a counterexample for $k = 2$ and $\alpha \in (0, 1/2)$ (counterexamples for $\alpha \in [1/2, 1)$ can be constructed in a similar manner). Consider the convex action domain $\mathsf{A} = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 \geq 0,\ x_2 \leq |x_1|\}$. Let $\mathcal{G}_2 \colon \mathbb{R} \to \mathbb{R}$ be a strictly convex twice differentiable function and define $G_1 \colon [0, \infty) \to \mathbb{R}$ via

$$G_1(s) = \mathcal{G}_2(-s)/\alpha, \quad s \geq 0\,.$$

Observe that the condition at (5.3) in Fissler and Ziegel (2016a) holds since for $(x_1, x_2) \in$ A

$$G_1'(x_1) + \mathcal{G}_2'(x_2)/\alpha = \big(\mathcal{G}_2'(x_2) - \mathcal{G}_2'(-x_1)\big)/\alpha \begin{cases} = 0, & \text{if } x_2 = -x_1, \\ > 0, & \text{if } x_2 > -x_1, \end{cases}$$

due to the fact that $\mathcal{G}_2$ is strictly convex. Consider the specific choice $\mathcal{G}_2 = \exp$ and $a \equiv 0$. This results in the score

$$\begin{aligned} S(x_1, x_2, y) &= (\mathbb{1}\{y \le x_1\} - \alpha)\exp(-x_1)/\alpha - \mathbb{1}\{y \le x_1\}\exp(-y)/\alpha \\ &\quad + \exp(x_2)\big(x_2 + (\mathbb{1}\{y \le x_1\} - \alpha)x_1/\alpha - \mathbb{1}\{y \le x_1\}y/\alpha\big) - \exp(x_2)\,. \end{aligned}$$

To demonstrate that the score fails to be consistent for the pair $T = (\mathrm{VaR}_\alpha, \mathrm{ES}_\alpha)$ with $\alpha = 0.05$ consider the following distributions: First, F a point-distribution in 0. Then $T(F) = (0, 0)$ and we obtain $\bar{S}(T(F), F) = S(0, 0, 0) = -2$ and, for example, $S(2, -1.8, 0) \approx -11.61$. Second, F a normal distribution with mean $\mu = 0.2$ and standard deviation $\sigma = 0.1$. Then $T(F) \approx (0.0355, -0.0063)$. A numerical integration yields that $\bar{S}(T(F), F) \approx -5.36$ and, for example, $\bar{S}(2, -1.8, F) \approx -8.76$.

2.2 A sufficient condition for Theorem 5.2(ii)

Proposition 2. *Let the conditions of Theorem 5.2 in Fissler and Ziegel (2016a)(ii) prevail. Moreover, assume that for any $x \in \mathsf{A}$ and any $t \in T(\mathcal{F})$, there exists a finite sequence $(z^{(n)})_{n=0,\dots,N} \subseteq \mathsf{A}$ such that*

1. *$z^{(0)} = x$;*
2. *$t_r, z_r^{(N)} \in \mathsf{A}'_{r,\min\{t_k, z_k^{(N)}\}}$ for all $r \in \{1, \dots, k-1\}$;*
3. *for all $n \in \{1, \dots, N\}$ there is an index $r \in \{1, \dots, k\}$ such that*
   a) *$t_r \le z_r^{(n)} \le z_r^{(n-1)}$ or $z_r^{(n-1)} \le z_r^{(n)} \le t_r$; and*
   b) *$z_m^{(n)} = z_m^{(n-1)}$ for all $m \in \{1, \dots, k\} \setminus \{r\}$;*
4. *if $x_k < t_k$ and $z_k^{(n)} \ne z_k^{(n-1)}$, then $z_k^{(n-1)} < z_k^{(n)} \le -B(z^{(n)}, t) = -B(z^{(n-1)}, t)$,*

where

$$B(x, t) = -t_k + \sum_{m=1}^{k-1} \frac{p_m}{q_m}(t_m - x_m)(q_m - \mathbb{1}\{t_m < x_m\}). \tag{3}$$

Then the scoring function S defined at (5.2) is $\mathcal{F}$-consistent for T. If additionally, the distributions in $\mathcal{F}$ have unique q_m-quantiles, $m \in \{1, \dots, k-1\}$, $\mathcal{G}_k$ is strictly convex and the functions given at (5.3) are strictly increasing, then S is strictly $\mathcal{F}$-consistent for T.

Proof. Let $F \in \mathcal{F}$, $t = T(F)$ and $x \in \mathsf{A}$. Let $(z^{(n)})_{n=0,\dots,N} \subseteq \mathsf{A}$ be a sequence that satisfies the above conditions. We will show that for all $n \in \{1, \dots, N\}$, we have $\bar{S}(z^{(n-1)}, F) - \bar{S}(z^{(n)}, F) \ge 0$ and $\bar{S}(z^{(N)}, F) - \bar{S}(t, F) \ge 0$ with one inequality being strict under the conditions for strict $\mathcal{F}$-consistency.

In the decomposition $\bar{S}(x,F)-\bar{S}(t,F)=R_1+R_2$ on top of page 1702, we were not completely precise if F is not continuous. For $a,b\in\mathbb{R}$, we define

$$I(a,b)=\begin{cases}(a,b], & \text{if } a\le b\\ (b,a], & \text{if } a\ge b,\end{cases}\qquad \bar{I}(a,b)=\begin{cases}[a,b], & \text{if } a\le b\\ [b,a], & \text{if } a\ge b.\end{cases}$$

For any $z,z'\in\mathsf{A}$ and any $w\in\mathsf{A}'_k$ it holds that $\bar{S}(z',F)-\bar{S}(z,F)=R_1+R_2$ with

$$\begin{aligned}R_1&=\sum_{r=1}^{k-1}(F(z'_r)-q_r)\left(G_r(z'_r)+\frac{p_r}{q_r}G_k(w)z'_r\right)-(F(z_r)-q_r)\left(G_r(z_r)+\frac{p_r}{q_r}G_k(w)z_r\right)\\&\quad-\operatorname{sgn}(z'_r-z_r)\int_{I(z'_r,z_r)}\left(G_r(y)+\frac{p_r}{q_r}G_k(w)y\right)\mathrm{d}F(y).\\R_2&=-\mathcal{G}_k(z'_k)+\mathcal{G}_k(z_k)+G_k(w)(z'_k-z_k)+(G_k(z'_k)-G_k(w))(z'_k+C(z',F))\\&\quad-(G_k(z_k)-G_k(w))(z_k+C(z,F))\end{aligned}$$

with

$$C(z,F)=\sum_{m=1}^{k-1}\frac{p_m}{q_m}\left(\int_{(-\infty,z_m]}(z_m-y)\mathrm{d}F(y)-q_mz_m\right).$$

If $t_r\le z_r\le z'_r$, and $y\mapsto G_r(y)+\frac{p_r}{q_r}G_k(w)y$ is increasing on $[z_r,z'_r]$ then the rth summand of R_1 is bounded below by

$$\begin{aligned}&(F(z'_r)-q_r)\left(G_r(z'_r)+\frac{p_r}{q_r}G_k(w)z'_r\right)-(F(z_r)-q_r)\left(G_r(z_r)+\frac{p_r}{q_r}G_k(w)z_r\right)\\&\qquad-(F(z'_r)-F(z_r))\left(G_r(z'_r)+\frac{p_r}{q_r}G_k(w)z'_r\right)\\&\quad=(F(z_r)-q_r)\left(G_r(z'_r)+\frac{p_r}{q_r}G_k(w)z'_r-G_r(z_r)-\frac{p_r}{q_r}G_k(w)z_r\right)\ge 0.\end{aligned}$$

If $z'_r\le z_r\le t_r$, and $y\mapsto G_r(y)+\frac{p_r}{q_r}G_k(w)y$ is increasing on $[z'_r,z_r]$ then the rth summand of R_1 can be rewritten as

$$\begin{aligned}&(F(z'_r)-q_r)\left(G_r(z'_r)+\frac{p_r}{q_r}G_k(w)z'_r\right)-(F(z_r)-q_r)\left(G_r(z_r)+\frac{p_r}{q_r}G_k(w)z_r\right)\\&\quad+\int_{[z'_r,z_r)}\left(G_r(y)+\frac{p_r}{q_r}G_k(w)y\right)\mathrm{d}F(y)-(F(z'_r)-F(z'_r-))\left(G_r(z'_r)+\frac{p_r}{q_r}G_k(w)z'_r\right)\\&\quad+(F(z_r)-F(z_r-))\left(G_r(z_r)+\frac{p_r}{q_r}G_k(w)z_r\right)\\&\ge(F(z'_r-)-q_r)\left(G_r(z'_r)+\frac{p_r}{q_r}G_k(w)z'_r\right)-(F(z_r-)-q_r)\left(G_r(z_r)+\frac{p_r}{q_r}G_k(w)z_r\right)\\&\quad+(F(z_r-)-F(z'_r-))\left(G_r(z'_r)+\frac{p_r}{q_r}G_k(w)z'_r\right)\\&=(F(z_r-)-q_r)\left(G_r(z'_r)+\frac{p_r}{q_r}G_k(w)z'_r-G_r(z_r)-\frac{p_r}{q_r}G_k(w)z_r\right)\ge 0.\end{aligned}$$

In case that $z'_r < z_r = t_r$ or $z'_r > z_r = t_r$, if $y \mapsto G_r(y) + \frac{p_r}{q_r} G_k(w) y$ is strictly increasing on $\bar{I}(z'_r, t_r)$, and F has a unique q_r-quantile, then even $R_1 > 0$. Indeed, if F does not put any mass on the open interval $\mathrm{int}(I(z'_r, t_r))$, then the last inequality is strict in both cases. Otherwise, the first inequality is strict because $y \mapsto G_r(y) + \frac{p_r}{q_r} G_k(w) y$ is strictly increasing. In summary, setting $w = \min\{z'_k, z_k\}$, exploiting condition 2 and the convexity of A, this implies that both $R_1 \geq 0$ for $z' = z^{(n-1)}$, $z = z^{(n)}$, $n \in \{1, \ldots, N\}$, and also for $z' = z^{(N)}$, $z = t$. For the latter case, we even have $R_1 > 0$ under the conditions for strict $\mathcal{F}$-consistency and if $(z_m^{(N)})_{m=1,\ldots,k-1} \neq (t_m)_{m=1,\ldots,k-1}$.

For the discussion of R_2, one can use the identity $-t_k = \sum_{m=1}^{k-1} (p_m/q_m) (\int_{(-\infty, t_m]} (t_m - y) \mathrm{d}F(y) - q_m t_m)$ to check that for any $z \in \mathsf{A}$

$$- t_k \leq C(z, F) \leq B(z, t), \tag{4}$$

where both inequalities are equalities for $z = t$. Let $z' = z^{(N)}$, $z = t$ and again $w = \min\{z'_k, z_k\}$. Then due to (4) and the monotonicity of G_k the term R_2 is bounded below by

$$-\mathcal{G}_k(z'_k) + \mathcal{G}_k(t_k) + G_k(w)(z'_k - t_k) + (G_k(z'_k) - G_k(w))(z'_k - t_k) \geq 0$$

invoking the convexity of $\mathcal{G}_k$. The inequality becomes strict if $\mathcal{G}_k$ is strictly convex and $z'_k \neq t_k$. Now, let $z' = z^{(n-1)}$ and $z = z^{(n)}$ for $n \in \{1, \ldots, N\}$. If $t_k \leq z_k \leq z'_k$, then we obtain

$$\begin{aligned}
R_2 &\geq -\mathcal{G}_k(z'_k) + \mathcal{G}_k(z_k) + G_k(z'_k) z'_k - G_k(z_k) z_k + (G_k(z'_k) - G_k(z_k))(-t_k) \\
&= -\mathcal{G}_k(z'_k) + \mathcal{G}_k(z_k) + G_k(z'_k)(z'_k - t_k) - G_k(z_k)(z_k - t_k) \\
&\geq -\mathcal{G}_k(z'_k) + \mathcal{G}_k(z_k) + G_k(z'_k)(z'_k - t_k) - G_k(z'_k)(z_k - t_k) \geq 0,
\end{aligned}$$

where the penultimate inequality is due to the fact that G_k is increasing and the last inequality follows due to the convexity of $\mathcal{G}_k$. The last inequality is strict if $\mathcal{G}_k$ is strictly convex and if $z_k \neq z'_k$. If $z'_k \leq z_k \leq t_k$ and $z_k \leq -B(z, t) = -B(z', t)$, then, using (4)

$$\begin{aligned}
R_2 &= -\mathcal{G}_k(z'_k) + \mathcal{G}_k(z_k) + G_k(z'_k) z'_k - G_k(z_k) z_k + (G_k(z'_k) - G_k(z_k)) C(z', F) \\
&\geq -\mathcal{G}_k(z'_k) + \mathcal{G}_k(z_k) + G_k(z'_k) z'_k - G_k(z_k) z_k + (G_k(z'_k) - G_k(z_k))(-z_k) \geq 0
\end{aligned}$$

by convexity of $\mathcal{G}_k$. Again, the inequality is strict if $\mathcal{G}_k$ is strictly convex and if $z_k \neq z'_k$. In summary, $R_2 \geq 0$ for $z' = z^{(n-1)}$, $z = z^{(n)}$, $n \in \{1, \ldots, N\}$, and also for $z' = z^{(N)}$, $z = t$. For the latter case, we even have $R_2 > 0$ if $\mathcal{G}_k$ is strictly convex and if $t_k \neq z_k^{(N)}$. □

We would like to remark that the additional condition stated in Proposition 2 holds in most practically relevant cases of action domains A. In fact, in the following situations, one can set $N = 0$ meaning that the original proof in Fissler and Ziegel (2016a) is applicable: $\mathsf{A} = \mathbb{R}^k$; $\mathsf{A} = \mathsf{A}_0$, where A_0 is the maximal sensible action domain for the functional T defined in Theorem 5.2 in Fissler and Ziegel (2016a) for $q_1 < \cdots < q_{k-1}$.[1]

[1] With 'maximal sensible' we mean that $T(\mathcal{F}) \subseteq \mathsf{A}_0$ for any class of distributions $\mathcal{F}$.

It is given by

$$\mathsf{A}_0 := \Big\{x \in \mathbb{R}^k \colon x_1 \le \cdots \le x_{k-1},\ x_k \le \sum_{m=1}^{k-1} p_m x_m\Big\}.$$

In particular, this construction retrieves the result of Corollary 5.5 in Fissler and Ziegel (2016a), considering the maximal action domain $\mathsf{A}_0 = \{x \in \mathbb{R}^2 \colon x_1 \ge x_2\}$ for the functional $T = (\mathrm{VaR}_\alpha, \mathrm{ES}_\alpha)$. Also, $\mathsf{A} = \mathsf{A}_0^+ := \mathsf{A}_0 \cap \mathbb{R}^k_+$ and $\mathsf{A} = \mathsf{A}_0^- := \mathsf{A}_0 \cap \mathbb{R}^k_-$ satisfy the condition. The action domain considered in Theorem C.3 in Nolde and Ziegel (2017), corresponding to $\mathsf{A} = \mathbb{R} \times (-\infty, 0)$ in our sign convention, also satisfies the condition given in Proposition 2.

For the action domain $\mathsf{A} = \{x \in \mathbb{R}^2 \colon x_2 \le x_1 \le x_2 + c\}$ of Proposition 4.10 in Fissler and Ziegel (2017) one generally needs $N > 0$. However, the existence and construction of the sequence $(z^{(n)})_{n=0,\dots,N}$ is obvious.

Acerbi and Szekely (2014) introduced a family of scoring functions S^W, $W \in \mathbb{R}$, with corresponding action domains $\mathsf{A}^W = \{x \in \mathbb{R}^2 \colon x_2 > W x_1\}$; see page 1697 in Fissler and Ziegel (2016a) for a discussion. The results are as follows: For $W > 1$ the additional condition of Proposition 2 is satisfied (note that the potentially problematic point $(0,0)$ is not in A^W then.) For $W = 1$, the condition is empty since $\mathsf{A}^1 \cap T(\mathcal{F}) \subseteq \mathsf{A}^1 \cap \mathsf{A}_0 = \emptyset$. For $W \in (0,1)$ the condition fails to be satisfied. For $W = 0$ the condition holds. For $W \in [(\alpha-1)/\alpha, 0)$ the condition fails to be satisfied. For $W < (\alpha-1)/\alpha$ the condition holds.

3 Remarks on Propositions 4.2 and 4.4

In Fissler and Ziegel (2016a) Proposition 4.2(i) we claimed that $g_m > 0$ and in Proposition 4.4(i) we claimed that the function $h(x)_{r,l=1,\dots,k}$ is positive definite for all $x \in \mathrm{int}(\mathsf{A})$. In the proof of Theorem 5.2(iii), we made similar claims about the function g_k and the derivatives of the functions appearing at (5.3). It turns out that we were slightly imprecise with these claims. Indeed, these functions are all non-negative (positive semi-definite). Moreover, we can show that the functions g_m in Proposition 4.2(i) (as well as the functions appearing in the proof of Theorem 5.2(iii)) are strictly positive almost everywhere.

We will first demonstrate that the function h appearing in Proposition 4.4(i) is positive semi-definite. Assume there is some $t \in \mathrm{int}(\mathsf{A})$ and some $v \in \mathbb{S}^{k-1}$ such that $v^\top h(t) v < 0$. Recall that due to the assumptions and Theorem 3.2, the function h is continuous. Hence, there is an open neighbourhood $t \in U \subseteq \mathrm{int}(\mathsf{A})$ such that $v^\top h(x) v < 0$ for all $x \in U$. Invoking the surjectivity of T, let $F \in \mathcal{F}$ such that $T(F) = t$. Then there is an $\varepsilon > 0$ such that – using the notation on top of page 1701 in Fissler and Ziegel (2016a) –

$$\frac{\mathrm{d}}{\mathrm{d}s}\bar{S}(t + sv, F) = \bar{q}(F) s v^\top h(t+sv) v \begin{cases} > 0, & \text{for } s \in (-\varepsilon, 0), \\ < 0, & \text{for } s \in (0, \varepsilon). \end{cases}$$

This is a contradiction to the $\mathcal{F}$-consistency of S.

Now, we will show that the functions g_m appearing in Proposition 4.2(i) are positive almost everywhere, the arguments for the remaining functions being similar. Let $m \in \{1, \ldots, k\}$. The argument at the top of page 1701 in Fissler and Ziegel (2016a) show that for all $x \in \mathrm{int}(\mathsf{A}'_m)$ there is an $\varepsilon > 0$ such that $g_m > 0$ on $(x-\varepsilon, x+\varepsilon) \setminus \{x\}$. Due to a compactness argument, g_m is positive on any compact set almost everywhere. Since $\mathrm{int}(\mathsf{A}'_m)$ is σ-compact, $g_m > 0$ almost everywhere. The continuity of g_m also implies that $g_m \geq 0$ everywhere.

References

C. Acerbi and B. Szekely. Backtesting expected shortfall. *Risk Magazine*, 27:76–81, 2014.

J. R. Brehmer. Elicitability and its application in risk management. Master's thesis, University of Mannheim, 2017. URL http://arxiv.org/abs/1707.09604.

T. Fissler and J. F. Ziegel. Higher order elicitability and Osband's principle. *Annals of Statistics*, 44:1680–1707, 2016a.

T. Fissler and J. F. Ziegel. Supplement to "Higher order elicitability and Osband's principle". *Annals of Statistics*, 44:1680–1707, 2016b.

T. Fissler and J. F. Ziegel. Order-sensitivity and equivariance of scoring functions. *Prepint*, 2017. URL https://arxiv.org/abs/1711.09628v1.

K. Königsberger. *Analysis 2*. Springer-Verlag, Berlin Heidelberg New York, 5th edition, 2004.

N. Nolde and J. F. Ziegel. Elicitability and backtesting: Perspectives for banking regulation. *Annals of Applied Statistics*, 11:1833–1874, 2017. Paper with discussion and rejoinder.